

# A NOTE ON THE ARITHMETIC OF RESIDUAL AUTOMORPHIC REPRESENTATIONS OF REDUCTIVE GROUPS

HARALD GROBNER

ABSTRACT. We prove an arithmeticity result for a class of cohomological residual automorphic representations of a general reductive group  $G$ . More precisely, we show that this class of residual representations is stable under the action of  $\text{Aut}(\mathbb{C})$ . This complements numerous results on the stability of cohomological cuspidal automorphic representations due by several people. We conclude by showing that the rationality field of such a cohomological residual automorphic representation is a number field.

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## INTRODUCTION

Let  $G$  be a connected reductive group over a number field  $F$ . Let  $\sigma \in \text{Aut}(\mathbb{C})$  and let  $\Pi$  be an automorphic representation of  $G(\mathbb{A})$ . Then,  $\Pi$  splits as  $\Pi = \Pi_\infty \otimes \Pi_f$  and we may define a new, “ $\sigma$ -twisted” representation  ${}^\sigma\Pi_f := \Pi_f \otimes_\sigma \mathbb{C}$  of  $G(\mathbb{A}_f)$ . A priori, it is by no means clear, if  ${}^\sigma\Pi_f$  appears again as the finite part of an automorphic representation of  $G(\mathbb{A})$ . However, for particular choices of the group  $G$  and the automorphic representation  $\Pi$ , there are numerous results in the related literature.

In any case, the key-technique, underlying all of these aforementioned results, is to assume that the automorphic representation considered is of *cohomological type*, i.e., that it contributes non-trivially to the relative Lie algebra cohomology of the space of automorphic forms  $\mathcal{A}(G)$ , twisted by an appropriate finite-dimensional coefficient system.

In order to render the above more precise, let  $E_\mu$  be a finite-dimensional, irreducible algebraic representation of  $G_\infty = R_{F/\mathbb{Q}}(G)(\mathbb{R})$  on a complex vector space. It is given by its highest weight  $\mu = (\mu_\iota)_{\iota \in I_\infty}$ . Here,  $R_{F/\mathbb{Q}}(G)$  denotes restriction of scalars and  $I_\infty$  stands for the set of field embeddings  $F \hookrightarrow \mathbb{C}$ . Similar to the case of an automorphic representation, on the level of  $G/F$ , we may form a new representation  ${}^\sigma E_\mu := E_\mu \otimes_\sigma \mathbb{C}$ . On the other hand, we let  $E_{\sigma\mu}$  be the representation of  $G_\infty$ , which has highest weight  $\sigma\mu := (\mu_{\sigma^{-1}\iota})_{\iota \in I_\infty}$ . If  $G$  is an inner form of a split algebraic group (which we assume it is from now on), then, as a representation of  $G(F)$ ,  ${}^\sigma E_\mu$  is isomorphic to  $E_{\sigma\mu}$ .

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As a consequence, if  $G$  is inner, we obtain a commutative diagram of  $G(\mathbb{A}_f)$ -module isomorphisms (with  $\sigma$ -linear columns),

$$\begin{array}{ccc} H^q(S, \mathcal{E}_\mu) & \xleftarrow{\cong} & H^q(\mathfrak{m}_G, K, \mathcal{A}(G) \otimes E_\mu) \\ \downarrow \sigma^* & & \downarrow \sigma^* \\ H^q(S, {}^\sigma \mathcal{E}_\mu) & \xrightarrow{\cong} & H^q(\mathfrak{m}_G, K, \mathcal{A}(G) \otimes E_{\sigma\mu}) \end{array} .$$

Using the key result of our recent paper [13], which shows that below a certain, sharp degree of cohomology  $q_{\max}$ , the space of automorphic cohomology  $H^q(\mathfrak{m}_G, K, \mathcal{A}(G) \otimes E_\mu)$  is isomorphic to the  $(\mathfrak{m}_G, K)$ -cohomology of the space of *square-integrable* automorphic forms, we can prove the main result of this short paper: We show that – under certain constraints – residuality is an *arithmetic property* of automorphic representations, or – in other words – that the action of  $\text{Aut}(\mathbb{C})$  preserves the class of these residual automorphic representations:

**Theorem.** *Let  $G$  be a connected reductive linear algebraic group over a number field  $F$ , which is an inner form of a split algebraic group, and let  $E_\mu$  be a finite-dimensional algebraic representation of  $G_\infty$ . Let  $\Pi$  be a residual automorphic representation of  $G(\mathbb{A})$ , with non-zero  $(\mathfrak{m}_G, K)$ -cohomology below degree  $q_{\max}$ , which does not appear in interior cohomology. Then for all  $\sigma \in \text{Aut}(\mathbb{C})$ , there exists a residual automorphic representation  $\Xi$  of  $G(\mathbb{A})$ ,  $(\mathfrak{m}_G, K)$ -cohomological with respect to  $E_{\sigma\mu}$ , such that  $\Xi_f \cong {}^\sigma \Pi_f$ , and which appears in the complement of interior cohomology.*

We would like to remark that in the case of  $G = GL_n/F$ , the general linear group over any number field  $F$ , our approach leads to a result (cf. Thm. 3.3) which complements Clozel [7], Thm. 3.13, Franke [8], Thm. 20 and Franke–Schwermer [9], Thm. 4.3. See also Grobner–Raghuram [14] Prop. 7.21 and Thm. 7.23.

Additionally, we also prove an algebraicity theorem on the field of rationality,

$$\mathbb{Q}(\Pi_f) := \{z \in \mathbb{C} \mid \sigma(z) = z \quad \forall \sigma \text{ satisfying } \Pi_f \cong {}^\sigma \Pi_f\},$$

**Corollary.** *Let  $\Pi$  be a cohomological residual automorphic representation as in the statement of the above theorem. Then  $\mathbb{Q}(\Pi_f)$  is a number field.*

We would like to remark that the above corollary generalizes a result which is well-known for cohomological cuspidal automorphic representations (cf., e.g., Shimura [19], Harder [15] p.80, Waldspurger [20], Cor. I.8.3 and first line of p. 153, Clozel [7] and Grobner–Raghuram [14], Thm. 8.1.).

## 1. SOME BASICS

**1.1. Number fields.** We let  $F$  be an algebraic number field. Its set of places is denoted  $V = V_\infty \cup V_f$ , where  $V_\infty$  stands for the set of archimedean places and  $V_f$  is the set of non-archimedean places. We shall use the letter  $I_\infty$  for the set of field embeddings  $\iota : F \hookrightarrow \mathbb{C}$ . The ring of adèles of  $F$  is denoted  $\mathbb{A}$ , the subspace of finite adèles is denoted  $\mathbb{A}_f$ .

**1.2. Algebraic groups.** In this paper,  $G$  is a connected, reductive linear algebraic group over a number field  $F$ . We assume to have fixed a minimal parabolic  $F$ -subgroup  $P_0$  with Levi decomposition  $P_0 = L_0 N_0$  and let  $A_0$  be the maximal  $F$ -split torus in the center  $Z_{L_0}$  of  $L_0$ . This choice defines the standard parabolic  $F$ -subgroups  $P$  with Levi decomposition  $P = L_P N_P$ , where  $L_P \supseteq L_0$  and  $N_P \subseteq N_0$ . We let  $A_P$  be the maximal  $F$ -split torus in the center  $Z_{L_P}$  of  $L_P$ , satisfying  $A_P \subseteq A_0$  and denote by  $\mathfrak{a}_P$  (resp.  $\mathfrak{a}_{P,\mathbb{C}}$ ) its Lie algebra (resp. its complexification  $\mathfrak{a}_{P,\mathbb{C}} = \mathfrak{a}_P \otimes \mathbb{C}$ ). We write  $H_P : L_P(\mathbb{A}) \rightarrow \mathfrak{a}_{P,\mathbb{C}}$  for the standard Harish-Chandra height function. The group  $L_P(\mathbb{A})^1 := \ker H_P$ , admits a direct complement  $A_P^{\mathbb{R}} \cong \mathbb{R}_+^{\dim \mathfrak{a}_P}$  in  $L_P(\mathbb{A})$  whose Lie algebra is isomorphic to  $\mathfrak{a}_P$ . With respect to a maximal compact subgroup  $K_{\mathbb{A}} \subseteq G(\mathbb{A})$  in good position, cf. [17] I.1.4, we obtain an extension  $H_P : G(\mathbb{A}) \rightarrow \mathfrak{a}_{P,\mathbb{C}}$  to all of  $G(\mathbb{A})$ .

**1.3. Lie groups and Lie algebras.** We put  $G_\infty := R_{F/\mathbb{Q}}(G)(\mathbb{R})$ , where  $R_{F/\mathbb{Q}}$  denotes the restriction of scalars from  $F$  to  $\mathbb{Q}$ . We let  $\mathfrak{m}_G := \mathfrak{g}_\infty/\mathfrak{a}_G = \text{Lie}(G(\mathbb{A})^1 \cap G_\infty)$  and denote by  $\mathfrak{Z}(\mathfrak{g})$  the center of the universal enveloping algebra  $\mathcal{U}(\mathfrak{g})$  of  $\mathfrak{g}_{\infty, \mathbb{C}}$ .

Let  $K_\infty \subset G_\infty$  be a maximal compact subgroup (the archimedean factor of the maximal compact subgroup  $K_\mathbb{A}$  of  $G(\mathbb{A})$  in good position) and set once and for all  $K := K_\infty^\circ$ . (Observe that this choice of a compact subgroup of  $G_\infty$  is in accordance with Franke [8], p. 184.) Please refer to Borel–Wallach [6], I, for the basic facts and notations concerning  $(\mathfrak{m}_G, K)$ -cohomology.

Furthermore, we let  $\mathfrak{h}_\infty$  be a Cartan subalgebra of  $\mathfrak{g}_\infty$  that contains  $\mathfrak{a}_{0, \infty}$ .

**1.4. Algebraic representations.** In this paper, we always let  $E_\mu$  be a finite-dimensional irreducible algebraic representation of  $G_\infty$  on a complex vector space, given by its highest weight  $\mu = (\mu_\iota)_{\iota \in I_\infty} \in \check{\mathfrak{h}}_\infty$ . We will assume that  $A_G^\mathbb{R}$  (and so  $\mathfrak{a}_G$ ) acts trivially on  $E_\mu$ . There is therefore no difference between the  $(\mathfrak{g}_\infty, K)$ -module and the  $(\mathfrak{m}_G, K)$ -module defined by  $E_\mu$ .

Take a field automorphism  $\sigma \in \text{Aut}(\mathbb{C})$ . Then we may form the new finite-dimensional irreducible algebraic representation of  $G_\infty$ , denoted  $E_{\sigma\mu}$ , which is given by its highest weight

$$\sigma\mu := (\mu_{\sigma^{-1}\iota}).$$

Clearly this amounts in a permutation of the components  $\mu_\iota$  of  $\mu$ . On the level of the algebraic group  $G/F$ , there is another representation, denoted  ${}^\sigma E_\mu := E_\mu \otimes_\sigma \mathbb{C}$ .

**1.5. Cohomology of locally symmetric spaces.** We let

$$S := G(F)A_G^\mathbb{R} \backslash G(\mathbb{A})/K$$

be the projective limit of the “locally symmetric spaces” attached to  $G$ . Starting from the algebraic representation  $E_\mu$ , one obtains a sheaf  $\mathcal{E}_\mu$  on  $S$  by letting  $\mathcal{E}_\mu$  be the sheaf with espace étalé  $G(\mathbb{A})/A_G^\mathbb{R}K \times_{G(F)} E_\mu$  with the discrete topology on  $E_\mu$ . In the same way, we obtain a sheaf  ${}^\sigma \mathcal{E}_\mu$  from  ${}^\sigma E_\mu$ .

**Proposition 1.1.** *Let  $G$  be a connected reductive linear algebraic group over a number field  $F$  and let  $E_\mu$  be a finite-dimensional algebraic representation of  $G_\infty$ . Then for all  $\sigma \in \text{Aut}(\mathbb{C})$  and all degrees  $q$  of cohomology, there is a  $G(\mathbb{A}_f)$ -equivariant,  $\sigma$ -linear isomorphism*

$$H^q(S, \mathcal{E}_\mu) \rightarrow H^q(S, {}^\sigma \mathcal{E}_\mu).$$

*If  $G$  is an inner form of a split algebraic group, then, as a representation of  $G(F)$ ,  ${}^\sigma E_\mu$  is isomorphic to  $E_{\sigma\mu}$ . This may fail, if  $G$  is an outer form.*

*Proof.* The first assertion is obvious by the definition of Betti cohomology  $H^q(S, \bullet)$ . For the other assertions see [10], Sect. 2.  $\square$

## 2. AUTOMORPHIC COHOMOLOGY

**2.1. Residual automorphic forms.** Our notion of an *automorphic form*  $f : G(\mathbb{A}) \rightarrow \mathbb{C}$  and of an *automorphic representation* of  $G(\mathbb{A})$  is the one of Borel–Jacquet [3], 4.2 and 4.6. Let  $\mathcal{A}(G)$  be the space of all automorphic forms  $f : G(\mathbb{A}) \rightarrow \mathbb{C}$ , which are constant on the real Lie subgroup  $A_G^\mathbb{R}$ . By its very definition, every automorphic form is annihilated by some power of an ideal  $\mathcal{J} \triangleleft \mathfrak{Z}(\mathfrak{g})$  of finite codimension. We fix such an ideal  $\mathcal{J}$ , once and for all: As we will only be interested in cohomological automorphic forms, we take  $\mathcal{J}$  to be the ideal which annihilates the contragredient representation  $E_\mu^\vee$  of  $E_\mu$ , cf. Sect.1.3, and denote by

$$\mathcal{A}_\mathcal{J}(G) \subset \mathcal{A}(G)$$

the space consisting of those automorphic forms which are annihilated by some power of  $\mathcal{J}$ . Clearly,  $\mathcal{A}_\mathcal{J}(G)$  carries a commuting  $(\mathfrak{g}_\infty, K_\infty)$  and  $G(\mathbb{A}_f)$ -action and hence defines a  $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$ -module.

The  $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$ -submodule of all square integrable automorphic forms in  $\mathcal{A}_\mathcal{J}(G)$  is denoted  $\mathcal{A}_{dis, \mathcal{J}}(G)$ . An irreducible subquotient of  $\mathcal{A}_{dis, \mathcal{J}}(G)$  will be called a *discrete series automorphic representation*, cf. Borel [2] 9.6. If  $\omega : Z_G(F) \backslash Z_G(\mathbb{A}) \rightarrow \mathbb{C}^*$  is a continuous character of the center  $Z_G$  of  $G$ , we let  $\mathcal{A}_{dis, \mathcal{J}}(G, \omega)$  be

the space of square-integrable automorphic forms with central character  $\omega$ .

We further recall that  $\mathcal{A}_{dis, \mathcal{J}}(G, \omega)$  decomposes as a direct sum of automorphic representations  $\Pi$

$$(2.1) \quad \mathcal{A}_{dis, \mathcal{J}}(G, \omega) \cong \bigoplus \Pi^{m(\Pi)}$$

with finite multiplicities  $m(\Pi)$ . One has  $m(\Pi) = m_{cusp}(\Pi) + m_{res}(\Pi)$ , where  $m_{cusp}(\Pi)$  is the multiplicity of  $\Pi$  in the space of cuspidal automorphic forms in  $\mathcal{A}_{\mathcal{J}}(G)$ . If  $m_{res}(\Pi) \neq 0$ , we call  $\Pi$  a *residual automorphic representation*. Clearly, it decomposes as  $\Pi = \Pi_{\infty} \otimes \Pi_f$ , where  $\Pi_{\infty}$  is an irreducible  $(\mathfrak{m}_G, K)$ -module and  $\Pi_f$  and irreducible  $G(\mathbb{A}_f)$ -module.

Apart from the action of automorphisms  $\sigma \in \text{Aut}(\mathbb{C})$  on finite-dimensional algebraic representations  $E_{\mu}$ , cf. Sect. 1.4, there is also an action of  $\sigma$  on the finite part  $\Pi_f$  of an automorphic representation: Given  $\sigma \in \text{Aut}(\mathbb{C})$  and  $\Pi_f$ , we define  ${}^{\sigma}\Pi_f$  to be the representation of  $G(\mathbb{A}_f)$  on  $\Pi_f \otimes_{\sigma} \mathbb{C}$  (analogous to  ${}^{\sigma}E_{\mu}$ ). This is in accordance with Waldspurger, [20], I.1.

**2.2. Parabolic and cuspidal supports.** Let  $\{P\}$  be the associate class of the parabolic  $F$ -subgroup  $P$ : It consists by definition of all parabolic  $F$ -subgroups  $Q = L_Q N_Q$  of  $G$  for which  $L_Q$  and  $L_P$  are conjugate by an element in  $G(F)$ . We denote by  $\mathcal{A}_{\mathcal{J}, \{P\}}(G)$  the space of all  $f \in \mathcal{A}_{\mathcal{J}}(G)$  which are negligible along every parabolic  $F$ -subgroup  $Q \notin \{P\}$ . There is the following decomposition of  $\mathcal{A}_{\mathcal{J}}(G)$  as a  $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$ -module, cf. [4] Thm. 2.4 or [2] 10.3, first established by Langlands:

$$(2.2) \quad \mathcal{A}_{\mathcal{J}}(G) \cong \bigoplus_{\{P\}} \mathcal{A}_{\mathcal{J}, \{P\}}(G).$$

The various summands  $\mathcal{A}_{\mathcal{J}, \{P\}}(G)$  can be decomposed even further. To this end, recall from [9], 1.2, the notion of an *associate class*  $\varphi_P$  of cuspidal automorphic representations  $\pi$  of the Levi subgroups of the elements in the class  $\{P\}$ . Given  $\varphi_P$ , represented by a cuspidal representation  $\pi$ , a  $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$ -submodule

$$\mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}(G)$$

of  $\mathcal{A}_{\mathcal{J}, \{P\}}(G)$  was defined in [9], 1.3 as follows: It is the span of all possible holomorphic values or residues of all Eisenstein series attached to  $\tilde{\pi}$ , evaluated at a certain point  $\lambda = d\Lambda$  determined by  $\pi$ , together with all their derivatives. (This definition is independent of the choice of the representatives  $P$  and  $\pi$ , thanks to the functional equations satisfied by the Eisenstein series considered.) For details, we refer the reader to [9] 1.2-1.4.

The following refined decomposition as  $(\mathfrak{m}_G, K, G(\mathbb{A}_f))$ -modules of the spaces  $\mathcal{A}_{\mathcal{J}, \{P\}}(G)$  of automorphic forms was obtained in Franke–Schwermer [9], Thm. 1.4:

$$\mathcal{A}_{\mathcal{J}, \{P\}}(G) \cong \bigoplus_{\varphi_P} \mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}(G).$$

**2.3. Automorphic cohomology.** We recall that the  $G(\mathbb{A}_f)$ -module

$$H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\mu})$$

is called the *automorphic cohomology* of  $G$  in degree  $q$ . From what we recalled in section 2.2 it inherits a direct sum decomposition

$$H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\mu}) \cong \bigoplus_{\{P\}} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}}(G) \otimes E_{\mu}) \cong \bigoplus_{\{P\}} \bigoplus_{\varphi_P} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}(G) \otimes E_{\mu})$$

The summand  $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{G\}}(G) \otimes E)$  attached to  $\{G\}$  consists precisely of all cuspidal automorphic forms in  $\mathcal{A}_{\mathcal{J}}(G)$  and will be denoted for short by  $H_{cusp}^q(G, E_{\mu})$

Apart from this, we will need the following result of Franke, [8] Thm. 18, which links automorphic cohomology with the sheaf cohomology of  $S$ :

**Theorem 2.3.** *There is an isomorphism of  $G(\mathbb{A}_f)$ -modules*

$$H^q(S, \mathcal{E}_\mu) \cong H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu).$$

We may therefore compare automorphic cohomology with the cohomology of  $S$ . Recall the Borel-Serre compactification of  $S$ , cf. [5], [18]: We will denote it by  $\bar{S}$  and by  $\partial\bar{S}$  its boundary. Let  $H_1^q(S, \mathcal{E}_\mu)$  be the space of *interior cohomology*, i.e., the kernel of the natural restriction map,

$$res^q : H^q(S, \mathcal{E}_\mu) = H^q(\bar{S}, \mathcal{E}_\mu) \rightarrow H^q(\partial\bar{S}, \mathcal{E}_\mu).$$

It is well-known that  $H_{cusp}^q(G, E_\mu) \subseteq H_1^q(S, \mathcal{E}_\mu)$ : This is a direct consequence of the fact that the inclusion of compactly supported differential forms into fast decreasing differential forms defines an isomorphism in sheaf cohomology.

**2.4. A diagram.** Assume now that  $G$  is an inner form of a split reductive algebraic group. Then, for every  $\sigma \in \text{Aut}(\mathbb{C})$ , we obtain the following commutative diagram of  $G(\mathbb{A}_f)$ -module isomorphisms (with  $\sigma$ -linear columns) by Prop. 1.1 and Thm. 2.3:

$$\begin{array}{ccc} H^q(S, \mathcal{E}_\mu) & \xleftarrow{\cong} & H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu) \\ \downarrow \sigma^* & & \downarrow \sigma^* \\ H^q(S, \sigma\mathcal{E}_\mu) & \xrightarrow{\cong} & H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\sigma\mu}) \end{array}$$

Observe the position of the automorphism  $\sigma$  in the various coefficient systems: This is the reason for our assumption that  $G$  is an inner form, since otherwise  ${}^\sigma E_\mu$  and  $E_{\sigma\mu}$  may differ as  $G(F)$ -modules, cf. Prop. 1.1. Moreover, observe that  $H_1^q(S, \mathcal{E}_\mu)$  is stable by  $\sigma^*$ , i.e.,  $\sigma^*(H_1^q(S, \mathcal{E}_\mu)) = H_1^q(S, \sigma\mathcal{E}_\mu)$ . This is due to the rationality of the map  $res^q$ , which we used in order to define interior cohomology.

**2.5. Certain bounds in cohomology.** Given an associate class  $\{P\}$  of the parabolic  $F$ -subgroups, represented by  $P$ , we define (with  $R = L_R N_R$  denoting a parabolic subgroup of  $G$ )

$$q_{\max}(P) := \min_{\substack{R \\ R \supseteq P}} \left( \sum_{v \in V_\infty} \left\lceil \frac{1}{2} \dim_{\mathbb{R}} N_R(F_v) \right\rceil \right).$$

Moreover we let

$$q_{\max} := \min_{P \text{ standard}} q_{\max}(P).$$

This definition is motivated by our own results in [13]: The two numbers  $q_{\max}(P)$  and  $q_{\max}$  will serve as certain ‘‘upper bounds’’ for the degrees of automorphic cohomology. Indeed, in [13], we proved the following result:

**Theorem 2.4.** *Let  $G$  be a connected, reductive group over a number field  $F$  and let  $E_\mu$  be an irreducible, finite-dimensional, algebraic representation of  $G_\infty$  on a complex vector space. Let  $\{P\}$  be an associate class of parabolic  $F$ -subgroups of  $G$  and let  $\varphi_P$  be an associate class of cuspidal automorphic representations of  $L_P(\mathbb{A})$ . Let  $L_{\mathcal{J}, \{P\}, \varphi_P}^2 := \mathcal{A}_{dis, \mathcal{J}}(G) \cap \mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}$  be the space of square-integrable automorphic forms in  $\mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}$ . Then, the inclusion  $L_{\mathcal{J}, \{P\}, \varphi_P}^2(G) \hookrightarrow \mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}(G)$  induces an isomorphism of  $G(\mathbb{A}_f)$ -modules*

$$H^q(\mathfrak{m}_G, K, L_{\mathcal{J}, \{P\}, \varphi_P}^2(G) \otimes E_\mu) \xrightarrow{\cong} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}(G) \otimes E_\mu),$$

*in all degrees  $q < q_{\max}(P)$ . In particular, below  $q_{\max}$ , automorphic cohomology  $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu)$  is identical to  $H^q(\mathfrak{m}_G, K, \mathcal{A}_{dis, \mathcal{J}}(G) \otimes E_\mu)$ .*

The latter theorem will be the key-result for the proof of the main result of this article detailed in the following section.

## 3. THE MAIN RESULTS

## 3.1. Arithmeticity of residual automorphic representations - The general case.

**Theorem 3.1.** *Let  $G$  be a connected reductive linear algebraic group over a number field  $F$ , which is an inner form of a split algebraic group, and let  $E_\mu$  be a finite-dimensional algebraic representation of  $G_\infty$ . Let  $\Pi$  be a residual automorphic representation of  $G(\mathbb{A})$ , with non-zero  $(\mathfrak{m}_G, K)$ -cohomology below degree  $q_{\max}$ , which does not appear in  $H_1(S, \mathcal{E}_\mu)$ . Then for all  $\sigma \in \text{Aut}(\mathbb{C})$ , there exists a residual automorphic representation  $\Xi$  of  $G(\mathbb{A})$ ,  $(\mathfrak{m}_G, K)$ -cohomological with respect to  $E_{\sigma\mu}$ , such that  $\Xi_f \cong \sigma\Pi_f$ , and which appears in the complement of  $H_1(S, \sigma\mathcal{E}_\mu)$ .*

*Proof.* Let  $\Pi$  be as in the statement of the theorem. Let  $q_0 < q_{\max}$  be a degree, where the  $(\mathfrak{m}_G, K)$ -cohomology of  $\Pi_\infty$  (with respect to  $E_\mu$ ) does not vanish. Clearly,  $H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu) \cong H^{q_0}(\mathfrak{m}_G, K, \Pi_\infty \otimes E_\mu) \otimes \Pi_f$  as  $G(\mathbb{A}_f)$ -modules, whence we may fix an injection  $\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$ . By Thm. 2.4 (resp. [13] Cor. 17), we hence obtain an injection

$$\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu).$$

In other words, we have realized  $\Pi_f$  as a submodule of the space of automorphic cohomology in degree  $q_0$ . Chasing the image of  $\Pi_f$  in  $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_\mu)$  through our diagram, see Sect. 2.4, we finally end up with an irreducible subrepresentation  $\sigma^*(\Pi_f)$  of  $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\sigma\mu})$ . This is the place, where the assumption that  $G$  is an inner form of a split algebraic group comes into play: As mentioned below our diagram in Sect. 2.4, otherwise,  $\sigma E_\mu$  (which defines the sheaf  $\mathcal{E}_\mu$ ) may not be isomorphic to  $E_{\sigma\mu}$  (which defines the coefficient module in  $(\mathfrak{m}_G, K)$ -cohomology) as a representation of  $G(F)$ .

By the definition of the  $\sigma$ -linear  $G(\mathbb{A}_f)$ -morphism  $H^q(S, \mathcal{E}_\mu) \rightarrow H^q(S, \sigma\mathcal{E}_\mu)$ , this subrepresentation  $\sigma^*(\Pi_f)$  is isomorphic to  $\sigma\Pi_f$ . By our Thm. 2.4 (resp. our Thm. 18 in [13]),

$$H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{dis, \mathcal{J}}(G) \otimes E_{\sigma\mu}) \cong H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\sigma\mu}),$$

so,  $\sigma^*(\Pi_f)$ , which by the above is isomorphic to  $\sigma\Pi_f$ , is contained as a subrepresentation of the former cohomology space  $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{dis, \mathcal{J}}(G) \otimes E_{\sigma\mu})$ . Now recall that the discrete automorphic spectrum  $\mathcal{A}_{dis, \mathcal{J}}(G)$  decomposes as a direct sum, cf. 2.1, (at least, if one fixes a central character). Therefore, there is also a direct sum decomposition of the  $G(\mathbb{A}_f)$ -module  $H^q(\mathfrak{m}_G, K, \mathcal{A}_{dis, \mathcal{J}}(G) \otimes E_{\sigma\mu})$ . As a consequence, there exists a square-integrable automorphic representation  $\Xi$  of  $G(\mathbb{A})$ , with non-vanishing  $(\mathfrak{m}_G, K)$ -cohomology with respect to  $E_{\sigma\mu}$ , such that  $\Xi_f \cong \sigma\Pi_f$ .

Since  $H_1^{q_0}(S, \mathcal{E}_\mu)$  is preserved by  $\sigma^*$  and contains  $H_{cusp}^{q_0}(G, E_\mu)$ , this representation  $\Xi$  can be chosen to be residual, i.e., such that  $m_{res}(\Xi) \neq 0$ , and even outside of  $H_1(S, \sigma\mathcal{E}_\mu)$ . This shows the claim.  $\square$

**Remark 3.2.** Let  $\Pi$  be a residual automorphic representation as in the statement of Thm. 3.1. Assume that  $\Pi$  has been obtained using the Langlands–Shahidi method, i.e., by determining the position of poles and zeros of certain automorphic  $L$ -functions, which are attached to those cuspidal automorphic representations  $\pi$  of Levi subgroups, which appear in the calculation of the constant term. Our main theorem above hence suggests that if such an  $L$ -function  $-L(s, \pi, r)$ , say  $-$  has a pole at the relevant point  $s = s_0$  (giving rise to a pole of some Eisenstein series, whose residues span  $\Pi$ ), then also  $L(s, \sigma\pi, r)$  should have a pole at this point. This may be of some use in the determination of poles of Eisenstein series, respectively for showing algebraicity results of residual automorphic  $L$ -functions.

**3.2. Arithmeticity of residual automorphic representations - The case of  $GL_n/F$ .** In the case of the split general linear group  $G = GL_n/F$ , we obtain the following refinement<sup>1</sup> of Thm. 3.1. At the same time this result complements the results of Clozel [7], Thm. 3.13, Franke [8], Thm. 20 and Franke–Schwermer [9], Thm. 4.3. See also Grobner–Raghuram [14] Prop. 7.21 and Thm. 7.23.

<sup>1</sup>As I learned from L. Clozel, stronger results than the ones of Sect. 3.2 may be proved using the fact that a residual automorphic representation of  $GL_n(\mathbb{A})$  is a Mœglin–Waldspurger quotient, cf. [16]. However, I decided to leave this section, in order to exemplify the technique of Thm. 3.1.

**Theorem 3.3.** *Let  $G = GL_n$  be the split general linear group over  $F$  and let  $E_\mu$  be a finite-dimensional algebraic representation of  $G_\infty$ . Let  $\Pi$  be a residual automorphic representation of  $GL_n(\mathbb{A})$ , with parabolic support  $\{P\}$  and with non-zero  $(\mathfrak{m}_G, K)$ -cohomology in degree  $q_0 < q_{\max}(P)$ . Then for all  $\sigma \in \text{Aut}(\mathbb{C})$ , there exists a residual automorphic representation  $\Xi$  of  $G(\mathbb{A})$ , with parabolic support  $\{P\}$ ,  $(\mathfrak{m}_G, K)$ -cohomological with respect to  $E_{\sigma\mu}$  and such that  $\Xi_f \cong \sigma\Pi_f$ . If  $P$  is maximal parabolic, then the condition  $q_0 < q_{\max}(P)$  can be dropped, i.e., the result holds for all cohomological residual automorphic representations  $\Pi$  supported in the maximal parabolic subgroup  $P = (GL_{n/2} \times GL_{n/2}) \cdot N$ .*

*Proof.* Let  $\Pi$  be as in the statement of the theorem. Let  $q_0 < q_{\max}(P)$  be a degree, where the  $(\mathfrak{m}_G, K)$ -cohomology of  $\Pi_\infty$  (with respect to  $E_\mu$ ) does not vanish. As,  $H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu) \cong H^{q_0}(\mathfrak{m}_G, K, \Pi_\infty \otimes E_\mu) \otimes \Pi_f$  as  $G(\mathbb{A}_f)$ -modules, we may fix an injection  $\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$ . By Thm. 2.4, we hence obtain an injection

$$\Pi_f \hookrightarrow H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}}(G) \otimes E_\mu).$$

We now use a result of Franke: In [8], Thm. 20, he proved that the decomposition of  $\mathcal{A}_{\mathcal{J}}(G)$  along the parabolic supports  $\{P\}$ , cf. (2.2), is respected by the  $\sigma$ -linear isomorphism  $\sigma^*$ . As a consequence of our diagram, cf. Sect. 2.4,  $\sigma^*(\Pi_f) \cong \sigma\Pi_f$  appears not only in  $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}}(G) \otimes E_{\sigma\mu})$ , but even in  $H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}}(G) \otimes E_{\sigma\mu})$ . By our Thm. 2.4,

$$H^{q_0}(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}}(G) \otimes E_{\sigma\mu}) \cong H^{q_0}(\mathfrak{m}_G, K, \bigoplus_{\varphi_P} L_{\mathcal{J}, \{P\}, \varphi_P}^2(G) \otimes E_{\sigma\mu}),$$

whence  $\sigma^*(\Pi_f) \cong \sigma\Pi_f$  appears as an irreducible submodule of the latter cohomology space. Again we recall that the discrete automorphic spectrum  $\mathcal{A}_{dis, \mathcal{J}}(G)$ , which contains  $\bigoplus_{\varphi_P} L_{\mathcal{J}, \{P\}, \varphi_P}^2(G)$ , decomposes as a direct sum, cf. 2.1, (at least, if one fixes a central character). Hence, we find a square-integrable automorphic representation  $\Xi$  of  $G(\mathbb{A})$ , with parabolic support  $\{P\}$ ,  $(\mathfrak{m}_G, K)$ -cohomological with respect to  $E_{\sigma\mu}$  and such that  $\Xi_f \cong \sigma\Pi_f$ . This representation must be residual by Multiplicity One for square-integrable automorphic representations of  $GL_n(\mathbb{A})$ : Indeed,  $\Pi$  being assumed to be residual implies that  $P \neq G$  by Multiplicity One in the discrete automorphic spectrum. Therefore, the parabolic support of  $\Xi$  also defers from  $G$ . In particular,  $\Xi$  is residual itself (and we could avoid the assumption that  $\Pi_f$  does not appear in interior cohomology  $H_1(S, \mathcal{E}_\mu)$ ). This shows the first claim.

We will now prove the last assertion, claiming that the condition  $q_0 < q_{\max}(P)$  can be dropped in the case of maximal parabolic subgroups. For a moment, let  $\Pi$  be any cohomological residual automorphic representation of  $GL_n(\mathbb{A})$  supported in a pair  $(\{P\}, \varphi_P)$ . In [12], Thm. 4.1, we showed that its cohomology  $H^q(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$  always injects into  $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}(G) \otimes E_\mu)$  in the lowest degree  $q = q_{\min}$ , where  $H^q(\mathfrak{m}_G, K, \Pi \otimes E_\mu)$  does not vanish identically. In particular, by our above considerations, resp. our diagram in Sect. 2.4,  $\sigma^*(\Pi_f) \cong \sigma\Pi_f$  appears as an irreducible submodule of  $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}(G) \otimes E_{\sigma\mu})$  for an appropriate cuspidal support denoted  $\sigma\varphi_P$  here. We use [12] again: In Sect. 6B therein, we showed that the lowest degree of cohomology  $q_{\min}$  is always smaller than a certain bound  $q_{\text{res}} = q_{\text{res}}(\{P\}, \varphi_P)$  of degrees of cohomology. (A precise definition of this degree may be found in [13], Sect. 6.1 and [12], Sect. 3C.) It is not important to know how this degree is defined in details; rather one should recall the main result of our paper [13], Thm. 15, which says that for all  $q < q_{\text{res}}$ ,

$$H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}^{(m)}(G) \otimes E) \xrightarrow{\cong} H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}(G) \otimes E),$$

where  $\mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}^{(m)}(G)$  is the deepest step in Franke's filtration of the space  $\mathcal{A}_{\mathcal{J}, \{P\}, \varphi_P}(G)$ . Again, it is not important to know precisely, how this filtration is defined (All details may be found in [13], Sect. 3); however, by our concrete calculations in [12], Sect. 6 (cf. Prop. 6.4), it is shown that

$$q_{\text{res}} = q_{\text{res}}(\{P\}, \varphi_P) = q_{\text{res}}(\{P\}, \sigma\varphi_P),$$

whence summarizing what we said above,  $\sigma^*(\Pi_f) \cong \sigma\Pi_f$  appears as an irreducible submodule of the cohomology of the deepest filtration step  $H^q(\mathfrak{m}_G, K, \mathcal{A}_{\mathcal{J}, \{P\}, \sigma\varphi_P}^{(m)}(G) \otimes E_{\sigma\mu})$ . We would like to point out that all of this still holds for all parabolic subgroups.

Let now  $P$  be maximal. Then it is shown in Grbac [11] Thm. 3.1, that the deepest step  $\mathcal{A}_{\mathcal{J}, \{P\}, \sigma\varphi_P}^{(m)}(G)$  in Franke's filtration is identical to  $L_{\mathcal{J}, \{P\}, \sigma\varphi_P}^2(G)$ , i.e., every automorphic form in  $\mathcal{A}_{\mathcal{J}, \{P\}, \sigma\varphi_P}^{(m)}(G)$  is square-integrable. This is wrong, if  $P$  is not maximal. As a consequence,  $\sigma\Pi_f$  must be the finite part of a square-integrable automorphic representation  $\Xi$ . By Multiplicity One,  $\Xi$  must be residual.  $\square$

**Example 3.4.** Let  $G = GL_4$  over  $F = \mathbb{Q}$ . Then,  $\mathfrak{m}_G = \mathfrak{sl}_4(\mathbb{R})$  and  $K = SO(4)$ . Let  $P$  be the standard maximal parabolic subgroup with Levi factor  $L \cong GL_2 \times GL_2$ . We take furthermore  $E_\mu = \mathbb{C}$ , the trivial representation and let  $D(2)$  be the discrete series representation of  $GL_2(\mathbb{R})$  of lowest non-negative  $O(2)$ -type 2. Then, there exists a cuspidal automorphic representation  $\pi$  of  $GL_2(\mathbb{A})$ , with  $\pi_\infty \cong D(2)$  and  $\Pi = |\cdot|^{1/2}\pi \times |\cdot|^{-1/2}\pi$  (Langlands quotient) is a residual automorphic representation of  $GL_4(\mathbb{A})$  with non-trivial  $(\mathfrak{sl}_4(\mathbb{R}), SO(4))$ -cohomology with respect to  $\mathbb{C}$  in degrees  $q = 3, 6$ . By Thm. 3.3, and a simple uniqueness-argument at the archimedean place,  $\sigma\Pi = \Pi_\infty \otimes \sigma\Pi_f$  is a residual automorphic representation of  $GL_4(\mathbb{A})$ , for all  $\sigma \in \text{Aut}(\mathbb{C})$ . Indeed, the archimedean component must be  $\Pi_\infty$  again, since this is the only irreducible unitary representation of  $GL_4(\mathbb{R})$ , which has non-trivial cohomology in degree  $q = 3, 6$ . By construction it is the Langlands quotient  $|\cdot|^{1/2}D(2) \times |\cdot|^{-1/2}D(2)$ . See, e.g., [14], Sect. 5.4.

**Remark 3.5.** Let  $G'$  be an inner form of  $GL_n/F$  and let  $E_\mu$  be an irreducible algebraic representation of  $G'_\infty$ , which is of regular highest weight. Then Thm. 7.23 in [14] together with Multiplicity One of the discrete automorphic spectrum of  $G'(\mathbb{A})$ , cf. [1], may be used to show the first part of Thm. 3.3 for residual automorphic representation of  $G'(\mathbb{A})$ , too. More precisely, if  $\Pi$  is a residual automorphic representation of  $G'(\mathbb{A})$ , with parabolic support  $\{P\}$  and with non-zero  $(\mathfrak{m}_{G'}, K)$ -cohomology in degree  $q_0 < q_{\max}(P)$ , then for all  $\sigma \in \text{Aut}(\mathbb{C})$ , there exists a residual automorphic representation  $\Xi$  of  $G'(\mathbb{A})$ , with parabolic support  $\{P\}$ ,  $(\mathfrak{m}_{G'}, K)$ -cohomological with respect to  $E_{\sigma\mu}$  and such that  $\Xi_f \cong \sigma\Pi_f$ .

**3.3. Rationality fields.** We conclude this article proving a result on the rationality field of a cohomological residual automorphic representation  $\Pi$ . Recall that this field is defined by

$$\mathbb{Q}(\Pi_f) := \{z \in \mathbb{C} \mid \sigma(z) = z \quad \forall \sigma \text{ satisfying } \Pi_f \cong \sigma\Pi_f\}.$$

**Corollary 3.6.** *Let  $G$  be a connected reductive linear algebraic group over a number field  $F$ , which is an inner form of a split algebraic group, and let  $E_\mu$  be a finite-dimensional algebraic representation of  $G_\infty$ . Let  $\Pi$  be a residual automorphic representation of  $G(\mathbb{A})$ , with non-zero  $(\mathfrak{m}_G, K)$ -cohomology below degree  $q_{\max}$ , which does not appear in  $H_1(S, \mathcal{E}_\mu)$ . Then  $\mathbb{Q}(\Pi_f)$  is a number field.*

*If  $G = GL_n/F$ , then we may lighten the assumptions on  $\Pi$  and may take  $\Pi$  to be a residual automorphic representation of  $GL_n(\mathbb{A})$ , with parabolic support  $\{P\}$  and with non-zero  $(\mathfrak{m}_G, K)$ -cohomology in degree  $q_0 < q_{\max}(P)$ . If  $P$  is maximal, then the condition  $q_0 < q_{\max}(P)$  may be dropped without replacement.*

*Proof.* Let  $\Pi$  be any of the representations in the statement of the corollary. Then, for all automorphisms  $\sigma$ , there is a residual automorphic representation  $\Xi$  of  $G(\mathbb{A})$ ,  $(\mathfrak{m}_G, K)$ -cohomological with respect to  $E_{\sigma\mu}$ , such that  $\Xi_f \cong \sigma\Pi_f$  by Thm. 3.1 or Thm. 3.3, respectively. Now fix an open compact subgroup  $K_f \subseteq G(\mathbb{A}_f)$ , such that  $\Pi_f^{K_f}$  is non-zero. Then also  $(\sigma\Pi)_f^{K_f}$  is non-zero. Since  $H^q(S, \mathcal{E}_\mu)^{K_f}$  is finite-dimensional, the result follows from the finiteness of the set  $\{E_{\sigma\mu} : \sigma \in \text{Aut}(\mathbb{C})\}$ .  $\square$

**Remark 3.7.** The above corollary generalizes a result which is well-known for cohomological cuspidal automorphic representations (cf., e.g., Shimura [19], Harder [15] p.80, Waldspurger [20], Cor. I.8.3 and first line of p. 153, Clozel [7] and Grobner–Raghuram [14], Thm. 8.1.).

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HARALD GROBNER: FAKULTÄT FÜR MATHEMATIK, UNIVERSITÄT WIEN, OSKAR-MORGENSTERN-PLATZ 1, A-1090 WIEN, AUSTRIA

*E-mail address:* harald.grobner@univie.ac.at