A NOTE ON THE ARITHMETIC OF RESIDUAL AUTOMORPHIC REPRESENTATIONS OF REDUCTIVE GROUPS

HARALD GROBNER

Abstract. We prove an arithmeticity result for a class of cohomological residual automorphic representations of a general reductive group G. More precisely, we show that this class of residual representations is stable under the action of Aut($\mathbb{C}$). This complements numerous results on the stability of cohomological cuspidal automorphic representations due by several people. We conclude by showing that the rationality field of such a cohomological residual automorphic representation is a number field.

Contents

Introduction

Let G be a connected reductive group over a number field F. Let $\sigma \in$Aut($\mathbb{C}$) and let Π be an automorphic representation of $G(\mathbb{A})$. Then, Π splits as $\Pi = \Pi_\infty \otimes \Pi_f$ and we may define a new, "σ-twisted" representation ${}^\sigma\Pi_f := \Pi_f \otimes_\sigma \mathbb{C}$ of $G(\mathbb{A}_f)$. A priori, it is by no means clear, if ${}^\sigma\Pi_f$ appears again as the finite part of an automorphic representation of $G(\mathbb{A})$. However, for particular choices of the group G and the automorphic representation Π, there are numerous results in the related literature.

In any case, the key-technique, underlying all of these aforementioned results, is to assume that the automorphic representation considered is of *cohomological type*, i.e., that it contributes non-trivially to the relative Lie algebra cohomology of the space of automorphic forms $\mathcal{A}(G)$, twisted by an appropriate finite-dimensional coefficient system.

In order to render the above more precise, let E_μ be a finite–dimensional, irreducible algebraic representation of $G_\infty = R_{F/\mathbb{Q}}(G)(\mathbb{R})$ on a complex vector space. It is given by its highest weight $\mu = (\mu_\iota)_{\iota\in I_\infty}$. Here, $R_{F/\mathbb{Q}}(G)$ denotes restriction of scalars and I_∞ stands for the set of field embeddings $F \hookrightarrow \mathbb{C}$. Similar to the case of an automorphic representation, on the level of G/F, we may form a new representation ${}^\sigma E_\mu := E_\mu \otimes_\sigma \mathbb{C}$. On the other hand, we let $E_{{}^\sigma\mu}$ be the representation of G_∞, which has highest weight ${}^\sigma\mu := (\mu_{\sigma^{-1}\iota})_{\iota\in I_\infty}$. If G is an inner form of a split algebraic group (which we assume it is from now on), then, as a representation of $G(F)$, ${}^\sigma E_\mu$ is isomorphic to $E_{{}^\sigma\mu}$.

2010 *Mathematics Subject Classification.* Primary: 11F70, 11F75, 22E47; Secondary: 11F67.

Key words and phrases. Automorphic cohomology, arithmetic of automorphic forms, residual automorphic representation, residual spectrum.

H.G. is supported by the Austrian Science Fund (FWF), project number P 25974-N25.

As a consequence, if G is inner, we obtain a commutative diagram of $G(\mathbb{A}_f)$-module isomorphisms (with σ-linear columns),

$$\begin{tikzcd} H^q(S,\mathcal{E}_\mu) \arrow[d, "\sigma^*"] & H^q(\mathfrak{m}_G,K,\mathcal{A}(G)\otimes E_\mu) \arrow[l, "\cong"'] \arrow[d, "\sigma^*"] \\ H^q(S,{}^\sigma\mathcal{E}_\mu) \arrow[r, "\cong"] & H^q(\mathfrak{m}_G,K,\mathcal{A}(G)\otimes E_{^\sigma\mu}) \end{tikzcd} .$$

Using the key result of our recent paper [13], which shows that below a certain, sharp degree of cohomology q_{max}, the space of automorphic cohomology $H^q(\mathfrak{m}_G,K,\mathcal{A}(G)\otimes E_\mu)$ is isomorphic to the $(\mathfrak{m}_G,K)$-cohomology of the space of *square–integrable* automorphic forms, we can prove the main result of this short paper: We show that – under certain constraints – residuality is an *arithmetic property* of automorphic representations, or – in other words – that the action of $\mathrm{Aut}(\mathbb{C})$ preserves the class of these residual automorphic representations:

Theorem. *Let G be a connected reductive linear algebraic group over a number field F, which is an inner form of a split algebraic group, and let E_μ be a finite-dimensional algebraic representation of G_∞. Let Π be a residual automorphic representation of $G(\mathbb{A})$, with non–zero $(\mathfrak{m}_G,K)$-cohomology below degree q_{max}, which does not appear in interior cohomology. Then for all $\sigma\in\mathrm{Aut}(\mathbb{C})$, there exists a residual automorphic representation Ξ of $G(\mathbb{A})$, $(\mathfrak{m}_G,K)$-cohomological with respect to $E_{^\sigma\mu}$, such that $\Xi_f\cong{}^\sigma\Pi_f$, and which appears in the complement of interior cohomology.*

We would like to remark that in the case of $G=GL_n/F$, the general linear group over any number field F, our approach leads to a result (cf. Thm. 3.3) which complements Clozel [7], Thm. 3.13, Franke [8], Thm. 20 and Franke–Schwermer [9], Thm. 4.3. See also Grobner–Raghuram [14] Prop. 7.21 and Thm. 7.23.

Additionally, we also prove an algebraicity theorem on the field of rationality,

$$\mathbb{Q}(\Pi_f):=\{z\in\mathbb{C}|\sigma(z)=z\quad\forall\sigma \text{ satisfying } \Pi_f\cong{}^\sigma\Pi_f\},$$

Corollary. *Let Π be a cohomological residual automorphic representation as in the statement of the above theorem. Then $\mathbb{Q}(\Pi_f)$ is a number field.*

We would like to remark that the above corollary generalizes a result which is well–known for cohomological cuspidal automorphic representations (cf., e.g., Shimura [19], Harder [15] p.80, Waldspurger [20], Cor. I.8.3 and first line of p. 153, Clozel [7] and Grobner–Raghuram [14], Thm. 8.1.).

1. Some basics

1.1. **Number fields.** We let F be an algebraic number field. Its set of places is denoted $V=V_\infty\cup V_f$, where V_∞ stands for the set of archimedean places and V_f is the set of non–archimedean places. We shall use the letter I_∞ for the set of field embeddings $\iota:F\hookrightarrow\mathbb{C}$. The ring of adeles of F is denoted $\mathbb{A}$, the subspace of finite adeles is denoted $\mathbb{A}_f$.

1.2. **Algebraic groups.** In this paper, G is a connected, reductive linear algebraic group over a number field F. We assume to have fixed a minimal parabolic F-subgroup P_0 with Levi decomposition $P_0=L_0N_0$ and let A_0 be the maximal F-split torus in the center Z_{L_0} of L_0. This choice defines the standard parabolic F-subgroups P with Levi decomposition $P=L_PN_P$, where $L_P\supseteq L_0$ and $N_P\subseteq N_0$. We let A_P be the maximal F-split torus in the center Z_{L_P} of L_P, satisfying $A_P\subseteq A_0$ and denote by $\mathfrak{a}_P$ (resp. $\mathfrak{a}_{P,\mathbb{C}}$) its Lie algebra (resp. its complexification $\mathfrak{a}_{P,\mathbb{C}}=\mathfrak{a}_P\otimes\mathbb{C}$). We write $H_P:L_P(\mathbb{A})\to\mathfrak{a}_{P,\mathbb{C}}$ for the standard Harish-Chandra height function. The group $L_P(\mathbb{A})^1:=\ker H_P$, admits a direct complement $A_P^{\mathbb{R}}\cong\mathbb{R}_+^{\dim\mathfrak{a}_P}$ in $L_P(\mathbb{A})$ whose Lie algebra is isomorphic to $\mathfrak{a}_P$. With respect to a maximal compact subgroup $K_{\mathbb{A}}\subseteq G(\mathbb{A})$ in good position, cf. [17] I.1.4, we obtain an extension $H_P:G(\mathbb{A})\to\mathfrak{a}_{P,\mathbb{C}}$ to all of $G(\mathbb{A})$.

1.3. **Lie groups and Lie algebras.** We put $G_\infty := R_{F/\mathbb{Q}}(G)(\mathbb{R})$, where $R_{F/\mathbb{Q}}$ denotes the restriction of scalars from F to $\mathbb{Q}$. We let $\mathfrak{m}_G := \mathfrak{g}_\infty/\mathfrak{a}_G = Lie(G(\mathbb{A})^1 \cap G_\infty)$ and denote by $\mathfrak{Z}(\mathfrak{g})$ the center of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ of $\mathfrak{g}_{\infty,\mathbb{C}}$.

Let $K_\infty \subset G_\infty$ be a maximal compact subgroup (the archimedean factor of the maximal compact subgroup $K_\mathbb{A}$ of $G(\mathbb{A})$ in good position) and set once and for all $K := K_\infty^\circ$. (Observe that this choice of a compact subgroup of G_∞ is in accordance with Franke [8], p. 184.) Please refer to Borel–Wallach [6], I, for the basic facts and notations concerning $(\mathfrak{m}_G, K)$-cohomology.

Furthermore, we let $\mathfrak{h}_\infty$ be a Cartan subalgebra of $\mathfrak{g}_\infty$ that contains $\mathfrak{a}_{0,\infty}$.

1.4. **Algebraic representations.** In this paper, we always let E_μ be a finite–dimensional irreducible algebraic representation of G_∞ on a complex vector space, given by its highest weight $\mu = (\mu_\iota)_{\iota \in I_\infty} \in \check{\mathfrak{h}}_\infty$. We will assume that $A_G^\mathbb{R}$ (and so $\mathfrak{a}_G$) acts trivially on E_μ. There is therefore no difference between the $(\mathfrak{g}_\infty, K)$–module and the $(\mathfrak{m}_G, K)$-module defined by E_μ.

Take a field automorphism $\sigma \in$Aut$(\mathbb{C})$. Then we may form the new finite–dimensional irreducible algebraic representation of G_∞, denoted $E_{^\sigma\mu}$, which is given by its highest weight

$$^\sigma\mu := (\mu_{\sigma^{-1}\iota}).$$

Clearly this amounts in a permutation of the components μ_ι of μ. On the level of the algebraic group G/F, there is another representation, denoted $^\sigma E_\mu := E_\mu \otimes_\sigma \mathbb{C}$.

1.5. **Cohomology of locally symmetric spaces.** We let

$$S := G(F)A_G^\mathbb{R}\backslash G(\mathbb{A})/K$$

be the projective limit of the "locally symmetric spaces" attached to G. Starting from the algebraic representation E_μ, one obtains a sheaf $\mathcal{E}_\mu$ on S by letting $\mathcal{E}_\mu$ be the sheaf with espace étalé $G(\mathbb{A})/A_G^\mathbb{R} K \times_{G(F)} E_\mu$ with the discrete topology on E_μ. In the same way, we obtain a sheaf $^\sigma\mathcal{E}_\mu$ from $^\sigma E_\mu$.

Proposition 1.1. *Let G be a connected reductive linear algebraic group over a number field F and let E_μ be a finite-dimensional algebraic representation of G_∞. Then for all $\sigma \in$ Aut$(\mathbb{C})$ and all degrees q of cohomology, there is a $G(\mathbb{A}_f)$-equivariant, σ-linear isomorphism*

$$H^q(S, \mathcal{E}_\mu) \to H^q(S, {^\sigma\mathcal{E}_\mu}).$$

If G is an inner form of a split algebraic group, then, as a representation of $G(F)$, $^\sigma E_\mu$ is isomorphic to $E_{^\sigma\mu}$. This may fail, if G is an outer form.

Proof. The first assertion is obvious by the definition of Betti cohomology $H^q(S, \bullet)$. For the other assertions see [10], Sect. 2. □

2. Automorphic cohomology

2.1. **Residual automorphic forms.** Our notion of an *automorphic form* $f : G(\mathbb{A}) \to \mathbb{C}$ and of an *automorphic representation* of $G(\mathbb{A})$ is the one of Borel-Jacquet [3], 4.2 and 4.6. Let $\mathcal{A}(G)$ be the space of all automorphic forms $f : G(\mathbb{A}) \to \mathbb{C}$, which are constant on the real Lie subgroup $A_G^\mathbb{R}$. By its very definition, every automorphic form is annihilated by some power of an ideal $\mathcal{J} \lhd \mathfrak{Z}(\mathfrak{g})$ of finite codimension. We fix such an ideal $\mathcal{J}$, once and for all: As we will only be interested in cohomological automorphic forms, we take $\mathcal{J}$ to be the ideal which annihilates the contragredient representation $E_\mu^\vee$ of E_μ, cf. Sect.1.3, and denote by

$$\mathcal{A}_\mathcal{J}(G) \subset \mathcal{A}(G)$$

the space consisting of those automorphic forms which are annihilated by some power of $\mathcal{J}$. Clearly, $\mathcal{A}_\mathcal{J}(G)$ carries a commuting $(\mathfrak{g}_\infty, K_\infty)$ and $G(\mathbb{A}_f)$-action and hence defines a $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-module.

The $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-submodule of all square integrable automorphic forms in $\mathcal{A}_\mathcal{J}(G)$ is denoted $\mathcal{A}_{dis,\mathcal{J}}(G)$. An irreducible subquotient of $\mathcal{A}_{dis,\mathcal{J}}(G)$ will be called a *discrete series automorphic* representation, cf. Borel [2] 9.6. If $\omega : Z_G(F)\backslash Z_G(\mathbb{A}) \to \mathbb{C}^*$ is a continuous character of the center Z_G of G, we let $\mathcal{A}_{dis,\mathcal{J}}(G, \omega)$ be the space of square-integrable automorphic forms with central character ω.

We further recall that $\mathcal{A}_{dis,\mathcal{J}}(G,\omega)$ decomposes as a direct sum of automorphic representations Π

$$\mathcal{A}_{dis,\mathcal{J}}(G,\omega) \cong \bigoplus \Pi^{m(\Pi)} \tag{2.1}$$

with finite multiplicities $m(\Pi)$. One has $m(\Pi) = m_{cusp}(\Pi) + m_{res}(\Pi)$, where $m_{cusp}(\Pi)$ is the multiplicity of Π in the space of cuspidal automorphic forms in $\mathcal{A}_\mathcal{J}(G)$. If $m_{res}(\Pi) \neq 0$, we call Π a *a residual automorphic representation*. Clearly, it decomposes as $\Pi = \Pi_\infty \otimes \Pi_f$, where Π_∞ is an irreducible $(\mathfrak{m}_G, K)$-module and Π_f and irreducible $G(\mathbb{A}_f)$-module.

Apart from the action of automorphisms $\sigma \in$Aut$(\mathbb{C})$ on finite-dimensional algebraic representations E_μ, cf. Sect. 1.4, there is also an action of σ on the finite part Π_f of an automorphic representation: Given $\sigma \in$Aut$(\mathbb{C})$ and Π_f, we define ${}^\sigma\Pi_f$ to be the representation of $G(\mathbb{A}_f)$ on $\Pi_f \otimes_\sigma \mathbb{C}$ (analogous to ${}^\sigma E_\mu$). This is in accordance with Waldspurger, [20], I.1.

2.2. **Parabolic and cuspidal supports.** Let $\{P\}$ be the associate class of the parabolic F-subgroup P: It consists by definition of all parabolic F-subgroups $Q = L_Q N_Q$ of G for which L_Q and L_P are conjugate by an element in $G(F)$. We denote by $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ the space of all $f \in \mathcal{A}_\mathcal{J}(G)$ which are negligible along every parabolic F-subgroup $Q \notin \{P\}$. There is the following decomposition of $\mathcal{A}_\mathcal{J}(G)$ as a $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-module, cf. [4] Thm. 2.4 or [2] 10.3, first established by Langlands:

$$\mathcal{A}_\mathcal{J}(G) \cong \bigoplus_{\{P\}} \mathcal{A}_{\mathcal{J},\{P\}}(G). \tag{2.2}$$

The various summands $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ can be decomposed even further. To this end, recall from [9], 1.2, the notion of an *associate class* φ_P of cuspidal automorphic representations π of the Levi subgroups of the elements in the class $\{P\}$. Given φ_P, represented by a cuspidal representation π, a $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-submodule

$$\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$$

of $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ was defined in [9], 1.3 as follows: It is the span of all possible holomorphic values or residues of all Eisenstein series attached to $\tilde{\pi}$, evaluated at a certain point $\lambda = d\Lambda$ determined by π, together with all their derivatives. (This definition is independent of the choice of the representatives P and π, thanks to the functional equations satisfied by the Eisenstein series considered.) For details, we refer the reader to [9] 1.2-1.4.

The following refined decomposition as $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$-modules of the spaces $\mathcal{A}_{\mathcal{J},\{P\}}(G)$ of automorphic forms was obtained in Franke–Schwermer [9], Thm. 1.4:

$$\mathcal{A}_{\mathcal{J},\{P\}}(G) \cong \bigoplus_{\varphi_P} \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G).$$

2.3. **Automorphic cohomology.** We recall that the $G(\mathbb{A}_f)$-module

$$H^q(\mathfrak{m}_G, K, \mathcal{A}_\mathcal{J}(G) \otimes E_\mu)$$

is called the *automorphic cohomology* of G in degree q. From what we recalled in section 2.2 it inherits a direct sum decomposition

$$H^q(\mathfrak{m}_G, K, \mathcal{A}_\mathcal{J}(G) \otimes E_\mu) \cong \bigoplus_{\{P\}} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\}}(G) \otimes E_\mu) \cong \bigoplus_{\{P\}} \bigoplus_{\varphi_P} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu)$$

The summand $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{G\}}(G) \otimes E)$ attached to $\{G\}$ consists precisely of all cuspidal automorphic forms in $\mathcal{A}_\mathcal{J}(G)$ and will be denoted for short by $H^q_{cusp}(G, E_\mu)$

Apart from this, we will need the following result of Franke, [8] Thm. 18, which links automorphic cohomology with the sheaf cohomology of S:

Theorem 2.3. *There is an isomorphism of $G(\mathbb{A}_f)$-modules*
$$H^q(S,\mathcal{E}_\mu)\cong H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J}}(G)\otimes E_\mu).$$

We may therefore compare automorphic cohomology with the cohomology of S. Recall the Borel-Serre compactification of S, cf. [5], [18]: We will denote it by $\overline{S}$ and by $\partial\overline{S}$ its boundary. Let $H^q_!(S,\mathcal{E}_\mu)$ be the space of *interior cohomology*, i.e., the kernel of the natural restriction map,
$$res^q: H^q(S,\mathcal{E}_\mu)=H^q(\overline{S},\mathcal{E}_\mu)\to H^q(\partial\overline{S},\mathcal{E}_\mu).$$
It is well-known that $H^q_{cusp}(G,E_\mu)\subseteq H^q_!(S,\mathcal{E}_\mu)$: This is a direct consequence of the fact that the inclusion of compactly supported differential forms into fast decreasing differential forms defines an isomorphism in sheaf cohomology.

2.4. **A diagram.** Assume now that G is an inner form of a split reductive algebraic group. Then, for every $\sigma\in$Aut($\mathbb{C}$), we obtain the following commutative diagram of $G(\mathbb{A}_f)$-module isomorphisms (with σ-linear columns) by Prop. 1.1 and Thm. 2.3:

$$\begin{tikzcd}
H^q(S,\mathcal{E}_\mu) \arrow[d, "\sigma^*"] & H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J}}(G)\otimes E_\mu) \arrow[l, "\cong"'] \arrow[d, "\sigma^*"] \\
H^q(S,{}^\sigma\mathcal{E}_\mu) \arrow[r, "\cong"] & H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J}}(G)\otimes E_{^\sigma\mu})
\end{tikzcd}$$

Observe the position of the automorphism σ in the various coefficient systems: This is the reason for our assumption that G is an inner form, since otherwise ${}^\sigma E_\mu$ and $E_{^\sigma\mu}$ may differ as $G(F)$-modules, cf. Prop. 1.1. Moreover, observe that $H^q_!(S,\mathcal{E}_\mu)$ is stable by σ^*, i.e., $\sigma^*(H^q_!(S,\mathcal{E}_\mu))=H^q_!(S,{}^\sigma\mathcal{E}_\mu)$. This is due to the rationality of the map res^q, which we used in order to define interior cohomology.

2.5. **Certain bounds in cohomology.** Given an associate class $\{P\}$ of the parabolic F-subgroups, represented by P, we define (with $R=L_RN_R$ denoting a parabolic subgroup of G)
$$q_{\mathsf{max}}(P):=\min_{\substack{R\ maximal\\ R\supseteq P}}\left(\sum_{v\in V_\infty}\left\lceil\tfrac{1}{2}\dim_{\mathbb{R}}N_R(F_v)\right\rceil\right).$$
Moreover we let
$$q_{\mathsf{max}}:=\min_{P standard}q_{\mathsf{max}}(P).$$
This definition is motivated by our own results in [13]: The two numbers $q_{\mathsf{max}}(P)$ and q_{max} will serve as certain "upper bounds" for the degrees of automorphic cohomology. Indeed, in [13], we proved the following result:

Theorem 2.4. *Let G be a connected, reductive group over a number field F and let E_μ be an irreducible, finite-dimensional, algebraic representation of G_∞ on a complex vector space. Let $\{P\}$ be an associate class of parabolic F-subgroups of G and let φ_P be an associate class of cuspidal automorphic representations of $L_P(\mathbb{A})$. Let $L^2_{\mathcal{J},\{P\},\varphi_P}:=\mathcal{A}_{dis,\mathcal{J}}(G)\cap\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}$ be the space of square-integrable automorphic forms in $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}$. Then, the inclusion $L^2_{\mathcal{J},\{P\},\varphi_P}(G)\hookrightarrow\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$ induces an isomorphism of $G(\mathbb{A}_f)$-modules*
$$H^q(\mathfrak{m}_G,K,L^2_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E_\mu)\xrightarrow{\cong}H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)\otimes E_\mu),$$
in all degrees $q<q_{\mathsf{max}}(P)$. In particular, below q_{max}, automorphic cohomology $H^q(\mathfrak{m}_G,K,\mathcal{A}_{\mathcal{J}}(G)\otimes E_\mu)$ is identical to $H^q(\mathfrak{m}_G,K,\mathcal{A}_{dis,\mathcal{J}}(G)\otimes E_\mu)$.

The latter theorem will be the key-result for the proof of the main result of this article detailed in the following section.

3. The main results

3.1. **Arithmeticity of residual automorphic representations - The general case.**

Theorem 3.1. *Let G be a connected reductive linear algebraic group over a number field F, which is an inner form of a split algebraic group, and let E_μ be a finite-dimensional algebraic representation of G_∞. Let Π be a residual automorphic representation of $G(\mathbb{A})$, with non–zero $(\mathfrak{m}_G, K)$-cohomology below degree $q_{\max}$, which does not appear in $H_!(S, \mathcal{E}_\mu)$. Then for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists a residual automorphic representation Ξ of $G(\mathbb{A})$, $(\mathfrak{m}_G, K)$-cohomological with respect to $E_{\sigma\mu}$, such that $\Xi_f \cong {}^\sigma\Pi_f$, and which appears in the complement of $H_!(S, {}^\sigma\mathcal{E}_\mu)$.*

Proof. Let Π be as in the statement of the theorem. Let $q_0 < q_{\max}$ be a degree, where the $(\mathfrak{m}_G, K)$-cohomology of Π_∞ (with respect to E_μ) does not vanish. Clearly, $H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu) \cong H^{q_0}(\mathfrak{m}_G, K, \Pi_\infty \otimes E_\mu) \otimes \Pi_f$ as $G(\mathbb{A}_f)$-modules, whence we may fix an injection $\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$. By Thm. 2.4 (resp. [13] Cor. 17), we hence obtain an *injection*

$$\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu).$$

In other words, we have realized Π_f as a submodule of the space of automorphic cohomology in degree q_0. Chasing the image of Π_f in $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu)$ through our diagram, see Sect. 2.4, we finally end up with an irreducible subrepresentation $\sigma^*(\Pi_f)$ of $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\sigma\mu})$. This is the place, where the assumption that G is an inner form of a split algebraic group comes into play: As mentioned below our diagram in Sect. 2.4, otherwise, ${}^\sigma E_\mu$ (which defines the sheaf $\mathcal{E}_\mu$) may not be isomorphic to $E_{\sigma\mu}$ (which defines the coefficient module in $(\mathfrak{m}_G, K)$-cohomology) as a representation of $G(F)$.

By the definition of the σ-linear $G(\mathbb{A}_f)$-morphism $H^q(S, \mathcal{E}_\mu) \to H^q(S, {}^\sigma\mathcal{E}_\mu)$, this subrepresentation $\sigma^*(\Pi_f)$ is isomorphic to ${}^\sigma\Pi_f$. By our Thm. 2.4 (resp. our Thm. 18 in [13]),

$$H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{dis,\mathcal{J}}(G) \otimes E_{\sigma\mu}) \cong H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\sigma\mu}),$$

so, $\sigma^*(\Pi_f)$, which by the above is isomorphic to ${}^\sigma\Pi_f$, is contained as a subrepresentation of the former cohomology space $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{dis,\mathcal{J}}(G) \otimes E_{\sigma\mu})$. Now recall that the discrete automorphic spectrum $\mathcal{A}_{dis,\mathcal{J}}(G)$ decomposes as a direct sum, cf. 2.1, (at least, if one fixes a central character). Therefore, there is also a direct sum decomposition of the $G(\mathbb{A}_f)$-module $H^q(\mathfrak{m}_G, K, \mathcal{A}_{dis,\mathcal{J}}(G) \otimes E_{\sigma\mu})$. As a consequence, there exists a square-integrable automorphic representation Ξ of $G(\mathbb{A})$, with non-vanishing $(\mathfrak{m}_G, K)$-cohomology with respect to $E_{\sigma\mu}$, such that $\Xi_f \cong {}^\sigma\Pi_f$.

Since $H^{q_0}_!(S, \mathcal{E}_\mu)$ is preserved by σ^* and contains $H^{q_0}_{cusp}(G, E_\mu)$, this representation Ξ can be chosen to be residual, i.e., such that $m_{res}(\Xi) \neq 0$, and even outside of $H_!(S, {}^\sigma\mathcal{E}_\mu)$. This shows the claim. □

Remark 3.2. Let Π be a residual automorphic representation as in the statement of Thm. 3.1. Assume that Π has been obtained using the Langlands–Shahidi method, i.e., by determining the position of poles and zeros of certain automorphic L-functions, which are attached to those cuspidal automorphic representations π of Levi subgroups, which appear in the calculation of the constant term. Our main theorem above hence suggests that if such an L-function – $L(s, \pi, r)$, say – has a pole at the relevant point $s = s_0$ (giving rise to a pole of some Eisenstein series, whose residues span Π), then also $L(s, {}^\sigma\pi, r)$ should have a pole at this point. This may be of some use in the determination of poles of Eisenstein series, respectively for showing algebraicity results of residual automorphic L-functions.

3.2. **Arithmeticity of residual automorphic representations - The case of GL_n/F.**

In the case of the split general linear group $G = GL_n/F$, we obtain the following refinement [1] of Thm. 3.1. At the same time this result complements the results of Clozel [7], Thm. 3.13, Franke [8], Thm. 20 and Franke–Schwermer [9], Thm. 4.3. See also Grobner–Raghuram [14] Prop. 7.21 and Thm. 7.23.

[1]As I learned from L. Clozel, stronger results than the ones of Sect. 3.2 may be proved using the fact that a residual automorphic representation of $GL_n(\mathbb{A})$ is a Mœglin–Waldspurger quotient, cf. [16]. However, I decided to leave this section, in order to exemplify the technique of Thm. 3.1.

Theorem 3.3. *Let $G = GL_n$ be the split general linear group over F and let E_μ be a finite-dimensional algebraic representation of G_∞. Let Π be a residual automorphic representation of $GL_n(\mathbb{A})$, with parabolic support $\{P\}$ and with non–zero $(\mathfrak{m}_G, K)$-cohomology in degree $q_0 < q_{\mathsf{max}}(P)$. Then for all $\sigma \in \mathrm{Aut}(\mathbb{C})$, there exists a residual automorphic representation Ξ of $G(\mathbb{A})$, with parabolic support $\{P\}$, $(\mathfrak{m}_G, K)$-cohomological with respect to $E_{\sigma\mu}$ and such that $\Xi_f \cong {}^\sigma\Pi_f$. If P is maximal parabolic, then the condition $q_0 < q_{\mathsf{max}}(P)$ can be dropped, i.e., the result holds for all cohomological residual automorphic representations Π supported in the maximal parabolic subgroup $P = (GL_{n/2} \times GL_{n/2}) \cdot N$.*

Proof. Let Π be as in the statement of the theorem. Let $q_0 < q_{\mathsf{max}}(P)$ be a degree, where the $(\mathfrak{m}_G, K)$-cohomology of Π_∞ (with respect to E_μ) does not vanish. As, $H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu) \cong H^{q_0}(\mathfrak{m}_G, K, \Pi_\infty \otimes E_\mu) \otimes \Pi_f$ as $G(\mathbb{A}_f)$-modules, we may fix an injection $\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$. By Thm. 2.4, we hence obtain an injection

$$\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\}}(G) \otimes E_\mu).$$

We now use a result of Franke: In [8], Thm. 20, he proved that the decomposition of $\mathcal{A}_{\mathcal{J}}(G)$ along the parabolic supports $\{P\}$, cf. (2.2), is respected by the σ-linear isomorphism σ^*. As a consequence of our diagram, cf. Sect. 2.4, $\sigma^*(\Pi_f) \cong {}^\sigma\Pi_f$ appears not only in $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\sigma\mu})$, but even in $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\}}(G) \otimes E_{\sigma\mu})$. By our Thm. 2.4,

$$H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\}}(G) \otimes E_{\sigma\mu}) \cong H^{q_0}(\mathfrak{m}_G, K, \bigoplus_{\varphi_P} L^2_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_{\sigma\mu}),$$

whence $\sigma^*(\Pi_f) \cong {}^\sigma\Pi_f$ appears as an irreducible submodule of the latter cohomology space. Again we recall that the discrete automorphic spectrum $\mathcal{A}_{dis,\mathcal{J}}(G)$, which contains $\bigoplus_{\varphi_P} L^2_{\mathcal{J},\{P\},\varphi_P}(G)$, decomposes as a direct sum, cf. 2.1, (at least, if one fixes a central character). Hence, we find a square-integrable automorphic representation Ξ of $G(\mathbb{A})$, with parabolic support $\{P\}$, $(\mathfrak{m}_G, K)$-cohomological with respect to $E_{\sigma\mu}$ and such that $\Xi_f \cong {}^\sigma\Pi_f$. This representation must be residual by Multiplicity One for square-integrable automorphic representations of $GL_n(\mathbb{A})$: Indeed, Π being assumed to be residual implies that $P \neq G$ by Multiplicity One in the discrete automorphic spectrum. Therefore, the parabolic support of Ξ also defers from G. In particular, Ξ is residual itself (and we could avoid the assumption that Π_f does not appear in interior cohomology $H_!(S, \mathcal{E}_\mu)$). This shows the first claim.

We will now prove the last assertion, claiming that the condition $q_0 < q_{\mathsf{max}}(P)$ can be dropped in the case of maximal parabolic subgroups. For a moment, let Π be any cohomological residual automorphic representation of $GL_n(\mathbb{A})$ supported in a pair $(\{P\}, \varphi_P)$. In [12], Thm. 4.1, we showed that its cohomology $H^q(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$ always injects into $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E_\mu)$ in the lowest degree $q = q_{\mathsf{min}}$, where $H^q(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$ does not vanish identically. In particular, by our above considerations, resp. our diagram in Sect. 2.4, $\sigma^*(\Pi_f) \cong {}^\sigma\Pi_f$ appears as an irreducible submodule of $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},{}^\sigma\varphi_P}(G) \otimes E_{\sigma\mu})$ for an appropriate cuspidal support denoted ${}^\sigma\varphi_P$ here. We use [12] again: In Sect. 6B therein, we showed that the lowest degree of cohomology q_{min} is always smaller than a certain bound $q_{\mathsf{res}} = q_{\mathsf{res}}(\{P\}, \varphi_P)$ of degrees of cohomology. (A precise definition of this degree may be found in [13], Sect. 6.1 and [12], Sect. 3C.) It is not important to know how this degree is defined in details; rather one should recall the main result of our paper [13], Thm. 15, which says that for all $q < q_{\mathsf{res}}$,

$$H^q(\mathfrak{m}_G, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E) \xrightarrow{\ \cong\ } H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G) \otimes E),$$

where $\mathcal{A}^{(m)}_{\mathcal{J},\{P\},\varphi_P}(G)$ is the deepest step in Franke's filtration of the space $\mathcal{A}_{\mathcal{J},\{P\},\varphi_P}(G)$. Again, it is not important to know precisely, how this filtration is defined (All details may be found in [13]), Sect. 3); however, by our concrete calculations in [12], Sect. 6 (cf. Prop. 6.4), it is shown that

$$q_{\mathsf{res}} = q_{\mathsf{res}}(\{P\}, \varphi_P) = q_{\mathsf{res}}(\{P\}, {}^\sigma\varphi_P),$$

whence summarizing what we said above, $\sigma^*(\Pi_f) \cong {}^\sigma\Pi_f$ appears as an irreducible submodule of the cohomology of the deepest filtration step $H^q(\mathfrak{m}_G, K, \mathcal{A}^{(m)}_{\mathcal{J},\{P\},{}^\sigma\varphi_P}(G) \otimes E_{{}^\sigma\mu})$. We would like to point out that all of this still holds for all parabolic subgroups.

Let now P be maximal. Then it is shown in Grbac [11] Thm. 3.1, that the deepest step $\mathcal{A}^{(m)}_{\mathcal{J},\{P\},{}^\sigma\varphi_P}(G)$ in Franke's filtration is identical to $L^2_{\mathcal{J},\{P\},{}^\sigma\varphi_P}(G)$, i.e., every automorphic form in $\mathcal{A}^{(m)}_{\mathcal{J},\{P\},{}^\sigma\varphi_P}(G)$ is square-integrable. This is wrong, if P is not maximal. As a consequence, ${}^\sigma\Pi_f$ must be the finite part of a square-integrable automorphic representation Ξ. By Multiplicity One, Ξ must be residual. $\square$

Example 3.4. Let $G = GL_4$ over $F = \mathbb{Q}$. Then, $\mathfrak{m}_G = \mathfrak{sl}_4(\mathbb{R})$ and $K = SO(4)$. Let P be the standard maximal parabolic subgroup with Levi factor $L \cong GL_2 \times GL_2$. We take furthermore $E_\mu = \mathbb{C}$, the trivial representation and let $D(2)$ be the discrete series representation of $GL_2(\mathbb{R})$ of lowest non-negative $O(2)$-type 2. Then, there exists a cuspidal automorphic representation π of $GL_2(\mathbb{A})$, with $\pi_\infty \cong D(2)$ and $\Pi = |.|^{1/2}\pi \times |.|^{-1/2}\pi$ (Langlands quotient) is a residual automorphic representation of $GL_4(\mathbb{A})$ with non-trivial $(\mathfrak{sl}_4(\mathbb{R}), SO(4))$-cohomology with respect to $\mathbb{C}$ in degrees $q = 3, 6$. By Thm. 3.3, and a simple uniqueness–argument at the archimedean place, ${}^\sigma\Pi = \Pi_\infty \otimes {}^\sigma\Pi_f$ is a residual automorphic representation of $GL_4(\mathbb{A})$, for all $\sigma \in$Aut$(\mathbb{C})$. Indeed, the archimedean component must be Π_∞ again, since this is the only irreducible unitary representation of $GL_4(\mathbb{R})$, which has non-trivial cohomology in degree $q = 3, 6$. By construction it is the Langlands quotient $|.|^{1/2}D(2) \times |.|^{-1/2}D(2)$. See, e.g., [14], Sect. 5.4.

Remark 3.5. Let G' be an inner form of GL_n/F and let E_μ be an irreducible algebraic representation of G'_∞, which is of regular highest weight. Then Thm. 7.23 in [14] together with Multiplicity One of the discrete automorphic spectrum of $G'(\mathbb{A})$, cf. [1], may be used to show the first part of Thm. 3.3 for residual automorphic representation of $G'(\mathbb{A})$, too. More precisely, if Π is a residual automorphic representation of $G'(\mathbb{A})$, with parabolic support $\{P\}$ and with non–zero $(\mathfrak{m}_{G'}, K)$-cohomology in degree $q_0 < q_{\mathsf{max}}(P)$, then for all $\sigma \in$ Aut$(\mathbb{C})$, there exists a residual automorphic representation Ξ of $G'(\mathbb{A})$, with parabolic support $\{P\}$, $(\mathfrak{m}_{G'}, K)$-cohomological with respect to $E_{{}^\sigma\mu}$ and such that $\Xi_f \cong {}^\sigma\Pi_f$.

3.3. **Rationality fields.** We conclude this article proving a result on the rationality field of a cohomological residual automorphic representation Π. Recall that this field is defined by

$$\mathbb{Q}(\Pi_f) := \{z \in \mathbb{C} | \sigma(z) = z \quad \forall\sigma \text{ satisfying } \Pi_f \cong {}^\sigma\Pi_f\}.$$

Corollary 3.6. *Let G be a connected reductive linear algebraic group over a number field F, which is an inner form of a split algebraic group, and let E_μ be a finite-dimensional algebraic representation of G_∞. Let Π be a residual automorphic representation of $G(\mathbb{A})$, with non–zero $(\mathfrak{m}_G, K)$-cohomology below degree q_{max}, which does not appear in $H_!(S, \mathcal{E}_\mu)$. Then $\mathbb{Q}(\Pi_f)$ is a number field.*
If $G = GL_n/F$, the we may lighten the assumptions on Π and may take Π to be a residual automorphic representation of $GL_n(\mathbb{A})$, with parabolic support $\{P\}$ and with non–zero $(\mathfrak{m}_G, K)$-cohomology in degree $q_0 < q_{\mathsf{max}}(P)$. If P is maximal, then the condition $q_0 < q_{\mathsf{max}}(P)$ may be dropped without replacement.

Proof. Let Π be any of the representations in the statement of the corollary. Then, for all automorphisms σ, there is a residual automorphic representation Ξ of $G(\mathbb{A})$, $(\mathfrak{m}_G, K)$-cohomological with respect to $E_{{}^\sigma\mu}$, such that $\Xi_f \cong {}^\sigma\Pi_f$ by Thm. 3.1 or Thm. 3.3, respectively. Now fix an open compact subgroup $K_f \subseteq G(\mathbb{A}_f)$, such that $\Pi_f^{K_f}$ is non-zero. Then also $({}^\sigma\Pi)_f^{K_f}$ is non-zero. Since $H^q(S, \mathcal{E}_\mu)^{K_f}$ is finite-dimensional, the result follows from the finiteness of the set $\{E_{{}^\sigma\mu} : \sigma \in \text{Aut}(\mathbb{C})\}$. $\square$

Remark 3.7. The above corollary generalizes a result which is well–known for cohomological cuspidal automorphic representations (cf., e.g., Shimura [19], Harder [15] p.80, Waldspurger [20], Cor. I.8.3 and first line of p. 153, Clozel [7] and Grobner–Raghuram [14], Thm. 8.1.).

References

[1] I. A. Badulescu, D. Renard, Unitary Dual of GL(n) at archimedean places and global Jacquet-Langlands correspondence, *Comp. Math.* **145** (2010) 1115–1164

[2] A. Borel, Automorphic Forms on Reductive Groups, in *Automorphic forms and Applications*, IAS/Park City Mathematics Series, vol. 12, eds. P. Sarnak and F. Shahidi, (Utah, 2002) AMS, 2007, pp. 5–40.

[3] A. Borel, H. Jacquet, *Automorphic forms and automorphic representations*, in: Proc. Sympos. Pure Math., Vol. XXXIII, part I, AMS, Providence, R.I., (1979), pp. 189-202

[4] A. Borel, J.-P. Labesse, J. Schwermer, On the cuspidal cohomology of S-arithmetic subgroups of reductive groups over number fields, *Comp. Math.*, **102** (1996), pp. 1-40

[5] A. Borel, J.-P. Serre, Corners and arithmetic groups, *Comm. Math. Helvet.* **48** (1973) 436–491

[6] A. Borel, N. Wallach, *Continuous cohomology, discrete subgroups and representations of reductive groups*, Ann. of Math. Studies, (Princeton Univ. Press, New Jersey, 2000)

[7] L. Clozel, Motifs et Formes Automorphes: Applications du Principe de Fonctorialité, in *Automorphic forms, Shimura varieties, and L-functions*, Vol. I, Perspect. Math., vol. 10, eds. L. Clozel and J. S. Milne, (Ann Arbor, MI, 1988) Academic Press, Boston, MA, 1990, pp. 77–159.

[8] J. Franke, Harmonic analysis in weighted L_2-spaces, *Ann. Sci. de l'ENS* 4^e série tome 31 **2** (1998) 181–279

[9] J. Franke, J. Schwermer, A decomposition of spaces of automorphic forms, and the Eisenstein cohomology of arithmetic groups, *Math. Ann.*, **311** (1998) pp. 765–790

[10] W. T. Gan, A. Raghuram, Arithmeticity for periods of automorphic forms, in *Automorphic Representations and L-functions*, Tata Inst. Fund. Res. Stu. Math. No. 22, Hindustan Book Agency, Mumbai (2013), pp. 187–229

[11] N. Grbac, The Franke filtration of the spaces of automorphic forms supported in a maximal proper parabolic subgroup, *Glas. Mat. Ser. III*, **47** (2012) pp. 351–372

[12] H. Grobner, A cohomological injectivity result for the residual automorphic spectrum of GL_n, *Pacific J. Math.*, **268** (2014) pp. 33–46

[13] H. Grobner, Residues of Eisenstein series and the automorphic cohomology of reductive groups, *Comp. Math.* **149** (2013) pp. 1061–1090

[14] H. Grobner, A. Raghuram, On some arithmetic properties of automorphic forms of GL_m over a division algebra, *Int. J. Number Theory* **10** (2014) pp. 963–1013

[15] G. Harder, General aspects in the theory of modular symbols, in: *Séminaire de Théorie des Nombres, Paris 1981-82*, ed M.-J. Bertin, Progress in Mathematics **38** (Boston, Basel, Stuttgart 1983), pp. 73-88

[16] C. Mœglin, J.-L. Waldspurger, Le Spectre Résiduel de $GL(n)$, *Ann. Sci. de l'ENS* 4^e série tome 22 **4** (1989) 605–674

[17] C. Mœglin, J.-L. Waldspurger, *Spectral decomposition and Eisenstein series*, Cambridge Univ. Press (1995)

[18] J. Rohlfs, Projective limits of locally symmetric spaces and cohomology, *J. reine angew. Math.* **479** (1996) 149–182

[19] G. Shimura, The special values of the zeta functions associated with Hilbert modular forms, *Duke Math. J.* 45 (1978), no. 3, 637–679.

[20] J.-L. Waldspurger, Quelques propriétés arithmétiques de certaines formes automorphes sur GL(2), *Comp. Math.* **54** (1985) 121–171

Harald Grobner: Fakultät für Mathematik, Universität Wien, Oskar–Morgenstern–Platz 1, A-1090 Wien, Austria

E-mail address: harald.grobner@univie.ac.at