Unusual Cyclic and Tangential Quadrilaterals – An Overview

By **Hans Humenberger**

Cyclic and tangential quadrilaterals are a famous, interesting and important topic of geometry (see e. g. De Villiers, 2020; Josefsson, 2011 and 2012; Worrall, 2004). In school geometry classes a question that is often explored is whether or not the well-known special quadrilaterals (squares, rectangles, rhombuses, parallelograms, kites, etc.) have a circumcircle or an incircle, respectively. But the more general question "What makes a quadrilateral have a circumcircle (incircle)?" is not asked so commonly, although there are easy answers that need only very basic facts: that isosceles triangles have equal angles, and from a point outside a circle two tangents of equal length can be drawn.

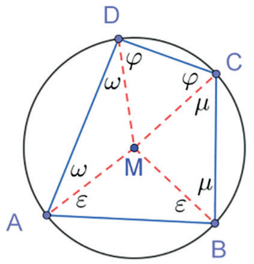


Fig. 1a Cyclic quadrilateral

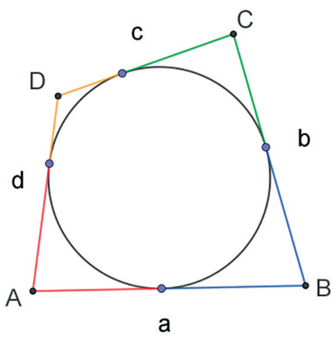


Fig. 1b Tangential quadrilateral

In Fig. 1a, there are four isosceles triangles with the radius as legs, and so in a convex cyclic quadrilateral the relation $\alpha + \gamma = \beta + \delta$ (Note 1) holds because both sides equal $\varepsilon + \mu + \varphi + \omega$. The case in which the circle centre M lies outside the quadrilateral would have to be treated separately. By looking at Fig. 1b one can see immediately, because of the equally coloured tangent line segments, that in a convex tangential quadrilateral, the relation $a + c = b + d$ must hold (Henri Pitot, 1725; here a denotes AB, etc.). It is also well-known that also the converse is true; both conditions are not only necessary but also sufficient (first proven by Jakob Steiner in 1846), i.e., convex quadrilaterals fulfilling these properties must be tangential (cyclic). There are several possible ways to prove it, both direct proofs and indirect ones.

We omit the case of a cyclic quadrilateral, we give two proofs for the case of a tangential quadrilateral. In both versions one has to distinguish two cases.

Theorem 1: A convex quadrilateral is tangential if it fulfils $a + c = b + d$.

Proof 1 (direct)

Case 1: $a = b$
Then also $c = d$ must hold and we have a kite, and kites are tangential.

Case 2: $a \neq b$

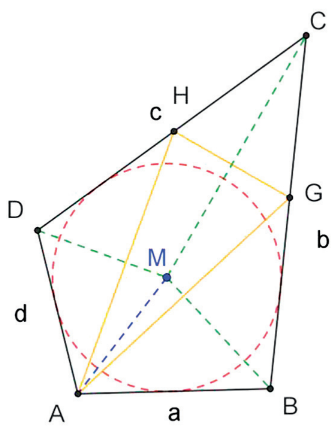


Fig. 2 Direct proof

Without loss of generality, we assume $b > a$ and thus $c > d$ (Fig. 2). We construct a point $G \in b$ with distance a from B and analogously a point $H \in c$ with distance d from D. Then three isosceles triangles arise ABG, AHD, HGC, the third one because $c - d = b - a$. Then we know that the angle bisectors at B, C, D are the perpendicular bisectors of the sides of ΔAGH and thus must meet in a single point (M). And this point must lie on the angle bisector of A, too (because its distance to a and d is equal), and hence is the centre of a circle touching all four sides of the quadrilateral $ABCD$.

Proof 2 (indirect): See e. g. De Villiers 2020.

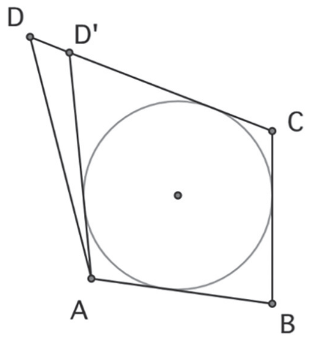


Fig. 3a D' nearer to C than D is to C

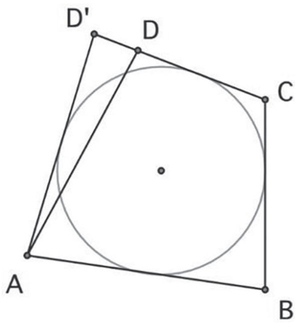


Fig. 3b D' farther away from C than D is to C

With the two angle bisectors at B and C one can always construct a circle touching the sides AB, BC, CD. Let us, indirectly, assume that AD is not tangent to that circle. Then there is a point $D' \neq D$, nearer to C or farther away from it than D (Figs. 3a, 3b) on the straight line CD such that AD' is tangent. Then we have on the one hand

$AB + CD = BC + AD$ (Theorem 1 pre-condition)

and on the other hand

$AB + CD' = BC + AD'$ (Pitot's Theorem).

Subtraction yields $\underbrace{CD - CD'}_{\pm DD'} = AD - AD'$ which is impossible in the triangle ADD' (triangle inequality), unless triangle ADD' degenerates (D=D').

Theorem 2: For **convex** quadrilaterals $ABCD$ the following two statements hold:

- $ABCD$ is cyclic $\Leftrightarrow \alpha + \gamma = \beta + \delta = 180°$
- $ABCD$ is tangential $\Leftrightarrow a + c = b + d$

In terms of "What if?" questions one could ask: What would this theorem look like if we omitted the restriction "convex"? Which **non-convex** cyclic or tangential quadrilaterals do exist? Can we establish in these cases also necessary and sufficient conditions? Such questions are typical for the process of doing mathematics, and for students such activities are important for getting proper conceptions of what mathematics is about. Especially when dealing with *crossed quadrilaterals*, students can think about concepts like *angle, side, diagonal, area, interior, exterior*, *etc.* twice and one step deeper.

In what follows, we take the reader to a short journey through all the cases of unusual (non-convex) cyclic and tangential quadrilaterals.

Cyclic quadrilaterals

Let us start with the shorter case of cyclic quadrilaterals. Which kinds of **non-convex cyclic** quadrilaterals are possible? It is easy to see that there are *no concave* cyclic quadrilaterals (no matter in which direction one moves the point D in the circle, one will always get either a convex or a crossed one, Fig. 4).

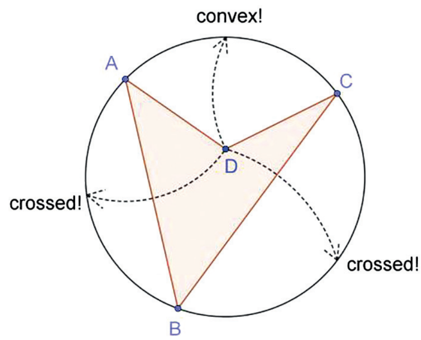


Fig. 4 There is no concave cyclic quadrilateral

But there are *crossed* cyclic quadrilaterals. In general, **crossed quadrilaterals** are somehow strange in terms of *area, interior*, *exterior*, *angle sum*, etc.

In non-crossed quadrilaterals it is easy to say what is the interior and the exterior. (If you imagine moving anticlockwise around the quadrilateral the interior is always to your left.) This is no longer true for crossed quadrilaterals and, therefore, it is not so easy to say what the *interior* or the *area* of a crossed quadrilateral should be. Following the same rule as in case of non-crossed quadrilaterals, one could mark the interior like in Fig. 5a.

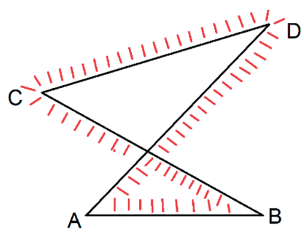


Fig. 5a "Interior" always left

On the other hand, one could also imagine the interior in another sense, like in Fig. 5b.

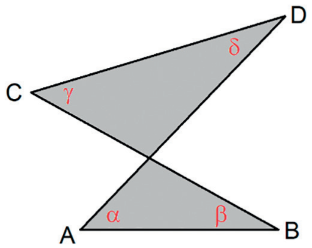


Fig. 5b Both triangles as "interior"

In terms of Fig. 5a it easy to see that the sum of the internal angles (oriented!) of a crossed quadrilateral is always 720°. With the angles $\alpha,\beta,\gamma,\delta$ of Fig. 5b we can write:

$$\alpha+\beta+(360°-\gamma)+(360°-\delta) = 720° + \underbrace{(\alpha+\beta)-(\gamma+\delta)}_{0}$$

In terms of Fig. 5b, the only rule for the sum of the internal angles $\alpha+\beta+\gamma+\delta$ would be: it is somewhere between 0° and 360°.

With the help of dynamic geometry software, one could also think of a crossed quadrilateral originating from an usual convex one where the point A is dragged beyond the side BC (Fig. 6). Then it seems more natural to call the marked angles *internal* (De Villiers 1999, p. 42ff; De Villiers 2015), all angles measured anticlockwise. Here the difference between *oriented* and *not oriented* angles becomes crucial, a good opportunity to discuss that in classroom.

Fig. 6 A crossed quadrilateral originating from a convex one

Using the inscribed angle theorem, it is easy to establish a necessary condition for a crossed quadrilateral to be cyclic (in terms of Fig. 5b): $\alpha=\gamma$ and $\beta=\delta$. And using the converse of the inscribed angle theorem it is also clear that this condition is sufficient. But in this formulation neither the analogy to the convex case (*sums* of angles) nor to cyclic tangential quadrilaterals (sums of sides) is that visible, much better in terms of the right part of Fig. 6 (oriented angles): For crossed quadrilaterals $ABCD$: $ABCD$ is cyclic $\Leftrightarrow \measuredangle A+\measuredangle C=360°=\measuredangle B+\measuredangle D$.

Tangential quadrilaterals

Which kinds of **non-convex tangential** quadrilaterals are possible? Maybe at first glance one is tempted to say there are no such quadrilaterals. How should the four sides of a concave or crossed quadrilateral touch a circle? Indeed, if one sticks to the concept of a *tangential quadrilateral* that the sides themselves need to be tangents to a circle, then clearly only convex ones are possible.

But if we don't stick to that and change to "the straight lines containing the sides are tangents" one can see easily that in this sense also a concave kite is a *tangential quadrilateral*, one just has to extend the sides c and d, then a convex kite $A'BC'D$ arises which, of course, has an incircle (Fig. 7).

In this sense, also the *concave kite* is a *tangential quadrilateral* and it fulfils $a+c=b+d$.

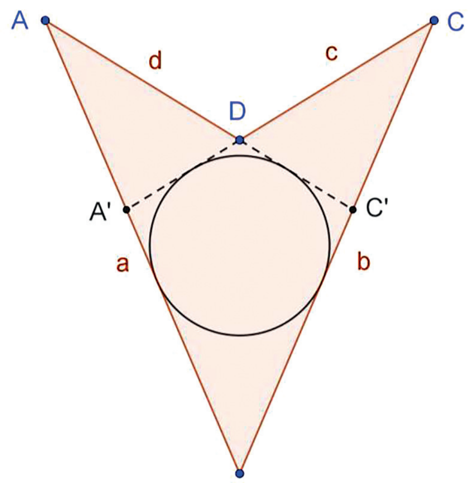


Fig 7. Concave kite as a *tangential quadrilateral*

It is easy to see that in this wider sense there are several other concave tangential quadrilaterals, not only concave kites. Imagine an arbitrary convex tangential quadrilateral $A'BC'D$ with non-parallel opposite sides. Then extend both pairs of opposite sides until they intersect (A,C). Then we have in principle the same situation as in Fig. 7.

But here we have to answer two questions:

1) Is $a+c=b+d$ fulfilled in such a situation? Is $a+c=b+d$ a necessary condition for being a concave tangential quadrilateral which contains the touched circle in its interior?

2) Is the condition $a + c = b + d$ also sufficient for being such a concave tangential quadrilateral?

In both cases the answer is "yes". The proof for 1) is easy, it just uses the fact several times that from a point outside a circle the two tangent line segments are equal (Fig. 8). Let P,Q,R,S be the points of tangency of the straight lines AB, BC, CD, DA.

$$\begin{aligned}\underbrace{BP + AP}_{a} + \underbrace{CD}_{c} &= BQ + AD + \cancel{DS} + CR - \cancel{DR}\\ &= BQ + AD + CQ\\ &= \underbrace{BC}_{b} + \underbrace{AD}_{d}\end{aligned}$$

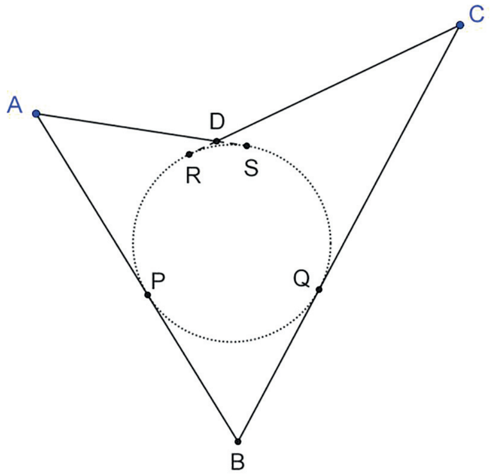


Fig. 8 $a + c = b + d$ is necessary here

The proof that also 2) can be answered with "yes" is completely analogous to the convex case (see above).

Proof 1 (direct)

Without loss of generality, we assume $a > d$ and $b > c$ (Fig. 9). We construct a point $G \in a$ with distance d from A and analogously a point $H \in b$ with distance c from C. Then three isosceles triangles arise AGD, CDH, BHG, the third one because of $a - d = b - c$. Then we know that the angle bisectors at A, B, C are the perpendicular bisectors of the sides of ΔGHD and thus must meet in a single point (M). And this point must lie on the angle bisector of D, too (because the distance of M to the straight lines containing c and d, respectively, is equal), and hence is the centre of a circle touching all four straight lines containing the sides of the quadrilateral $ABCD$.

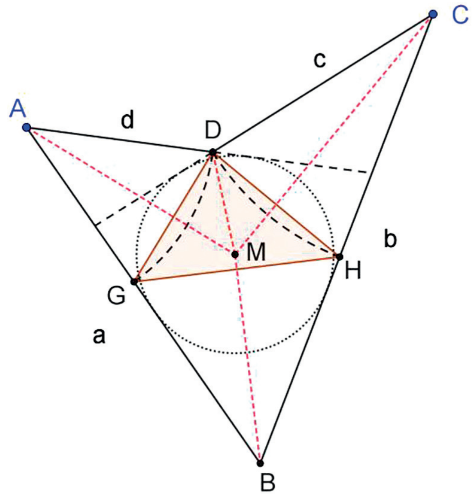


Fig. 9 Direct proof

Proof 2 (indirect)

With the two angle bisectors at B and C one can always construct a circle touching the sides AB, BC, and the straight line CD. The only difference to before is that the point of tangency S is this time not on the line segment CD but on its extension, and the proof can be done literally as above. This technique of a proof also works in the cases below.

In a sense, only a bit wider than usual, the corresponding circle could be called the *incircle* of such a concave quadrilateral.

Once familiar with this situation, it is not a big step to realize that the circle could touch the straight lines containing c, d also from the exterior (in terms of Fig. 8 the circle "moves upwards"); then one could call it the *excircle* (Fig. 10).

This time we don't have $a + c = b + d$ but, rather, $a + d = b + c$. (Here a, b denote the sides not involved in the reflex angle, c, d their adjacent sides.)

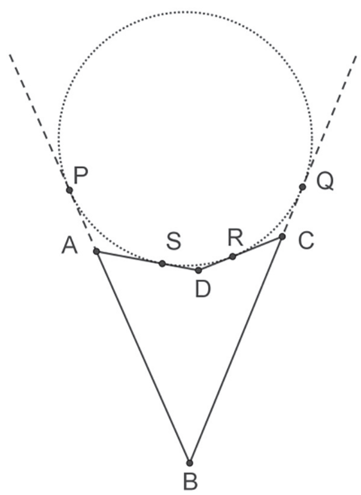


Fig. 10 Another concave tangential quadrilateral (with an "excircle")

Again by using the fact several times that from a point outside of a circle the two tangent line segments are equal we get:

$$\begin{aligned}\underbrace{AB}_{a}+\underbrace{AD}_{d}&=BP-\cancel{AP}+\cancel{AS}+SD\\&=BQ+DR\\&=\underbrace{BC}_{b}+\underbrace{CR+DR}_{c}\end{aligned}$$

And once more, this condition is also sufficient; this can be proved with the same technique as above.

The concave kite has a peculiarity among the concave quadrilaterals: it is the only one which has both a sort of incircle in the sense of Fig. 7, 8 and an excircle in the sense of Fig. 10. This is clear because

$$\left.\begin{aligned}a+c&=b+d\\a+d&=b+c\end{aligned}\right\}\Rightarrow a=b, c=d.$$

And if one draws Fig. 10 in a bit different way (extending AD and CD gives new points A, C in Fig. 11) one even has a *convex* quadrilateral with the property that the four extensions of the sides touch a circle. (Here B denotes the vertex beyond which no extension takes place; the extensions must yield new points of intersection). Again:

1. Such a circle could be called a sort of *excircle*. Also kites which are not rhombuses have such an excircle.
2. We have $a+d=b+c$, necessarily, in this situation, and this condition (together with $ABCD$ does not have parallel opposite sides) is also sufficient for $ABCD$ having such an excircle, analogously to prove.

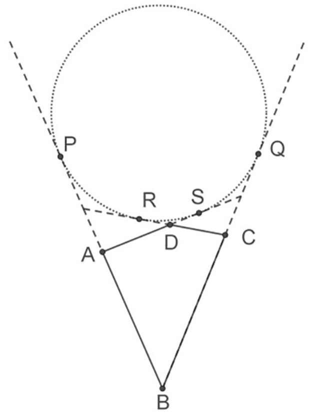


Fig. 11 Another convex tangential quadrilateral (with an excircle)

For us *convex quadrilaterals with an excircle* were a very astonishing issue in the whole "story" of systematizing all the possibilities of tangential quadrilaterals.

The convex kite which is not a rhombus has a peculiarity among the convex quadrilaterals: It is the only one which has both an incircle and an excircle in the sense of Fig. 11. This is clear because

$$\left.\begin{aligned}a+c&=b+d\\a+d&=b+c\end{aligned}\right\}\Rightarrow a=b, c=d.$$

If one changes Fig. 8 accordingly (extending CD and AD gives new points B, D in Fig. 12) one gets a *crossed* tangential quadrilateral where a circle touches the two crossed sides and the extensions of the two other sides beyond B and D. In Fig. 12a these *extensions intersect each other* and, in a sense, the circle lies in the area where the extensions get *nearer and nearer*, so this circle could be viewed as sort of an *incircle* (but this is also possible when $a||c$ (Fig. 12b)).

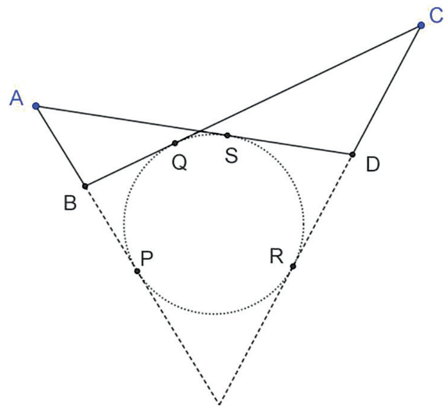


Fig. 12a *Crossed* tangential quadrilateral

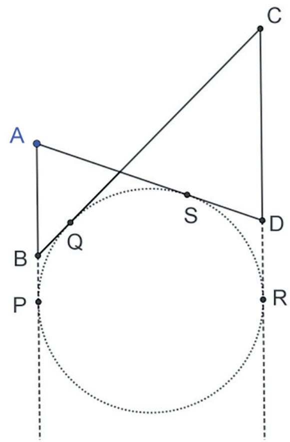


Fig. 12b $a||c$

We have $a+b=c+d$ in this situation (a, c denote the sides not crossing):

$$\begin{aligned}\underbrace{AB}_{a}+\underbrace{BC}_{b}&=AP-\cancel{BP}+\cancel{BQ}+QC\\&=AS+CR\\&=AS+DS+CD\\&=\underbrace{CD}_{c}+\underbrace{AD}_{d}\end{aligned}$$

This condition is also sufficient, and this is proved analogously.

If one looks at Fig. 11 differently one gets immediately a *crossed* tangential quadrilateral where a circle is touching the two crossed sides and the extensions of the other two sides beyond A and C (Fig. 13a; again: this would also be possible with $a||c$). But this is essentially the same as in Fig. 12. The only difference is that the circle in Fig 12a lies on the side where the extensions intersect, whereas in Fig. 13a it is on the side where the *extensions do not intersect* (Note 2). Since it is not so clear what the interior of a crossed quadrilateral is, these circles are neither obviously incircles nor excircles.

We have $a + d = b + c$ in this situation (a, c denote the sides not crossing):

$$\begin{aligned}\underbrace{AB}_{a} + \underbrace{AD}_{d} &= BP - AP + AS + SD \\ &= BQ - \cancel{AS} + \cancel{AS} + DR \\ &= BQ + DC + CQ \\ &= \underbrace{BC}_{b} + \underbrace{CD}_{c}\end{aligned}$$

This condition is also sufficient, and this is proved in an analogous way to previously.

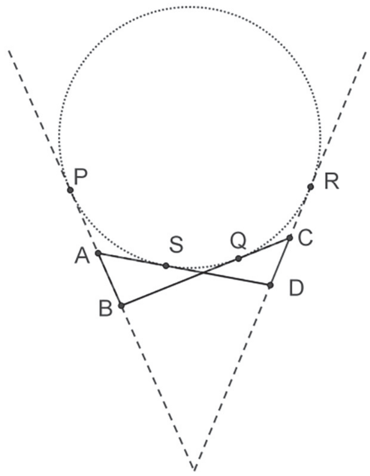


Fig. 13a *Crossed* tangential quadrilateral

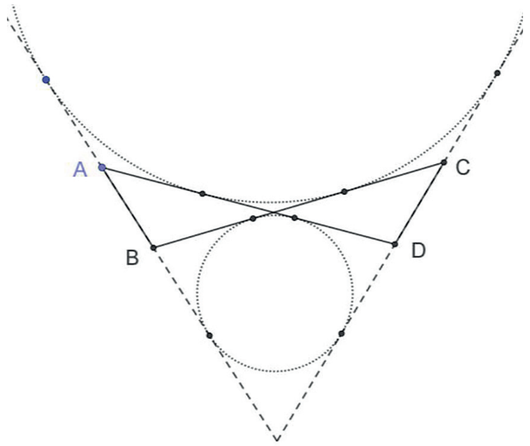


Fig. 13b Symmetric crossed quadrilateral

The *symmetric* crossed quadrilateral (Fig. 13b) has a peculiarity among the crossed quadrilaterals: it is the only one which has two touching circles (similar to Fig. 12a and similar to Fig. 13a). This is clear because

$$\left.\begin{aligned} a + b &= c + d \\ a + d &= b + c \end{aligned}\right\} \Rightarrow b = d, a = c.$$

This is the analogous situation to the one with concave kites, and with convex kites that are not rhombuses.

When reading a first draft of this article, Berthold Schuppar (TU Dortmund University), had the following idea for an overview concerning the six mentioned cases. Starting with a circle and four tangents we can distinguish two cases:

1. All adjacent radii to the points of tangency have a central angle $\leq 180°$ (here we mean *oriented* angles, Figs. 14a, 14b, 14c).
2. There are two adjacent radii to the points of tangency with a reflex central angle (again, *oriented*; Figs. 15a, 15b, 15c; all four points of tangency are on *one half of the circle*).

In both of these cases one can choose four intersection points as vertices of a quadrilateral in three different ways (convex, concave, crossed).

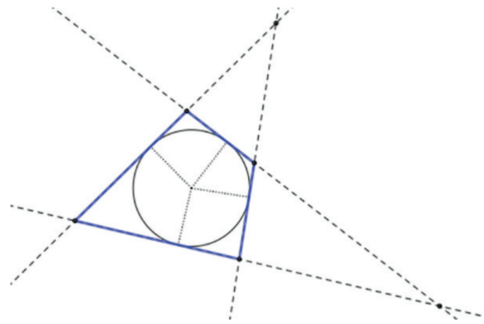
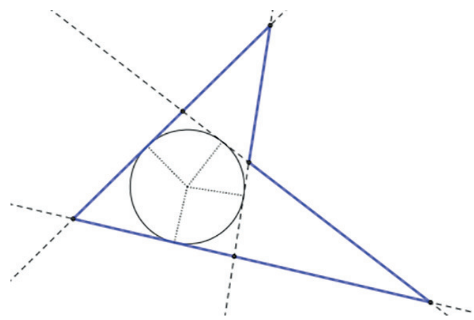
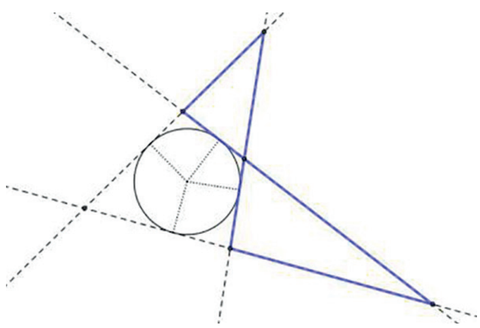


Fig. 14a convex



14b concave



14c crossed

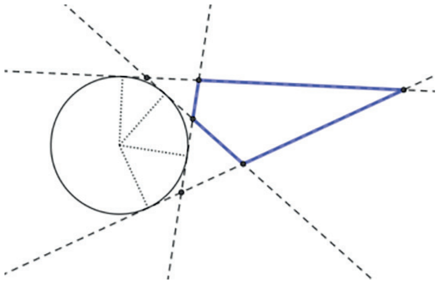
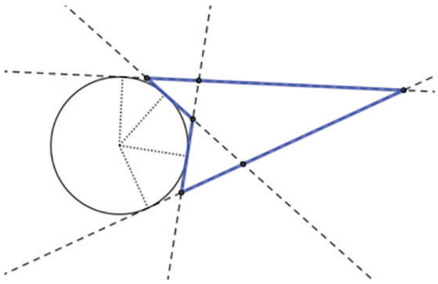
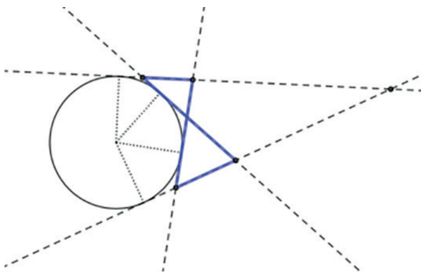


Fig. 15a convex



15b concave



15c crossed

With a pure *geometrical* view, it is clear that there are only the three mentioned types of quadrilaterals. But this could also be confirmed by *combinatorial* and *logical* considerations which students could contribute for reasons of linking different mathematical fields: 4 straight lines in general position (no parallels) have $\binom{4}{2} = 6$ intersection points. Out of these 6 points (A, B, C, D, E and F Fig. 16) we can choose 4 points in $\binom{6}{4} = \binom{6}{2} = 15$ ways, but only in 3 cases these 4 points form a quadrilateral (no 3 points collinear):

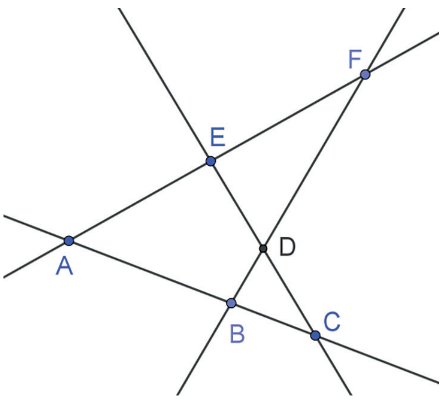


Fig. 16 Four straight lines with six intersection points

If A is one of the points B or C must be also in the set of the 4 points (because in $ADFE$ there would be 3 collinear points).

1. If one takes A, B one necessarily has to take also D, E ($ABDE$, *convex*)
2. If one takes A, C one necessarily has to take also D, F ($ACDF$, *concave*)
3. If one does not take A one must not take D (otherwise 3 collinear points), and this yields the *crossed* case ($BCEF$).

Conclusion

Altogether, we have five (or six) cases of tangential quadrilaterals. How do we know that there are not more? In the convex/concave case the circle may be in the interior/exterior, this makes four possibilities (Fig. 14ab, 15ab); in the crossed case one cannot distinguish properly between interior and exterior, hence here we have only one case. (If one distinguishes between Figs. 14c and 15c there are 6 cases altogether.)

All cases have a common property: The sum of two appropriate sides equals the sum of the two others, and this condition is in every case also sufficient.

With cyclic quadrilaterals there are only two possibilities, the usual convex one and the crossed one (the diagonals in the one case are sides in the other and vice versa).

Before we did these explorations we knew that there are non-convex cyclic and tangential quadrilaterals but we did not have a systematic overview, therefore – at least for us – these were very interesting explorations, and it seems that they are not so well-known. We think that they deserve to be better known because there are only few prerequisites and students can find one direction of the statements (necessary conditions: using that from a point outside of a circle the two tangent line segments are equal) probably by themselves. For the converse (conditions are sufficient) the proofs are not so easy to find by autonomous work – this has to be done, at least in one case, together with the help of the teacher. But here one can choose between a direct way of proving and an indirect one. Students can experience mathematics as a vivid process and seemingly clear concepts of interior, exterior, area, internal angle, etc. can be thought of in a deeper way (via crossed quadrilaterals). Of course, this does not mean that every school student should meet that topic. (It is more important that they are confronted with the more general concept of cyclic and tangential quadrilaterals, especially the necessary and sufficient conditions in the convex case). But in courses for interested students, in teacher education programmes, in geometry books etc. this topic should be given more attention. And if proofs and active systematisation should not play a really crucial role students could be given (in different groups?) pictures of the 5 (6?) possible cases of tangential quadrilaterals (Fig. 14, Fig. 15) with the challenge to find out in which way the sides can be grouped in pairs of equal sum. Explorations with dynamic geometry software are possible here.

We are sure that such thoughts cannot be new – and indeed we were pointed to Hadamard (Saul 2010, p. 76ff). Perhaps there are many examples in the literature where such ideas provide an overview of the possibilities for not so common cyclic and tangential quadrilaterals. And if one has the idea to start with a circle and four touching tangent lines – no matter where the circle touches (Note 3) – the way to an overview is shorter and can be combined with combinatorial and logical aspects. Where necessary and sufficient conditions should be a central theme, we hope that this article will be a guide.

Notes

1. Here α denotes the internal angle at A, etc.
2. In a sense – the circle lies in the area where the extensions get *farther and farther away from each other* – this circle could be viewed as sort of an *excircle*. One could distinguish these two cases, then there would be altogether 6 cases. This distinction would somehow fit to the convex and concave case where we had also two cases, one incircle and one excircle.
3. But this way of thinking may not be very easy and natural because many persons usually have in mind the *sides* (line segments) of quadrilaterals not their extensions, in other words the straight lines containing the sides.

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THE 'HAT POLYKITE'

a real first in aperiodic tiling

Many shapes can tile the plane in a repeating pattern and there are some that can tile the plane in a strange aperiodic way, without pattern. Until recently, the most famous aperiodic tiling required two units, the dart and kite that Roger Penrose used to build his tiles. The 'hat polykite' is the first aperiodic monotile (etymologically dubbed an 'einstein'), built from eight congruent kites (60°, 90°, 120°, 90°). It was discovered by David Smith, Joseph Samuel Myers, Craig S. Kaplan and Chaim Goodman-Strauss and made public in March. They show that it admits of uncountably many tilings. Isn't it astonishing that despite being so simple, it has taken mathematicians until well into the twenty-first century to find it?