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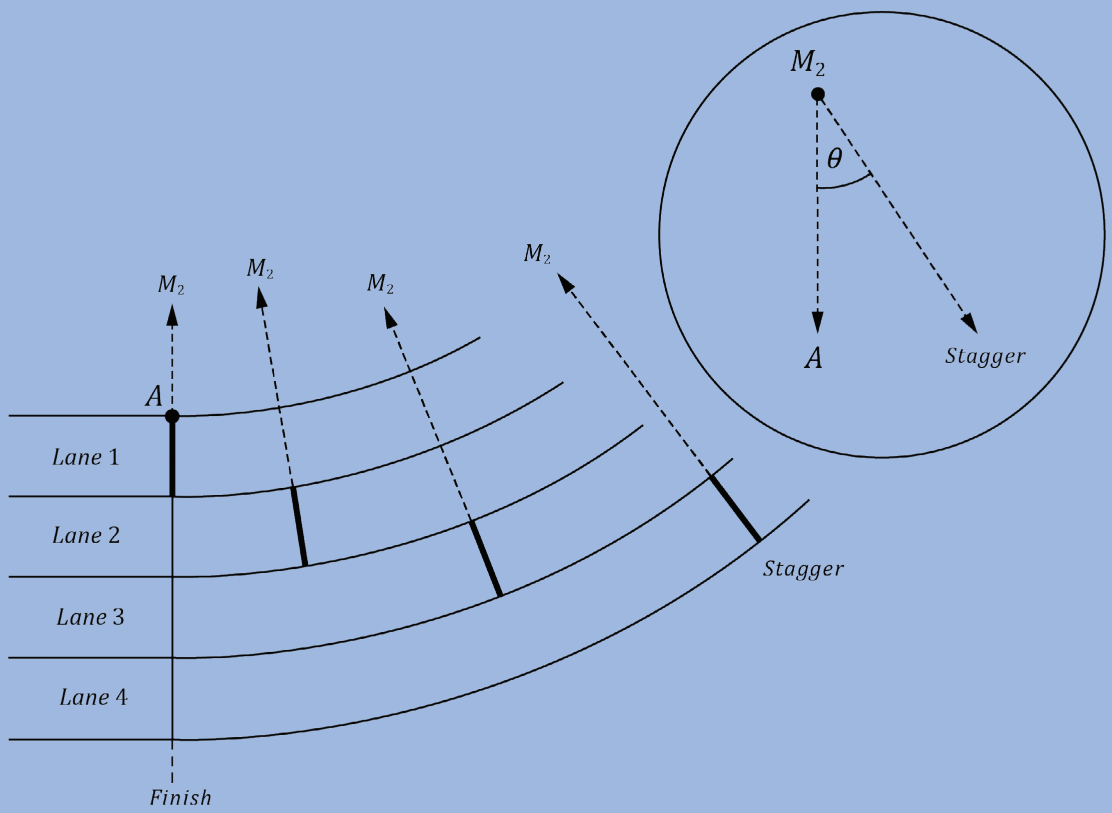


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# Learning and Teaching Mathematics



*Learning and Teaching Mathematics is a journal of the Association for Mathematics Education of South Africa (AMESA). This journal is aimed at mathematics teachers at primary and secondary school level and it provides a medium for stimulating and challenging ideas, offering innovation and practice in all aspects of mathematics teaching and learning in school. Learning and Teaching Mathematics aims to inform, enlighten, stimulate, challenge, entertain and encourage mathematics educators. Its emphasis is on addressing the challenges that arise in the mathematics classroom. It presents articles that describe or discuss mathematics teaching and learning through the eyes of practising teachers and learners. While this journal 'listens' to research and considers it in the activities, lesson ideas, and teaching strategies that it publishes, it is not a research publication.*

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Articles submitted will be reviewed by the editors and members of the Editorial Board. The Board will ensure that the papers make a contribution to our understanding of mathematics learning and teaching, that the mathematics presented is correct, and that the language and layout used is user friendly. Support will be provided by the editors to contributors in relation to meeting the above requirements.

The main criterion of acceptance is that the article should make a contribution to the improvement of school mathematics teaching and learning. See the inner back cover for more information on the submission of materials and articles for publication.

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# From the Editor

Dear LTM readers

In the first article of LTM 32, Craig Pournara, Yvonne Sanders and Shikha Takker identify four important issues that underpin learners' difficulties with algebraic linear equations and provide practical recommendations along with examples of tasks for teaching and learning. In the second article in this issue, Duncan Samson, Osnat Pinto and Moshe Stupel explore scenarios which involve conservation of similarity in triangles and then extend the exploration to quadrilaterals. The third article, by Peter Bishop, provides a few classroom activities inspired by the palindromic date 22022022 (22nd February 2022), while in the fourth article Hanna Savion and Michal Seri show how the physical use of bowls and coins can be used as a didactic approach to illustrating and explaining the long division algorithm. Duncan Samson then explores proofs of Stewart's theorem and Van Aubel's theorem for triangles, showing how basic concepts and techniques learnt at school find application beyond the confines of the classroom.

In the sixth article, Hans Humenberger presents a problem solving activity which highlights the practicality of using circular plates to quickly draw circular arcs without the need for construction, while in the seventh article Yiu-Kwong Man explores the area of the largest rectangle that can be inscribed inside a right-angled triangle. Moshe Stupel then proves a geometric property of a triangle containing a single 60° angle, after which James Metz describes a practical scenario involving tan graph transformations. In the tenth article, Alan Christison introduces us to some specifics of the IAAF 400 metre standard athletics track and the embedded mathematics that goes into ensuring the required levels of accuracy. Issue 32 of LTM concludes with a review of Peter Bishop's book "Give Meaning to Maths".

We hope you enjoy the diverse array of articles in this issue, and remind you that we are always eager to receive submissions. Suggestions to authors, as well as a breakdown of the different types of article you could consider, can be found at the end of this journal. If you have an idea but aren't sure how to structure it into an article, you are welcome to email the editor directly – we'd be happy to engage with you about turning your idea into a printed article.

Duncan Samson

# Learners' Errors with Linear Equations – and How to Solve Them!


## Craig Pournara, Yvonne Sanders & Shikha Takker

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### INTRODUCTION

Learners are introduced to algebraic linear equations in Grade 7 through very simple examples that can be solved by inspection, such as $p + 5 = 12$. At high school they progress to questions like $2(m + 3) = 10$, and then to examples where the unknown (or variable) appears on both sides, such as $2x + 1 = x - 7$. Many learners are able to figure out the solution to an equation like $2x - 3 = 9$. Some learners might reason "what times 2, then decreased by 3 gives me 9?" Others might use a process of 'undoing' or 'reversing' by starting with 9, adding 3, and then dividing by 2. Equations like this can be solved without operating on the unknown, but this is not generally true for equations with unknowns on both sides. International research over the past 40 years suggests that the transition from equations with unknowns on one side only to equations with unknowns on both sides is a big jump for learners. The research is unanimous that learners need to be explicitly taught a method for solving equations with the unknown on both sides.

The Wits Maths Connect Secondary (WMCS) project has done a great deal of research and development work on the teaching and learning of algebraic linear equations. We have tested thousands of learners on various items that require them to solve linear equations. All our results point to learners' difficulties with equations when the variable appears on both sides. We have also analysed very thoroughly the Grade 7, 8 and 9 CAPS documentation on equations and it is clear that CAPS does not pay attention to what research has been saying for nearly half a century! For example, the illustrative examples in the curriculum document do not address the important shift from equations with variables on one side to equations with variables on both sides. Furthermore, CAPS pays too much attention to simple examples that can be solved by inspection and hence do not create the need for working with inverses and inverse operations. There is also content listed under equations which is not fundamentally about equations, such as using substitution to populate tables of values – a task more suitable to learning functions.

In this article we identify four issues that underpin learners' difficulties with algebraic linear equations. They are:

(1) not seeing the equal sign as a balance

(2) difficulties in working with negatives and subtraction

(3) difficulties with algebraic simplification

(4) difficulties in working with inverse operations

In this article we expand on each of these issues. Thereafter we provide recommendations for teaching and we include examples of tasks to deal with the abovementioned difficulties which have already proved to be useful for teachers.

Before proceeding, we must be explicit about our position on the teaching of linear equations. We strongly advocate that learners are taught to use inverse operations (additive and multiplicative) to solve equations. This supports the idea that an equation is a statement where two expressions (left side and right side) are equal to each other. By applying inverse operations, we maintain the balance of both sides of the equation. We do not support "short-cut" methods such as transposition, "change sides, change the sign", etc. because these hide the fundamental mathematical ideas on which equation solving is based.

### SEEING THE EQUAL SIGN AS A BALANCE

When learners are first introduced to the equal sign, they treat it as signal to do something. This is not surprising because in the early grades they encounter examples such as:

$$4 + 10 = \_\_ \qquad 4 + \_\_ = 10 \qquad \_\_ - 6 = 10 \qquad 10 = 4 + \_\_$$

In each of these examples, learners can reason that adding/subtracting the numbers "gives me" the answer. However, learners also need to view the equal sign as a balance, showing that the left side and the right side are equivalent, e.g. $4 + 10 = \_\_ - 5$. This means they need to see that the result on the left side (in this case 14) "is the same as" the result on the right side. In this example, learners who continue to operate with a do-something view of the equal sign are likely to write 14 in the blank space, thus ignoring the "subtract 5" on the right side. Learners with an equivalence view of the equal sign will treat it as a balance and will recognise that the correct answer is 19. There is lots of evidence, both locally and internationally, that learners in Grades 7 – 9 don't have an equivalence view of the equal sign. This can become an obstacle in making sense of equations.

### WORKING WITH NEGATIVES AND SUBTRACTION

One of the big transitions in Senior Phase mathematics is the need to view the minus symbol as both sign and operation. Up to Grade 6, learners treat the minus symbol ($-$) as an operation: subtract or take away. When they are introduced to negative numbers, the symbol takes on an additional meaning because now it also represents a sign, e.g. $-2$ (negative two). This is confusing for learners. Even more confusing is that the meaning of the symbol can shift from operation to sign (and vice versa) in a single question, as in the example $4 + (-5) = 4 - 5 = -1$. We can read this as "four add negative five" which is then simplified to "four subtract 5" and then the answer is "negative one". The same symbol was treated as negative, then as subtract and then as negative again. It takes time to get used to these new ideas.

When working with equations, we cannot avoid dealing with negatives and subtraction irrespective of whether we speak about "taking to the other side and changing the sign" or working with inverse operations. Consider the following example which a teacher might offer learners:

| | | |
|---|---|---|
| Solve: | $-5 + 2x = x + 6$ | (line 1) |
| The teacher then writes: | $\mathbf{5} - 5 + 2x = x + 6 \mathbf{+ 5}$ | (line 2) |

The teacher is collecting variables on the left and constants on the right. So she adds 5 on both sides. On the left, she simply writes 5 and chooses to write it on the left of the expression so that it is easy to compute "5 subtract 5". However, in doing this, the minus symbol (in $-5$) shifts from representing a sign (negative) in line 1 to an operation (subtract) in line 2. Learners who are comfortable working with integers will ignore these subtle distinctions. But for learners still grappling with different meanings of the minus symbol, this layout on the left side may be a source of difficulty. It might be more helpful to make the changes on the right of each expression (i.e. $-5 + 2x \mathbf{+ 5} = x + 6 \mathbf{+ 5}$) because this would not affect the interpretation of the minus symbol on the left side. It is also worth noting that the operation of adding 5 is now explicit on both sides of the equation whereas this was not the case when 5 was appended to the front of the expression.

### DIFFICULTIES WITH ALGEBRAIC SIMPLIFICATION

In the context of linear equations, algebraic simplification is typically limited to: (1) distinguishing like and unlike terms; (2) collecting like terms; and (3) applying the distributive law – and all of this is done with only one variable. Yet, learners still make many errors with such apparently simple algebraic simplification. We conducted a study with over 800 Grade 9 learners from top-performing quintile 5 schools in Gauteng which revealed many basic algebraic errors in learners' attempts to solve linear equations (Pournara, 2020). Most of these errors involved subtraction/negatives and many involved working with *like* terms. This is surprising because we often assume that errors arise from incorrectly combining *unlike* terms such as $x + 5 = 5x$. The following errors were common:

| | |
|---|---|
| $5x - x = 5$ | Learner seemingly focuses on $x - x$ which eliminates the variable and so $5$ remains. |
| $5x - x = 5x$ | Learner may be treating $x$ as $0x$, hence $5 - 0 = 5$ and retain $x$. |
| $-5x + x = -6x$ | Learner likely isolates the leading negative, adds $5x$ and $1x$, then brings back the negative |

Algebraic simplification tasks in Grades 8 and 9 typically involve far more complex expressions than those shown above. However, it appears that more attention should be given to these seemingly simple combinations of terms which arise in linear equations. We provide suggestions of such expressions in the last section of the article.

### BALANCING AND INVERSE OPERATIONS

An equivalence view of the equal sign is closely linked to the idea of balancing an equation. As previously mentioned, we strongly advocate for the use of inverse operations when manipulating equations to promote the idea of balance. We would therefore expect learners to be familiar with teacher utterances such as "*if you add* 2 *on the left side, then you must add* 2 *on the right side*"; "*if you divide the left side by* –3*, then you must divide the right side by* –3"; "*whatever you do on the left, you do on the right*". However, irrespective of the approach used, learner errors arise in their attempts to manipulate equations. Of course, we can add/subtract any number (or variable) from both sides of an equation and it will remain balanced. We can also multiply/divide both sides by the same value. The point is to make wise choices when adding, subtracting, multiplying or dividing so as to make a sum of zero or a product of one.

Typical learner errors that don't maintain balance include: (1) 'moving a term across the equal sign' without changing sign; (2) subtracting a constant (or a term with a variable) from one side but adding it on the other side; and (3) applying the wrong inverse (e.g. learners might simplify $2x = 6$ by subtracting 2 from both sides rather than dividing by 2, leading to the incorrect solution $x = 4$).

On more careful reflection, we have come to realise that learners' difficulties may stem from "having too many options" in the first step of solving an equation. For example, given $2x - 3 = 5x + 9$, learners could begin by collecting terms with the variable on the left or the right. Therefore they could subtract $2x$ or $5x$ from both sides. Similarly, they must decide whether to collect constants on the left or the right which means they could add 3 to both sides or subtract 9 from both sides. They therefore have four options when they begin to solve the equation. For equations with the unknown on one side only, they have only two options, and these likely reduce to keeping the variable where it is and then isolating it. We return to this issue in the next section.

We have also come to appreciate subtle differences in applying the additive inverse and the multiplicative inverse. When applying the additive inverse, the goal is to eliminate a particular term on one side. By contrast, when applying the multiplicative inverse, the goal is to reduce the coefficient of the variables to 1, but not to eliminate the entire term. In the earlier example ($2x - 3 = 5x + 9$), if we want to eliminate the constant on the right side, we apply the additive inverse of 9 which gives $2x - 3 - 9 = 5x + 9 - 9$, and then simplifies to $2x - 12 = 5x$. We might then apply the additive inverse of $2x$, which is $-2x$ and not just $-2$, yielding $-12 = 3x$. Given that the goal is to solve for "one $x$", we must apply the multiplicative inverse of $3$, not $3x$, which of course is $\frac{1}{3}$ not $\frac{1}{3x}$. We also see learners subtracting $2x$ from $3x$ in order to get $1x$ on the right. If they subtract correctly on both sides, then they end up with the unknown on both sides again which is a step away from, and not towards, solving for the unknown.

## RECOMMENDATIONS FOR TEACHING

In this final section we make five recommendations for teaching and provide examples of tasks to illustrate each recommendation. But first, recommendation zero: avoid spending too much time on algebraic equations with letters on one side only, e.g. $2x = 15$ and even $2 - x = 15$. As we have already explained, learners can solve these without having to operate on the unknown (even if they may have to work a bit harder on the second example listed here).

### 1. EMPHASISE AN EQUIVALENCE OR BALANCE VIEW OF THE EQUAL SIGN

Fill in the missing number:

a) $6 + 4 = \_\_$
b) $6 + 4 = \_\_ + 7$
c) $6 + \_\_ = 10 + 5$
d) $6 + 4 = \_\_ - 5$
e) $\_\_ + 6 = 6 \times 5$

This task begins with a do-something view of the equal sign in (a) and then immediately shifts to an equivalence view with the subtle introduction of "$+7$" in (b). Learners need to reason "What added to 7 is the same as 6 add 4?" We also introduce subtraction and multiplication on the right side (d and e) to indicate to learners that any operations can be used. Note that the position of the unknown is varied too.

### 2. GIVE MORE ATTENTION TO ALGEBRAIC EXPRESSIONS THAT LEARNERS ENCOUNTER WHEN SOLVING EQUATIONS

Simplify:

a) $3x - x - 3 =$
b) $3 - x - 3 =$
c) $2x + 6 - x - 6 =$
d) $-x + 5 + x - 3 =$
e) $-2x + 3x + 6 - 6 =$
f) $2(x + 3) - x - 2 =$

This task deals with like and unlike terms involving $x$ and constants. There are several instances of adding and subtracting a constant or term in $x$ to create zero (c, d, e). We also recommend that teachers vary the order of the terms. For example, in (e) the expression begins with $-2x + 3x$. Our research findings show that this is far more difficult for learners to cope with than the equivalent form of $3x - 2x$.

### 3. EXPLORE DIFFERENT SEQUENCES OF APPLYING INVERSES

In the matrix below we focus on a single example, $2x + 4 = x - 5$. We apply different additive inverses and show the new terms which are highlighted by shading them. In A and B, we start with the constants and then move to the variables. In C and D, we start with the terms containing the variable. In A and C, we begin on the left side, subtracting 4 and $2x$ respectively. In B and D, we begin on the right, adding 5 and subtracting $x$ respectively. We deliberately stop after applying the first inverse. This shows the range of simplifications that learners will encounter before applying the next inverse. The matrix also reveals how the partially solved equations differ, depending on what operations have already been performed. For example, in A we get $2x = x - 9$ while in C we get $4 = -x - 5$. While the partially solved equations are equivalent, learners may not yet realise this. However, they should ultimately see that they all the equations reduce to $x = -9$.

We recommend that teachers take time to show all four options to learners. We further recommend that teachers specify which approach to use on particular examples so that learners become comfortable to isolate the variable on the left side or right side, and to work with negative coefficients and constants on either side.

| | **Start on the left** | **Start on the right** |
|---|---|---|
| **Start with constant term** | **A.** $2x + 4 = x - 5$<br>$2x + 4 - 4 = x - 5 - 4$<br>$2x = x - 9$ | **B.** $2x + 4 = x - 5$<br>$2x + 4 + 5 = x - 5 + 5$<br>$2x + 9 = x$ |
| **Start with term with variable** | **C.** $2x + 4 = x - 5$<br>$2x - 2x + 4 = x - 2x - 5$<br>$4 = -x - 5$ | **D.** $2x + 4 = x - 5$<br>$2x - x + 4 = x - x - 5$<br>$x + 4 = -5$ |

### 4. EMPHASISE DIFFERENT PLACEMENTS OF TERMS WHEN APPLYING INVERSES

We have already noted the importance of practice in simplifying the kinds of algebraic expressions that learners encounter when solving linear equations. Learners need to get used to working with like and unlike terms in different positions. The matrix below shows different possibilities of placing the "new terms" when applying the additive inverse of terms with variables. Note that we have copied C and D from the matrix above and continued to number E and F. In C and D the new terms are placed immediately after the terms with variables so the like terms are adjacent to each other on both sides of the equal sign. In E and F, we place the new terms on the extreme right of each expression. For the given example, this means that like terms are not adjacent on each side. The same can be done with constants.

| | **Start on the left** | **Start on the right** |
|---|---|---|
| **Writing terms with variables together** | **C.** $2x + 4 = x - 5$<br>$2x - 2x + 4 = x - 2x - 5$<br>$4 = -x - 5$ | **D.** $2x + 4 = x - 5$<br>$2x - x + 4 = x - x - 5$<br>$x + 4 = -5$ |
| **Writing 'new term' at end of each expression** | **E.** $2x + 4 = x - 5$<br>$2x + 4 - 2x = x - 5 - 2x$<br>$4 = -x - 5$ | **F.** $2x + 4 = x - 5$<br>$2x + 4 - x = x - 5 - x$<br>$x + 4 = -5$ |

### 5. DEALING DIRECTLY WITH LEARNER'S ERRORS

We consider learners' errors as opportunities for learning, not failures to be avoided. For this reason we recommend confronting learners with typical errors and then dealing explicitly with the erroneous thinking that leads to the error. Returning to the previous example: $2x + 4 = x - 5$. Consider the following learner response:

| | |
|---|---|
| Line 1 | $2x + 4 = x - 5$ |
| Line 2 | $2x + 4 - 4 = x - 5 + 5$ |
| Line 3 | $2x = x$ |
| Line 4 | $x = 1$ |

In line 2, the learner has subtracted 4 on the left side but added 5 on the right side. Both moves are partially correct since they eliminate the constant term on each side. However, they disrupt the balance of the equation. Line 3 follows logically based on the errors in line 2. But line 4 does not follow from line 3. In our research we have seen many instances where the last line does not follow from the line above. We suspect that this stems from learners' knowing what the final line should look like (i.e. $x = \cdots$) and so learners force the final line into this familiar form without a logical connection to the previous line.

In dealing with these errors, we recommend that teachers begin by asking learners to check the solution. Learners should then see that they get different results on the left side and the right side:

| Left side | Right side |
|---|---|
| $2x + 4$<br>Substituting $x = 1$:<br>$2(1) + 4$<br>$= 6$ | $x - 5$<br>Substituting $x = 1$:<br>$1 - 5$<br>$= -4$ |

Since the left side does not yield the same result as the right side, we know that $x = 1$ is not the solution to the equation. This process reinforces the meaning of solution – that the solution is the value which makes the left side equal to the right side, or the value that gives the same result on the left side as the right side. This also promotes an essential practice in solving equations – checking the solution by substitution. It is likely that many learners will have identified the errors in line 2. They can then be encouraged to correct the errors, to obtain a new solution and to check this solution. Note the layout of the two substitutions above. We deliberately write "$= 6$" and "$= -4$" below the line of substitution, rather than alongside. We do so to further emphasise that we are working separately with each expression in the equation and are not setting up another equation. Clearly, there is no mathematical difference in writing the final line next to the line of substitution.

In the WMCS equation materials we provide a range of worksheets with varying foci and different levels of difficulty. We include a large collection of worksheets on numeric equations, some deal only with whole numbers while others require an understanding of integers. Many tasks tackle learners' errors head on, asking learners to identify the errors and to correct them. The extract which follows comes from a worksheet on additive inverses. It shows how we encourage learners to pay attention to what happens to each side of the equation when they apply the additive inverse. The worksheets on equations and several other topics can be freely downloaded from www.witsmathsconnectsecondary.co.za/resources.

**Questions**

1) Make 5 pairs of additive inverses from the list of terms below. If the additive inverse does not appear in the list, provide it.

   $-4 \quad \frac{1}{4} \quad -x \quad 6x \quad 6 \quad \frac{1}{6} \quad -\frac{1}{2} \quad 4 \quad x \quad 0{,}5$

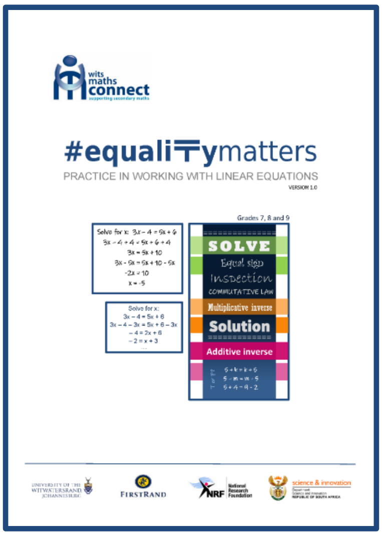
2) In the table below, apply the additive inverse that is indicated. Then write down the new form of the equation after applying the inverse. The first one has been done for you.

| Equation | Apply additive inverse of … | Equation after applying inverse |
|---|---|---|
| $x-4=2x$ | $-4$ | $x=2x+4$ |
| | $x$ | |
| | $2x$ | |

3) This equation has $k$'s on both sides of the equal sign: $4k=k+6$
   a) If you apply the additive inverse of 6 to both sides, will there still be $k$'s on both sides?
   b) Remember that the additive inverse of $k$ is $-k$.
      i) Apply this additive inverse to both sides.
      ii) Are there still $k$'s on both sides?
      iii) What remains on the right side?
   c) Continue to solve the equation and show that the solution is 2.
   d) What type of inverse did you use to continue solving the equation in Q3c?

4) Solve the following equations by applying the additive inverse of the variable.

   a) $3p=p-2$ b) $p=3p-2$ c) $7b=5b+4$ d) $20b=50b-10$

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# Exploring Intersecting Line Segments in Polygons


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### CONSERVATION OF SIMILARITY

Consider triangle $ABC$ with angles $\alpha$, $\beta$ and $\theta$ as illustrated in Figure 1. From each vertex of the triangle, moving in a counter-clockwise direction, draw a line segment at an angle $\delta$ to the adjacent side. This results in the formation of an inner triangle $DEF$ (Figure 2).

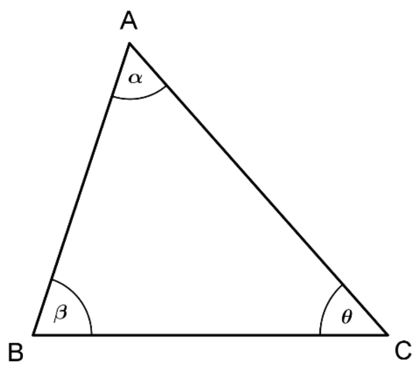


**FIGURE 1:** Triangle $ABC$ with angles $\alpha$, $\beta$ and $\theta$.

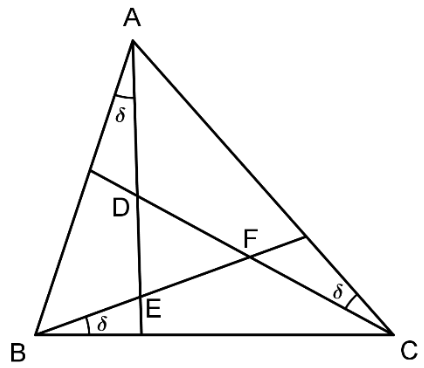


**FIGURE 2:** Triangle $ABC$ with three line segments.

The interesting property of this scenario is the conservation of similarity of the triangles $ABC$ and $DEF$. This can readily be shown as follows (Figure 3). Since the angle at vertex $A$ is $\alpha$, then $D\hat{A}C = \alpha - \delta$. With reference to triangle $ADC$, exterior angle $E\hat{D}F = \alpha - \delta + \delta = \alpha$. Similarly we can show that $D\hat{E}F = \beta$ and $E\hat{F}D = \theta$. Thus $\Delta ABC$ is similar to $\Delta DEF$ (equiangular).

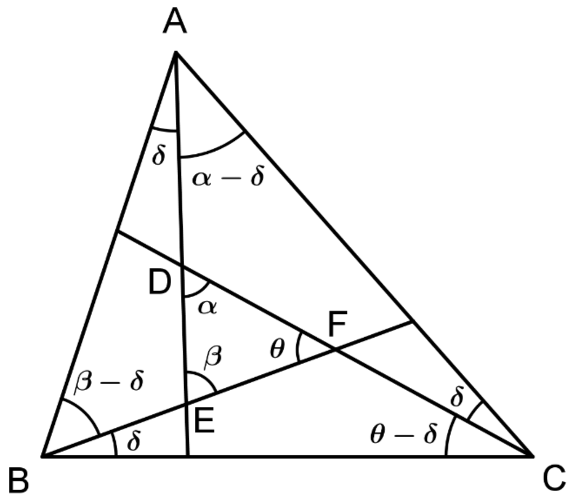


**FIGURE 3:** Conservation of similarity between triangles $ABC$ and $DEF$.

This conservation of similarity can be explored dynamically here:
https://www.geogebra.org/m/mtrqmhpm

## EXTENDING THE IDEA TO QUADRILATERALS

Let us now extend the idea to quadrilaterals. By way of example, consider quadrilateral $ABCD$ with angles $\alpha$, $\beta$, $\theta$ and $\varphi$ as illustrated in Figure 4. From each vertex of the quadrilateral, moving in a counter-clockwise direction, draw straight line segments at an angle $\delta$ to the adjacent side. These intersecting line segments form an inner quadrilateral $EFGH$.

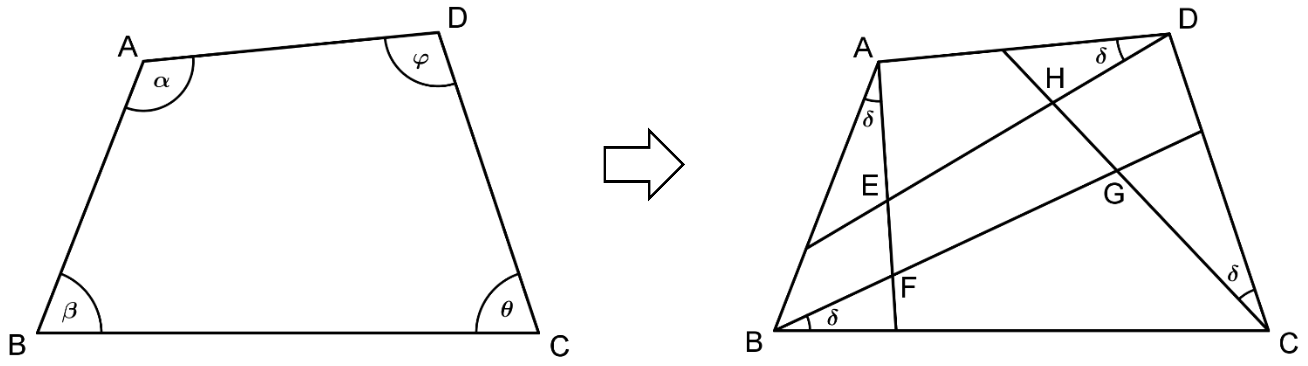


**FIGURE 4:** Quadrilateral $ABCD$ with four straight line segments at an angle $\delta$.

Quadrilaterals $ABCD$ and $EFGH$ can be proved to be equiangular as before by considering the exterior angles of triangles $EAD$, $FBA$, $GCB$ and $HDC$ (Figure 5). Note that while the two quadrilaterals are equiangular, they are not however similar. Similarity between the outer and inner quadrilaterals only occurs when the original quadrilateral is a square, i.e. a regular polygon.

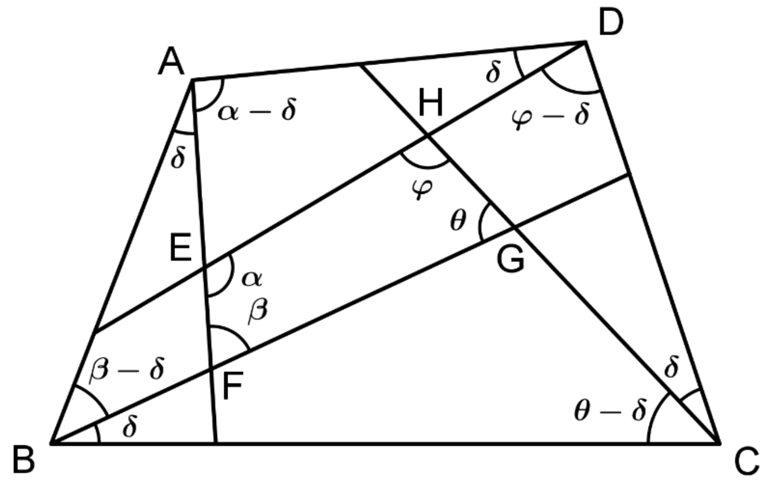


**FIGURE 5:** Equiangular quadrilaterals $ABCD$ and $EFGH$.

Depending on the angle $\delta$, the inner quadrilateral may be a crossed quadrilateral. It is left to the interested reader to explore the relationship of the angles of the original quadrilateral to those of the crossed quadrilateral in such cases. Note also that some of the vertices of the 'inner' quadrilateral may fall outside the original quadrilateral.

## EXPLORING RECTANGLES FURTHER

The rectangle represents an interesting case. Not surprisingly, when line segments are passed through the vertices of a rectangle at an angle $\delta$ to the sides as before, then the inner quadrilateral formed is also a rectangle (Figure 6).

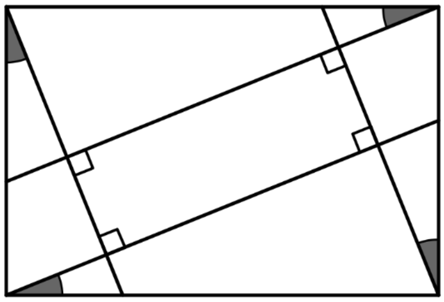

**FIGURE 6:** The case of the rectangle.

However, when the line segments form an angle $\delta = 45°$ to the sides, then the inner quadrilateral formed is a square. If the rectangle has sides in the ratio 1:2 then the square formed will have two vertices lying on the longer side of the rectangle (Figure 7). If the side lengths are in a ratio either smaller or greater than 1:2 then these two vertices will lie respectively either inside or outside the original rectangle (Figure 8).

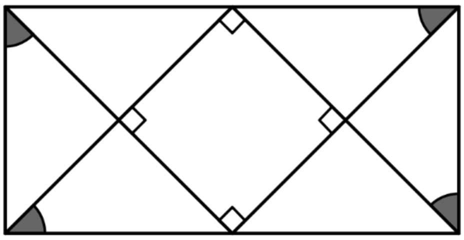

**FIGURE 7:** Rectangle with side lengths in the ratio 1:2 with $\delta = 45°$.

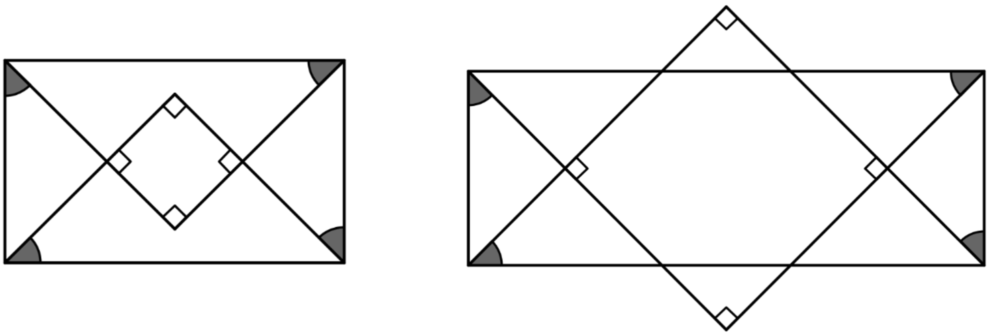

**FIGURE 8:** Rectangle with side lengths in the ratio either smaller or greater than 1:2 with $\delta = 45°$.

## ANGLE BISECTORS

With reference to Figure 7 and Figure 8, the line segments drawn with $\delta = 45°$ are in fact the angle bisectors of the original quadrilateral. Let us now look at angle bisectors more generally. Returning to the triangle, the three angle bisectors intersect at a common point known as the incentre of the triangle.

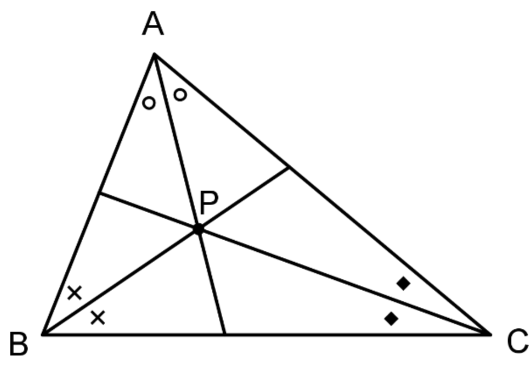


**FIGURE 9:** Triangle $ABC$ with incentre $P$.

What about the angle bisectors of a quadrilateral? Consider an arbitrary quadrilateral $ABCD$ with angles $\alpha$, $\beta$, $\theta$ and $\varphi$ as illustrated in Figure 10. The four angle bisectors result in the formation of the inner quadrilateral $EFGH$.

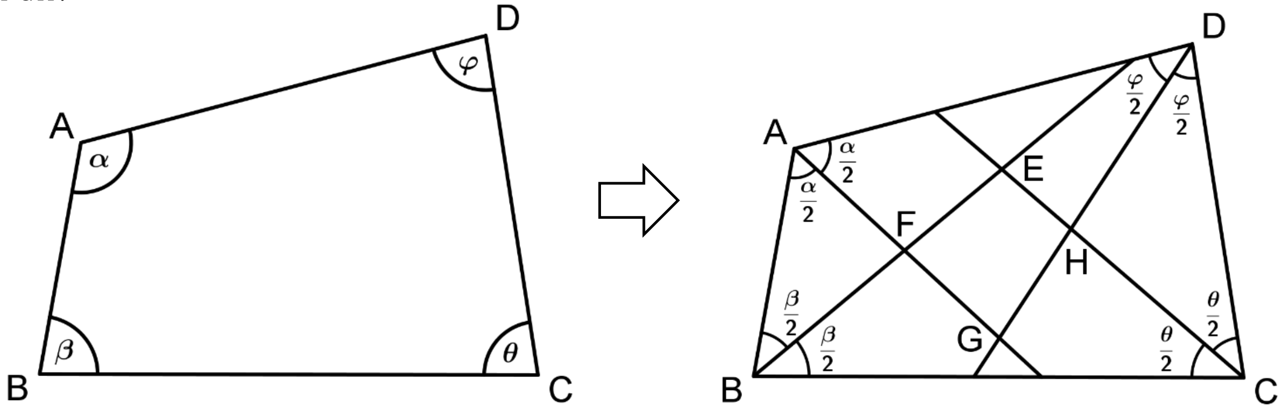


**FIGURE 10:** Quadrilateral $ABCD$ with four angle bisectors.

The interesting property of this scenario is that the resulting inner quadrilateral $EFGH$ is cyclic. This can be readily proved as follows.

In $\Delta CEB$: $\quad C\hat{E}B = 180° - \left(\frac{\beta}{2} + \frac{\theta}{2}\right)$

In $\Delta AGD$: $\quad A\hat{G}D = 180° - \left(\frac{\alpha}{2} + \frac{\varphi}{2}\right)$

$\therefore \quad C\hat{E}B + A\hat{G}D = 360° - \left(\frac{\beta}{2} + \frac{\theta}{2} + \frac{\alpha}{2} + \frac{\varphi}{2}\right)$

But $\beta + \theta + \alpha + \varphi = 360°$, $\therefore \quad \frac{\beta}{2} + \frac{\theta}{2} + \frac{\alpha}{2} + \frac{\varphi}{2} = 180°$

$\therefore \quad C\hat{E}B + A\hat{G}D = 360° - (180°) = 180°$

$\therefore \quad EFGH$ is cyclic (opposite angles supplementary)

As a final observation, note that if the original quadrilateral was itself cyclic, then not only is the resulting inner quadrilateral $EFGH$ cyclic as before, but its diagonals are perpendicular to one another – i.e. it is an orthodiagonal quadrilateral. It is left to the interested reader to explore this further.

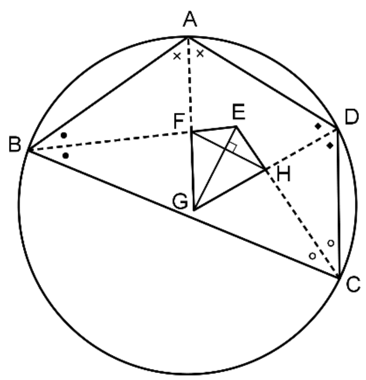


**FIGURE 11:** Orthodiagonal quadrilateral

## CONCLUDING COMMENTS

The ideas explored in this article could readily be worked into an interesting classroom investigation – either on paper (by construction and measurement) or with dynamic geometry software.

Use of dynamic geometry software would lend itself well to exploring special cases of triangles (equilateral, isosceles, right-angled) and quadrilaterals (parallelograms, rhombuses, kites). While we have focused on triangles and quadrilaterals, the investigation could readily be extended to higher order polygons as well.

# Palindromic Dates – A Classroom Exercise


## Peter Bishop[1]

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### INTRODUCTION

We all know what a palindrome is – words like mom, dad, level and rotator – that read the same both forwards and backwards. Sentences can also be palindromic, such as 'Madam, I'm Adam', and the mathematical example 'never odd or even'. This year we had an interesting palindromic date on Tuesday 22nd February – 22022022 when written in the form DDMMCCYY. To explore numerical palindromes you may wish to share the following exercises with your class. The activities are geared towards Grade 7 to 9, and pupils should ideally work in groups of two or three so as to encourage discussion and sharing of ideas.

### ACTIVITY 1

As an initial activity, introduce the idea of a palindrome and then get each group to come up with five or so palindromic words. Next get them to calculate each of the following:

| | | | | |
|---|---|---|---|---|
| (a) $11^2$ | (b) $11^3$ | (c) $11^4$ | (d) $121^2$ | (e) $212^2$ |
| (f) $111^2$ | (g) $1111^2$ | (h) $11111^2$ | (i) $202^2$ | (j) $101^2$ |

### ACTIVITY 2

Now get each group to explore the following questions. Each question relates to palindromic dates written in the form DDMMCCYY. Don't provide any direction or scaffolding, just let each group explore and mathematise in their own way. As you wonder around the class listening to the mathematical discussions you will be amazed!

(a) After the 22nd February 2022 (i.e. 22022022), what are the next two palindromic dates?

(b) What are the two nearest palindromic dates prior to 1st January 2000?

(c) How many dates in the 21st century are palindromic?

### DISCUSSION AND SOLUTIONS

Activity 1 is meant as a warm-up activity, but hopefully it will get pupils wondering about why these answers are all palindromic. Sowing the seeds of curiosity is such an important thing! For activity 2, I would encourage you to explore these three questions yourself before giving them to your pupils. This will give you some insight into how they might approach the questions, and the kinds of discussions you should listen out for.

(a) Trying 2023 as 32022023 doesn't work, and this should hopefully lead to 2030 as the next viable year. The two dates are 03022030 and 13022031.

(b) With a bit of trial and error it should soon become apparent that the month needs to be November. This gives 29111192 and 19111191.

(c) The 'Aha!' moment should come when one realises that we need only consider February. The answer is thus 29 (since 2092 is a leap year).

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[1] Peter Bishop is a retired mathematics teacher. His recently published book *Give Meaning to Maths* is reviewed at the end of the journal.

# Long Division: A Conceptual Approach using Bowls and Coins


## Hanna Savion & Michal Seri

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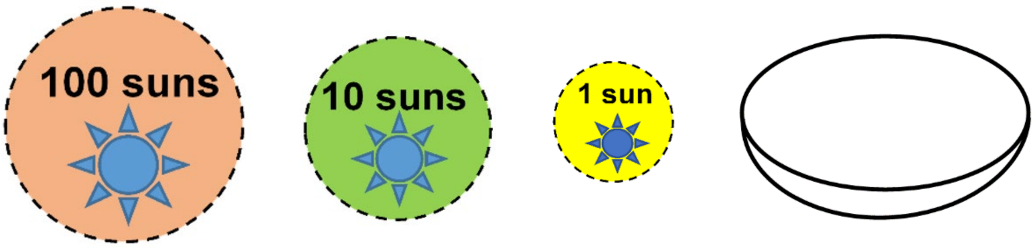
### INTRODUCTION

Mastering mathematical skills and developing a sound understanding of fundamental mathematical properties and relationships between concepts is one of the most important didactic goals in teaching primary school mathematics. When teaching procedural techniques (algorithms) it is important that we make the links to previous knowledge clear in order to ensure a sound conceptual understanding of the algorithm. In this article we present a hands-on approach to introducing the long division algorithm which clearly demonstrates the mathematical foundations of the technique. This article offers clear and didactic explanations for each step of the long division algorithm by using physical coins and bowls. The coins are representative of the dividend, while the bowls represent the divisor. Illustrating the long division technique through this hands-on approach helps to develop the underlying concepts associated with the long division algorithm so that the decimal structure of the numbers and the arithmetic operations in each stage of the process become clearer.

### MATERIALS

For the purposes of this article we will demonstrate the idea using three different coins known as "Suns". The coins, "100 suns", "10 suns" and "1 sun" represent the decimal structure of the dividend. The value of each coin is denoted by its inscription, size and colour. In addition, a number of bowls are used to represent the divisor.

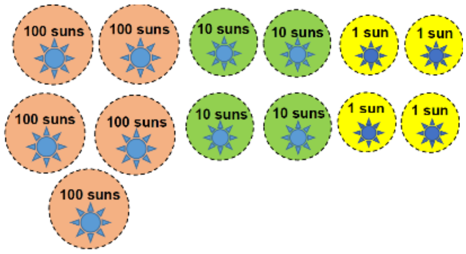



Although the coins, or "Suns", cannot be broken up, they can nonetheless be exchanged for smaller value coins. So, a "10 suns" coin can be replaced with ten "1 sun" coins, and a "100 suns" coin can be replaced with ten "10 suns" coins. For a given calculation, each pupil (or pair of pupils) should be given coins to represent the dividend and bowls to represent the divisor. Pupils will then physically distribute the dividend value (coins) over the divisor (bowls). This hands-on process then serves to enable the transition to the arithmetic long division algorithm. Two examples follow to illustrate the use of the coins and bowls.

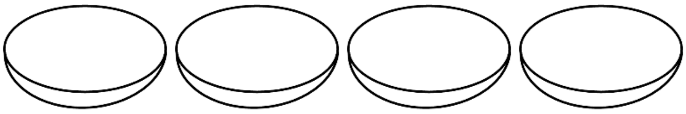
### Example 1: $544 \div 4$

The "Sun" coins represent the decimal structure of the dividend 544. Thus, we use five "100 suns" coins, four "10 suns" coins and four "1 sun" coins:

$$544 = (5 \times 100) + (4 \times 10) + (4 \times 1)$$

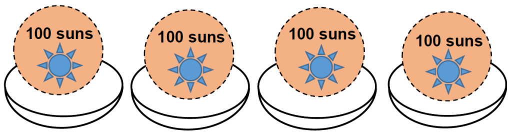



The divisor of 4 is represented by four bowls:

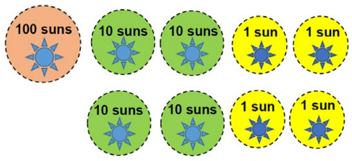


To divide 544 by 4 using coins and bowls, our aim is distribute the 544 Suns evenly between the four bowls. We begin with the highest value (hundreds) and continue through to the lowest value without skipping any digits. If a larger value cannot be divided evenly, then it can be replaced (exchanged) with an equivalent value made up of smaller coins. It is important to maintain the value of the place in the quotient by paying attention to every digit of the dividend in each stage of the process. The following flowchart illustrates the process:

A1:
In the **hundreds stage** we have five 100-Sun coins. We can distribute four of them evenly amongst the four bowls, one coin per bowl.

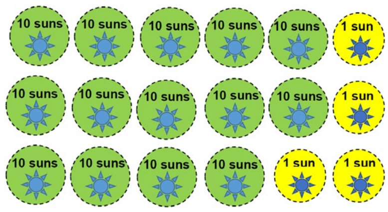



A2:
We now have one 100-Sun coin remaining, plus 44 more Suns: $(1 \times 100) + (4 \times 10) + (4 \times 1) = 144$.
(400 Suns have already been distributed)

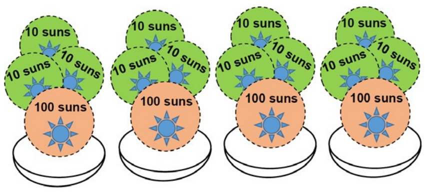



This single 100-Sun coin cannot be distributed evenly between the four bowls, therefore…

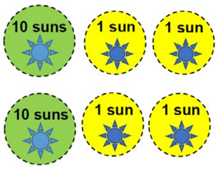
A3:
We replace the 100-Sun coin with ten 10-Sun coins. We now have $(14 \times 10) + (4 \times 1) = 144$ in the dividend.




A4:
In the **tens stage** we have fourteen 10-Sun coins. We can distribute twelve of them evenly amongst the four bowls, three coins per bowl. Each bowl now has 130 Suns.

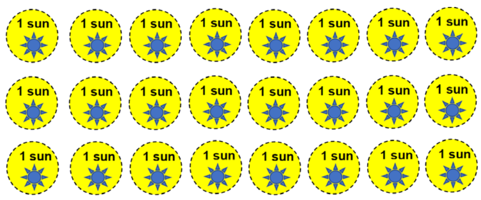



A5:
There are now two 10-Sun coins remaining, plus four 1-Sun coins: $(2 \times 10) + (4 \times 1) = 24$.
(520 Suns have already been distributed)



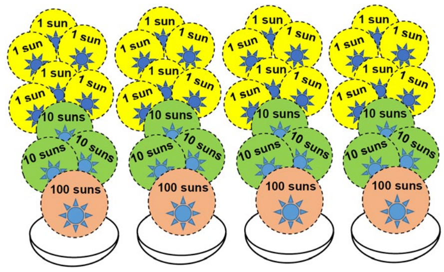

The two 10-Sun coins cannot be distributed evenly between the four bowls, therefore…

A6:
We replace the two 10-Sun coins with twenty 1-Sun coins. We now have 24 Suns (all 1-Sun coins) remaining.




A7:
In the **units stage** we have twenty-four 1-Sun coins. We can distribute them evenly between the four bowls, six coins per bowl.

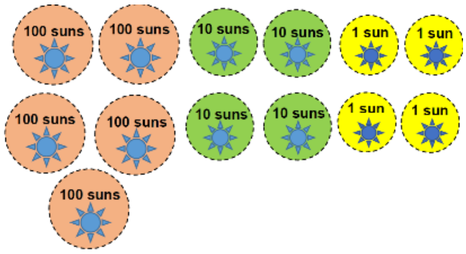
Each bowl now has 136 Suns. This is the **quotient**.



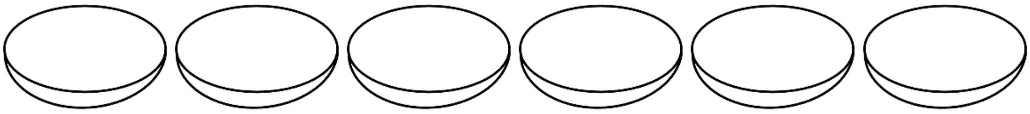
**Example 2: $544 \div 6$**

The “Sun” coins represent the decimal structure of the dividend 544. Thus, we use five “100 suns” coins, four “10 suns” coins and four “1 sun” coins as before:

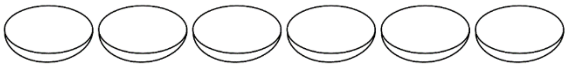
$$544 = (5 \times 100) + (4 \times 10) + (4 \times 1)$$




The divisor of 6 is represented by six bowls:

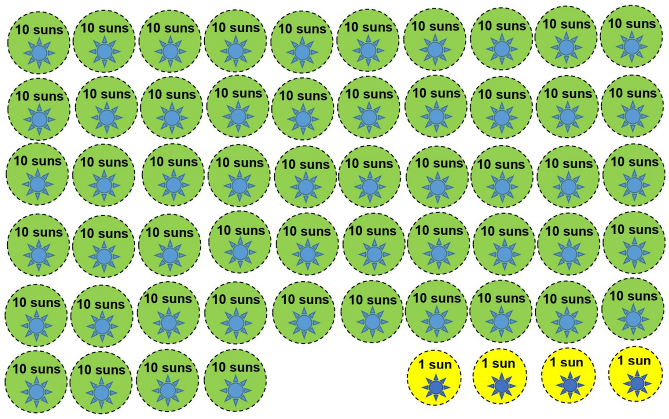


A1:
In the **hundreds stage** we have five 100-Sun coins. We wish to distribute them evenly between the six bowls.

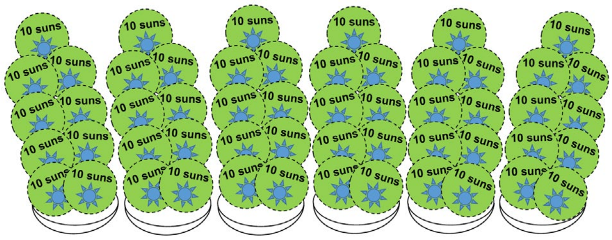


These five 100-Sun coins cannot be distributed evenly between the six bowls, therefore…

A2:
We replace the five 100-Sun coins with fifty 10-Sun coins: $500 = 50 \times 10$.

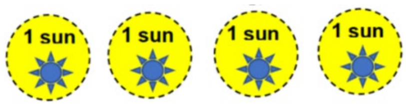


A3:
In the **tens stage** we have fifty-four 10-Sun coins. We can distribute all of them evenly between the six bowls, nine coins per bowl. Each bowl now has 90 Suns.



⇩

A4:
There are now four 1-Sun coins remaining.
(540 Suns have already been distributed)




⇩

A5:
In the **units stage**, the four remaining 1-Sun coins cannot be evenly distributed between the six bowls.

Each bowl has 90 Suns, and this represents the **quotient**.

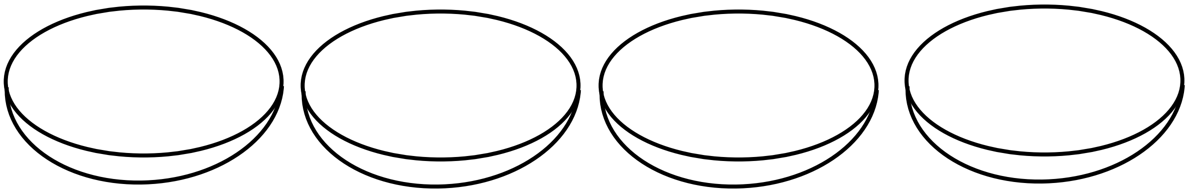
The 4 Suns that are left over represent the **remainder.**

## CONCLUDING COMMENTS

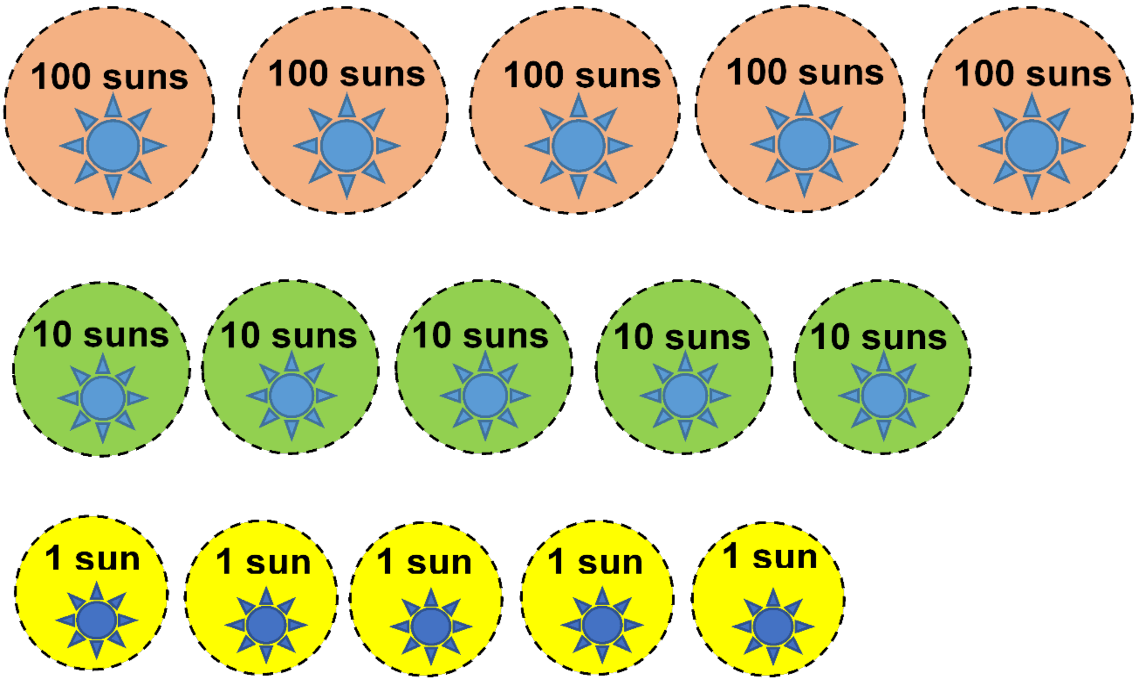
The approach of using coins and bowls as a physical counterpart to the more formal long-division algorithmic procedure is a powerful way to introduce long division to primary school pupils. The bowls and coins help develop the idea of sharing equally, and then determining what is left of the dividend. Replacing coins (that cannot be distributed equally between the bowls) with an equivalent value of smaller coins deepens conceptual understanding that will strengthen the underlying process of the formal long-division algorithm. In the South African context, the bowls and coins could be used as a strategy to informally introduce long division in Grade 4 or 5, and then used to support the arithmetic notation when it is introduced later in Grade 6.

## APPENDIX – MATERIALS

**Bowls**



**Coins**



# Exploring Cevians


## Duncan Samson

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### INTRODUCTION

A cevian is a line segment joining a vertex of a triangle to a point on the opposite side (or its extension) as illustrated in Figure 1. Cevians are named after the Italian mathematician Giovanni Ceva (1647-1734) who is probably best known for what is generally referred to as Ceva's theorem – the condition for three general cevians to be concurrent. Medians, altitudes and angle bisectors are special cases of cevians.

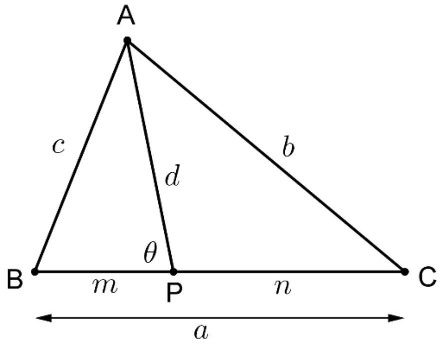



**FIGURE 1:** Triangle $ABC$ with cevian $AP$ of length $d$.

### STEWART'S THEOREM

The length of a cevian can be determined by Stewart's theorem. With reference to Figure 1, the cevian length $d$ is given by the following formula:

$$b^2m + c^2n = a(d^2 + mn)$$

Stewart's theorem can readily be proved using the cosine rule. In triangles $APB$ and $ACP$ we have respectively:

$$c^2 = d^2 + m^2 - 2dm\cos\theta \ \dots \text{(1)}$$

$$b^2 = d^2 + n^2 - 2dn\cos(180° - \theta)$$

$$= d^2 + n^2 + 2dn\cos\theta \ \dots \text{(2)}$$

Multiplying equation (1) by $n$ and equation (2) by $m$ and then adding allows us to eliminate $\cos\theta$:

$$c^2n = d^2n + m^2n - 2dmn\cos\theta$$

$$b^2m = d^2m + n^2m + 2dmn\cos\theta$$

$$\therefore \ b^2m + c^2n = d^2m + d^2n + n^2m + m^2n$$

Factorising the expression on the right-hand side, and replacing $m + n$ with $a$, gets us to the required result:

$$b^2m + c^2n = d^2(m + n) + mn(n + m)$$

$$= (m + n)(d^2 + mn)$$

$$= a(d^2 + mn)$$

While Stewart's theorem will be unfamiliar to most high school pupils, both it and its proof are readily accessible. The proof is in fact rather pleasing as it incorporates many aspects of high school mathematics – the cosine rule, trigonometric reductions, elimination, and factorisation of a four-term expression involving grouping. Stewart's theorem can also be proved using Pythagoras's theorem directly by dropping a perpendicular from vertex $A$ to base $BC$ and writing the distances $b$, $c$ and $d$ in terms of this altitude. It is left to the interested reader to complete this proof.

### VAN AUBEL'S THEOREM FOR TRIANGLES

Henri Van Aubel (1830-1906) taught pre-university mathematics at the *Koninklijk Atheneum Antwerpen* in Belgium. Given triangle $ABC$ with three cevians intersecting at a common point $P$, as illustrated in Figure 2, then Van Aubel's theorem for triangles states that:

$$\frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}$$

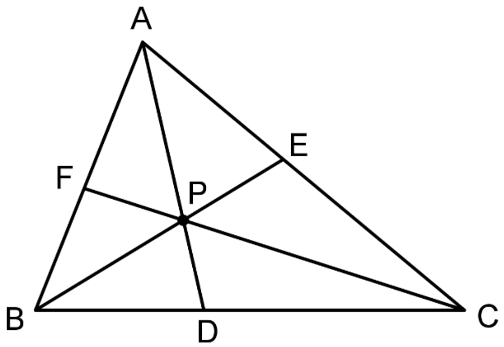



**FIGURE 2:** Triangle $ABC$ with three cevians concurrent at point $P$.

The proof of this interesting result is also very accessible to the high school pupil. There are two basic ideas we will make use of in the proof:

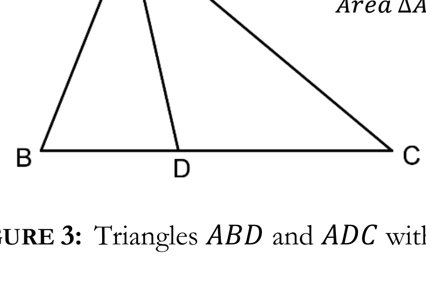
- If $\frac{a}{b} = \frac{c}{d}$ then $\frac{a}{b} = \frac{c}{d} = \frac{a+c}{b+d} = \frac{a-c}{b-d}$

  This can readily be understood as follows:

  If $\frac{a}{b} = \frac{c}{d}$ then $c = ka$ and $d = kb$ where $k$ is some scalar constant. Thus:

  $\frac{a+c}{b+d} = \frac{a+ka}{b+kb} = \frac{a(1+k)}{b(1+k)} = \frac{a}{b}$ and $\frac{a-c}{b-d} = \frac{a-ka}{b-kb} = \frac{a(1-k)}{b(1-k)} = \frac{a}{b}$

- The areas of triangles with equal altitudes (perpendicular heights) are in the same proportion as the lengths of their bases.




$$\frac{Area\ \Delta ABD}{Area\ \Delta ADC} = \frac{\frac{1}{2} \times BD \times \perp h}{\frac{1}{2} \times DC \times \perp h} = \frac{BD}{DC}$$

**FIGURE 3:** Triangles $ABD$ and $ADC$ with areas in the ratio $BD\colon DC$.

We can now prove Van Aubel's theorem for triangles as follows:

$$\left.\begin{aligned}\frac{Area\ \Delta ACF}{Area\ \Delta BCF} = \frac{AF}{BF} \\ \frac{Area\ \Delta APF}{Area\ \Delta BPF} = \frac{AF}{BF}\end{aligned}\right\} \quad \therefore \quad \frac{AF}{FB} = \frac{Area\ \Delta ACF - Area\ \Delta APF}{Area\ \Delta BCF - Area\ \Delta BPF} = \frac{Area\ \Delta APC}{Area\ \Delta BPC} \quad \ldots (1)$$

$$\left.\begin{aligned}\frac{Area\ \Delta ABE}{Area\ \Delta CBE} = \frac{AE}{EC} \\ \frac{Area\ \Delta APE}{Area\ \Delta CPE} = \frac{AE}{EC}\end{aligned}\right\} \quad \therefore \quad \frac{AE}{EC} = \frac{Area\ \Delta ABE - Area\ \Delta APE}{Area\ \Delta CBE - Area\ \Delta CPE} = \frac{Area\ \Delta ABP}{Area\ \Delta CBP} \quad \ldots (2)$$

Adding equations (1) and (2) gives:

$$\frac{AF}{FB} + \frac{AE}{EC} = \frac{Area\ \Delta APC}{Area\ \Delta BPC} + \frac{Area\ \Delta ABP}{Area\ \Delta CBP} = \frac{Area\ \Delta APC + Area\ \Delta ABP}{Area\ \Delta BPC}$$

But: $$\frac{AP}{PD} = \frac{Area\ \Delta ABP}{Area\ \Delta DBP} = \frac{Area\ \Delta APC}{Area\ \Delta DPC}$$

$$\therefore \quad \frac{AP}{PD} = \frac{Area\ \Delta ABP + Area\ \Delta APC}{Area\ \Delta DBP + Area\ \Delta DPC} = \frac{Area\ \Delta ABP + Area\ \Delta APC}{Area\ \Delta BPC}$$

$$\therefore \quad \frac{AP}{PD} = \frac{AF}{FB} + \frac{AE}{EC}$$

## CONCLUDING COMMENTS

While Stewart's theorem and Van Aubel's theorem for triangles will both be unfamiliar to most high school pupils (other than those who have received special Olympiad training), both theorems are readily accessible. More importantly, their *proofs* rely only on concepts and techniques covered within the school Mathematics syllabus. Exploring such proofs in the classroom (with appropriate scaffolding and guidance) can expose pupils to interesting theorems and results beyond the confines of the school syllabus, and at the same time show how basic concepts and techniques learnt at school find application beyond the walls of the classroom.

# 'Lenses' with Fixed Radius – A Problem Solving Activity with Circular Plates

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### INTRODUCTION

In most instances, circles are drawn using a pair of compasses. This makes sense since compasses can easily be adjusted to draw circles of any desired radius. However, if one wanted to draw a number of identical circles, then an alternative approach would be to use a *circular plate* (made from metal or plastic). In the problem solving activity discussed in this article, the use of circular plates is recommended since if one instead made use of compasses or dynamic geometry software, the need to first construct each circle's centre would make the process unnecessary laborious. Using circular plates, each circle (or circular arc) can be drawn quickly and efficiently.

The inspiration for this article is the German book *Wege zu geometrischen Sätzen* (Haag, 2003). The activity uses the shape we will refer to as a *lens* or *circlar 2-gon*. There are a number of possible approches to solving the problem, and three of these are discussed in this article. Problems that allow for a variety of solution strategies are very useful for exploration as they allow students, whether working alone or in groups, to work autonomously and independently to a greater degree. The amount of scaffolding a teacher might need to supply will of course differ in every case, but it is important that this does not detract from the sense that the students are *doing* mathematics, and are engaged in the *process* of mathematical exploration.

### THE PROBLEM: CIRCULAR 2-GONS WITH FIXED RADIUS

Using a circular plate, draw a circular 2-gon $AB$ shaped like a lens, i.e. comprising two circular arcs. On one of these arcs an arbitrary point $C$ is chosen. Using the same circular plate (i.e. with the same fixed radius), draw two further arcs, through $A$ and $C$ and through $B$ and $C$ respectively. Label the point of intersection of these two further arcs as $D$.

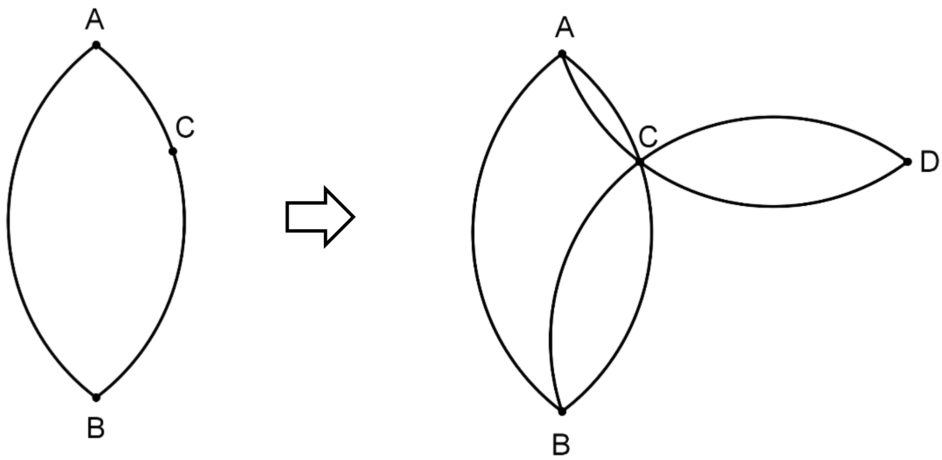


**FIGURE 1:** Drawing the problem context.

Given the setup illustrated in Figure 1, then independent of the position of point $C$ on the arc $AB$, we can make the following claims (Figure 2):

1. All circular 2-gons $CD$ drawn in this way are congruent.
2. The 'diagonal' $AB$ and the extension of the 'diagonal' $CD$ are perpendicular.
3. Point $D$ will always lie on the circle of which $AB$ is an arc.

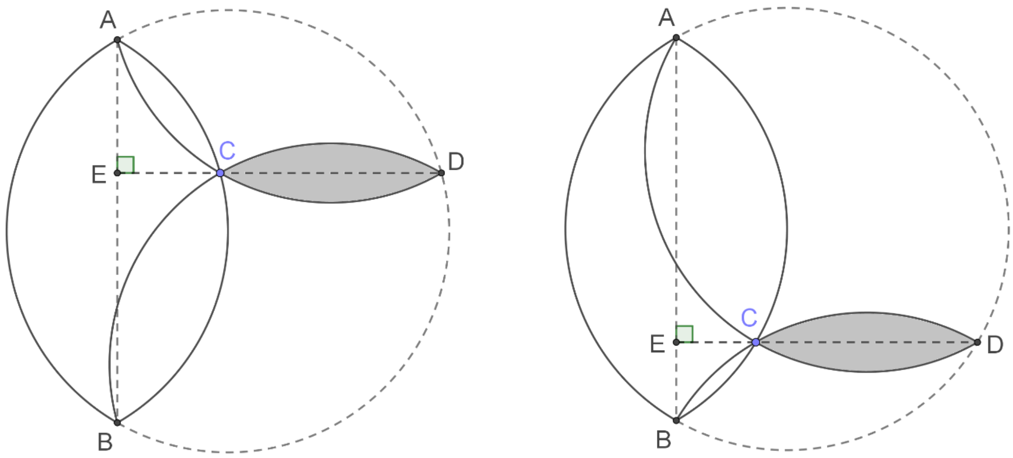


**FIGURE 2:** Congruent circular 2-gons $CD$.

By using circular plates (rather than a pair of compasses) one can quickly draw several such situations and conjecture that all three claims seem to hold true. The question now arises that *if* this is true, *why* is it so? What we require is to be able to *prove* these claims using *verification* and *explanation* (De Villiers, 2012).

As an initial observation, which could be used as general background irrespective of the solution approach, note that a circular 2-gon of given radius is uniquely determined by the opening angle $\varphi$ at the vertices, i.e. the angle formed by the tangents to the arcs at the vertex (Figure 3). Thus, to prove the congruence of two circular 2-gons with the same radius, it suffices to show the equality of their opening angles.

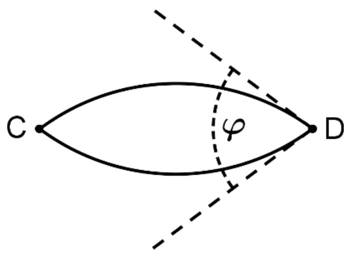


**FIGURE 3**

**SOLUTION 1**

This solution is inspired by Haag (2003, pp. 11-14). As an initial prompt one could ask the question: What is the relation between a *circular 2-gon's opening angle at its vertices* and the *central angle of the corresponding arc*? With a bit of exploration, and due to the fact that the two angles marked $\frac{\varphi}{2}$ are complementary to $B\hat{A}N$ (rhombus $AMBN$, see Figure 4), it should not be too difficult to discover that the opening angle of the 2-gon and the central angle of the corresponding arc are *equal*.

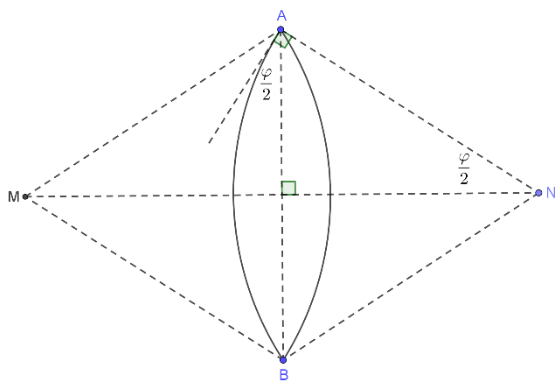


**FIGURE 4**

We can now use these observations as the basis to prove the three claims.

1. In the large circular 2-gon $AB$, two smaller circular 2-gons are inscribed with common point $C$ on arc $AB$ (Figure 5). The sum of the opening angles of the two smaller circular 2-gons ($\alpha$ and $\beta$) must equal the opening angle of the large circular 2-gon since the sum of the two central angles of the smaller arcs equals the central angle of the bigger arc. We thus see that the sum $\alpha + \beta$ must be constant. By focusing on the angles created by the three tangents at point $C$, we see that $\alpha + \beta + \gamma = 180°$. And since we have already established that $\alpha + \beta$ is constant, this means that $\gamma$ must also be constant, thus proving the claim (since circular 2-gons with the same radius are congruent if their opening angles are the same).

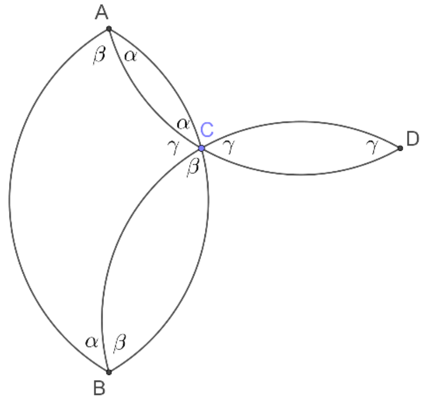


**FIGURE 5:** Constant angle $\gamma$.

2. With reference to Figure 6, note that in triangle $AEC$ the angle at $C$ is $\frac{\alpha}{2} + \frac{\gamma}{2}$. Furthermore, the angle at $A$ is $\frac{\alpha+\beta}{2} - \frac{\alpha}{2} = \frac{\beta}{2}$. And since $\alpha + \beta + \gamma = 180°$, this means that $\frac{\alpha}{2} + \frac{\gamma}{2} + \frac{\beta}{2} = 90°$, and thus that the angle at $E$ is indeed $90°$.

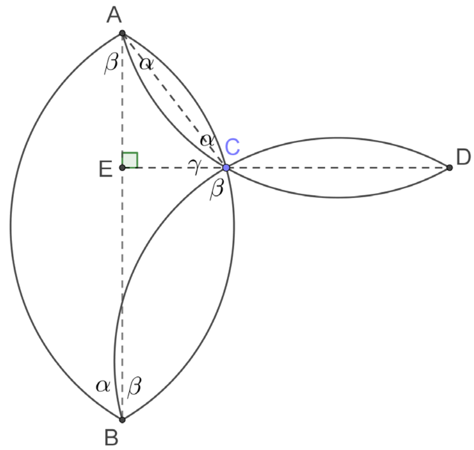


**FIGURE 6:** Right angle at $E$.

3. If we reflect the arc $ACD$ about $AD$ and the arc $BCD$ about $BD$ we create the dashed arcs with central angles of $\alpha + \gamma$ and $\beta + \gamma$ respectively (Figure 7). These two arcs, along with the original arc $AB$, thus have a total central angle of $360°$, since $\alpha + \beta + \gamma = 180°$. And since these three arcs all have the same radius it follows that they form the circumcircle of triangle $ABD$, and thus that arc $ADB$ lies on the circle containing arc $AB$.

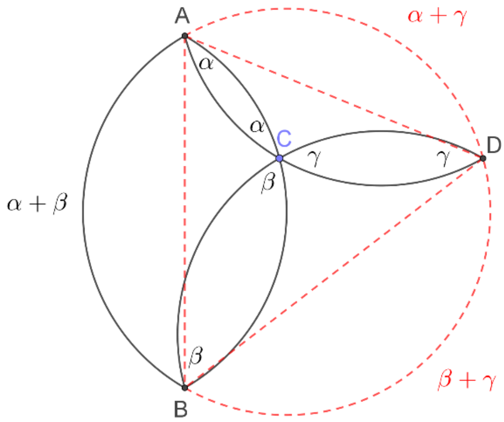


**FIGURE 7**

**SOLUTION 2**[2]

This solution makes use of the inscribed angle theorem, and this could be suggested as an initial useful 'hint' to students.

1. From the inscribed angle theorem it follows that $A\hat{C}B = \frac{\alpha}{2} + \frac{\beta}{2} + \gamma$ is a constant value irrespective of the position of point $C$ on arc $AB$. And since $\alpha + \beta + \gamma = 180°$ (tangents at point $C$) it immediately follows that $\frac{\alpha}{2} + \frac{\beta}{2}$, and hence also $\alpha + \beta$ and $\gamma$ must also be constant.

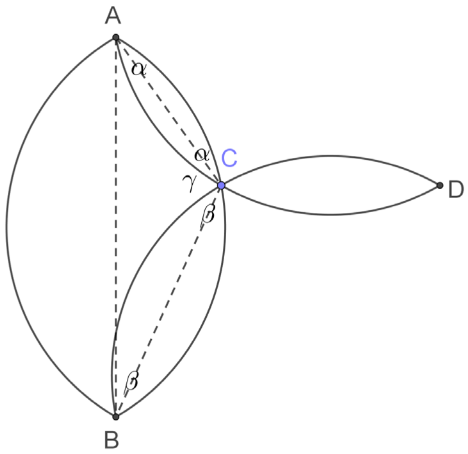


**FIGURE 8:** Constancy of $\gamma$ using the inscribed angle theorem.

2. Can be proved as per Solution 1.

3. Here one can argue slightly differently to the proof shown in Solution 1, using the converse of the inscribed angle theorem. Note that $A\hat{C}B = \frac{\alpha}{2} + \frac{\beta}{2} + \gamma$, and this would be the same if point $C$ lay on the other arc $AB$. Thus, if we can show that $A\widehat{D}B = \frac{\alpha+\beta}{2}$ then we are done with proving that $D$ lies on the circle of which arc $AB$ forms part. To show that $A\widehat{D}B = \frac{\alpha+\beta}{2}$ we need to return to the principle used in Solution 1(a). Using this approach we see that an angle $\frac{\alpha+\gamma}{2}$ is formed at the vertices of the circular 2-gon $AD$, and that an angle $\frac{\beta+\gamma}{2}$ is formed at the vertices of the circular 2-gon $BD$. Thus, $A\widehat{D}B = 180° - \left(\frac{\alpha+\gamma}{2} + \frac{\beta+\gamma}{2}\right) = \frac{\alpha+\beta}{2}$ (tangent at $D$, Figure 9).

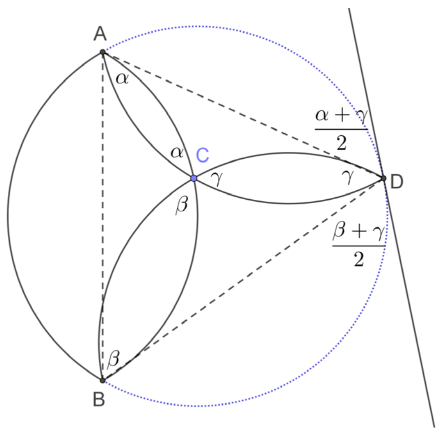


**FIGURE 9:** $A\widehat{D}B = 180° - \left(\frac{\alpha+\gamma}{2} + \frac{\beta+\gamma}{2}\right) = \frac{\alpha+\beta}{2}$.

[2] My thanks to Ch. Dorner for stimulating discussions concerning this particular solution.

### SOLUTION 3[3]

As an initial prompt or hint, one could suggest the following. Draw the centres $M$ and $N$ of the circles corresponding to the arcs $AB$ and $ACB$, and then draw quadrilateral $AMBN$. Also draw the centres $F$ and $G$ of the circles corresponding to the arcs $ACD$ and $BCD$, and then draw quadrilateral $CGFD$. With reference to Figure 10, what observations can you make about the two quadrilaterals?

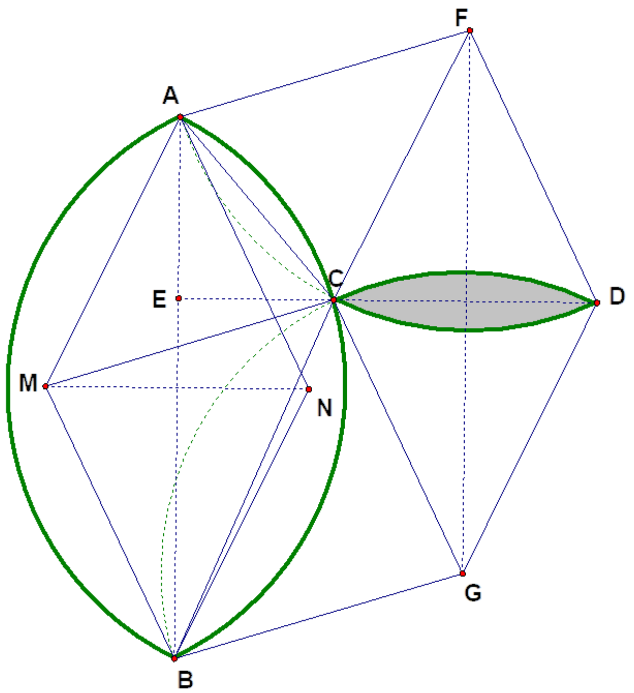


**FIGURE 10:** Congruent rhombuses $AMBN$ and $CGDF$.

The important observation is that the two quadrilaterals are congruent rhombuses. Note that quadrilateral $AMBN$ is a rhombus since its sides are the radii of equal arcs. A similar argument holds for quadrilateral $CGDF$ where $F$ and $G$ are the centres of the arcs (circles) with the original radius. $AF$, $BG$ and $MC$ are also such radii, thus $AMCF$, and analogously $MBGC$, are also rhombuses. This yields $AF \parallel MC \parallel BG$. Since $AF$ and $BG$ are equal (radii), it follows that $ABGF$ is a parallelogram, and this means that $AB$ and $FG$ are equal. Thus, $AMBN$ and $CGDF$ are congruent rhombuses (equal sides, one diagonal equal).

Note that in this particular solution the centres of the arcs play a crucial role. The use of a circular plate, while useful in the experimental phase of validating observations, does not suffice in terms of working towards a proof.

We can now make use of the above observations and results to prove the original three claims in yet another way:

1. The opening angle of the circular 2-gon $AB$ is the obtuse angle of the rhombus (central angle, see above). In the circular 2-gon $CD$, it is the acute angle. These two angles add up to $180°$, thus the opening angle $\gamma$ of the circular 2-gon $CD$ is fixed, and thus so is the circular 2-gon.

2. The diagonals $CD$ and $FG$ of the rhombus are perpendicular. Since $AMBN$ is simply a translation of $CGDF$ it follows that $DC$ extended to $DE$ is perpendicular to $AB$.

3. Proved as per Solution 1 or Solution 2.

[3] My thanks to B. Schuppar for stimulating discussions concerning this particular solution.

### FURTHER POSSIBILITIES TO EXPLORE

One could expand on these ideas by exploring related scenarios such as the 'clover-leaf theorem'. If one begins at a point on a circle and then draws a 'closed' set of three arcs with the same radius as the circle, then these three arcs will be concurrent at a single point and form a clover-leaf pattern (Figure 11) that has the following properties:

- The sum of the angles at the three vertices is $180°$, with the total angle sum of the whole clover-leaf being $360°$.
- The perimeter of the clover-leaf is the same as the circumference of the circle.

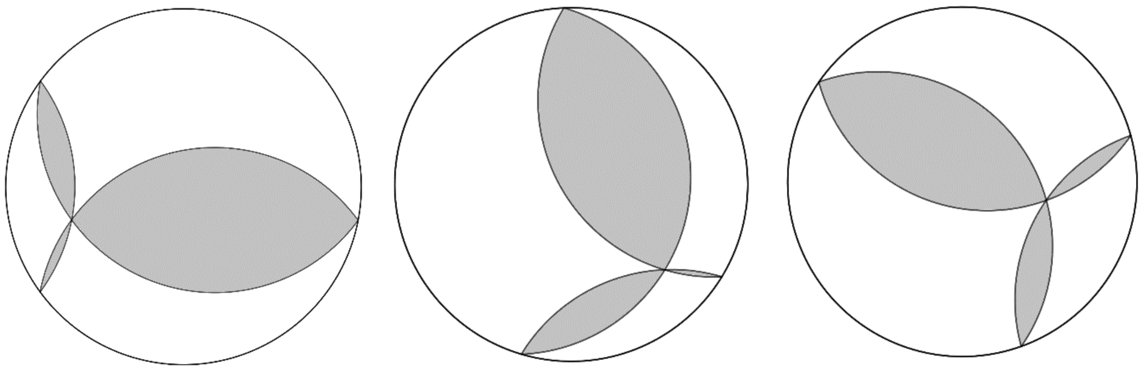

**FIGURE 11:** Clover-leaf diagrams.

### CONCLUDING COMMENTS

The problems posed in this article open many avenues for exploration. The use of a circular plate, as opposed to a pair of compasses, is an interesting departure from the norm. The crucial figures in our context (circular 2-gons) are rich and comprehensive, and while they are not a standard topic in geometry education, they are sufficiently easy geometric objects to enable students to work autonomously on interesting problems. The important thing is that students have the opportunity to experience mathematics as a process. The mathematics itself is not complex, but students may struggle to make progress as they might not know where to begin. In such instances it is up to the teacher to assist by offering helpful prompts that assist in a meaningful way without revealing too much. Even if students don't manage to come to a complete solution, they will nonetheless learn a great deal along the way.

### REFERENCES

De Villiers, M. (2012). *Rethinking Proof with The Geometer's Sketchpad.* Key Curriculum Press, Emeryville.
Haag, W. (2003). *Wege zu geometrischen Sätzen* (German). Klett, Stuttgart.

# Area of the Largest Rectangle Inscribed in a Right-Angled Triangle


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### INTRODUCTION

Given a right-angled triangle $ABC$, we are interested in finding the area of the largest rectangle $CDFE$ with all four vertices lying on the sides of triangle $ABC$, as illustrated in Figure 1.

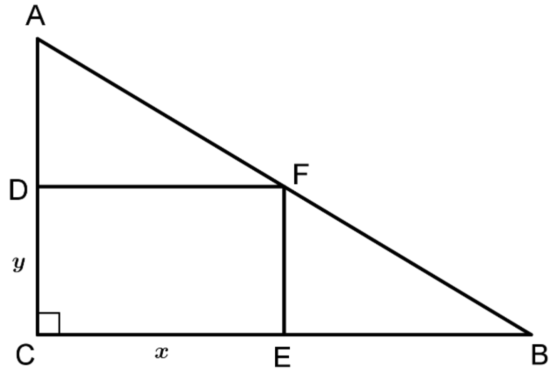


**FIGURE 1:** Finding the largest rectangle inscribed in triangle $ABC$.

Let $BC = a$, $AC = b$, $CE = x$ and $CD = y$. Since triangles $ADF$ and $ACB$ are similar, we have:

$$\frac{AD}{AC} = \frac{DF}{CB} \quad \Rightarrow \quad \frac{b-y}{b} = \frac{x}{a}$$

This can be simplified to $ay + bx = ab$.

In this short article we discuss two different approaches to solving this problem, and offer them as alternatives to the trigonometric approach described by Pillay and Bizony (2009). The first approach assumes a basic knowledge of the process of completing the square, while the second approach assumes knowledge of basic differential calculus. For convenience we will use $W$ to denote the area of rectangle $CDFE$.

### AN ALGEBRAIC APPROACH

Using the relationship arrived at above, namely $ay + bx = ab$, we have $x = (ab - ay)/b$. We can now express the area of the rectangle in terms of $y$ as follows:

$$W = xy = \frac{(ab-ay)y}{b} = \frac{a}{b}(by - y^2) = \frac{a}{b}\left(\frac{b^2}{4} - \left(\frac{b}{2} - y\right)^2\right) = \frac{ab}{4} - \frac{a}{b}\left(\frac{b}{2} - y\right)^2$$

From this we can see that the maximum area of $W = \frac{ab}{4}$ occurs when $y = \frac{b}{2}$. Since $ay + bx = ab$ it also follows that the maximum area occurs when $x = \frac{a}{2}$. The area of the largest rectangle $CDFE$ is thus half the area of triangle $ABC$.

### A CALCULUS APPROACH

Since $W = xy = \frac{a}{b}(by - y^2)$, we have $\frac{dW}{dy} = \frac{a}{b}(b - 2y)$. Since the second derivative $\frac{d^2W}{dy^2} = \frac{-2a}{b} < 0$ and the first derivative $\frac{dW}{dy} = 0$ occurs when $y = \frac{b}{2}$, we can similarly conclude that the maximum area of $W$ is $\frac{ab}{4}$ and occurs when $y = \frac{b}{2}$ and $x = \frac{a}{2}$.

### A RELATED PAPER-FOLDING ACTIVITY

A way of introducing this activity to pupils is through a related paper-folding activity. Provide pupils with right-angled triangles and ask them to explore how to construct a rectangle from the triangle by folding over the vertices of the triangle in such a way that the vertices of the rectangle lie on the sides of the triangle. They should hopefully accomplish this by folding vertices $A$ and $B$ onto $C$ as illustrated in Figure 2. Next ask pupils to explore whether the constructed rectangle is the largest possible rectangle. This then leads into asking pupils to find the area of the largest rectangle, and an introduction to possible solution methods.

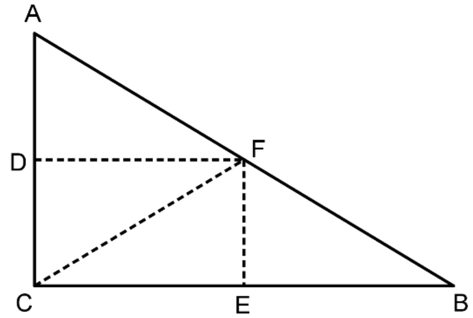


**FIGURE 2:** Constructing the largest inscribed rectangle by paper-folding.

It is possible during the paper-folding activity that some pupils find a different largest rectangle (Figure 3a). However, it is not difficult to show that the area of this alternative rectangle is also half the area of the triangle. Dropping a perpendicular $CH$ reveals two smaller right-angled triangles, $CBH$ and $CAH$, each with largest rectangles, $FEJH$ and $GDJH$ respectively, inscribed as before (Figure 3b).

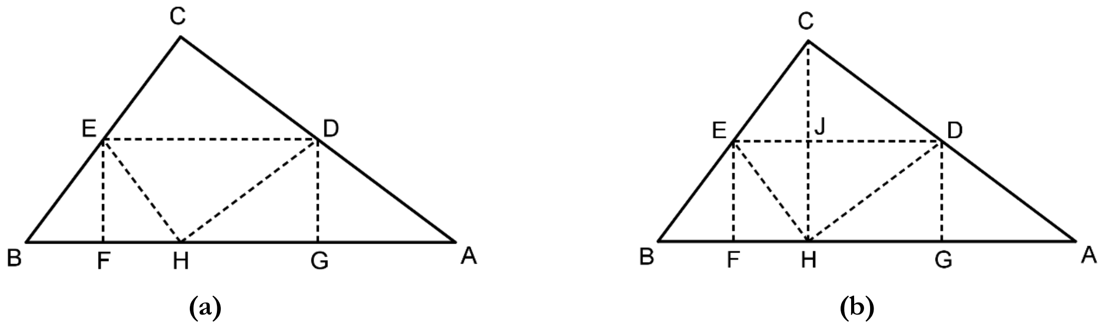


**FIGURE 3:** Another possible rectangle obtained by paper-folding.

### REFERENCES

Pillay, P., & Bizony, M. (2009). Rectangles inscribed in a triangle. *Learning and Teaching Mathematics*, *7*, 13-15.

# A Short Proof of a Geometric Property of a Triangle with Only One 60° Angle


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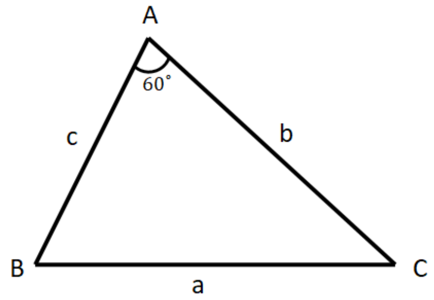
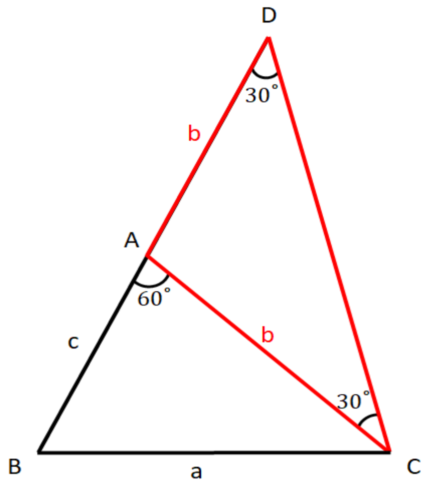
Consider a triangle that contains exactly one angle equal to 60° as shown alongside. For such a triangle, the following property holds true for the three sides:

$$a > \frac{b+c}{2}$$




Without loss of generality, let $\hat{B} > 60°$ and $\hat{C} < 60°$. Now extend $BA$ to $D$ with $AD = AC = b$. In the isosceles triangle $ADC$ we now have $\hat{D} = A\hat{C}D = 30°$.




Using the sine rule in triangle $BDC$ we have:

$$\frac{a}{\sin 30°} = \frac{b+c}{\sin B\hat{C}D} \quad \Rightarrow \quad a = \frac{b+c}{2\sin B\hat{C}D}$$

Now, since $B\hat{C}D < 90°$ this means that $\sin B\hat{C}D < 1$ from which it follows that $a > \frac{b+c}{2}$. Equality is obtained only in the case of an equilateral triangle.

Additional geometric properties of triangles with only one 60° angle can be found in Stupel, Oxman and Sigler (2021).

## References


Stupel, M, Oxman, V. & Sigler, A. (2021). Special properties of a triangle with an angle of 60°. *Resonance – Journal of Science Education, 26*(8), 1141-1152.

# Tan Graph Transformations – a Practical Application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The counter top in Dr Lee's office, along with certain dimensions, is illustrated in Figure 1. Determine the length of $y$. Readers are encouraged to solve this problem before reading on.

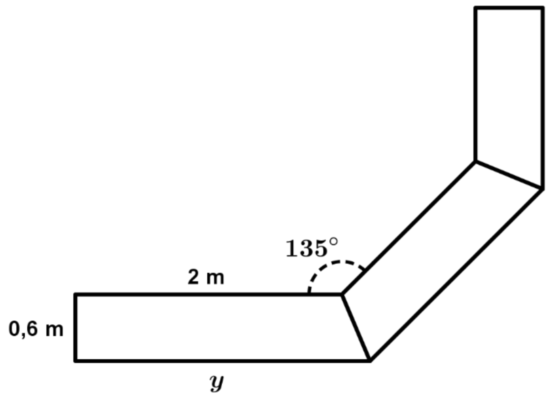


**FIGURE 1:** The counter top in Dr Lee's office.

With reference to Figure 2, by constructing perpendiculars $AB$ and $AD$ we see that the two triangles $ABC$ and $ADC$ are congruent (90°HS). From this it follows that $\theta = 22{,}5°$, and hence $t = 0{,}6\tan(22{,}5°)$. We thus have $y = 2 + 0{,}6\tan(22{,}5°)$ which is approximately 2,25 m. If the original angle of turn (135°) is $x$, then we have the following relationship: $y = 2 + 0{,}6\tan[(180° - x)/2]$. This can be interpreted graphically as the graph of $y = \tan x$ having undergone a number of successive transformations (Figure 3).

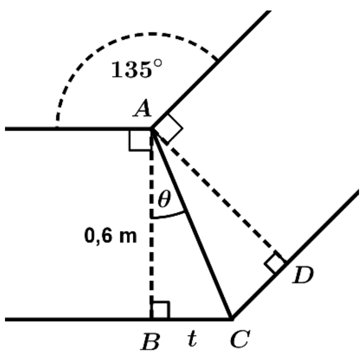


**FIGURE 2**

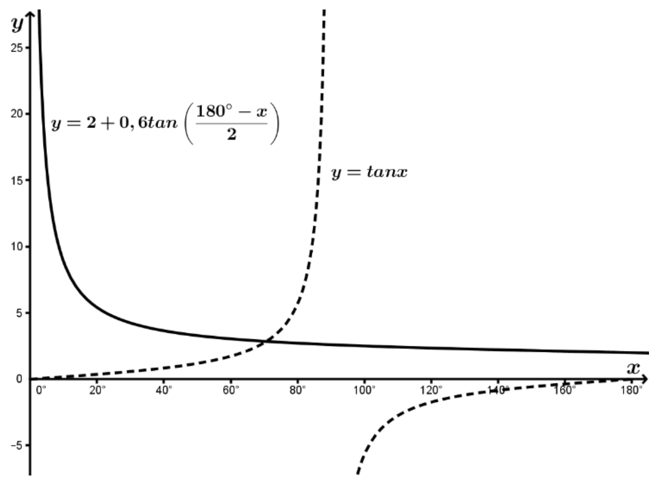


**FIGURE 3**

In general, if the width of the counter top is $w$, the shorter horizontal length is $v$, the angle of turn is $x$, and the longer horizontal length is $y$, then $y = v + w\tan[(180° - x)/2]$. It is left to the interested reader to explore this further.

# The IAAF 400 Metre Standard Athletics Track: "It's All About Mathematics"


## Alan Christison

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### INTRODUCTION

Excluding the field events, there are in excess of five hundred paint markings on an IAAF 400m Standard Athletics Track surface[4]. These markings set out positions of the lanes, starts, finish, hurdles, water jump, and relay change-overs. Each of these markings is mathematically defined and must be positioned within accuracy tolerances set out in the IAAF Competition Rules. The IAAF Certification System comprises an extremely comprehensive Track and Field Facilitation Measurement Report which must be completed and signed off by an appropriately registered surveyor.

This article initially introduces the dimensions of the IAAF 400m Standard Athletics Track and the important invisible running line. Thereafter the mathematics behind the positions of the 400m starting staggers and the 5000m curved start and group start lines are discussed. The latter also represent the starting positions of the 1000m and 3000m races, and introduces the use of a break line to ensure that all athletes run the same distance.

### THE IAAF 400 METRE STANDARD ATHLETICS TRACK

The track comprises two straights and two semi-circles as shown below in Figure 1.

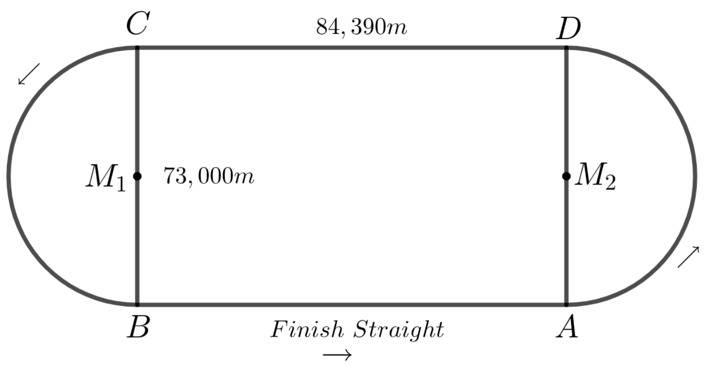



**FIGURE 1:** Dimensions of the IAAF 400m standard athletics track.

The straights are 84,390 metres, and the radii of the semi-circles 36,500 metres to the outside of the kerb edge on the inside of lane 1. $M_1$ and $M_2$ are permanently marked points, typically stainless-steel bolts, placed underground in concrete tubes and 84,390 metres apart.

This article considers an eight-lane track. A maximum of nine lanes is permissible, after which the runner in the outside lane of a 200m race will have an unfair advantage over the runner in the inside lane. Lanes are 1,22 metres wide and within each lane lies an invisible running line on which all mathematical calculations are based. These running lines do not affect the 100m, 100m hurdles, and the 110m hurdles, which are all run in a straight line down the finishing straight. The lines defining the lanes are painted white and are 0,05 metres wide, with the exception of the inside of lane 1 which is defined by a white kerb, typically 0,05 metres wide and 0,05 metres high.

[4] http://dansk-atletik.dk/media/1634050/iaaf-track-and-field-facilities-manual-2008-edition-marking-plan-400m-standard-track-1-.pdf

In lanes 2 – 8 the running line is 0,20 metres from the outside edge of the lane line, while in lane 1, due to the psychological nature of running adjacent to a raised kerb, the running line is 0,30 metres from the outside edge of the kerb.

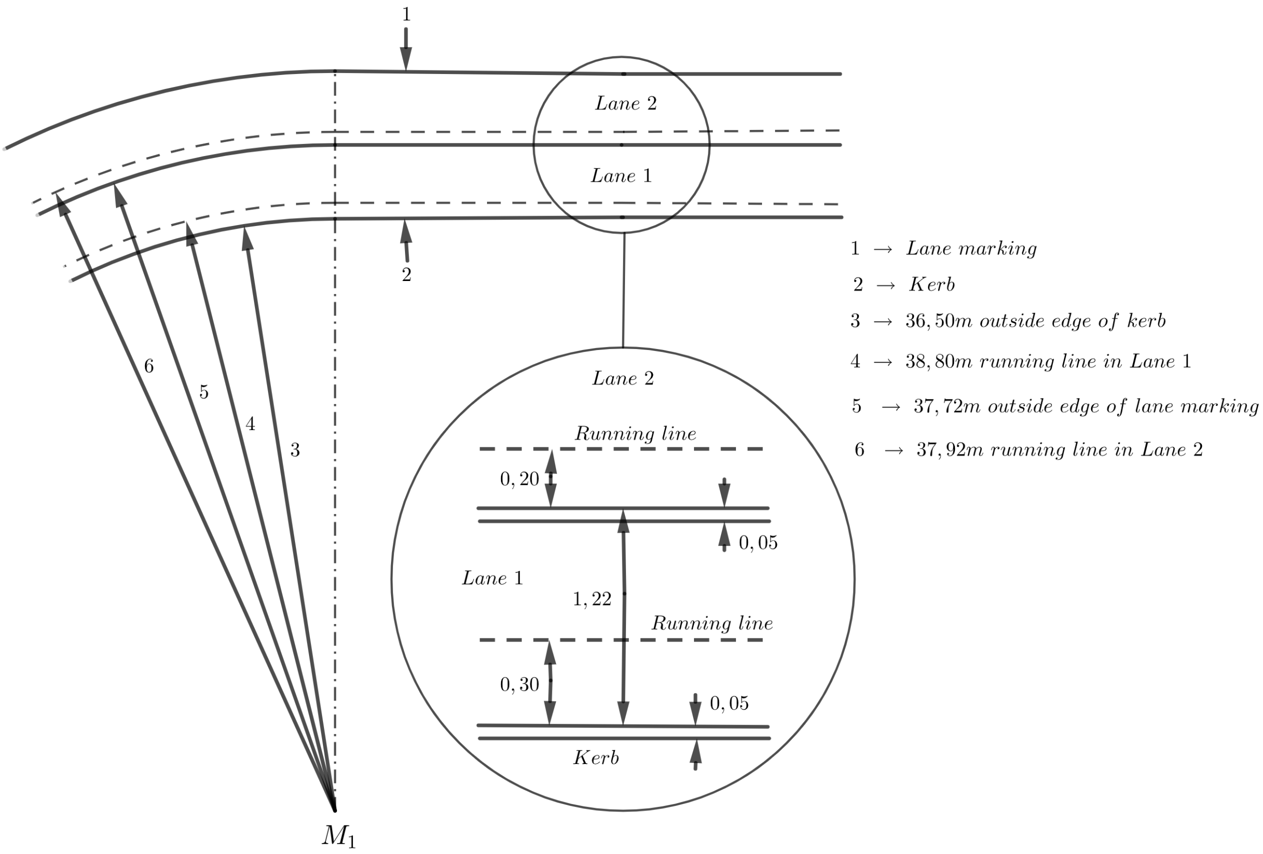



**FIGURE 2:** Kerb, lane marking, and running line – lanes 1 and 2.

The radius of the running line in lane 1 is $36{,}500 + 0{,}300 = 36{,}800$ metres, and thus the distance around this lane is given by:

$$2 \times 84{,}390 + 2\pi \times 36{,}800 = 400{,}001 \text{ metres}$$

The distance tolerance, $T$, in races related to all start lines is given by $0 \leq T \leq 0{,}0001L$ where $L$ is the race length in metres. *Clearly, no short running is permitted at all.*

### THE 400 METRE STAGGERED START

As shown above, the running distance in lane 1 is 400,001 metres, and thus the athlete starts on the finish line. As the radius increases in lanes 2 – 8, the distance around the track increases and therefore the starting positions are staggered to compensate for this and to ensure that all athletes run an identical distance.

By way of example, in lane 2 the radius is $36{,}500 + 1{,}220 + 0{,}200 = 37{,}920$ metres, and the full distance around the track in this lane is then given by:

$$400{,}001 + 2\pi(37{,}920 - 36{,}800) = 407{,}038 \text{ metres}$$

The stagger in lane 2 therefore is set at 7,038 metres *along the running line.*

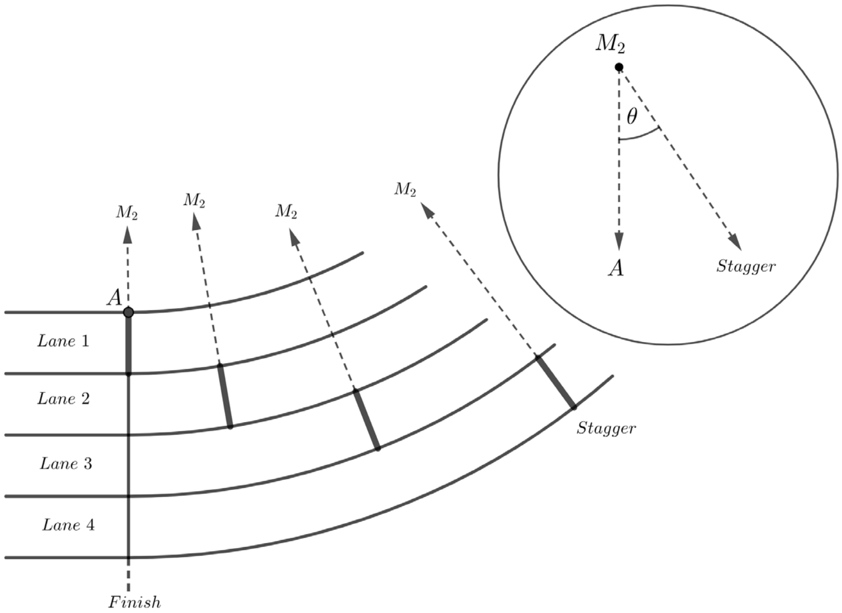



**FIGURE 3:** Typical 400m race staggers.

The stagger offset angle, $\theta$, may then be calculated using $\theta = \frac{d}{2\pi r} \times 360^{\circ}$ , where $d$ is the length of the arc along the running line between the finish line and the stagger, and $r$ is the radius of the running line with centre $M_2$. In lane 2, we have:

$$\theta_2 = \frac{180°}{\pi} \times \frac{7{,}038}{37{,}920} = 10^{\circ}\, 38'\, 03''$$

Table 1 below gives the full details for lanes 1 – 8, noting that, where applicable, a round-up of $0{,}001$m has been applied to the staggers in order to agree exactly with the official IAAF figures. Using a theodolite set up at $M_2$, the seven lane staggers may be exactly set out by applying the offset angle $\theta$ as applicable to each lane. *In all cases, the measurement is to the side of the* $0{,}05m$ *stagger line closest to the finish line working backwards in a clockwise direction.*

| Lane | Radius ($r$) | Distance ($m$) | Stagger ($d$) | Offset angle $\theta$ ($DMS$) |
|---|---|---|---|---|
| 1 | 36,80 | 400,001 | 0 | 0° |
| 2 | 37,92 | 407,038 | 7,038 | 10° 38′ 03″ |
| 3 | 39,14 | 414,704 | 14,704 | 21° 31′ 29″ |
| 4 | 40,36 | 422,370 | 22,370 | 31° 45′ 25″ |
| 5 | 41,58 | 430,034 | 30,034 | 41° 23′ 09″ |
| 6 | 42,80 | 437,700 | 37,700 | 50° 28′ 07″ |
| 7 | 44,02 | 445,366 | 45,366 | 59° 02′ 52″ |
| 8 | 45,24 | 453,032 | 53,032 | 67° 09′ 51″ |

**TABLE 1:** 400m lane stagger ($d$) and offset angle ($\theta$).

Note that the $200$ metre race staggers may be similarly calculated and set out from $M_1$. Due to the fact that only a single bend is run, each lane stagger and offset angle will be half that of the $400$ metre race. There is similarly no stagger applied in Lane 1, and the athlete lines up on the running line adjacent to point $C$ in Figure 1.

### THE 5000 METRE START AND GROUP START

The 5000m start (1) and group start (2) are shown in Figure 4. $T_2 - T_8$ each represent a tangent point on a lane running line measured from the associated running line at the end of the straight at $C$, while $T_9 - T_{11}$ each represent a tangent point on the running line measured from a perpendicular radial line $0{,}134$ metres in front of the 200m stagger in lane 5. $R_1$ and $R_8$ are the running lines in lanes 1 and 8 respectively. By way of example, the distance from the point of intersection of the running line in lane 8 with curve 1 to the tangent point $T_8$ must equal the distance run by the athlete in lane 1, i.e. the running line arc $R_1$ to $T_8$. In order to determine the mathematical points of the intersections of the running lines with curves 1 and 2, the associated arc lengths are required to be calculated. *It is important to note that curves 1 and 2 are not circular.*

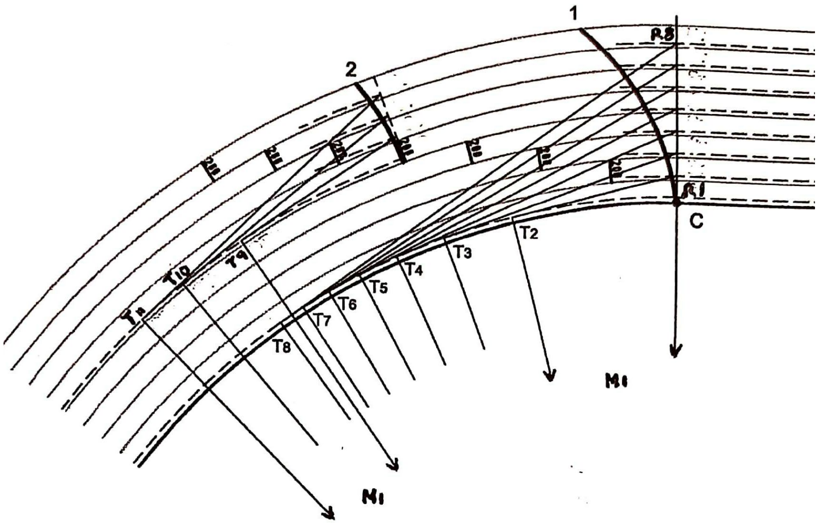



**FIGURE 4:** 5000m start and group start.

Notwithstanding the mathematical calculations below, in practice these curved lines are laid out on the running surface as described, and shown for the 5000m start in Figure 5.

A tape is laid out *on the running line* from $R_1$ to an appropriate tangent distance around the bend, and anchored at that tangent point, shown here as $T_8$. The tape at $R_1$ is then moved outwards towards lane 8, simultaneously marking the arc described on the track surface with chalk. This arc will define the curved start line as applicable to the race distance. The straight distance from the intersection of the lane running line (dotted) and the curved start line to the associated tangent point in lane 1 will equal the arced running distance of the athlete in lane 1. Clearly, in the case of lane 8, this straight distance will equal $d$.

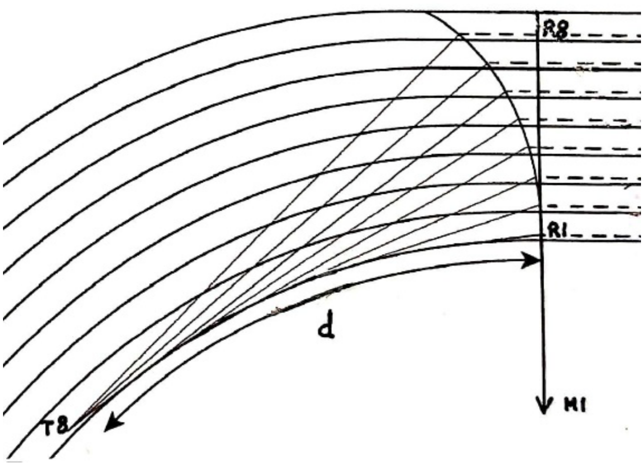



**FIGURE 5**

We will now determine the minimum distance $d$ applicable to the 5000m start, as well as the mathematical starting positions of each lane on the curved start line.

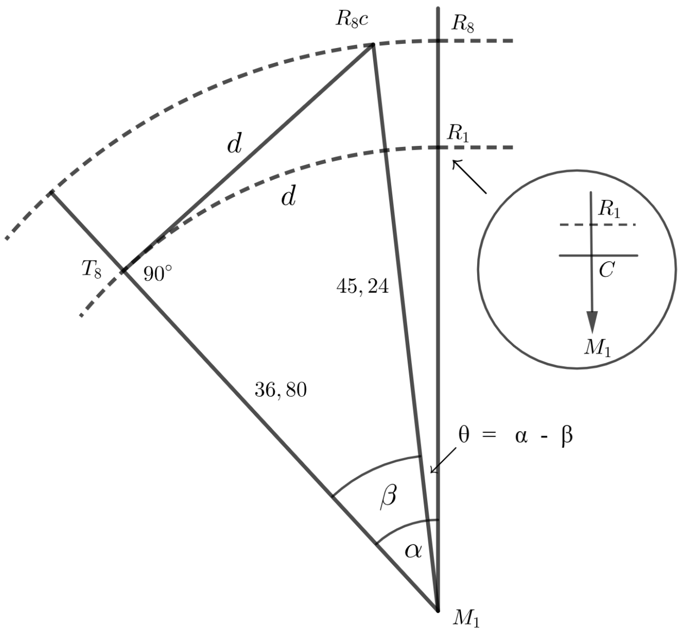



**FIGURE 6:** Lane 8 – determination of distance $d$.

With reference to Figure 6, $d^2 = 45{,}24^2 - 36{,}80^2$, leading to $d = 26{,}31382906$. From this it follow that:

$$\alpha = \frac{180d}{36{,}80\pi} = 40{,}96933011^{\circ}$$

We also have $\beta = cos^{-1}\left(\frac{36{,}80}{45{,}24}\right) = 35{,}56665236^{\circ}$, from which $\theta = 5{,}402677759^{\circ}$. The offset distance on the running line in lane 8 is thus $R_8\ R_8c = \frac{45{,}24\pi\theta}{180} = 4{,}266$ metres. Similar calculations may be carried out for lanes 2 – 7, and the results are detailed in Table 2.

| Lane ($L$) | Radius ($m$) | Offset angle $\theta_L$ ($DMS$) | $R_L\ R_Lc$ ($m$) |
|---|---|---|---|
| 1 | 36,80 | 0° | 0 |
| 2 | 37,92 | 0° 16′ 59″ | 0,187 |
| 3 | 39,14 | 0° 50′ 33″ | 0,575 |
| 4 | 40,36 | 1° 33′ 32″ | 1,098 |
| 5 | 41,58 | 2° 23′ 35″ | 1,737 |
| 6 | 42,80 | 3° 19′ 17″ | 2,481 |
| 7 | 44,02 | 4° 19′ 42″ | 3,325 |
| 8 | 45,24 | 5° 24′ 10″ | 4,266 |

**TABLE 2:** 5000m start – offset angle and running line stagger to curved start line.

The exact intersection points of the lane running lines and the curved start line may then be set out using a theodolite and tape measure or an electronic distance measuring device set up at $M_1$. The offset angle $\theta$ and the radius applicable to each of lanes 1 – 8 are then applied. Thereafter the curved 0,05m start line may be marked out on the track through these points.

Similar mathematics is used for the group start in lanes 5 – 8, however all athletes must remain at best in lane 5 until the end of the bend before again running (at best) a diagonal distance to a new tangent point just past the finish line. This is a longer distance than that applicable to the athletes in lane 1 who lined up on the start line, and this additional distance must be taken into account.

In Figure 7, the group start athletes run $NT$, where $N$ is a break line point on the running line in lane 5 at the end of the curve and $T$ is a tangent point on the running line in lane 1. The athletes who lined up on the start line run $PRT$ where $RT$ is a circular arc on the running line in lane 1. We therefore require the length of the additional running distance $NT - PRT$.

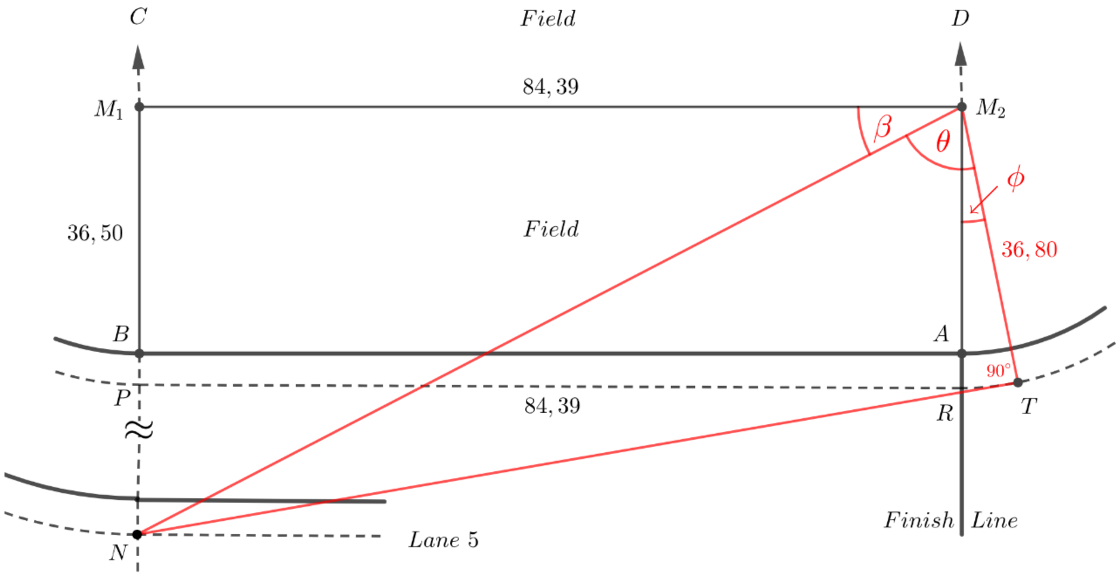



**FIGURE 7:** Break line – 5000m group start.

$$NM_2 = \sqrt{84{,}39^2 + 41{,}58^2} = 94{,}07746011 \rightarrow NT = \sqrt{94{,}07746011^2 - 36{,}80^2} = 86{,}58134037$$

$$\beta = tan^{-1}\left(\frac{41{,}58}{84{,}39}\right) = 26{,}23003974^\circ \text{ and } \theta = cos^{-1}\left(\frac{36{,}80}{94{,}07746011}\right) = 66{,}97286551^\circ$$

$$\varphi = \theta + \beta - 90^\circ = 3{,}202905254^\circ$$

$$Arc\ RT = \frac{3{,}202905254\pi}{180} \times 36{,}80 = 2{,}05717 \text{ and } PRT = 84{,}39 + 2{,}05717 = 86{,}44717$$

$$\text{Thus: } NT - PRT = 86{,}58134 - 86{,}44717 = 0{,}13417 \text{ metres}$$

This is an arc length on the curved running line in lane 5, thus the athlete lines up on the radial line from $M_1$, 0,134 metres on the running line ahead of the 200m stagger in this lane. This 0,134m offset from the 200m stagger on the running line in lane 5 is clearly shown in Figure 8.

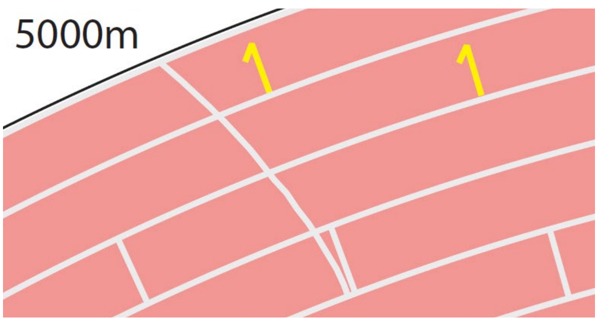



**FIGURE 8:** 5000m group start in lanes 5 – 8.

With reference to Figure 9, $d^2 = 45{,}24^2 - 41{,}58^2$ , leading to $d = 17{,}82585762$. From this it follows that:

$$\alpha = \frac{180d}{41{,}58\pi} = 24{,}56340568^\circ$$

We also have $\beta = cos^{-1}\left(\frac{41{,}58}{45{,}24}\right) = 23{,}20540928^\circ$, from which $\theta = 1{,}357996401^\circ$. The offset distance on the running line in lane 8 is thus $R_8\, R_8c = \frac{45{,}24\pi\theta}{180} = 1{,}072m$. Similar calculations may be carried out for lanes 6 – 7, and the results are detailed in Table 3.

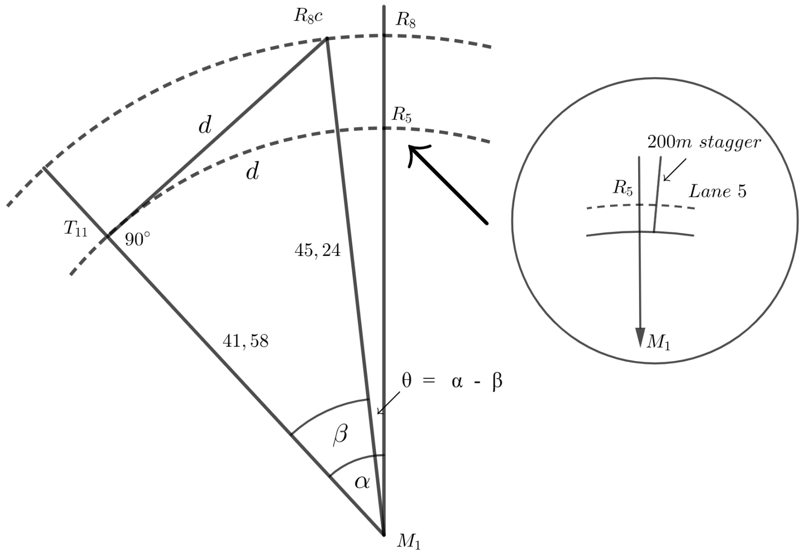



**FIGURE 9:** Group start, lane 8 – determination of distance $d$.

In this formula, $\theta$ refers to the offset angle measured anti-clockwise from the radial line $M_1$ through the point on the running line in lane 5 being 0,134 metres ahead of the 200m stagger in that lane. $R_L\, R_Lc$ is then the arc distance on the applicable lane running line measured anti-clockwise *from this radial line.* For setting out purposes, the final offset angle will relate to that used in the start line, i.e. anti-clockwise from the radial line $M_1$ through $C$ (see Figure 4).

As previously stated, the 200m offset angles are half those of the 400m, therefore the anti-clockwise offset angle of the 200m stagger in lane 5 is 20° 41′ 34″. In lane 5 the angle between the radial line $M_1$ through the 200m stagger and the starting position of the athlete in this lane is given by:

$$\frac{0{,}13417 \times 180}{41{,}58\pi} = 0^\circ\, 11'\, 06''$$

The final group start offset angle in lane 5, anti-clockwise from the line $M_1C$ is therefore:

$$\theta_{G5} = 20^\circ\, 41'\, 34'' + 0^\circ\, 11'\, 06'' = 20^\circ\, 52'\, 40''$$

Similar calculations may be carried out for lanes 6 – 8, and the results are detailed in Table 3.

| Lane | Radius $(m)$ | $\theta_L$ $(DMS)$ | $R_L\, R_Lc$ $(m)$ | $\theta_{GL}$ $(DMS)$ |
|---|---|---|---|---|
| 5 | 41,58 | 0 | 0 | 20° 52′ 40″ |
| 6 | 42,80 | 0° 16′ 05″ | 0,200 | 21° 08′ 45″ |
| 7 | 44,02 | 0° 44′ 54″ | 0,575 | 21° 37′ 34″ |
| 8 | 45,24 | 1° 21′ 29″ | 1,072 | 22° 14′ 09″ |

**TABLE 3:** 5000m group start – final offset angle $\theta_{GL}$ and lane radius.

The exact intersection points of the lane running lines and the curved group start line may then be set out using a theodolite and tape measure or an electronic distance measuring device set up at $M_1$. The offset angle $\theta_{GL}$ and the radius applicable to each of lanes 5 – 8 are then applied. Thereafter the curved $0{,}05$m start line may be marked out through these points. Additional points may be calculated and set out to facilitate accurate marking, for example a point at the outside of lane 8. For the 5000m start this point has a radius of $46{,}26$ metres with offset angle $\theta_L = 6° \, 20' \, 46''$, while for the 5000m group start the radius is $46{,}26$ metres with offset angle $\theta_{GL} = 22° \, 49' \, 17''$.

Note that when using the practical method shown in Figure 5 to position the curved starts, in practice it is difficult to accurately lay a tape out on the invisible arc defining the running line. Instead, to determine the arc for the 5000m start, nails are placed on the running line $0{,}3$m radially from the outside of the raised kerb at various equal intervals between $R_1$ and a point just beyond $T_8$. The tape is then braced against the nails and the start line swung out as described. As previously noted, this is *not a circular arc*. In the case of the 5000m group start, the nails are set out between $R_5$ and a point just beyond $T_{11}$, $0{,}2$m from the outside of the painted lane line between lanes 4 and 5.

The line that is swung out needs to reach the outside of lane 8 in each case. From this outside point, the 5000m start arc distance to the tangent point is $d = 28{,}031$m, while for the 5000m group start $d = 20{,}275$m. Due to a number of chords being created, the sum of the chord lengths will be slightly shorter than the arc length. In the case of the 5000m start, using 10 equal chords over the arc length of $d = 28{,}031$m, the reader may wish to confirm that the sum of the chord lengths is $28{,}024$m, or $7$mm short. Similarly for the 5000m group start, where $d = 20{,}275$m, the sum of the chord lengths is $2$mm short. 20 equal chords reduces these to $2$mm and $0$mm respectively.

In order to keep the number of chord lengths to a reasonable number, and at the same time ensuring no short running per the IAAF Rules, it is recommended that minimum tape lengths of $29$m and $21$m are used for the 5000m start and group start respectively, together with 10 equal chords. As the short distance indicated above is at its *maximum* on the outside of lane 8, and *zero* on the running line in lanes 1 and 5 respectively, the paint marking machine could be set to mark out the start lines say $1$cm behind the scored out arcs commencing in lane 8, tapering in towards the exact scored out line on the respective running lines in lanes 1 and 5. Note that as the shortest distance run from these curved starts is 1000 metres, the IAAF distance tolerance is $0 \leq T \leq 0{,}1$ metres.

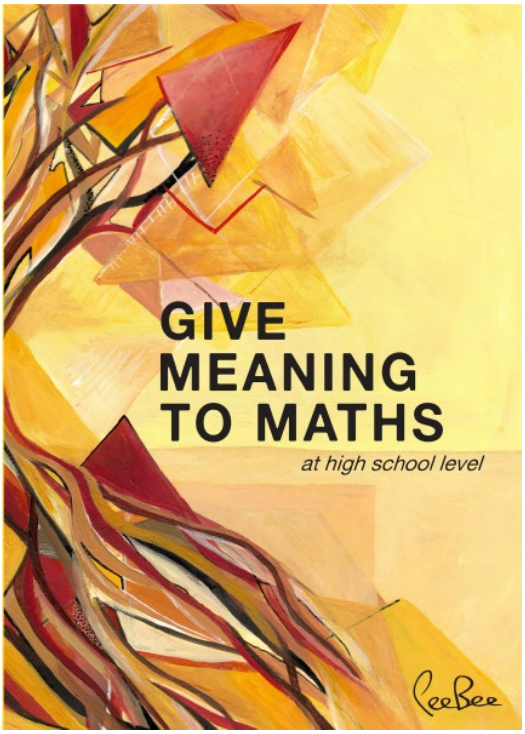
## CONCLUDING COMMENTS

The marking of an IAAF 400m standard athletics track is a major exercise, with every aspect embedded in mathematics. This article has identified the mathematics involved in calculating and setting out a typical radial stagger start, a curved start and group start, together with a break line. Similar and additional mathematics may be used for every other painted mark on the track, and indeed on the field. The IAAF Track and Field Facilitation Measurement Report is an extremely comprehensive document, and its goal is to ensure that every aspect of the track surface and track and field markings comply with the applicable dimensions and gradients as set out in the Rules. Mathematics plays a wonderful part in ensuring that this Report may be successfully completed, ensuring that no athlete is prejudiced, and all times, lengths and heights are accurate and stand up to any verification, as may be the case for national or world records.

# Book Review

## Give Meaning to Maths
## Peter Bishop
**ISBN: 978-0-620-98225-2 (softcover, pp. 491)**

***Reviewed by Duncan Samson***




Mathematics curriculums at school level differ from one country to the next – not only in terms of the scope and depth of content coverage, but also in terms of specific teaching and learning styles. However, at the core of any mathematics educational endeavour must surely be the hope of developing in pupils not only a toolbox of useful skills and techniques, but a growth mindset related to mathematical exploration, of being excited and motivated by new contexts and ideas, and being able to problematize in creative and reflective ways. In short, in the development of pupils' ability to 'mathematise'. And that is what *Give Meaning to Maths* is all about – a celebration of 46 years of teaching high school mathematics with precisely this aim at heart.

Ostensibly written for the high school pupil, *Give Meaning to Maths* is a collection of themes, ideas and investigations that have been developed and refined over a lifetime of teaching. This book is for anyone who considers themselves a life-long learner.
The book inspires one to investigate and to challenge oneself with new contexts (or old contexts looked at in new ways). By sparking our curiosity, and hence our creativity, the book engages pupil and teacher alike to dig deeper and to become immersed in the learning process itself.

*Give Meaning to Maths* comprises 150 topics broadly grouped under the following 17 themes:

- Getting to know you
- Prime numbers
- Palindromic & other unique dates
- Patterns emerging in numbers
- The golden ratio – the divine proportion
- Investigating the circle and the story of $\pi$
- More than a peek into trigonometry
- Pythagoras
- Patterns, patterns and yet more patterns
- When is the impossible possible?
- The art of thinking
- Bits and pieces
- Thinking deeper
- Modelling and measuring
- New dimensions
- Playing games by the rules
- Ending with the unorthodox

*Give Meaning to Maths* takes us on a mathematical journey from past to present, and shows how mathematics remains a central part of human consciousness and evolution. For teachers, the book is a treasure trove of ideas for maths club activities, classroom investigations, portfolio projects, pupil enrichment, or simply interesting and diverting lessons. For pupils, the book represents a smorgasbord of exciting ideas, historical insights and real-world applications where each pupil should be able to discover the beauty of mathematics by finding something that resonates with their particular interests.

*Give Meaning to Maths* is much more than just another collection of interesting activities and mathematical ideas. It is a celebration of a lifetime's commitment to captivating, inspiring and challenging young minds. The pages resonate with the contagious academic curiosity of a teacher who clearly loves to learn with his pupils. One cannot but come away from this book thinking – now that's a teacher I would have enjoyed being taught by!

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# Suggestions to writers

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**What type of submissions are we calling for?**
The types of articles considered for publication in *Learning and Teaching Mathematics* are:

- *Ideas for teaching and learning*: articles in this section report on classroom activities and good ideas for teaching various mathematics topics. This includes worksheets, activities, investigations etc.
- *Letters to the editors*: discussion pieces that raise important issues on the teaching and learning of mathematics and current curriculum innovations. Views and news on current initiatives
- *Kids say and do the darndest things*: personal anecdotes of something mathematical that has happened in a classroom
- *Window on a Child's Mind*: description of a classroom event that you want the Journal to respond to.
- *A day in the life of* ... includes stories about a head of department, a maths teacher, an NGO worker etc.; it could also be an account of a visit to another mathematics classroom... another school... another country...
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