Modelling and optimising regarding swings and swinging monkeys

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Two examples of mathematical modelling are explored. Firstly, optimising the distance travelled when jumping from a swing is discussed. The second concerns optimising the distance travelled by monkeys when swinging from branch to branch.

Parabolas can be considered in several contexts, e.g., in the context of quadratic functions (especially the vertex of a parabola) or in the context of projectile motion by splitting a motion into a horizontal and a vertical component. Furthermore, it plays an important role—implicitly or explicitly—when students work on several modelling tasks (e.g., penalty shots in basketball, goal kicks in soccer—with or without considering the effects of air resistance). Thus, it seems natural to combine somehow the parabola of projectile motion with circular motion which appears regarding swings or pendulums.

Indeed, there are substantial possibilities without facing the risk of dealing with only 'faked' applications of mathematics. Here the use of technology will play a crucial role (see below). Surprisingly for the case that seems to be more complicated there is an exact algebraic solution to the problem, in the case that seems to be the easier one this will not be true. One also may wonder about the mentioned exact solution itself; it is constant $\alpha_{opt} = \frac{\pi}{4} = 45°$ (see below, Swinging Monkeys), maybe one would not conjecture that.

Figure 1. Jumping off the swing.

'Long jumps' using swings

Swinging is something that most children are fond of. Also, most adults have used swings in their childhood and maybe they tried to jump—after leaving the swing—as far as possible. In my childhood there were even sort of competitions, among the children of the neighbourhood, concerning this kind of 'sport'. The question that comes up here immediately in a natural way is: "How can one manage to jump as far as possible?" Of course, children on a swing do not think of mathematics in that situation, they rather act intuitively.

However, one can use mathematics to confirm the solutions which are derived on an intuitive level. When swinging, children learn early how to increase the angles of excursion (to the rear and to the front), but in the last movement from the rear to the front (before jumping off) we assume that nothing is done actively for getting additional acceleration. In this last movement one has an angle of excursion φ'to the rear' and one has to decide at which angle of excursion α 'to the front' it is best to leave the swing (jump off).

Here we assume that only gravity works and nobody gives a push, thus we have here $0 \leq \alpha \leq \varphi$. The best angle for jumping off is surely not $\alpha = \varphi$ (so to speak at the front dead centre) because in this case the swing does not give any 'kick' to the person (neither upwards nor forward—one would simply drop down to earth). Also, $\alpha = 0$ is surely not the best idea, because directly beneath the hanging of the swing (at the point next to earth) one has high velocity but its direction is purely forward, it has no upwards component.

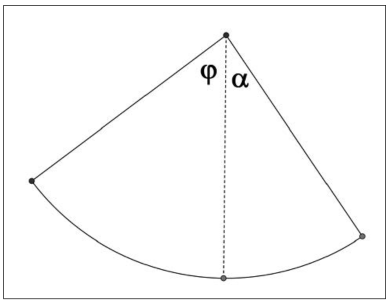

Figure 2. Angle of excursion to the rear (φ) before the last movement forward, and the angle of excursion to the front (α) where the jump off takes place.

Somewhere in between there will be the best angle α_{opt} for jumping off, and the first guess could be the "midpoint": $\alpha_{opt} \approx \frac{\varphi}{2}$. How good would this first estimation be? Is this guess $\frac{\varphi}{2}$ rather above the exact value α_{opt} or below? Are there cases in which $\alpha_{opt} \approx \frac{\varphi}{2}$ holds exactly? With these questions in mind one can start a mathematical analysis of the situation. Of course, we will not consider the air resistance, we will model the human body as a mass point. More exactly, we will consider the 'centre of gravity' of the body—this will be approximately on the seat of the swing.

With these assumptions we have done first idealisations. This is the first step in almost all modelling cycles—concerning the different steps in solving modelling tasks. In the literature there are many different such modelling cycles (e.g., Blum & Leiß, 2007 or Niss & Blum, 2020).

From the mathematical/physical point of view we have the following situation (Figure 3). After a ‘pendulum motion’ (on the swing) there will be somehow a flight phase (parabola of projectile motion), and we are interested in the maximum of the total distance.

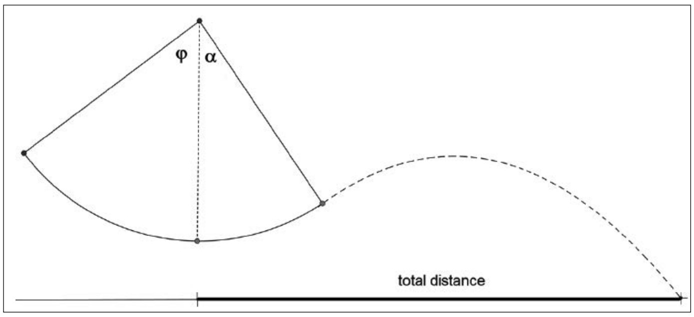

Figure 3. Parabola of projectile motion after leaving the swing.

In Figure 3, the centre of gravity lands on the ground, but in reality that is not the case, because the centre of gravity at the moment of the person’s landing is—roughly estimated —at the height of the seat (inactive, at rest). Thus, a better sketch of the situation is displayed in Figure 4. (In reality the jumping distance *W* is, of course, measured on the ground, but here—to simplify—we shifted it by the height of the seat or by the height of the person’s centre). We could also assume that the centre of gravity is a bit lower or higher in the moment of the person’s landing, but this would not change the situation in principle.

We have to maximise the total jumping distance *W*, more exactly: In case of given values for the angle φ and the length *L* of the swing’s rod (or chain) we are looking for the special value of *α* which maximises *W*. Therefore, our aim must be to establish a formula for *W* in terms of φ, *L* (constant parameters; also the gravity acceleration $g = 9.81\frac{m}{s^2}$ will be involved) and the variable *α* . Establishing this function is a challenging task for students. Once having done this job one will probably use technology (e.g., calculating derivatives and the corresponding zeroes, drawing graphs), and the question whether the resulting equation $W'(\alpha) = 0$ has an explicit (exact) solution or only a numerical one (approximate solution by iterative methods after fixing the values of *L* and φ) is not so crucial when using technology. In former days this would have been a knock-out criterion for dealing with a problem in teaching situations, nowadays this situation has changed.

For establishing an equation of the parabola of projectile motion one needs the initial conditions, that is information about the (initial) velocity $\vec{v}$, namely its absolute value $v = |\vec{v}|$, and the corresponding initial angle. Since the ‘take off’ happens tangentially to the circle (motion of the swing) the initial angle can be determined easily, but how can we get $v = |\vec{v}|$?

Probably the easiest way here is to consider the ‘energy’, an often-used concept in physics. Generally, the gained kinetic energy $\frac{mv^2}{2}$ of a pendulum results from *losing potential energy mgh* (if no other forces like pushes from somebody else exist). The equation $\frac{mv^2}{2}$ =mgh yields in general $v = \sqrt{2gh}$. This also holds in the moment of leaving the swing, the *kinetic energy* (caused by the velocity) results from losing some *potential energy*. The difference in height between the initial position of the swing (excursion φ backwards) and the jump off position is given by $h = L\cos(\alpha) - L\cos(\varphi)$ (Figure 4) and this finally results in

$$v = \sqrt{2gL(\cos(\alpha) - \cos(\varphi))} \ (*),$$

where *g* denotes the acceleration due to gravity on earth (the mass *m* can be cancelled).

The well-known representation of the parabola of projectile motion in terms of *v* , α is: $y(x) = \tan(\alpha)x - \frac{g}{2v^2\cos^2(\alpha)}x^2$. Here the point of take-off is the origin point of the coordinate system.

We can use this equation for calculating the distance (‘*x*’) for which we have the ‘negative height’ $L\cos(\alpha) - L = L\left(\cos(\alpha) - 1\right)$. (If one estimates the position of the person’s centre of gravity a bit higher or lower this value would have to be changed slightly). Adding $L\sin(\alpha)$ to this value of *x* yields the wanted total jump distance *W*. Thus, the first step is to solve the equation

$$L\left(\cos(\alpha) - 1\right) = \tan(\alpha)x - \frac{g}{2v^2\cdot\cos^2(\alpha)}x^2 \text{ for } x \text{ (quadratic equation in } x).$$

The positive solution is here: $x = \frac{v^2\sin(\alpha)\cos(\alpha)}{g} + \frac{v\cos(\alpha)}{g}\sqrt{v^2\sin^2(\alpha) - 2Lg\cos(\alpha) + 2Lg}$

Then we have: $$W = \frac{v^2\sin(\alpha)\cos(\alpha)}{g} + \frac{v\cos(\alpha)}{g}\sqrt{v^2\sin^2(\alpha) - 2Lg\cos(\alpha) + 2Lg} + L\sin(\alpha)$$

Here *v* is not a constant but depends on α via (*). It is easy to see that substituting *v* by the corresponding term of (*) yields a function *W* (in terms of α) with the property: the zero of the first derivative can rarely be found algebraically.

However, in case of given values of L, φ and using technology one can solve the equation $W'(\alpha) = 0$ numerically (approximately) in the interval $0 \le \alpha \le \varphi$ (every computer algebra system has such commands). The solution α_{opt} yields the optimal angle for jumping off and $W(\alpha_{opt})$ the corresponding total jump distance.

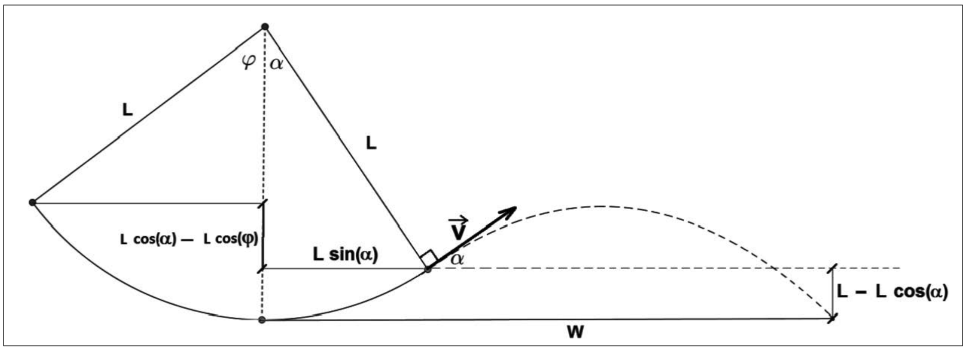

Figure 4. A better sketch of the situation.

One result could be a table like Table 1 for the values of α_{opt} and the corresponding total jump distances W using different values of φ (here we used $L = 3m$, $g = 9.81\,\frac{m}{s^2}$).

Table 1. Optimal angles for jumping off and corresponding total jump distances.

φ[°]	10	15	20	25	30	35	40	45	50	55	60	65	70	75	80	85	90
α_{opt}[°]	9.8	14.5	19	23	26	29	31	33	35	36	37	38	39	40	40	41	41
W [m]	0.5	0.8	1.1	1.4	1.7	2.1	2.4	2.8	3.2	3.7	4.1	4.6	5.1	5.6	6.1	6.7	7.2

Remark: If one substitutes v in the function W using (*) one can see two interesting aspects. First, one can see that g can be cancelled, and second that L can be factored out of the whole term. That means on the one hand that the jumping distance does not depend on the earth's gravity, it would be the same on the moon! On the other hand, it means that W and L are proportional (double L yields double W; probably this holds only in the case of our assumption: height of the body's centre of gravity = height of the seat). Moreover, by this it is clear that the optimal angle α_{opt} is independent of L. These aspects are not crucial for continuing the work with a computer algebra system, they only describe phenomena which may be regarded as interesting by themselves.

As one can see in Table 1 there seems to be an angle φ (approximately 80°) with $\alpha_{opt} = \frac{\varphi}{2}$ exactly. Using technology (computer algebra system) this angle can be calculated with approximately 80.5°. In Table 1 one can also see that the first guess from above is nearly always too small (for values of beyond 80°). A corresponding plot illustrates this graphically (Figure 5).

For a human long jumper using swings always the question arises whether one can to have a 'well balanced' position during the flight (i.e., head above, feet below), because nobody likes to land on the back (or even worse—on the head). However, it would be an interesting project—but not easy—to analyse some video material concerning jumping children and see how near/distant the 'real' and the 'theoretical' solutions are. It is somehow quite astonishing what computer algebra systems can perform. Although every single point in the curve of α_{opt} is itself a result of a tedious procedure (numerical solution) it is possible to plot the graph! This task could not be part of a learning process in former days because then equations could not be solved numerically by 'pressing one button'. Nowadays the complexity of an equation to be solved numerically is not relevant (the work of computing is done by technology), it just depends on whether or not the approach is elementary enough for students to be found (if they work autonomously) or to be understood (if the teacher tells them the underlying relations).

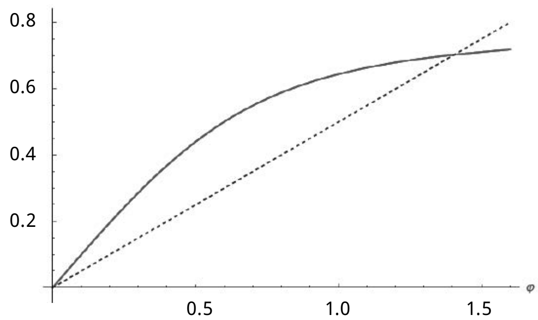

Figure 5. The graphs of α_{opt} (curved line) and $\frac{\varphi}{2}$ (dashed, straight line), are not a good fit!

In the next section we deal with a problem that seems in the beginning more advanced because the approach and the context are a bit more complex. However, the resulting equation is surprisingly much simpler (it could be solved even without technology) on the one hand, and on the other hand one can be astonished by the solution itself.

Figure 6. Gibbon monkey.
(Source: www.istockphoto.com)

Swinging monkeys

Many monkeys with long arms, for example, Gibbon monkeys, are somehow born to move with sort of 'swinging techniques', especially from one branch to another, in cages also on the ceiling (grid) of the cage. For this way of moving a special term was established, namely 'brachiation'. Doing their special moves using their swinging techniques along rods (ropes, branches, grids) these monkeys can have also phases of real flights (between the phases in which they use their arms like a swing's rod). In zoos or films (TV, cinema) one can see these techniques sometimes impressively. Also, for this case (maximising the distance from one 'grip' to the next) one can establish a mathematical model:

a) When should the monkey unhand?

b) How far and how high is its 'flight phase'? Is there always such a flight phase in the optimal case?

c) How far can the monkey get in such a step (from one grip to the next)?

Monkeys do not need such considerations—they do this instinctively, gathering experiences simply using the principle 'learning by doing'. For a mathematical/physical analysis from the human point of view such questions and considerations are highly relevant.

We think of the monkey's centre of gravity and assume that the monkey has two arms equally long (length L; this length corresponds to the length of the swing's rod above) and that the monkey can move them independently. Furthermore, we assume that the point of unhanding (before the flight phase) and the point of gripping (after the flight phase) have equal height (for the reason see below). The situation then can be sketched approximately as in Figure 7.

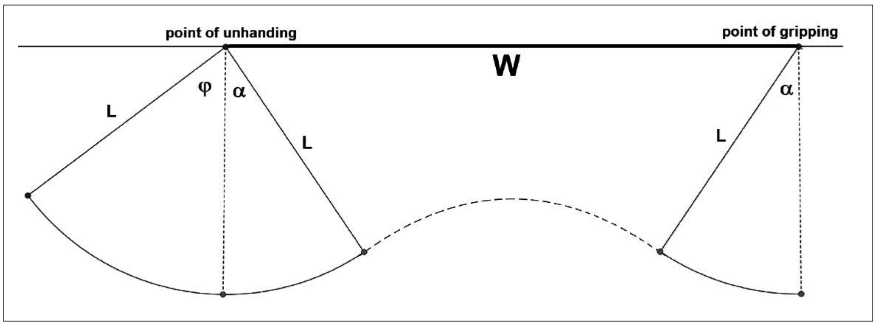

Figure 7. Sketch for swinging monkeys. The lines labelled with L should represent the monkey's arms (with the hands on the above ends, and the monkeys themselves thought of as a mass point at the bottom end). The points of unhanding and gripping are shown on the top line (rod on the ceiling).

In Figure 7, the situation of gripping and unhanding are symmetrical to each other, especially the point of gripping and the point of unhanding have equal height and equal angle α to the vertical direction. Is that the best for maximising the monkey's distance? Is it possible that by gripping somewhat earlier or later the distance could be increased? Intuitively one will probably say "no", but what is the reason for that phenomenon?

In the situation of unhanding, the circle ('pendulum motion') and the curve of the projectile motion (parabola) are tangent. In the symmetrical situation of gripping again (Figures 7, 8) this is the case, too. Thus, the monkey is at no point of its curve of projectile motion (parabola) nearer to the symmetrical point of gripping than in the symmetric situation (parabola and circle have only one point in common!), hence, when gripping earlier the monkey loses distance. Also, when gripping a bit later (so to speak following the parabola a little further) the distance grows to this symmetrical point of gripping and hence the monkey will not reach it—and any other point to the right of it—anymore.

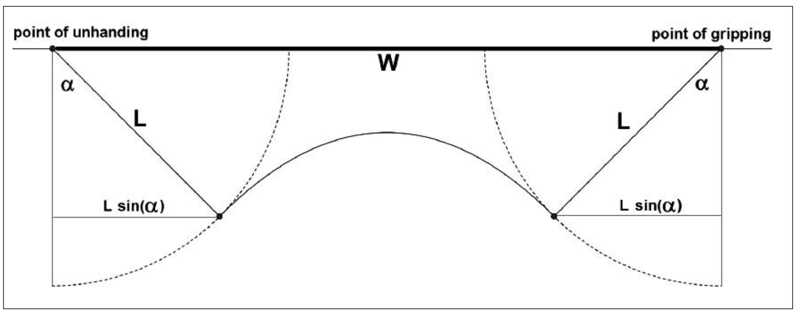

Figure 8. Parabola is tangent to both circles.

Therefore, the optimal (maximum) distance W between the point of unhanding and the point of gripping consists on the one hand of the horizontal distance in the parabola of projectile motion and on the other hand of two pieces with length $L\sin(\alpha)$.

For calculating the horizontal distance in the parabola of projectile motion one must solve the equation: $0 = \tan(\alpha)x - \frac{g}{2v^2 \cdot \cos^2(\alpha)}x^2$

This is easier than in the long jump with swings because of the zero on the left hand side of the equation. Even without a computer algebra system one gets:

$$x = \frac{2v^2 \sin(\alpha)\cos(\alpha)}{g} \quad \text{With } v^2 = 2gL(\cos(\alpha) - \cos(\varphi)) \text{ from (*) this yields}$$
$$x = 4L(\cos(\alpha) - \cos(\varphi))\sin(\alpha)\cos(\alpha)$$

Thus, we get for the distance: $W(\alpha) = 4L(\cos(\alpha) - \cos(\varphi))\sin(\alpha)\cos(\alpha) + 2L\sin(\alpha)$ and we are interested in the maximum value of that function. Presumably, one will use a computer algebra system to compute the first derivative and to compute its zero. The solution of $W'(\alpha) = 0$ is surprisingly $\alpha_{opt} = \frac{\pi}{4} = 45°$, apparently independent of φ (the independence of *L* is clear from the very beginning)!

Now this result must be interpreted. It is clear that α can have the value $\frac{\pi}{4} = 45°$ only if $\varphi \geq \frac{\pi}{4}$, because $0 \leq \alpha \leq \varphi$ holds (if the monkey starts swinging at a special initial angle, then in terms of angles, it cannot swing farther to the front than it has started in the rear). In case of $\varphi \geq \frac{\pi}{4}$ we have $\alpha_{opt} = \varphi$ (i.e. in these cases it is best to unhand at the dead centre and to simultaneousl≤rip the rod or rope with the other hand—the phase of 'flying' does not occur, Figure 9), because $W'(\alpha) > 0$ holds for $0 < \alpha < \varphi < \frac{\pi}{4}$.

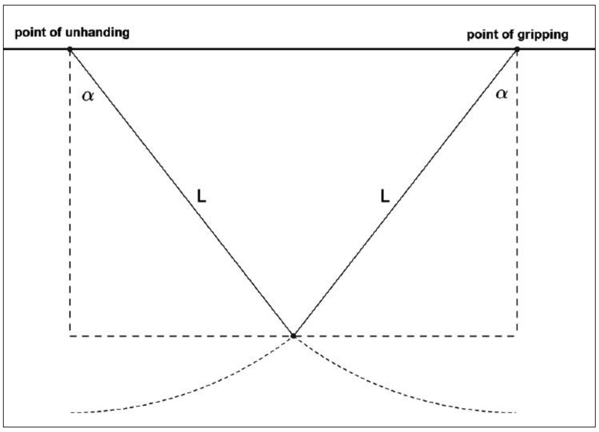

Figure 9. Changing at the dead centre.

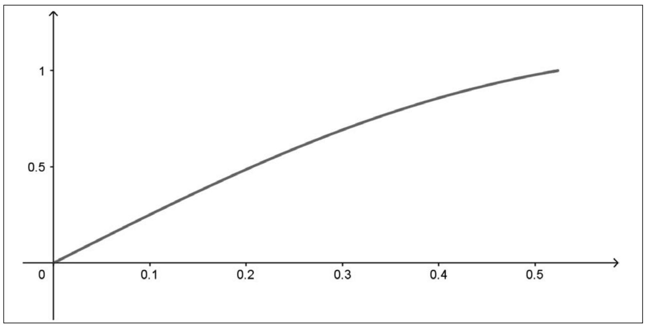

Figure 10. Graph of *W*(α) for *L* = 1*m* and φ = 30°.

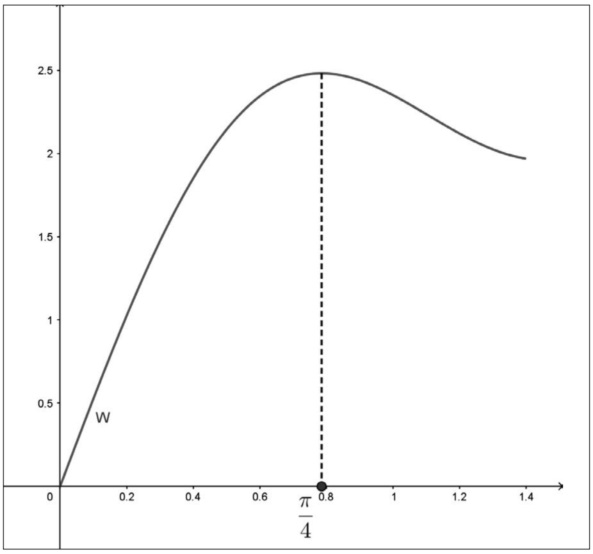

Figure 11. Graph of *W*(α) for *L* = 1*m* and φ = 80°.

Let us start with the case $\varphi < 45°$, for instance φ = 30° (for the arm length we take *L* = 1*m*). The graph of *W* (in terms of *α*) looks like in Figure 10. That means the bigger the value of *α* the bigger is the value of *W*, that is, the best strategy in this case is to unhand as late as possible. (At the dead centre *α* = φ and there is no phase of flying and the optimal distance is given by 2*L* sin(φ)).

For $\varphi \geq \frac{\pi}{4}$ we have always $\alpha_{opt} = \frac{\pi}{4} = 45°$; we take again *L* = 1*m* and this time $\varphi = 80°$ (Figure 11). For $\varphi \geq \frac{\pi}{4}$ the maximum value of *W* is given by $2L(\sqrt{2} - \cos(\varphi))$. The corresponding maximum value of the 'flight distance' (parabola) is $L(\sqrt{2} - 2\cos(\varphi))$ and the corresponding maximum value of the 'flight altitude' is given by $\frac{L}{4}(\sqrt{2} - 2\cos(\varphi))$. Here one can see the bigger the values of $\varphi \geq \frac{\pi}{4}$ the bigger are the values of W, flight distance, and flight altitude. As long as $\varphi \leq \frac{\pi}{2} = 90°$ the centre of gravity of the monkey cannot get higher than the height of the rod (rope). But if Gibbon monkeys jump from a position higher than the rod (rope) and grip it in their 'descent' and then have a phase of swinging on it before they unhand and have again a phase of flying (in order to move forward on this rod or rope), then it is possible that they come—during their second phase of flight, parabola of projectile motion—even higher than the mentioned rod which may be gripped again during the descent in their second flight phase. However, in such a situation they do not have zero velocity in the situation of the initial excursion φ but they have some kinetic energy in such a situation which helps them to exceed the height of the rods during their flight.

Now for the solution $\alpha_{opt} = \frac{\pi}{4} = 45°$. This beautiful and maybe surprising solution may give reason for the conjecture that it may be achieved also without using a computer algebra system. This is actually the case! If one substitutes—in the equation $W'(\alpha) = 0$ we have to solve—in the usual way, namely if we define $x = \cos(\alpha)$ and $c = \cos(\varphi)$, then we get a polynomial equation with degree 3, $6x^3 - 4cx^2 - 3x + 2c = 0$, and this equation can be solved also without the use of Cardano's formula: $3x(2x^2 - 1) - 2c(2x^2 - 1) = 0 \Leftrightarrow (2x^2 - 1)(3x - 2c) = 0$. There we have the restrictions $0 \leq \varphi \leq 180°$, $0 \leq \alpha \leq 90°$ and $\alpha \leq \varphi$.

Looking at the first bracket we get—after undoing the mentioned substitution—our solution $\alpha_{opt} = \frac{\pi}{4} = 45°$, and the second bracket vanishes in the relevant range only for $\alpha = 90° = \varphi$ (equivalently $x = 0 = c$), but this yields a local minimum at the right boundary, not the global maximum.

In retrospect and for some persons perhaps also in advance, the optimal angle $\alpha_{opt} = \frac{\pi}{4} = 45°$ is highly plausible because the parabola of the monkey's flight phase has equally high points of 'starting' and 'landing' (see above), and in such a case it is well-known in advance that the flight distance (given constant initial velocity) is a maximum with $\alpha_{opt} = \frac{\pi}{4} = 45°$. Seeing it this way it is not really astonishing that $\varphi \geq \frac{\pi}{4}$ always yields $\alpha_{opt} = \frac{\pi}{4} = 45°$.

Conclusions

The long jump using swings is an authentic problem, although children will not use mathematics in such situations but rather the famous principle learning by doing. Nevertheless, corresponding mathematical aspects (optimisation, extreme value problems) can be treated reasonably in mathematics education. Hereby a meaningful use of technology may play a crucial role. That should not mean: Teachers should deal with that topic at school because a meaningful use of technology is possible. No, the technology should not determine what is dealt with in mathematics education, this selection should be done by only mathematical (professional) criteria. We should not look for the right mathematics for some sort of technology, it should be the other way round: we should look for the right technique or technology for those parts of mathematics which seem to be important at school. It may well happen that no technology is used in such a process.

The mathematical cores of the topic long jump with swings are a substantial combination of for example pendulum motion and the parabola of projectile motion, functional thinking, optimising (extreme value problems), actually within a problem (context) that is known to everybody since the early days of childhood. The corresponding steps of modelling are not easy, and the context is dominated by physics, so it seems appropriate to work together with the physics teacher. The function (which should be maximised) is rather complex but using technology the problem can be solved. The second topic (swinging monkeys) is not authentic anymore in the living environment of the students, but it has its preferences: First, the symmetric solution is again the best one, second the resulting function (to be maximised) is much easier than in the case of the long jumps using swings, as it can be dealt with without computer algebra systems, which may be surprising at the first glance. And third, the following result itself may be astonishing, at least at the first glance: for $\varphi \geq \frac{\pi}{4}$ the best angle for unhanding is always $\alpha_{opt} = \frac{\pi}{4} = 45°$. Taking a second glance this may be regarded as not astonishing at all, and such insights and explanations (see above) are important opportunities for learning. Of course, these problems are not quick tasks or even tasks for exams or the like, they are problems for learning and they need time in the teaching and learning process. However, this time is not wasted but used reasonably. The degree of autonomy in the work of the students is highly variable. In the course of a special modelling event (modelling day, modelling week) it will be rather high, within a normal lesson it may be lower.

Acknowledgement
This task comes from an initial idea of Daniel Berger (student teacher, University of Vienna) who thought of such a task for a students' 'modelling day'. He quickly realised that it would be too complex for Grade 10–11 students to be solved within 2–3 hours, so he did not proceed.

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