Elliptical billiard tables: An easy proof for a special convergence

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Hans Humenberger presents an interesting and elementary geometric phenomenon concerning ellipses: Using elliptical billiard tables the path of a billiard ball quickly converges to the major axis when started at a focus. In the literature there can be found rather complex proofs for that phenomenon; here the author presents an elementary one.

In *Geometry By Its History* (Ostermann & Wanner, 2012, p. 230f) I read of the following phenomenon for the first time:

> Consider a billiard table of elliptical form. Then start a ball running (it must be seen as a mathematical ball which can run "eternally") from one of the foci in an arbitrary direction. Then you can observe that the path (orbit) of the ball will converge to the major axis (Fig. 1).

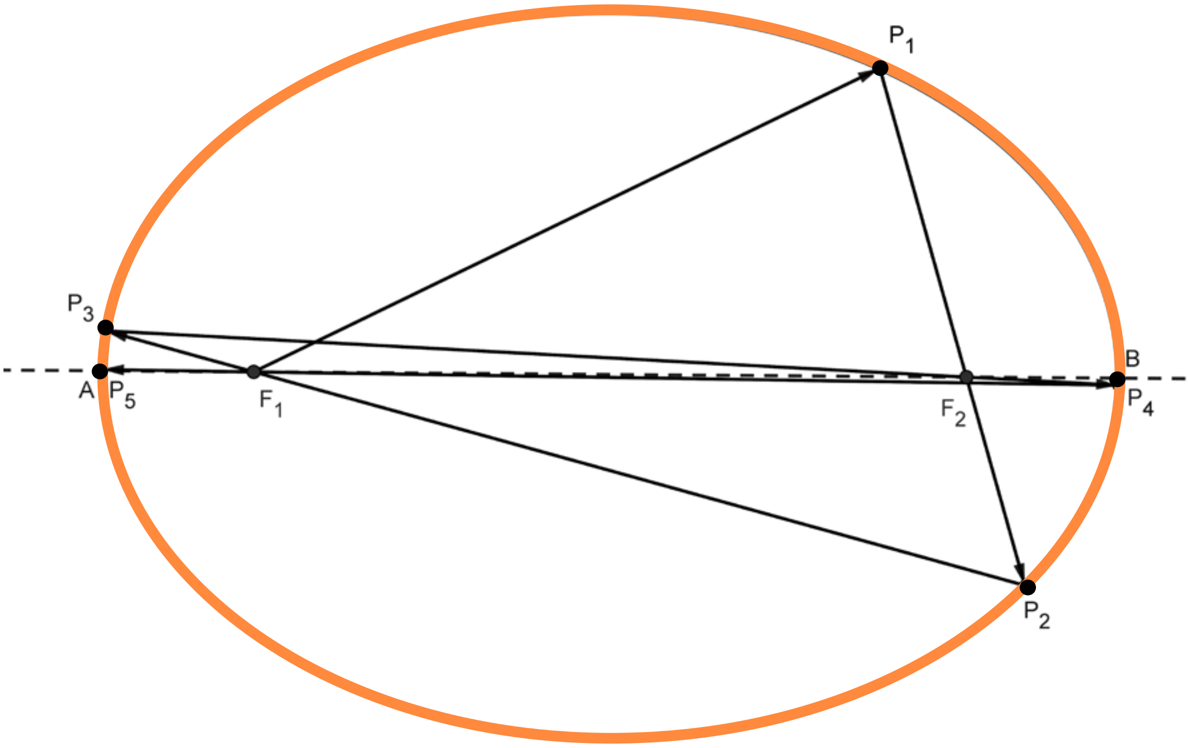

Figure. 1: Ball starting in F_1 and being reflected in the points P_1, P_2, P_3, P_4, P_5, ... (each reflection directs the ball through the other focus). Here we see $P_5 \approx A$. (With A we denote the left vertex of the ellipse, with B the right one.)

Ostermann & Wanner (2012) discuss a possible proof using the cosine function, Möbius transformations, eigenvectors and eigenvalues of matrices etc. It is a rather complex proof using very advanced mathematics. The second reference that I found by chance online concerning this phenomenon, was *Sun Woo Park* (2014), where a less complex proof is given (pp. 10–12).

The Wolfram Demonstrations Project (http://demonstrations.wolfram.com/DynamicBilliardsInEllipse/) has an app with which one can produce figures and patterns concerning elliptic billiard tables.

I wondered whether there might be a shorter way to prove the phenomenon and the solution I found follows.

Let us start at F_1 and have a closer look at special angles α_n; the angles of the rays with the negative *x*-axis (the major axis of the ellipse lies on the *x*-axis, the center of the ellipse is the origin). We have $\alpha_1 = \measuredangle P_1F_1A$ and $\alpha_2 = \measuredangle P_1F_2A$. It is clear that $\alpha_1 \geq \alpha_2$ (exterior angle theorem). Then one can see immediately $\alpha_2 = \measuredangle P_2F_2B$ (vertical angles) and again using the exterior angle theorem, yields $\alpha_2 \geq \measuredangle P_2F_1B = \measuredangle AF_1P_3 = \alpha_3$. One can continue this argumentation and generally get $\alpha_{n+1} \geq \alpha_n$,that means the sequence (α_n) is monotonically decreasing and bounded below (by 0). Thus (α_n) must converge: $\lim_{n\to\infty} \alpha_n =: \bar{\alpha}$. Why must $\bar{\alpha} = 0$ hold? Let us assume $\bar{\alpha} \neq 0$,then there would exist a point $P \neq A$ on the ellipse with $\measuredangle PF_1A = \bar{\alpha} = \measuredangle PF_2A$. This is obviously impossible (Figure 2, exterior angle theorem). Therefore, in the existing limit the ball must be reflected in the same straight line as it comes (on the major axis). So we have $\bar{\alpha} = 0$ and this completes the proof.

Remarks

- One could also consider the angles β_n to the positive *x*-axis $(\beta_n = \pi - \alpha_n)$ and prove in an analogous way: The sequence (β_n) is monotonically increasing and bounded above by π, the limit is π (then in the proof *B* takes the role of *A*).
- It is remarkable that in the above argumentation one does not have to do any calculations; even the equation of an ellipse is not needed. The only thing used of the ellipse is: If a ray starts at a focus of the ellipse, it is reflected through the other focus. This is a famous property of the ellipse with elementary and well-known proofs.
- In *Sun Woo Park* (2014) this convergence is formulated as an exercise for readers in *Tabachnikov* (2005, Exercise 4.3, p. 52). This is another hint that it was meant to be more elementary than the other proofs.

This topic can be explored in learning groups who know that bounded and monotonic sequences converge. This particular convergence is unusual.

Generally calculus problems come to mind when thinking of convergence. But there are also interesting geometric problems that have to do with convergence (e.g., De Villiers, 2014; Humenberger & Embacher, 2018). If this topic is dealt with in a teaching situation it is highly recommended to use or construct an app such as the 'wolfram' one mentioned above.

Another important prerequisite is that students know that any ray coming from a focus of an ellipse is reflected by the ellipse (or its tangent) through the other focus. This is used in 'whispering galleries' (in many cases they are constructed in form of elliptic walls or ceilings: when a person stands in one focus, say F_1, and whispers gently, then another standing in F_2, can clearly understand it), and in 'lithotripsy' (a medical procedure involving the physical destruction of hardened masses like kidney stones or gallstones: In one focus "shock waves" are generated, in the other focus is the stone that needs to be destroyed). This can be proven in well-known ways, either geometrically or analytically (Berendonk, 2014).

References

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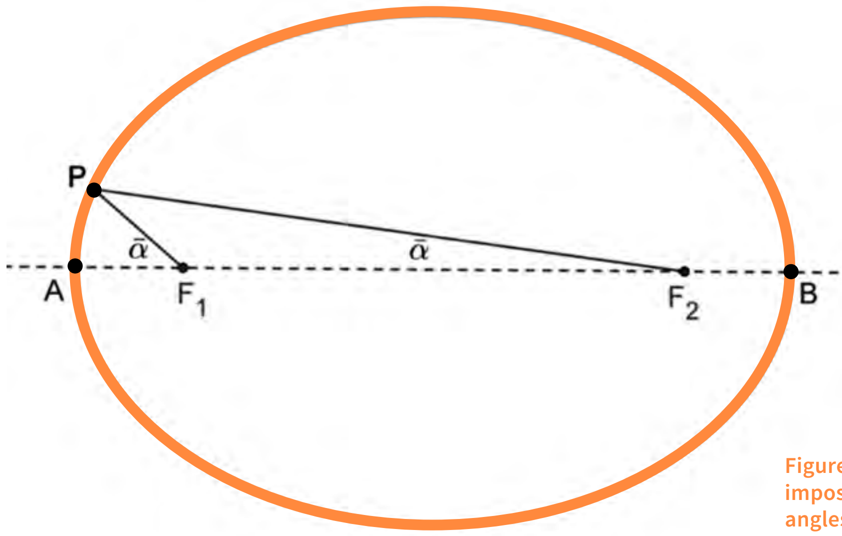

Figure 2. For $P \neq A, B$ it is impossible that both marked angles are equal.