NOTES ON THE REPRESENTATION THEORY OF LOCAL GROUPS

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1. Algebraic groups and local fields

1.1. Linear algebraic groups.

Definition 1.1. A group $G$ is called linear algebraic (or simply algebraic) if $G$ is a Zariski closed subgroup of $GL_n(\mathbb{C})$ for some $n \geq 1$. I.e., $G$ is the set of zeros of some polynomials. We may think of $G$ as being given by this set of polynomials. In particular, there is an ideal $a$ of $\mathbb{C}[X_{11}, \ldots, X_{nn}, \det((X_{ij}_{i,j})^{-1}] (n^2 + 1 \text{ variables})$ of all polynomials vanishing of $G$.

Definition 1.2. Given a subfield $K \subseteq \mathbb{C}$, we say $G$ is defined over $K$ (or for short symbolically: $G/K$), if $a \cap K[X_{11}, \ldots, X_{nn}, \det((X_{ij}_{i,j})^{-1}$ spans $a$ as an ideal.

Example 1.3. $G = GL_n$ or $G = SL_n := \{g \in GL_n | \det(g) = 1\}$ are defined over $\mathbb{Q}$ (or even any subfield of $\mathbb{C}$).

Assume that $G/K$. If $A$ is some $K$-algebra, then we may consider $G(A)$, the set of solutions of $a$ in $A$. Hence, we may also view $G$ as a functor from the category of $K$-algebras to the category of groups.

1.2. Local and global fields. Let $K$ be a number field, i.e., a finite extension of $\mathbb{Q}$ (e.g., $K = \mathbb{Q}(\sqrt{d})$). It comes with its ring of integers $\mathcal{O} := \{x \in K | f(x) = 0 \text{ for some monic polynomial in } \mathbb{Z}[X]\}$, i.e., the integral closure of $\mathbb{Z}$ in $K$.

Fact (Cf. [11] Thm. 3.1). The ring $\mathcal{O}$ is a Dedekind domain.

In particular, for every $x \in K^*$, there is a unique decomposition $x\mathcal{O} = \prod_p p^{\nu_p(x)}$, the product ranging over the prime ideals $p$ of $\mathcal{O}$. By uniqueness, $\nu_p(x) \in \mathbb{Z}$ is well-defined for every $x \in K^*$ and we may put $\|x\|_p := [\mathcal{O} : p]^{-\nu_p(x)}$. We set in addition $\|0\|_p := 0$, for all prime ideals $p$. The resulting map $\|\|_p : K \to \mathbb{R}$
is called the $p$-adic absolute value.

Now, let $\sigma : K \hookrightarrow \mathbb{C}$ be a field embedding (they are finite in number). In analogy to the case above, we define for $x \in K$, $\|x\|_\sigma := |\sigma(x)|$. It is immediate that with this definition $\|x\|_\sigma = \|\bar{\sigma}(x)\|_\sigma$.

We set

$$S_f := \{p|\text{prime ideal of } \mathcal{O}\}$$

and

$$S_\infty := \{\sigma : K \hookrightarrow \mathbb{C} \text{ up to complex-conjugation}\}$$

and

$$S := S_\infty \cup S_f.$$ 

The set $S$ is called the set of places of $K$. Clearly, its elements correspond 1-to-1 to the above absolute values $\|.,.\|_p$ and $\|.,.\|_\sigma$ and hence $s \in S$ defines a metric on $K$:

$$d_s(x,y) := \|x - y\|_s.$$ 

We consider the topological completion $K_s$ of $K$ with respect to $d_s(.,.)$.

**Fact** (Cf. [14] I §3).

1. $K_s$ is again a field.
2. $K_s \cong \mathbb{R}$, if $s$ corresponds to a real embedding $\sigma : K \hookrightarrow \mathbb{C}$.
3. $K_s \cong \mathbb{C}$, if $s$ corresponds to a complex embedding $\sigma : K \hookrightarrow \mathbb{C}$.
4. $K_s$ is a finite extension of the field of $p$-adic numbers $\mathbb{Q}_p$, if $s$ corresponds to a prime ideal $p$ dividing $p$.

Fields of the above form $K_s$ are called local fields (in contrast to number fields $K$, which are also called global fields). If $s \in S_\infty$, one says $K_s$ is an archimedean local field, while if $s \in S_f$, then one says that $K_s$ is a non-archimedean local field.

Let now $s$ be non-archimedean, i.e., in $S_f$. We may then form the topological completion $\mathcal{O}_s$ of the ring $\mathcal{O}$ as well.

**Fact** (Cf. [11] Prop. 3.8 and Prop. 5.1).

1. $\mathcal{O}_s$ is again a ring. It is compact and open in $K_s$.
2. There exists a unique maximal ideal $p_s$ in $\mathcal{O}_s$.
3. The residue field $k_s := \mathcal{O}_s/p_s$ is finite.

1.3. **Combine Sections 1.1 and [12].** Let $K$ be a number field and $G/K$. Since $K_s$ is a $K$-algebra for all $s \in S$, we may form the group $G(K_s)$. In analogy to $K$, such groups are called local groups. Now, recall that $S = S_\infty \cup S_f$. According to this phenomenon, the groups $G(K_s)$ also “decompose” into two different “worlds” of groups, having different properties:

1. If $s \in S_\infty$, then we get the groups $G(\mathbb{R})$ or $G(\mathbb{C})$, which are real Lie groups ($\hookrightarrow$ manifolds; real analysis)
2. If $s \in S_f$, then we obtain a group $G(K_s)$, which is locally profinite ($\hookrightarrow$ totally disconnected groups; $p$-adic analysis)

The representation theory of these types of groups is of highest importance in many different fields of mathematics.

In a way, they serve as the key examples of locally profinite groups as well as of Lie groups.

2. **Locally profinite groups and basics on their representations**

2.1. **The very basics.** In this section we will consider a class of groups (and their representations), which contains the family of groups of type $G(K_s)$, $s$ being a non-archimedean place of $K$. As mentioned above, these groups are all - what is called - locally profinite.
Definition 2.1. A topological group $G$ is called locally profinite, if the identity element has a neighborhood base consisting of compact, open subgroups.

Fact.
(1) Equivalently, we could have said, a topological group is locally profinite if it is locally compact and totally disconnected. (Cf. [S] Thm. 1)
(2) A closed subgroup of a locally profinite group is locally profinite.
(3) A quotient of a locally profinite group by a closed normal subgroup is locally profinite.

Example 2.2.
(1) $G(\mathbb{K}_s)$, $s \in S_f$, are all locally profinite. In fact, $G(\mathbb{K}_s) \subseteq GL_n(\mathbb{K}_s)$ is a closed subgroup and
\[ K := GL_n(O_s) \]
\[ K_j := 1 + p_j^i M_{n}(O_s) \]
1 being the identity matrix in $GL_n$, $j \geq 1$ and $M_{n}(O_s)$ the $n$-times-$n$ matrices with entries in $O_s$, are all compact and open and form a neighborhood base of the identity $1 \in GL_n(\mathbb{K}_s)$. Hence, $G(\mathbb{K}_s)$ is locally profinite by what we said above.
(2) Every finite and every discrete group is locally profinite.
(3) If $K$ is a compact open subgroup of a locally profinite group, then $K$ is locally profinite.

From now on in this section, unless otherwise stated, $G$ is a locally profinite group and $K$ a compact open subgroup.

2.2. Smooth representations.

Definition 2.3.
(1) A representation of $G$ is a pair $(\pi, V)$ of a $\mathbb{C}$-vector space $V$ (possibly infinite-dimensional) and a group homomorphism $\pi : G \to Aut_{\mathbb{C}}(V)$.
(2) A vector $v \in V$ is called smooth, if its orbit map $c_v : G \to V, g \mapsto \pi(g)v$ is smooth, i.e., there is a compact, open subgroup $K$ such that $c_v(g) = c_v(gk)$ for all $g \in G$ and $k \in K$. A representation $(\pi, V)$ of $G$ is called smooth, if every $v \in V$ is smooth.
(3) A smooth representation $(\pi, V)$ of $G$ is called admissible, if for all compact open subgroups $K$ of $G$, the space of $\pi(K)$-fixed vectors $V^K$ is of finite dimension.
(4) A representation $(\pi, V)$ of $G$ is called irreducible, if $V \neq 0$ and if $V$ has no proper $G$-invariant subspaces.
(5) A representation $(\pi, V)$ of $G$ is said to be finitely generated, if there are vectors $v_i, 1 \leq i \leq n$ in $V$ such that $V$ is spanned over $\mathbb{C}$ by the $\pi(g)v_i$ for $g \in G$ and $1 \leq i \leq n$.
(6) Two representations $(\pi_1, V_1)$ and $(\pi_2, V_2)$ of $G$ are called equivalent (or isomorphic), if there is a vector-space isomorphism $f : V_1 \to V_2$, which is $G$-equivariant (i.e., $f \circ \pi_1(g) = \pi_2(g) \circ f$ for all $g \in G$).

Fact.
(1) Subrepresentations and quotients of smooth representations are smooth.
(2) Any representation $(\pi, V)$ has a maximal smooth subrepresentation $(\pi^\infty, V^\infty)$, by letting $V^\infty$ be the space of all smooth vectors in $V$. (Exercise!)
(3) A character $\chi$ of $G$ is a group homomorphism $\chi : G \to \mathbb{C}^\ast$ with open kernel. Hence, the characters of $G$ exhaust precisely the isomorphism classes of 1-dimensional smooth representations of $G$. 
Proposition 2.4. Let $(\pi, V)$ be a representation of $G$. Then the following conditions are equivalent:

1. $V$ is the sum of its irreducible $G$-subrepresentations
2. $V$ is the direct sum of some irreducible $G$-subrepresentations
3. any $G$-subrepresentation of $V$ has a $G$-invariant complement in $V$

Proof. Before we start off, recall Zorn’s Lemma (in which we shall believe):

If $A$ is a non-empty ordered set, in which every non-empty totally ordered subset has an upper bound, then $A$ has a maximal element.

$(1) \Rightarrow (2)$: Let $V = \sum_{i \in I} U_i$, $U_i$ irreducible. We let $\mathcal{J}$ be the set of subsets $J$ of $I$, such that $\sum_{j \in J} U_j$ is direct. Clearly, $\mathcal{J}$ is non-empty and ordered by inclusion. We show that every non-empty totally ordered subset of $\mathcal{J}$ has an upper bound: So, let $\{J_r, r \in R\}$ be some totally ordered set in $\mathcal{J}$. Put $J := \bigcup_{r \in R} J_r$. If $\sum_{j \in J} U_j$ is not direct, then there is a (finite) subset $J_0$ of $J$ such that $\sum_{j \in J_0} U_j$ is not direct. As $\{J_r, r \in R\}$ is totally ordered, $J_0$ must be contained in some $J_r$. But this is impossible, since all $J_r$ are in $\mathcal{J}$ and hence $\sum_{j \in J_0} U_j$ would have to be direct. Therefore, $J \in \mathcal{J}$ and so every non-empty totally ordered subset of $\mathcal{J}$ has an upper bound. Hence, by Zorn’s Lemma, there exists a maximal element $J_{\text{max}}$ in $\mathcal{J}$. If $\oplus_{j \in J_{\text{max}}} U_j$ were not all of $V$, then there would be an $i \in I/J_{\text{max}}$, such that $\oplus_{j \in J_{\text{max}}} U_j + U_i$ is bigger then the original $\oplus_{j \in J_{\text{max}}} U_j$. But $U_i$ is irreducible, hence the intersection with all $U_j$, $j \in J_{\text{max}}$ must be trivial and so $\oplus_{j \in J_{\text{max}}} U_j + U_i$ would be direct. But this would contradict the maximality of $J_{\text{max}}$, whence $V = \oplus_{j \in J_{\text{max}}} U_j$ as desired.

$(2) \Rightarrow (3)$: Write $V$ as a direct sum, $V = \oplus_{i \in I} U_i$, with $U_i$ irreducible for all $i \in I$, and assume that $W$ is some $G$-subrepresentation of $V$. We let $\mathcal{J}$ be the set of subsets $J$ of $I$, such that $W \cap \oplus_{j \in J} U_j$ is trivial. Again, $\mathcal{J}$ is non-empty and ordered by inclusion and one checks as before that every non-empty totally ordered subset of $\mathcal{J}$ has an upper bound. Hence, there is a maximal element $J_{\text{max}}$ of $\mathcal{J}$. By definition, $W + \sum_{j \in J_{\text{max}}} U_j$ is direct and its intersection with every $U_i$, $i \in I$, must be either trivial or all of $U_i$ (by the irreducibility of the $U_i$). But this implies that it must be $U_i$, since otherwise $J_{\text{max}} \cup \{i\}$ would be an element of $\mathcal{J}$, contradicting the maximality of $J_{\text{max}}$. Thus, $V = W \oplus \oplus_{j \in J_{\text{max}}} U_j$ and so $W$ has a $G$-invariant complement in $V$.

$(3) \Rightarrow (1)$: Let $V_0$ be the sum of all irreducible $G$-subrepresentations of $V$. By (3), there is a $G$-invariant complement $W$ of $V_0$ in $V$. According to the definition of $V_0$ and since $V = V_0 \oplus W$ is direct, $W$ contains no irreducible subrepresentation of $V$. Assume that $W \neq \{0\}$, in order to obtain a contradiction. We first show that $W$ has a non-trivial irreducible subquotient, i.e., there are subrepresentations $W_0 \subset W_1 \subseteq W$ such that $W_1/W_0$ is irreducible. In fact, pick some non-zero $v \in W$ and let $X$ be the finitely generated subrepresentation spanned by $\pi(G)v$: Observe that $X$ cannot be irreducible, since otherwise $W$ would contain an irreducible subrepresentation of $V$. So, $X$ is not irreducible. We apply Zorn’s Lemma to the (then non-empty) set of proper $G$-subrepresentations of $X$, ordered by inclusion: Assume that $\{X_r, r \in R\}$ be a totally ordered subset and consider $X_0 = \bigcup_{r \in R} X_r$ (same trick as before). This is a $G$-subrepresentation of $X$ and it is again proper. Namely, if $X = X_0$ then the generators $v_1, \ldots, v_n$ of $X$ would all be in $X_0$, and so for each $\ell$, $1 \leq \ell \leq n$, there would be an index $r_\ell$ in $R$ such that $v_\ell$ belongs to $X_{r_\ell}$. By the fact that $\{X_r, r \in R\}$ is totally ordered, there is one of the subspaces $X_{r_\ell}$, $1 \leq \ell \leq n$, which contains all the other $X_r$’s. Hence, it would also contain $X$, which leads us to a contradiction, since all spaces $X_r$ are proper. As a consequence, $X$ contains a
maximal proper $G$-subrepresentation $X_{\text{max}}$ and by its sheer maximality, the quotient $X/X_{\text{max}}$ is irreducible. Hence, $W$ has a non-trivial irreducible subquotient $W_1/W_0$ by setting $W_1 = X$ and $W_0 = X_{\text{max}}$. Again, by (3), there is a $G$-invariant complement $W_2$ of $W_0$ in $W$, whence we obtain $V = V_0 \oplus W_0 \oplus W_2$ and hence a $G$-equivariant projection $V \to W_2$. But the image of $W_1$ under this projection is irreducible (because it is isomorphic to $W_1/W_0$) and so $W_2 \subseteq W$ would contain an irreducible subrepresentation of $V$. This is a contradiction and so $W = \{0\}$ as desired.

**Definition 2.5.** A representation $(\pi, V)$ of $G$ satisfying one of the equivalent conditions in Prop. 2.4 is called **completely reducible**.

We shall need the following result from the representation theory of finite groups, saying that finite-dimensional representations are always completely reducible:

**Lemma 2.6.** Let $(\sigma, W)$ be a finite-dimensional representation of a finite group $H$. Then $W$ is the sum of its irreducible subrepresentations.

**Proof.** By Prop. 2.4 we only need to show that every subrepresentation $X$ of $W$ has an $H$-invariant complement $W_0$. Since $W$ is finite dimensional, we may choose a vector space complement $W'$ of $X$ in $W$. Clearly, it comes with a projection $p : W \to X$. Consider

$$p_0 := \frac{1}{|H|} \sum_{h \in H} \sigma(h) \cdot p \cdot \sigma(h)^{-1}.$$  

Since $p$ maps $W$ onto $X$ and $\sigma(h)$ preserves $X$, $p_0$ maps $W$ into $X$. As $\sigma(h)^{-1} x \in X$ for all $x \in X$, $p(\sigma(h)^{-1} x) = \sigma(h)^{-1} x$ and so $p_0$ is even a projection of $W$ onto $X$. (Here we use the normalization factor $1/|H|$). Hence, it comes with a vector space complement $W_0$ to $X$. We claim that $W_0$ is $H$-invariant. Indeed, for arbitrary $h' \in H$

$$\sigma(h') \cdot p_0 \cdot \sigma(h')^{-1} = \frac{1}{|H|} \sum_{h \in H} \sigma(h') \sigma(h) \cdot p \cdot \sigma(h)^{-1} \sigma(h')^{-1} = \frac{1}{|H|} \sum_{h \in H} \sigma(h' h) \cdot p \cdot \sigma(h' h)^{-1} = p_0,$$

whence $p_0 \cdot \sigma(h') = \sigma(h') \cdot p_0$. If now $w_0 \in W_0$, $h' \in H$, then $p_0(\sigma(h')w_0) = \sigma(h')p_0(w_0) = 0$, since $p_0(w_0) = 0$. In other words, $\sigma(h')w_0 \in W_0$ as claimed.

**Lemma 2.7.** Let $(\pi, V)$ be a smooth representation of $G$. Then the space $V$ is the sum of its irreducible $K$-subrepresentations.

**Proof.** Let $v \in V$. Since $V$ is smooth as a representation of $K$, $v$ is fixed by an open subgroup $K'$ of $K$. Further, $K'' = \bigcap_{k \in K/K'} kK'k^{-1}$ is an open normal subgroup of $K$, which, since $K$ is compact, has finite index in $K$ and acts trivially on $W = \langle \pi(k)v, k \in K/K' \rangle$. Observe that, $K/K'$ being finite, implies that $W$ is finite-dimensional. As $K''$ acts trivially on $W$, this representation factors through the finite quotient group $K/K''$. Hence, by Lemma 2.6, $W$ is the sum of its irreducible $K/K''$-subrepresentations. In fact, since $K''$ acts trivially on $W$, the same holds true for $K/K''$ being replaced by $K$ itself. Since $v$ was chosen randomly, the lemma follows.

We may now define the notion of a $K$-isotypic component in a smooth $G$-representation $(\pi, V)$. Therefore, let $K$ be the set of all equivalence classes of irreducible smooth representations of $K$. 

Definition 2.8. If $\rho \in \hat{K}$ and $(\pi, V)$ is a smooth $G$-representation, we let $V^\rho$ be the sum of all irreducible $K$-subrepresentations of $V$, which are isomorphic to $\rho$. The space $V^\rho$ is called the $\rho$-isotypic component of $V$ (or simply a $K$-isotypic component, if one does not want to specialize the particular representation $\rho$).

With this definition, it is clear that $V^1 = V^K$. Here, $1$ is the trivial representation of $K$: $1 : K \to \mathbb{C}^*$, $1(k) = 1$ for all $k \in K$.

Proposition 2.9. Let $(\pi, V)$ be a smooth representation of $G$. Then the space $V$ is the direct sum of its $K$-isotypic components.

Proof. Combining Prop. 2.4 and Lem. 2.7 we may write

$$V = \bigoplus_{\rho \in \hat{K}} U(\rho),$$

for a family of irreducible $K$-subrepresentations of $V$. Putting $U(\rho)$ the sum of those $U_i$, which are of class $\rho$, we obtain

$$V = \bigoplus_{\rho \in \hat{K}} U(\rho).$$

Moreover, if $W$ is an irreducible $K$-subrepresentation of $V$ of class $\rho$, then $W$ must be contained in $U(\rho)$. Indeed, otherwise there would be a non-trivial $K$-homomorphism $W \to U(\tau)$ for some $\tau \neq \rho$. Hence, $V^\rho = U(\rho)$ and the result follows.

2.3. Smoothly induced representations and Frobenius reciprocity. Let $H$ be a closed subgroup of $G$. It is hence locally profinite. Let $(\sigma, W)$ be a smooth representation of $H$. Define the space $\text{Ind}_H^G[\sigma]$ of functions $f : G \to W$, which satisfy

1. $f(hg) = \sigma(h)f(g)$ for all $h \in H$ and all $g \in G$.
2. $f(gk) = f(g)$ for some compact open subgroup $K$ of $G$.
   (This $K$ may depend on $f$.)

Together with the homomorphism

$$\pi : G \to \text{Aut}_\mathbb{C}(\text{Ind}_H^G[\sigma])$$

$$\pi(g)(f) : x \to f(xg)$$

this defines a smooth representation of $G$, called the representation (smoothly) induced from $\sigma$, or just the induced representation of $\sigma$. Moreover, the evaluation of functions $f \in \text{Ind}_H^G[\sigma]$ at the identity $e \in H$, gives a $H$-equivariant homomorphism

$$ev : \text{Ind}_H^G[\sigma] \to W$$

$$ev(f) = f(e).$$

(Exercise!)

The following result is fundamental.

Proposition 2.10 (Frobenius reciprocity). Let $(\pi, V)$ be a smooth representation of $G$ and let $(\sigma, W)$ be a smooth representation of $H$. Composition with the evaluation map induces an isomorphism of vector spaces

$$\text{Hom}_G(\pi, \text{Ind}_H^G[\sigma]) \to \text{Hom}_H(\pi, \sigma)$$

$$\varphi \mapsto ev \circ \varphi.$$
Proof. For $\psi \in \text{Hom}_H(\pi, \sigma)$, we declare $\psi_G \in \text{Hom}_G(\pi, \text{Ind}_H^G[\sigma])$ in the following way: If $v \in V$, then $\psi_G(v)$ shall be the function $g \mapsto \psi(\pi(g)v)$ inside $\text{Ind}_H^G[\sigma]$. Hence, 
$$
(ev \circ \psi_G)(v) = \left[\psi_G(v)\right](e) = \left[\psi(\pi(g)v)\right](e) = \psi(v).
$$
In particular, the assignment $\psi \mapsto \psi_G$ is an inverse to the map $\varphi \mapsto ev \circ \varphi$ and the result follows.

The principle of smooth induction is highly important for classifying the irreducible admissible representations of locally profinite groups of type $G(\mathbb{K}_s)$. Indeed, if $G$ is an algebraic group, which is reductive (i.e., its maximal unipotent normal subgroup is trivial), then the so-called Langlands classification (although mostly due to Harish-Chandra) lists all irreducible admissible representations in terms of triples $(P, \sigma, \nu)$. Here, $P$ is a certain closed subgroup of $G(\mathbb{K}_s)$ (called parabolic), $\sigma$ a special irreducible admissible representation of the maximal reductive subgroup of $P$ (called tempered) and $\nu$ is a vector-parameter. It’s beyond the scope of these notes to give a detailed presentation of this theory, but we refer the interested reader to [2] XI.2.

2.4. Schur’s Lemma. In this section we make the following additional assumption on our locally profinite group $G$:

**Hypothesis.** For every open compact subgroup $K$, the quotient $G/K$ is a countable set of sets.

**Remark 2.11.** If $G/K$ is countable for one open compact subgroup $K$, then it is countable for all open compact subgroups. Indeed, if $K'$ is another open compact subgroup, then $K \cap K'$ is open, compact and of finite index in $K$. Hence, $G/(K \cap K') \twoheadrightarrow G/K$ has finite fibers and so $G/(K \cap K')$ is countable. In particular, $G/K'$ is countable.

The above hypothesis holds for groups of type $G(\mathbb{K}_s)$, $G$ reductive. This follows from the Cartan decomposition of such groups. See [12], II.1.3.

On direct consequence of Hypothesis 2.4 the following

**Fact.** Let $(\pi, V)$ be an irreducible smooth representation of $G$. Then $V$ is of countable dimension. (Exercise!)

This simple observation enables us to show the following, much more fundamental result.

**Proposition 2.12 (Schur’s Lemma).** Let $(\pi, V)$ be an irreducible smooth representation of $G$. Then $\text{End}_G(V)$ is one-dimensional.

**Proof.** For every element $\varphi \in \text{End}_G(V)$, the image and the kernel of $\varphi$ are $G$-invariant subrepresentations of $V$. Hence, by the irreducibility of $V$, every non-trivial $\varphi$ must be bijective, whence $\text{End}_G(V)$ is a complex division algebra. Now, take $0 \neq v \in V$. Again by the irreducibility of $V$, the set $\{\pi(g)v, g \in G\}$ spans $V$, so every $\varphi \in \text{End}_G(V)$ is uniquely determined by its value $\varphi(v)$. Therefore, as $V$ is of countable dimension, also $\text{End}_G(V)$ is of countable dimension. However, if $\varphi \in \text{End}_G(V)$ is not simply given as multiplication by a complex number, then $\varphi$ is transcendental over $\mathbb{C}$. As $\{(\varphi - a)^{-1}, a \in \mathbb{C}\}$ is an uncountable set of linear independent morphisms, $\text{End}_G(V)$ would contain a subspace of uncountable dimension. By what we just said, this is impossible, hence $\text{End}_G(V)$ only consists of those endomorphisms, which are given as multiplication by a complex number. Hence, $\text{End}_G(V)$ is one-dimensional. □
Corollary 2.13. Let \((\pi, V)\) be an irreducible smooth representation of \(G\). Then the center \(Z\) of \(G\) acts by a character, i.e., \(\pi(z)v = \omega_{\pi}(z)v\) for every \(z \in Z\), \(v \in V\) and \(\omega_{\pi}: Z \to \mathbb{C}^*\) a character. In particular, if \(G\) is abelian, then every irreducible smooth representation of \(G\) is one-dimensional.

Proof. For \(z \in Z\), \(\pi(z)\) is an element in \(\text{End}_G(V)\). Hence, by Schur’s Lemma, \(\pi(z)\) is given by the multiplication by a unique, non-trivial complex number \(\omega_{\pi}(z)\). Clearly, \(\omega_{\pi}\) defines a group homomorphism \(Z \to \mathbb{C}^*\). Since \(V\) is smooth, there must be an open compact subgroup \(K\) such that \(VK \neq \{0\}\). Hence, \(\omega_{\pi}\) must be trivial on the open subgroup \(K \cap Z\) of \(Z\). So, \(\omega_{\pi}\) is a character. \(\square\)

The character \(\omega_{\pi}\) is called the central character of \((\pi, V)\).

Further readings: Casselman [5], Bushnell–Henniart [4].

3. Real Lie groups and basics on their representations

3.1. The very basics. We now turn to a different class of groups (and their representations), which contains the family of local groups \(G(K_s)\), \(s\) being an archimedean place of \(K\). As pointed out, the latter groups are all real Lie groups - a notion to be defined now.

Definition 3.1. A group \(G\) is called a real Lie group, if \(G\) is a smooth, real manifold and if the group multiplication \(G \times G \to G\), \((g, h) \mapsto g \cdot h\) is a smooth map. A homomorphism from a real Lie group \(G\) to a real Lie group \(H\) is a smooth map \(G \to H\), which is a group homomorphism.

Remark 3.2. As with manifolds, the most fundamental idea to study the internal nature of a Lie group and its representations, is to pass over to a simpler (namely linear) object: the tangent space. In the case of a Lie group \(G\), the right tangent space to look at is the one at the identity element \(e\) of the group \(G\) in question: \(\mathfrak{g} := T_eG\). In fact, the group structure on \(G\) defines an algebraic structure on \(\mathfrak{g}\), which makes \(\mathfrak{g}\) into a real Lie algebra. For completeness, we shall give the definition.

Definition 3.3. A real algebra \(\mathfrak{g}\) is called a real Lie algebra, if its multiplication \(\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}\), \((X, Y) \mapsto [X, Y]\) satisfies

1. \([X, Y] = -[Y, X]\)
2. \([[[X, Y], Z] + [[Y, Z], X] + [[Z, X], Y] = 0\) (the Jacobi-identity)

for all \(X, Y, Z \in \mathfrak{g}\).

From now on in this section, unless otherwise stated, \(G\) denotes a real Lie group. For simplicity, we shall also assume that \(G\) has only finitely many connected components (which is no restriction in practise).

Fact.

1. It is automatic by the definition of a Lie group that the inversion map \(G \to G\), \(g \mapsto g^{-1}\) is smooth, too. (Exercise!)

2. If \(\varphi: G \to H\) is a homomorphism of Lie groups, then its derivative at the identity element \(\varphi': T_eG \to T_eH\) is a homomorphism of Lie algebras.

3. A closed subgroup \(H\) of \(G\) is a Lie group. (Cartan’s Theorem)

4. The quotient of a Lie group by a closed, normal subgroup is a Lie group.

Example 3.4.

1. Every group of type \(G(K_s)\), \(s \in S_{\infty}\) is a real Lie group. In fact, \(G(K_s)\) being the set of zeros of polynomials in the entries of the matrix group \(GL_n(\mathbb{C})\) is closed in \(GL_n(\mathbb{C})\) (since it is the intersection of preimages of continuous maps of the closed set \(\{0\}\), cf. Def. [1]). The group \(GL_n(\mathbb{C})\) is obviously a real Lie group. Hence, \(G(K_s)\) is a real Lie group by Fact (3) above.
(2) $GL_n(\mathbb{R})$ and $SL_n(\mathbb{R})$ are (hence) real Lie groups, cf. Ex. 1.3

3.2. **General Lie group representations.** One of the main basic differences between the representation theory of locally profinite groups and real Lie groups is the occurrence of functional analysis in the various definitions. This extra ingredient is forced on us by the need to use real analysis, when doing real representation theory.
Definition 3.5.

(1) A representation of \( G \) is a pair \((\pi, V)\) of a locally convex, complete topological \( C \)-vector space \( V \) and a group homomorphism \( \pi : G \to \text{Aut}_C(V) \) such that \( G \times V \to V \) is continuous.

(2) A character of \( G \) is a one-dimensional representation.

(3) A representation \((\pi, V)\) of \( G \) is called irreducible, if \( V \neq 0 \) and if \( V \) has no proper, closed \( G \)-invariant subspaces.

(4) Two representations \((\pi_1, V_1)\) and \((\pi_2, V_2)\) of \( G \) are called equivalent (or isomorphic), if there is a bi-continuous vector-space isomorphism \( f : V_1 \to V_2 \), which is \( G \)-equivariant (i.e., \( f \circ \pi_1(g) = \pi_2(g) \circ f \) for all \( g \in G \)).

(5) A vector \( v \in V \) is called smooth, if its orbit map \( c_v : G \to V, \ g \mapsto \pi(g)v \) is smooth. (Convergence in the seminorms.)

(6) A representation \((\pi, V)\) of \( G \) is called smooth, if every \( v \in V \) is smooth and if the map \( V \to C^\infty(G; V), \ v \mapsto c_v \) is a topological isomorphism onto its image. (Here, as usual \( C^\infty(G; V) \) carries the \( C^\infty \)-topology, i.e., the initial topology with respect to the linear maps \( c^\ast : C^\infty(G; V) \to C^\infty(\mathbb{R}; V), \ f \mapsto f \circ c \). It makes \( C^\infty(G; V) \) into a locally convex vector space.)

Let \( V^\infty \) be the space of all smooth vectors in \( V \). It is certainly a \( G \)-invariant subspace of \( V \), however, in general it is not closed in \( V \), whence not complete. So, it is not a representation for \( G \). However, endowed with the \( C^\infty \)-topology, \( V^\infty \) becomes closed. This is the reason for the additional condition in Def. 3.3 (4). Hence, from every representation \((\pi, V)\), one obtains a smooth representation \((\pi^\infty, V^\infty)\), provided that \( V^\infty \) was equipped with the \( C^\infty \)-topology. We remark that, if \( V \) was a Fréchet space (i.e. countably seminormed), so will be \( V^\infty \) (with its new topology). See, e.g., [2] §0.2.3 and [13] Lem. 1.6.4.

In order to give a definition of an admissible representation of a real Lie group, let \( K \) be a maximal compact subgroup of \( G \) (in terms of inclusion) and let \((\rho, W)\) be a finite-dimensional representation of \( K \). Consider the assignment

\[
\text{Hom}_K(W, V) \otimes W \to V
\]

\[
\tau \otimes w \mapsto \tau(w).
\]

(Here, "\( \text{Hom}^\ast \)" stands for the space of continuous linear maps.) Its image \( V^\rho \) is the \( \rho \)-isotypic component of \((\pi, V)\).

Definition 3.6. A representation \((\pi, V)\) of \( G \) is called admissible, if \( V^\rho \) is finite-dimensional for all \( \rho \).

One can show that if a \( \rho \)-isotypic component of \( V \) is finite-dimensional, then it is contained in \( V^\infty \). In particular, \( V = V^\infty \), if \((\pi, V)\) is admissible.

Exercise 3.7. For a second, let us go back to irreducible smooth representations of a locally profinite group \( G \). Assume that \( K_{\text{max}} \) is a maximal open compact subgroup of \( G \). Show that the notion of admissibility given in Def. 3.3 formally coincides with the notion of admissibility for a real Lie group: I.e., an irreducible smooth representation is admissible, if and only if all of its \( \rho \)-isotypic components, \( \rho \in K_{\text{max}} \), are finite-dimensional.

3.3. Maybe the most important class of representations: Unitary representations. Having given the relevant definitions in full generality, it is convenient to restrict one's attention to unitary representations of \( G \). That means that instead of a general locally convex, complete vector space \( V \), we only allow a (separable) Hilbert space and the operator \( \pi(g) \) has to be unitary for all \( g \in G \).
Remark 3.8. In a way, the family of all unitary representations – although already a very important object of research itself! – almost exhausts the set of all “potentially interesting” representations (at least from the automorphic point of view): Indeed, if $G(\mathbb{K}_s)$, $s \in S_\infty$, is the group of $\mathbb{K}_s$-points of a reductive group, then one can show that an irreducible unitary representation is automatically admissible (Harish-Chandra).

Proposition 3.9. Let $(\pi, V)$ be a unitary representation of $G$. Then $V$ is the orthogonal sum of its $K$-isotypic components.

Strategy of the proof. Formally, the idea of the proof works analogously to the proof of the corresponding result in the situation of a locally profinite group, see Prop. 2.9. One uses again Zorn’s Lemma, in order to choose a maximal family of orthogonal, irreducible, finite-dimensional $K$-subrepresentations of $V$. Assuming that their orthogonal sum $U$ is not all of $V$ leads once again to a contradiction, by constructing a finite-dimensional $K$-subrepresentation in the non-trivial orthogonal complement $U^\perp$ of $U$. However, the methods used to show this in the case of real Lie groups are different: One uses spectral theory for unitary operators on Hilbert spaces and the Peter-Weyl Theorem. For a detailed proof see [9], IX Thm. 9.4. □

Corollary 3.10. An irreducible unitary representation of a compact group is finite-dimensional.

Proof. This is immediate from Prop. 3.9 since every compact group is its own maximal compact subgroup. □

3.4. Unitarily induced representations. Recall the notion of a left-invariant measure on $G$: This is a positive measure $dg$ on $G$ such that

$$\int_G f(xg) dg = \int_G f(g) dg,$$

for all $x \in G$ and all compactly supported, continuous functions $f \in C_c(G)$. Such a measure always exists for $G$ by Haar’s Theorem and is furthermore unique up to a scalar. Consider the new measure $d(gh)$ for $h \in G$. Obviously, it is again left-invariant, so by uniqueness there is a (actually positive) real number $\delta(h)$ such that $d(gh) = \delta(h) dg$. If the modular function $\delta$ is constantly 1, one says that $G$ is unimodular.

Let us from now on assume in this subsection that $G$ is unimodular. Fix some closed subgroup $P$ of $G$ such that $G = PK$. We let $\delta_P$ be the modular function of $P$ and assume to have fixed a left-invariant measure $dp$ on $P$. On $K$, we choose a normalized invariant measure $dk$, i.e., the volume of $K$ with respect to this measure is 1. One can show that in this setup, there is an invariant measure $dg$ on $G$ such that

$$\int_G f(g) dg = \int_{P \times K} f(pk) dp dk,$$

for all $f \in C_c(G)$. Furthermore, we may extend $\delta_P$ to all of $G$ by setting $\delta_P(g) := \delta_P(pk) := \delta_P(p)$ for a decomposition $g = pk$. This is OK, since $\delta_P(K \cap P) = \{1\}$.

Let $(\sigma, W)$ be a unitary representation of $P$. Consider the space of continuous functions $f : G \to W$

$$\text{ind}_P^G[\sigma] := \{ f : G \to W | f(pg) = \delta_P(p)^{1/2} \sigma(p) f(g) \},$$

for all $p \in P$ and all $g \in G$. We equip it with an inner product

$$\langle f_1, f_2 \rangle := \int_K \langle f_1(k), f_2(k) \rangle W dk,$$
being the inner product making $W$ into a Hilbert space. Finally, we let \( \Ind_G^P[\sigma] \) be the Hilbert space completion of \( \text{ind}_G^P[\sigma] \). It is acted upon by $G$ by right-translation of functions:

$$\pi_{\sigma}(g)f(h) := (g \cdot f)(h) := f(gh).$$

We obtain

**Proposition 3.11.** For all $g \in G$, $\pi_{\sigma}(g)$ is a bounded operator on the Hilbert space $\Ind_G^P[\sigma]$. Moreover, the pair $(\pi_{\sigma}, \Ind_G^P[\sigma])$ is a unitary representation of $G$.

**Proof.** See Wallach [13], Lem. 1.5.3. □

The so-defined unitary representation $(\pi_{\sigma}, \Ind_G^P[\sigma])$ is called the representation unitarily induced from $\sigma$. It is for the sake of keeping the resulting induced representation unitary that we had to introduce the modular function $\delta_P$ in its definition (although it is suppressed in the notation).

As in the case of locally profinite groups, the principle of unitary induction is of high importance in the classification of the irreducible admissible representations of real Lie groups of type $G(\mathbb{K}_s)$, $G$ reductive. Indeed, if $G$ is a reductive algebraic group, then the so-called Langlands classification lists all irreducible admissible representations in terms of triples $(P, \sigma, \nu)$. Here, $P$ is a certain closed subgroup of $G(\mathbb{K}_s)$ (called parabolic), $\sigma$ a special irreducible admissible representation of the maximal reductive subgroup of $P$ (called tempered) and $\nu$ is a vector-parameter. It’s beyond the scope of these notes to give a detailed presentation of this theory, but we refer the interested reader to [2] IV. As a final remark let us point out that it is still unclear, which Langlands triple $(P, \sigma, \nu)$ precisely parameterize the unitary representations among the admissible ones.

### 3.5. Schur’s Lemma

**Proposition 3.12** (Schur’s Lemma). Let $(\pi, V)$ be an irreducible unitary representation of $G$. Then the space of continuous linear, $G$-equivariant endomorphisms on $V$, $\text{End}_G(V)$, is one-dimensional.

**Proof.** This is easily proved using spectral theory for unitary operators on Hilbert spaces. See [13], Lem. 1.2.1 for details. □

**Corollary 3.13.** Let $(\pi, V)$ be an irreducible unitary representation of $G$. Then the centre $Z$ of $G$ acts by a character, i.e., $\pi(z)v = \omega_{\pi}(z)v$ for every $z \in Z$, $v \in V$ and $\omega_{\pi} : Z \to \mathbb{C}^*$ a character. In particular, if $G$ is abelian, then every irreducible unitary representation of $G$ is one-dimensional.

**Proof.** For $z \in Z$, $\pi(z)$ is an element in $\text{End}_G(V)$. Hence, by Schur’s Lemma, $\pi(z)$ is given by the multiplication by a unique, non-trivial complex number $\omega_{\pi}(z)$. Clearly, $\omega_{\pi}$ defines a group homomorphism $Z \to \mathbb{C}^*$. □

The character $\omega_{\pi}$ is called the central character of $(\pi, V)$.

**Further readings:** Wallach [13], Knapp [9], Borel-Wallach [2].
4. A short remark on the representation theory of global groups

Let us now try to combine the (at the same time similar and different) theories of representations of non-archimedean and archimedean local groups $G(K_s)$. One may form a so-called global object, $G(A)$, the group of adelic points of $G$, a topological group, which contains each local group $G(K_s)$ as a closed subgroup. It hence consists out of two factors $G(A) = G_\infty \times G(A_f)$, where $G_\infty$ is the product over all real Lie groups $G(K_s)$, $s \in S_\infty$ and $G(A_f)$ is a totally disconnected, locally compact group, which contains each non-archimedean local group $G(K_s)$, $s \in S_f$. Therefore, the representation theory of this (huge) group $G(A)$ combines both of the representation-theories, which we have just shortly considered.

Assume that $G$ is reductive, e.g., $G = GL_n$. Then it is at the heart of the theory of automorphic forms to study the representations of $G(A)$. This field deals at the same time with (generalizations) of the Riemannian Conjecture, as well as of Fermat's Last Theorem.


References


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