104.08 Finding triangles with given circum-medial triangle

Let $\Delta A_1B_1C_1$ be an arbitrary triangle and P an arbitrary point not on its circumcircle Σ. The 'other' intersection points of the 'cevians' A_1P, B_1P, C_1P with Σ, the points A, B, C, make the so-called *circumcevian triangle* of $\Delta A_1B_1C_1$ with respect to P. If P is the centroid of $\Delta A_1B_1C_1$ then ΔABC is called the *circum-medial triangle* * of $\Delta A_1B_1C_1$ (see Figure 1: P = centroid G_1).

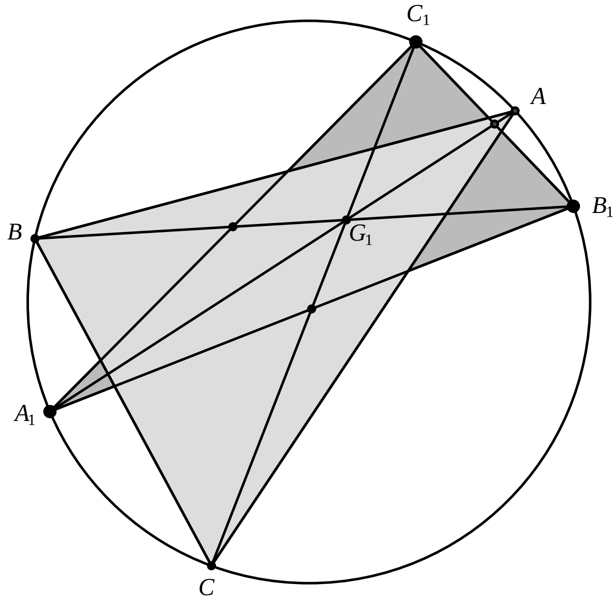

FIGURE 1: How to reverse the process of constructing the circum-medial triangle?

A recent paper [1] was concerned with iterating the construction of the circum-medial triangle (it turned out that the shape of the triangles converges to equilaterality). Here, we reverse the process. Specifically, given ΔABC we look for $\Delta A_1B_1C_1$ such that ΔABC is the circum-medial triangle of $\Delta A_1B_1C_1$. It turns out that there are in general two solutions, namely, the circumcevian triangles of ΔABC with respect to the foci of the Steiner inellipse of ΔABC.

Theorem 1: Given ΔABC and a point G_1 not on its circumcircle Σ, let $\Delta A_1B_1C_1$ be its circumcevian triangle with respect to G_1. Then G_1 is the centroid of $\Delta A_1B_1C_1$ if, and only if, it is a focus of the Steiner inellipse of ΔABC.

Remark: Since an ellipse has two foci, this yields the two mentioned solutions (see Figure 2), except in the case where ΔABC is equilateral, when

* Unfortunately, not every language has such short and precise terms for the corresponding triangles. For instance in German there is no such term, one would have to describe the whole process with many words.

the ellipse becomes a circle and the foci coincide at the circumcentre. And so, in this case, the two solution triangles also coincide, $\Delta A_1B_1C_1$ being also equilateral, with A_1, B_1, C_1 the points on Σ diametrically opposite A, B, C, respectively.

Searching in the literature for references concerning Theorem 1 and its proof gave us the impression that it is not very widely spread or known. This is the reason for this paper in which we want to give a short and straightforward proof. A second proof involves only Euclidean geometry, but it is rather long compared to the first proof. This is the reason why it is

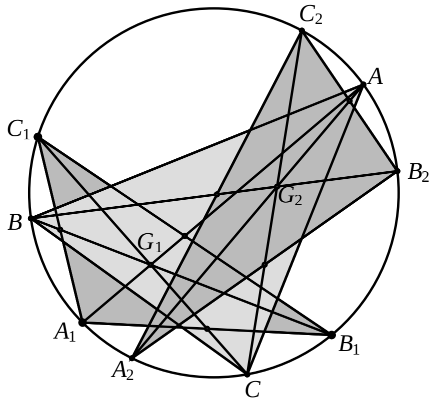

FIGURE 2: G_1, G_2 are the foci of the Steiner inellipse of ΔABC

placed in an online appendix to this printed article. Why should one be interested in another proof when there is a straightforward, elegant, and short one?

The answer is: This is a question of the used means. The first proof uses complex numbers and a strong and beautiful result. On the one hand this connection is beautiful itself (one can be fascinated how complex numbers 'fit to geometry'), on the other hand the problem (Theorem 1) is definitely a problem of Euclidean geometry. So it is nearby to look for a proof that sticks to the area of Euclidean geometry. Unfortunately, I did not find a short and straightforward approach. I am happy that I succeeded at all in finding a proof that uses only Euclidean geometry. I tried it quite a long time without success, then I found by chance a special lemma in the www from which I could work further on, I established two other lemmas, and these altogether brought a proof of Theorem 1 with only Euclidean geometry. Still another proof using completely different means, and many calculations using trilinear coordinates, is due to Embacher in [2].

A short proof using Marden's theorem

This proof uses a strong result involving complex numbers. It was a personal email from A. V. Akopyan that informed me about this connection (see also [3]). The mentioned strong and beautiful result is called Marden's theorem* (cf. [5], [6, p. 52]; the proof is not hard to understand):

Suppose the zeros a, b, c of the third-degree polynomial $p(z) = (z - a)(z - b)(z - c)$ are non-collinear in the complex plane†. Then the foci g_1, g_2 of the Steiner inellipse of the triangle ΔABC are the zeros of the derivative $p'(z)$.

For a proof of Theorem 1 let $p(z) = (z - a)(z - b)(z - c)$, then $\dfrac{p'(z)}{p(z)} = \dfrac{1}{z - a} + \dfrac{1}{z - b} + \dfrac{1}{z - c}$. By Marden's theorem, G_1 is a focus of the Steiner inellipse of ΔABC if, and only if,

$$\frac{1}{g_1 - a} + \frac{1}{g_1 - b} + \frac{1}{g_1 - c} = 0. \tag{1}$$

Then $g_1 - a = \lambda(g_1 - a_1)$ for some $\lambda \in \mathbb{R}$, and if $\varepsilon = \operatorname{sign}(\lambda)$, then the *power* P_1 of the point G_1 with respect to Σ is given by $P_1 = \varepsilon|g_1 - a||g_1 - a_1|$. Thus

$$P_1 = \varepsilon|\lambda||g_1 - a_1|^2 = \lambda(g_1 - a_1)\overline{(g_1 - a_1)} = (g_1 - a)\overline{(g_1 - a_1)},$$

and so $\dfrac{1}{g_1 - a} = \dfrac{\overline{g_1 - a_1}}{P_1}$. Similarly, $\dfrac{1}{g_1 - b} = \dfrac{\overline{g_1 - b_1}}{P_1}$, and $\dfrac{1}{g_1 - c} = \dfrac{\overline{g_1 - c_1}}{P_1}$, so that (1) is equivalent to

$$\frac{\overline{g_1 - a_1}}{P_1} + \frac{\overline{g_1 - b_1}}{P_1} + \frac{\overline{g_1 - c_1}}{P_1} = 0,$$

or, multiplying by P_1 and conjugating, $(g_1 - a_1) + (g_1 - b_1) + (g_1 - c_1) = 0$, that is, $g_1 = \frac{1}{3}(a_1 + b_1 + c_1)$, which just says that G_1 is the centroid of $\Delta A_1B_1C_1$, and this completes the proof. Similarly, this proof would work for $G_2(g_2)$ and the triangle $\Delta A_2B_2C_2\ (a_2, b_2, c_2)$.

For me it is fascinating how things come and fit together here in this proof (complex numbers, derivatives, zeros, geometry). We know that it is not the case, but one could have the impression that Mardens's theorem was established for the purpose to have a short proof of Theorem 1. I have to thank A. V. Akopyan for the corresponding hint by e-mail concerning the connection of Theorem 1 to Marden’s theorem. I also want to thank the reviewer of this paper who made a lot of useful suggestions to make things clearer (formulations, notations etc.).

But, at least for me, the question is interesting: Is there a possibility to argue for Theorem 1 just by means of Euclidean geometry? I struggled a lot for this question, and the interested reader can have a look at the web-reference.

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* This theorem is set an exercise in [4, p. 497, Ex. 9]. D. Kalman calls it in the very first line of his paper [5] ‘one of my favorite results in mathematics.’

† The complex numbers a, b, c, a_1, b_1, c_1, g_1 correspond to the points A, B, C, A_1, B_1, C_1, G_1 in the Euclidean plane.

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