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A Note on the Stammler Hyperbola

Franz Embacher and Hans Humenberger

Abstract. An alternative characterization of the Stammler hyperbola of a triangle is given.

The Stammler hyperbola of a scalene triangle is the unique conic section on which the incenter, the three excenters, and the circumcenter lie [2–4]. (In case of an isosceles but nonequilateral triangle the hyperbola collapses to a pair of straight lines.) In this note, we show how the Stammler hyperbola appears in a different context.

An arbitrary interior point P of an acute or rectangular triangle ΔABC gives rise to a partition of the triangle into six pieces, separated by the line segments from P to the vertices and the perpendiculars from P to the sides. We will call this a P-*induced partition* of the triangle.

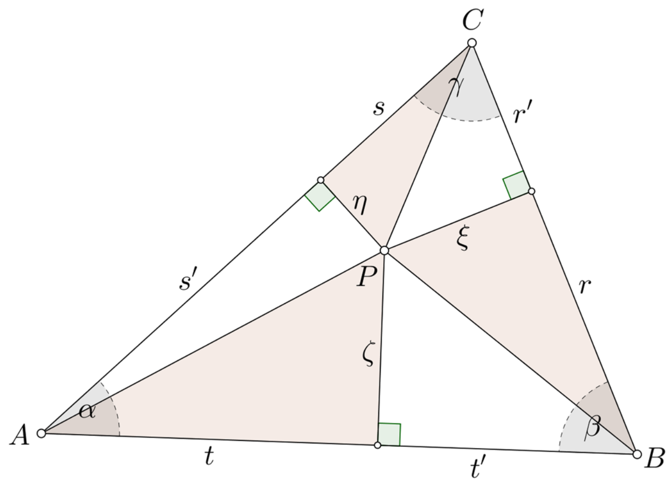

Figure 1. P is an arbitrary interior point of the acute triangle ΔABC, inducing a partition of the triangle into six pieces. The two groups of alternate pieces are distinguished by different shading. The true trilinear coordinates of P—i.e., its normal distances to the triangle sides—are denoted by ξ, η, and ζ. The triangle sides are divided into two subsegments each, their lengths being denoted by r, r', s, s', t, and t'. The angles of the triangle are denoted by α, β, and γ.

As illustrated in Figure 1, the six pieces of the original triangle fall into two groups (distinguished by different shading), each group consisting of three subtriangles sharing no common line segment. We may then ask for which locations of P the sums of the areas of the pieces of each group are equal. We will call the P-induced partition *fair* if this is the case.

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Theorem 1. *The P-induced partition of an acute or right scalene triangle is fair if and only if P lies on the Stammler hyperbola of that triangle.*

Proof. We use the notation shown in Figure 1. The condition for the partition to be fair obviously reads

$$\frac{1}{2} r\,\xi + \frac{1}{2} s\,\eta + \frac{1}{2} t\,\zeta = \frac{1}{2} r'\,\xi + \frac{1}{2} s'\,\eta + \frac{1}{2} t'\,\zeta, \tag{1}$$

which may be rewritten as

$$(r - r')\,\xi + (s - s')\,\eta + (t - t')\,\zeta = 0. \tag{2}$$

Some elementary trigonometry reveals that

$$r = \frac{\zeta + \xi\ \cos\beta}{\sin\beta}, \tag{3}$$

$$r' = \frac{\eta + \xi\ \cos\gamma}{\sin\gamma}, \tag{4}$$

and three similar expressions for s, s', t, and t' obtained by cyclic permutation of the respective quantities. Inserting these into (2), we first collect all expressions containing only the angle α:

$$-s'\,\eta + t\,\zeta = -\frac{\zeta\,\eta + \eta^2 \cos\alpha}{\sin\alpha} + \frac{\eta\,\zeta + \zeta^2 \cos\alpha}{\sin\alpha} = \left(\zeta^2 - \eta^2\right)\cot\alpha. \tag{5}$$

The contributions $r\,\xi - t'\,\zeta$ and $-r'\,\xi + s\,\eta$ now follow simply by cyclic permutation. Collecting everything, the fairness condition (2) takes the form

$$\left(\cot\beta - \cot\gamma\right)\xi^2 + \left(\cot\gamma - \cot\alpha\right)\eta^2 + \left(\cot\alpha - \cot\beta\right)\zeta^2 = 0. \tag{6}$$

This is the equation of a conic section in trilinear coordinates (see [**5**, pp 154–158] or [**6**]). In order to identify it, we allow exterior locations of P as well, i.e., trilinear coordinates $\xi : \eta : \zeta$ that are not necessarily all positive. Four points lying on this conic section are readily found:

(i) the incenter I of the triangle whose trilinear coordinates are $1 : 1 : 1$;

(ii) the excenters of the triangle whose trilinear coordinates are $-1 : 1 : 1$, $1 : -1 : 1$, and $1 : 1 : -1$, respectively.

All these points satisfy $\xi^2 = \eta^2 = \zeta^2$, thus rendering the left-hand-side of (6) identically zero. A fifth point meeting the condition (6) is the circumcenter O of the triangle. Its trilinear coordinates are given by [**1**, **6**]

$$\cos\alpha : \cos\beta : \cos\gamma. \tag{7}$$

Inserting them into (6) and using $\alpha + \beta + \gamma = \pi$ shows that (6) also holds in this case. Since five distinct points uniquely determine a conic section, (6) is the equation of the Stammler hyperbola.

Thus we have shown that the set of all points P inducing a fair partition of the triangle are exactly those points on the Stammler hyperbola that lie in the interior of the given triangle. ■

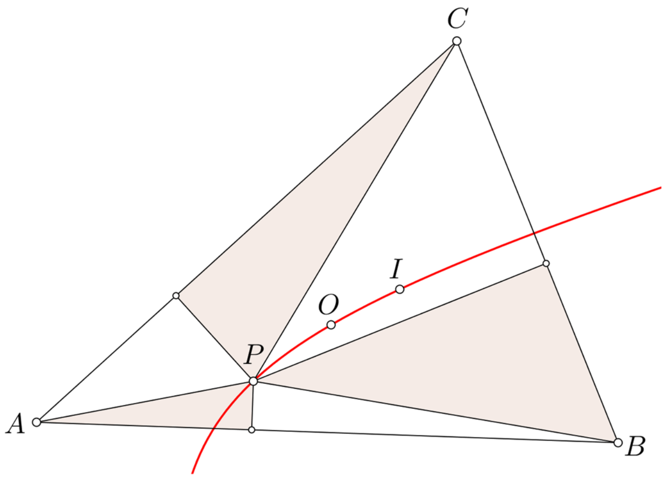

Figure 2. The P-induced partition of the acute scalene triangle ΔABC is fair if and only if P lies on the Stammler hyperbola.

In Figure 2 a configuration with P on the Stammler hyperbola is shown.

Remark (Generalization to arbitrary scalene triangles). So far, we have assumed the triangle ΔABC to be acute or rectangular. Allowing ΔABC to have an obtuse angle, we face the problem that one of the perpendiculars from P to the side lines may partially fall outside the triangle. This spoils the beautiful concept that every interior point P gives rise to a partition of the triangle into six pieces as shown in Figure 1. A simple possibility to generalize the previous result to arbitrary scalene triangles is simply to exclude all interior points for which that problem occurs. Hence, we call an interior point P *admissible* if all three perpendiculars from P to the side lines lie inside the triangle. Then

(i) any admissible point P induces a partition of the triangle, and

(ii) if this partition is fair, then P lies on the Stammler hyperbola of that triangle.

In order to include the reverse direction we must take into account that if the triangle has an obtuse angle the Stammler hyperbola contains nonadmissible interior points. Hence

(iii) an admissible point P induces a fair partition if and only if it lies on the (obviously defined) admissible part of the Stammler hyperbola.

Another way of generalizing our alternative characterization of the Stammler hyperbola to arbitrary scalene triangles—on a formal level—is to retain (3), (4), and the analogous expressions for s, s', t, and t' for every interior point P. However, since one of these quantities may become negative, they should be interpreted as *oriented* lengths. As a consequence, the fairness condition (1) may be interpreted as a balance equation in terms of *oriented* areas instead of areas. If, for example, the angle γ is obtuse, and P is chosen such that $r' < 0$ (see Figure 3), then the quantity $\frac{1}{2}r'\xi$ in (1) is negative. It represents the oriented area of the rectangular triangle ΔPCD, where D is the intersection of the perpendicular from P to BC with BC (i.e., the pedal point corresponding to the trilinear coordinate ξ). All the other five terms in (1) correspond to subtriangles of ΔABC and contribute positive areas. Since D lies outside of ΔABC, the triangle ΔPCD is not a subtriangle of ΔABC. Thus, although in such a case the

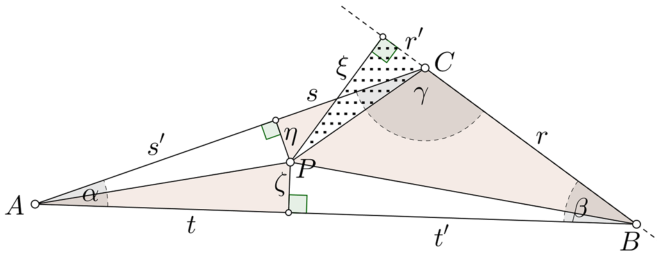

Figure 3. If the angle γ is obtuse, and P is chosen such that $r' < 0$ then the dotted area, represented by the quantity $\frac{1}{2}r'\,\xi$, counts as negative. In such a case the fairness condition (1) still holds in a formal sense, but it is no longer associated with a partition of ΔABC.

fairness condition is rescued in a formal sense and still leads to the Stammler hyperbola, it is no longer associated with a partition of ΔABC. In much the same way, this scheme may be generalized to points P outside the original triangle (in which case one or two of the trilinear coordinates ξ, η, and ζ are nonnegative), leading to a characterization of its complete Stammler hyperbola.

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