Finding Triangles With Given Circum-medial Triangle

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Online Appendix to a note in *The Mathematical Gazette 104, 559, 164 – 168, March 2020 DOI: 10.1017/mag.2020.23*

A proof of Theorem 1 based on Euclidean Geometry

Theorem 1: Given ΔABC and a point G_1 not on its circumcircle Σ, let $\Delta A_1B_1C_1$ be its circumcevian triangle w.r.t. G_1. Then G_1 is the centroid of $\Delta A_1B_1C_1$ if, and only if, it is a focus of the Steiner inellipse of ΔABC.

In this proof we want to restrict the methods essentially to Euclidean geometry, because this is obviously the area of Theorem 1.

We found in the www a helpful statement with solution (Lemma 1) and we build on this "basis" a proof of Theorem 1 with formulating two further lemmas (2 and 3). At the first glance these lemmas have very little to do with Theorem 1, the things will be brought together at the end. The reader needs a bit of patience (especially after reading the one page proof in the printed article), because in all three following lemmas one cannot see immediately their connection to Theorem 1. I would be pleased to hear or read a shorter and more elementary proof from a reader, using only Euclidean geometry.

Lemma 1[1]: The circumcenter O of ΔABC is the centroid of the antipedal triangle $\Delta A'B'C'$ of the symmedian point[2] L of ΔABC (see Fig. 3).

We found this statement and a solution in (there one can find also a short proof):

https://artofproblemsolving.com/community/c6h440326

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[1] For reasons of clarity one can split up this Lemma 1 into the Lemmas 1a – 1d (see below for interested readers). Fig. 1 and 2 are in the printed version (The Mathematical Gazette, March 2020), therefore the list of figures here in this online appendix starts with number 3.

[2] This point is also called *Lemoine point*, therefore the letter L.

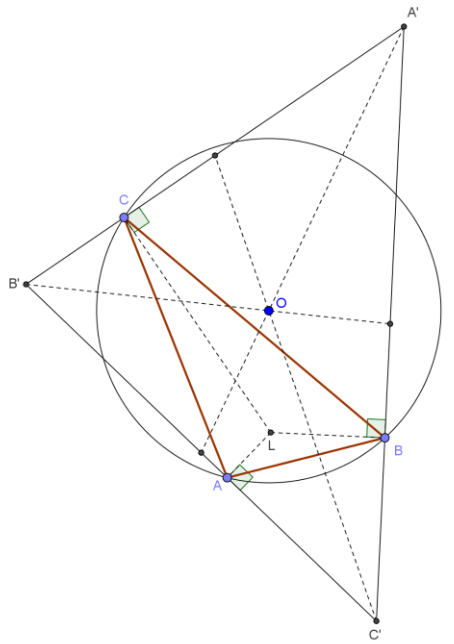

Fig. 3: O is the centroid of $\Delta A'B'C'$

Lemma 2: Let ΔPQR be a triangle with centroid G_1 and symmedian point L. The circum-medial triangle is ΔABC and the pedal triangle of G_1 w.r.t. ΔABC is ΔXYZ. Then G_1 is the symmedian point of ΔXYZ (see Fig. 4).

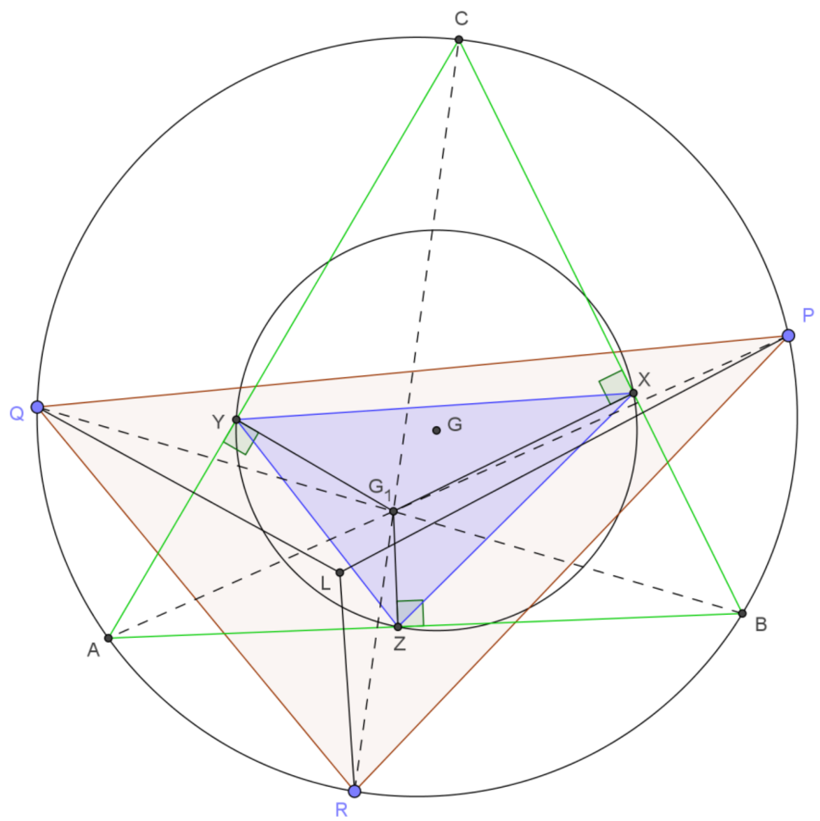

Fig. 4: G_1 is symmedian point of ΔXYZ, centroid G of ΔABC is circumcenter of ΔXYZ

ΔPQR is the circumcevian triangle of ΔABC w.r.t. G_1. We will show that $\Delta RPL \sim \Delta ZXG_1$ and $\Delta PQL \sim \Delta XYG_1$. Then we have the similarity $\Delta XYZ \sim \Delta PQR$ and G_1 is the symmedian point.

For $\Delta RPL \sim \Delta ZXG_1$ we have to prove that $\measuredangle XZG_1 = \measuredangle PRL$ and $\measuredangle G_1XZ = \measuredangle LPR$.

From the cyclic quadrilateral $ZBXG_1$ and the inscribed angle theorem we get $\measuredangle XZG_1 = \measuredangle XBG_1 = \measuredangle CBQ = \measuredangle CRQ = \measuredangle G_1RQ \underset{\text{symmedian!}}{\overset{RL\,\text{is}}{=}} \measuredangle PRL$, analogously we get $\measuredangle G_1XZ = \measuredangle G_1BZ = \measuredangle QBA = \measuredangle QPA = \measuredangle QPG_1 \underset{\text{symmedian!}}{\overset{PL\,\text{is}}{=}} \measuredangle LPR$.

Analogously one can prove $\Delta PQL \sim \Delta XYG_1$.

Lemma 3: It is well known and easy to see by analytical means that the locus of the points X with the property "the tangents from X onto an ellipse and the line segment from X to a focus of the ellipse are perpendicular at X" is a circle with radius = semi-major axis of the ellipse and center = center of the ellipse (see Fig. 5, this circle is also called the *pedal curve* of the ellipse w.r.t. the *focus*, also *pedal circle*).

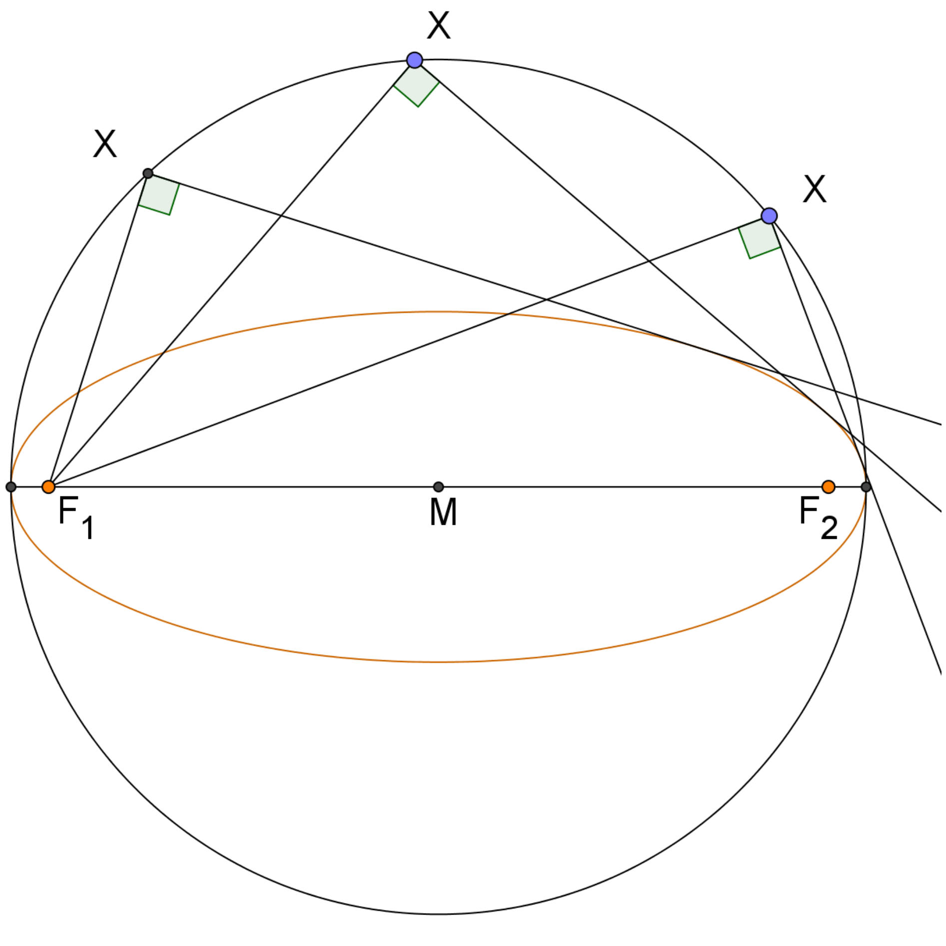

Fig. 5: Pedal curve w.r.t. F_1, pedal circle

Now we put these three lemmas – seemingly unconnected to the matter in hand – together and can **prove Theorem 1:** Let ΔPQR be a ("unknown") triangle with centroid G_1, symmedian point L and the property "its circum-medial triangle is ΔABC". Then by Lemma 2 we know that G_1 is the symmedian point of ΔXYZ (Fig. 4). The antipedal triangle of G_1 w.r.t. ΔXYZ is ΔABC by construction. Then Lemma 1 implies that the centroid G of ΔABC is the circumcenter of ΔXYZ (Fig. 4), i.e. the center of the pedal circle of G_1 w.r.t. ΔABC. According to Lemma 3 G_1 is a focus of an ellipse with center G and tangent to the sides AB, AC, BC. Therefore, it must be the Steiner inellipse of the triangle ΔABC tangent to its sides through their midpoints (an ellipse tangent to the triangle sides with center G can be mapped by an affine transformation to the incircle of an equilateral triangle). The possible points for G_1 are the two foci of this special ellipse. For the "unknown" triangle ΔPQR there are two "solutions", the *circumcevian* triangles of G_1 and G_2 (see Fig. 2 in printed article, The Mathematical Gazette, March 2020, here again):

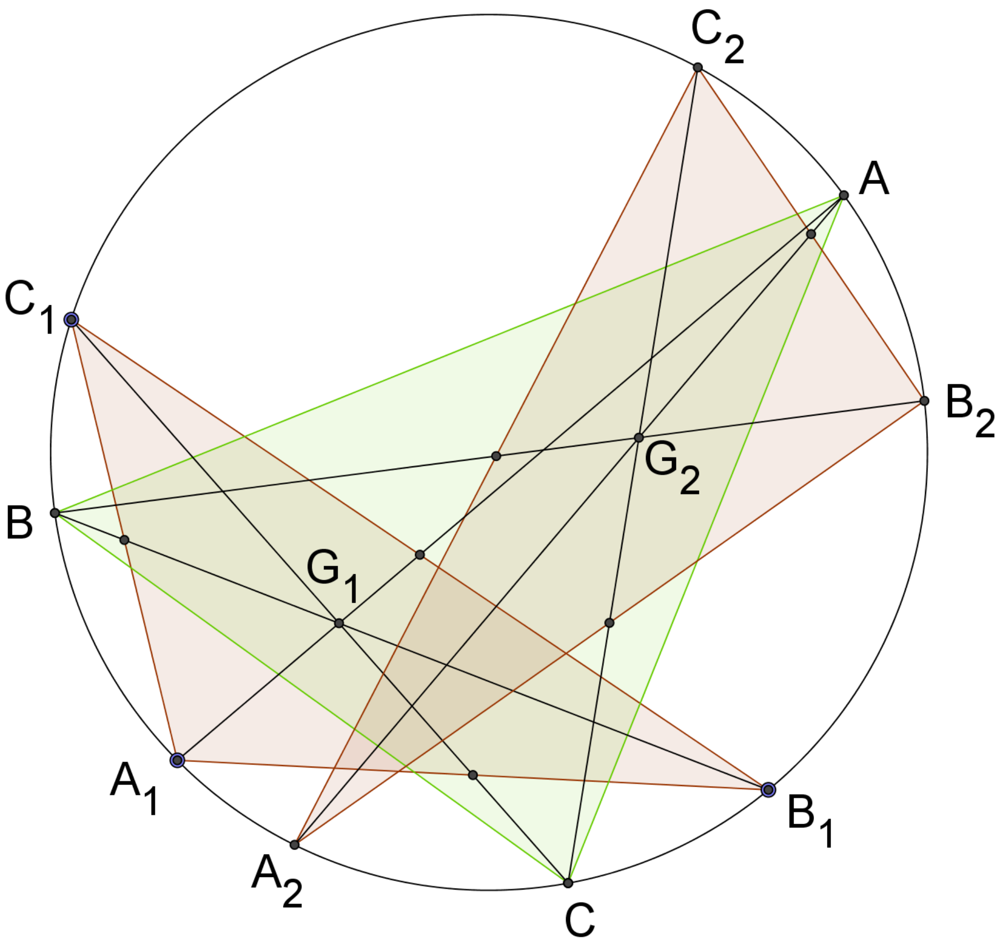

Fig. 2: G_1, G_2 are the foci of the Steiner inellipse of ΔABC

Here a split up version of Lemma 1, in order to make the short solution that can be found at https://artofproblemsolving.com/community/c6h440326 more easily to follow.

Lemma 1a: A point X lies on the C-median of a triangle if, and only if, the distances to the sides BC and AC are reciprocal to the side lengths themselves: $\frac{a}{b}=\frac{j}{i}$ (see Fig. 6)

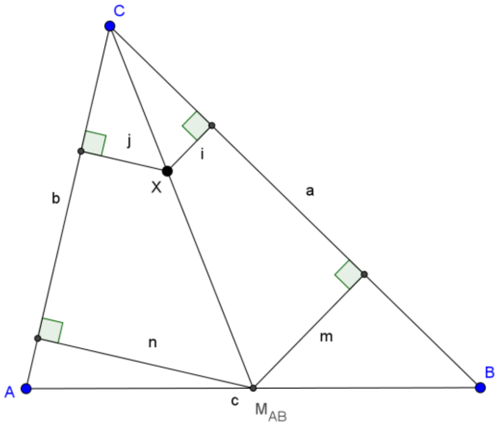

Fig. 6: Point on the C-median

The proof follows immediately by the fact that the median bisects the area of the triangle and $\frac{n}{m}=\frac{j}{i}$.

Lemma 1b: Given a triangle ΔABC. The pedal triangle ΔXYZ of the centroid G and the antipedal triangle $\Delta A'B'C'$ of the symmedian point L are homothetic (see Fig. 7).

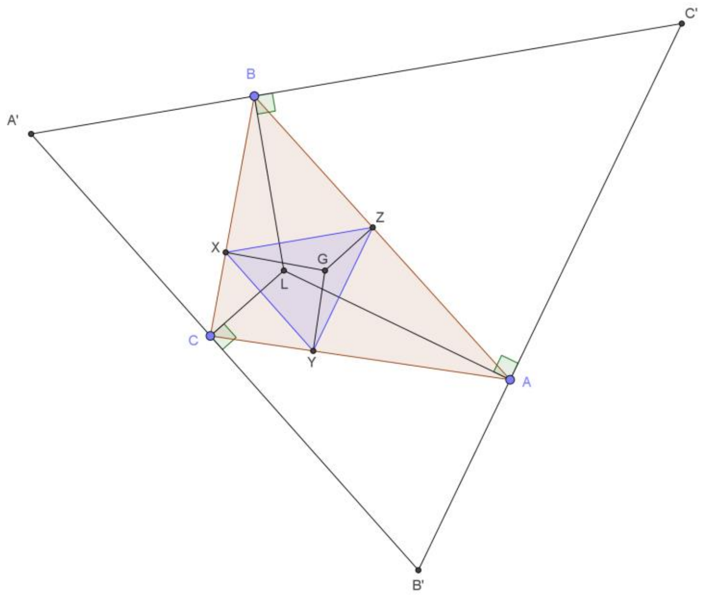

Fig. 7: Homothetic triangles

This is clear because XZ is perpendicular to BL (see [5, p. 14f, Theorem 6], see also [6, p. 64f], therefore $XZ \parallel A'C'$ (analogous for the other sides).

Lemma 1c: Let ΔABC be a triangle, P an arbitrary interior point and X, Y, Z the orthogonal projections of P onto the triangle sides. Then the following equation holds: $\frac{|AB|}{|AC|}\cdot\frac{|PC|}{|PB|}=\frac{|XY|}{|XZ|}$ (see Fig. 8)

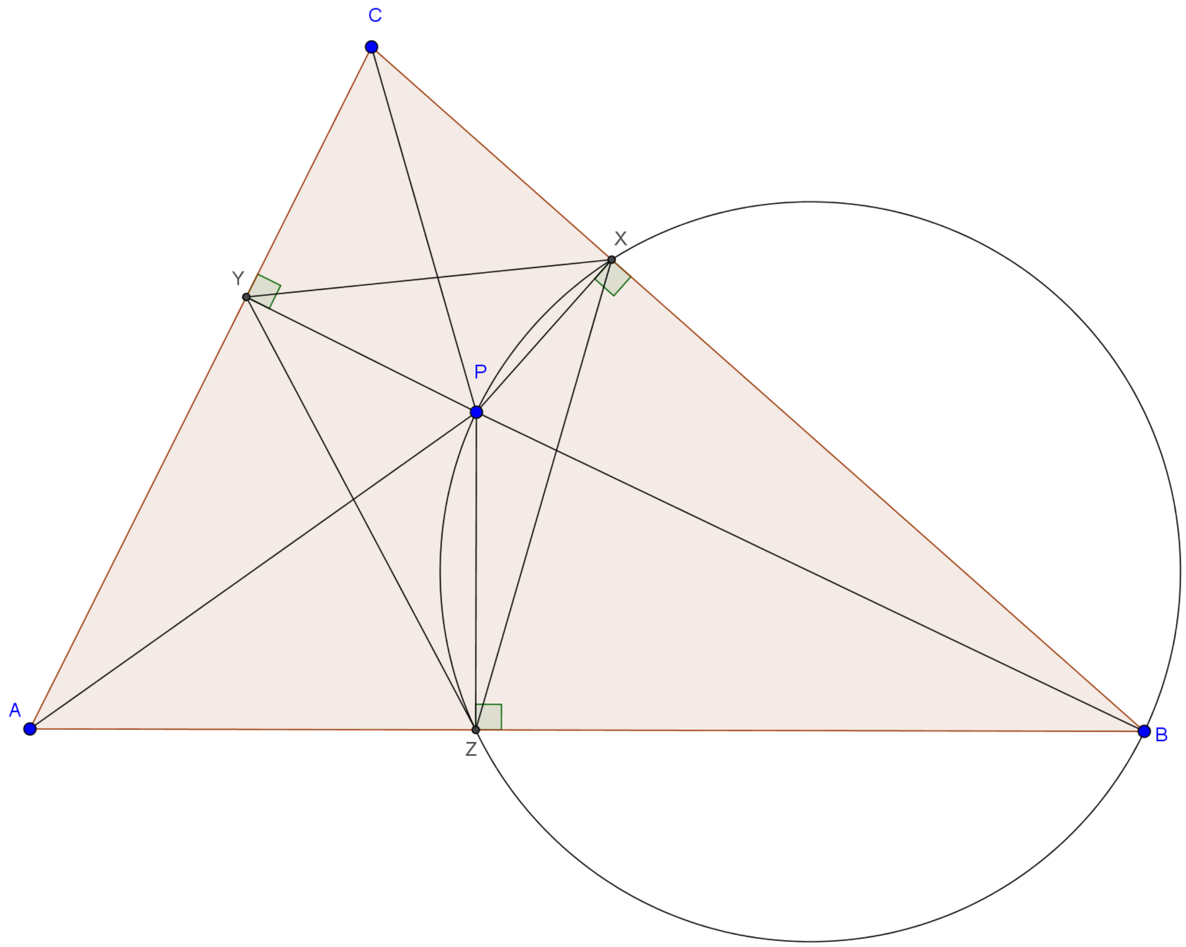

Fig. 8: Cyclic quadrilateral

$ZBXP$ is a cyclic quadrilateral, PB is a diameter of the circle and the angle $\measuredangle ZBX$ is an inscribed angle to the chord ZX. From the inscribed angle theorem we get $\frac{|XZ|}{|PB|}=\sin(\beta)$, and analogously $\frac{|XY|}{|PC|}=\sin(\gamma)$. According to the law of sines we have $\frac{\sin(\gamma)}{\sin(\beta)}=\frac{|AB|}{|AC|}$ and this yields the equation we want to prove.

Lemma 1d: Let G, O, L be the centroid, circumcenter and symmedian point of ΔABC. E, F are the midpoints of AC, AB and $\Delta A'B'C'$ is the antipedal triangle of the symmedian point L of ΔABC. Then we have $\measuredangle OBA' = \measuredangle BEA$ and $\measuredangle OCA' = \measuredangle CFA$ (see Fig. 9).

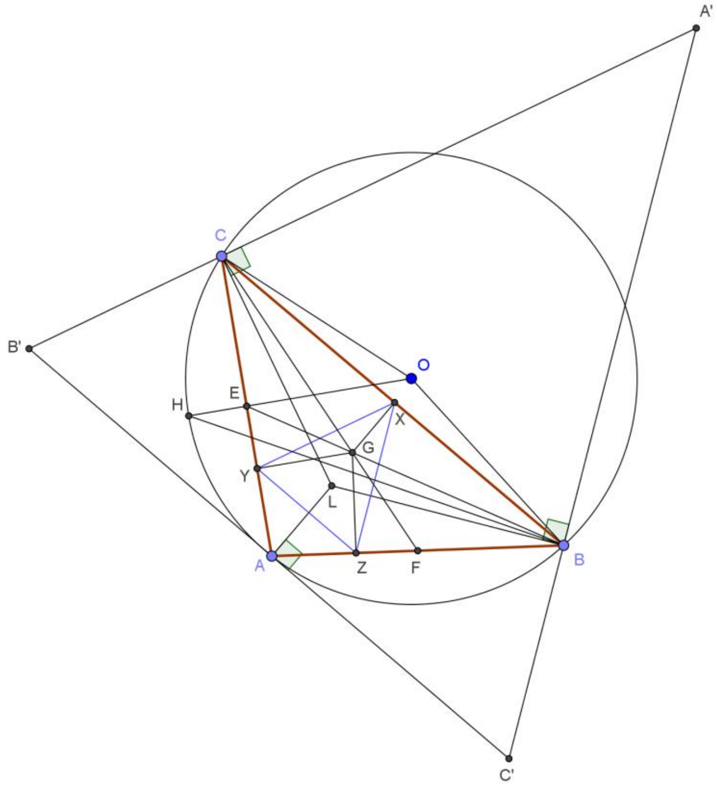

Fig. 9: Lemma 1d

The angle bisector at B meets the perpendicular bisector of AC at the point H on the circumcircle of ΔABC. Then we have

$$\begin{aligned}\measuredangle OBA' &= 90° - (\measuredangle OBH + \measuredangle HBL) = \\ &= 90° - (\measuredangle OHB + \measuredangle EBH) = \\ &= 90° - \measuredangle OEB = \measuredangle BEA\end{aligned}$$

and analogously $\measuredangle OCA' = \measuredangle CFA$.

Then we can prove Lemma 1 step by step: With $\mathrm{d}(O, A'C')$ we denote the distance between the point O and the line segment $A'C'$:

$$\frac{\mathrm{d}(O,A'C')}{\mathrm{d}(O,A'B')} \overset{\text{Lemma 1d}}{=} \frac{\sin(\measuredangle GEY)}{\sin(\measuredangle GFZ)} = \frac{|GY|}{|GZ|} \cdot \frac{|GF|}{|GE|} = \frac{|AB|}{|AC|} \cdot \frac{|GC|}{|GB|} \overset{\text{Lemma 1c}}{=} \frac{|XY|}{|XZ|} \overset{\text{Lemma 1b}}{=} \frac{|A'B'|}{|A'C'|}$$

With Lemma 1a we can conclude that $A'O$ is the A'-median of $\Delta A'B'C'$. Similarly $B'O$, $C'O$ are the medians issuing from B', C'. Thus we have proven that O is the centroid of $\Delta A'B'C'$.

References:

[5] Grinberg, D.: Isogonal conjugation with respect to a triangle. http://www.cip.ifi.lmu.de/~grinberg/geometry2.html

[6] Honsberger, R. (1995): Episodes in Nineteenth and Twentieth Century Euclidean Geometry. The Mathematical Association of America.