Generators of $L^p(\mathbb{R})$ by translations in time and frequency.

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Introducció

Lets begin with an example.

We root the interval $[-\pi, \pi]$ and we think of a space of clear-cut functions in this interval. We will not be very strict for the time being. We think for comfort that our space are the continuous functions (and/or the bounded ones). Now we want to represent somehow a function of this space. A form of making it is to give all the values of the function, but it does not seem much good.

Another way is to take a few concrete functions, and aproximate our function for linear combinations of these. This already has better aspect. If, for example, we want to guard our function in a computer (or in a sheet of paper), just we have to keep the coefficients, and to know what are the functions that which we have chosen for representing. Or also it can be useful for us, from a more theoretical point of view, to see how a linear operator operates about our function. Applying linearity the image of our function will be the sum of the images of the combination that we are using for representing it. Like this it is just necessary to us to know the image of the set of generators.

Well, without wanting already the first important notion has escaped to us. When we want to represent functions, generators are the one that we are searching. As they will be a set of functions, we can call them in all joints a system of generators. Let's not describe yet it already because we have said that we are being informal.

Very well, we are then searching systems of generators of the space of continuous functions of $[-\pi, \pi]$. Now we have to think of which class functions we want to form our system. This selection depends on which class of representation we wish, but we can give some simple example? We are not the first that we consider this problem, Fourier already made this with their series (or perhaps he was somebody another?, well, this does not interest us now).

We go to look at this example. We choose the complex exponential ones as functions. They are not as nice as the sine and cosine, but its are simplest of explaining. We take the set:

$$\mathcal{B} = \left\{ e^{in\xi} \ : \ n \in \mathbb{Z}, \xi \in [-\pi, \pi] \right\}$$

Already we have our first system of generators. Now we go to represent functions. We take a function of our space and the we write like a linear combination of elements of the system:

$$f(\xi) \approx \sum_{n=-N}^{N} a_n e^{in\xi}$$

Here we will have to be carefull about what it means this approach. In general it will mean that the norm in the space of the difference of both functions will be small. Some times even we will be able to think that it have sense the infinite sum:

$$f(\xi) = \sum_{n\in\mathbb{Z}} a_n e^{in\xi}$$

where this summation will be a limit of the partial sums. One of the main problems that appear to us in this context is to know if we will be able to represent any function of our space. That is, to know if a set of functions generates our space. Let's look again at our example, but changing the space. Instead of continuous functions we think of square integrable functions. It is known that $L^2(-\pi, \pi)$ is a Hilbert space, where we have a scalar product and a practically identical structure to a vectorial space of finite dimension. In this space the system that we have given is an orthonormal basis. The idea is the same than in finite dimension. We can represent any function of this space like an infinite linear combination of elements of the basis, and besides only way. Normally we will not obtain so good worked out.

But we continue with our problem. We go to give it a turn. We take inverse Fourier transform. For comfort we think in $L^2(\mathbb{R})$. The functions of the straight line that they have Fourier transform with support to $[-\pi, \pi]$ are the space of functions that appears to us. This space is knowed as the band-limited functions, and they all have a whole extension in the complex plane, so that we obtain the Paley-Wiener space. As the Fourier transform is an isomorphism, we should sustain the systems of generators and the basis.

If we calculate the transform of the system we see that it ends up to the following set of functions:

$$\widetilde{\mathcal{B}} = \left\{ \frac{\sin(t-n)}{t-n} \ : t \in \mathbb{R}, n \in \mathbb{Z} \right\}$$

This set is an orthonormal basis of the space of band-limited and square integrable functions in the straight line. If we notice it more we can observe that all functions are of a same type. All come of taking a concrete function and translate it. These are the class of systems that we are searching.

Already we have found one for this space. But this is not the space that we will treat in this memory. We want to make this same but for all $L^2(\mathbb{R})$. This will be a more complicated problem.

The generators that we search will be from the same type that we have shown now. We will take a fixed function and we will translate it. Many questions turn up in a natural way about this problem. First of everything we will have to attempt to know which functions we will be able to use. We will not be able to generate all the space with translations of any function. Also we can ask ourselves which set of translations we can use. And this question depends on the function that we choose. Then we can ask for which sets there is a function that it causes a system of generators, or fixed a concrete function, which sets will work with this.

And also we can attempt to search $L^2(\mathbb{R})$ bases. But here we find a very severe problem. There is not any. Instead of doing anything we can opt to add something more than translations. The two paths that we will follow will be to take translations and modulations of a same function, or translations and dilatations. And we can also make a heap of questions, which functions, which sets...

But enough of making you ask yourself and we go to give some answer.

Structure of the memory.

This memory is structured in two parts. The first studies the systems of generators for translations, which have sense in any $L^p(\mathbb{R})$. We focus especially on the case of $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, also although we give results in the rest of spaces.

The second part studies the frames (a generalization of the bases) in $L^2(\mathbb{R})$ that come of translated and modulated of a function (Gabor), or translated and dilated (wavelets). We go to explain this a bit more.

In the first chapter we give the main definitions of the concepts that will be turning up along the work. Also we give the problems that we consider and for which they have sense resulted from their helping us to understanding and the questions that we make are natural. Some of these results are known and other are new, but the idea is that they have a general sense in order to face the concrete problems. In the case of wavelets and Gabor also we give varying results about regular nets of points (already we will explain the one that they are). These nets have been the more studied ones and there is an extense knowledge of the problem in this case. Even though we will not use it, we add this case to informative title. We will focus more on irregular nets.

The last section of this chapter is about a concept of great importance during all this study, the phase spacese. These spaces will be, informally speaking, the set of all the continuous transforms of all functions. And it serves as link of union to us between both parts.

It will be the usual way to work as the cases in any:

- To Choose a function to analyze.
- To describe more or less its phase space.
- From the properties of it to attempt to search the sets that interest us (of uniqueness or sampling).

This outline will repeat with small variations in any of the problems that we consider.

The first part will study the generators by translations. The three first sections of the preliminaries give us the necessary notions, and from here we can go directly to the corresponding subpart aside in the definition of phase space.

The second chapter is dedicated to give conditions for attempting to characterize the functions that can be generators by translations. For the time being this continues being an open problem. We study a bit the state of the question and we dedicate to two concrete subsets of generators, the quasi–analytical ones and the analytical ones. For these we give very next necessary and sufficient conditions for characterizing them and we study a little how they can be the sets of points that afterwards will cause systems of generators.

Chapters 3 and 4 dedicate themselves to studying two concrete cases of generators, the function of Poisson and the Gaussian one respectively. These two functions have a phase space with good properties, which it allows us to make a good study of the sets that we can use for generating. In the case of Poisson there was a characterization of these, and what we make is to generalize it to similar cases. In the case of the Gaussian the only one that we can make is to improve the conditions that were already known, since a complete characterization seems difficult.

Studying frames and bases only has sense in $L^2(\mathbb{R})$. The basic definitions we will find in sections 1.4 and 1.5, like this like a brief summary of the situation when the net is regular. Also we will have to read the corresponding subpart of the phase spaces.

After this we can jump indistintament to chapter 5 or 6. The first takes care of Gabor frames and the second of those of wavelets. The study is similar in both cases. First we explain how to discretize the continuous transform. Afterwards we give concrete examples in very particular cases where the space of phase is composat by holomorfic functions, and the problem is simpler and has already been solved. In both cases we see that the fact that there is a phase space composed by holomorfic functions is practically an accident.

For the rest of cases we achieve to give some results of sampling that allow us to give enough conditions for obtaining a frame. Also we achieve some result of interpolation (the dual problem).

Finally, in the case of Gabor, we open a new way of study, restricting us to a special class of functions. We achieve to see that the phase space has an extension to entire functions in two variables. We give heights about the variety of zeros which it seems can cause results of sampling better of those that are known until now.

Chapter 1

Preliminaries.

1.1 Systems of Generators.

In this work we will study several ways to generate the $L^p(\mathbb{R})$ spaces. These spaces are the set of p-integrable functions with respect to the Lebesgue measure, with the usual norm:

$$\|f\|_p = \int_{\mathbb{R}} |f(t)|^p \, dt < \infty$$

with $1 \leqslant p < \infty$. We will not study the case $L^\infty(\mathbb{R})$ since it is a no separable space. In these spaces we can think that the functions take real or complex values. Both cases are quite similar, even though sometimes we find differences wich are not only technical and we have to remember which case we are treating.

When we speak about generating our goal is to describe the functions of the space by means of linear combinations of a reduced set of functions. By describe we will mean in some cases that every function can be approximate as well as we wish by these linear combinations, or in others that we can enen write every function as an infinite linear combination.

We formalize the notion of a generator, which we will use throughout a big part of this work:

Definition. Given a set $\mathcal{F} = \{f_i\}_{i \in I}$ of functions of $L^p(\mathbb{R})$ we will say that $\mathcal{F}$ is a **generator system** of $L^p(\mathbb{R})$ if for every function $g \in L^p(\mathbb{R})$ and every ε there exists a (finite) linear combination of elements of $\mathcal{F}$ that approximate the function g with error minor than ε:

$$\left\| g(t) - \sum{}' a_i f_i(t) \right\|_p \leqslant \varepsilon$$

This is equivalent to saying that the finite linear combinations of elements of $\mathcal{F}$ are dense in $L^p(\mathbb{R})$.

In our work we will focus in sets formed by actions applied to a fixed function and we will search countable sets.

The former definition is not very useful when we intend to find out if a family of functions is a generator system or not. We have an equivalent definition, which is the one that we will use more often.

Proposition 1.1. *A family $\mathcal{F} = \{f_i\}_{i\in I}$ generates $L^p(\mathbb{R})$ if and only if given $g \in L^{p'}(\mathbb{R})$ (the dual space of $L^p(\mathbb{R})$),*

$$\int_{\mathbb{R}} g(t)\overline{f_i(t)}\,dt = 0 \quad \forall i \in I \tag{1.1}$$

it implies that $g = 0$.

This result is a consequence of the *Hahn-Banach* theorem.

One of the more important basic tools that we will use throughout this work is the Fourier transform, which, to avoid confusion, we will define in the following way:

$$\widehat{f}(\xi) = \int_{\mathbb{R}} f(t)e^{-2\pi it\xi}\,d\xi$$

1.2 Generators for translations.

The first generator systems that we consider are families formed by translations of a fixed function. We are interested on studying when a set of translations of a fixed function φ generates $L^p(\mathbb{R})$. We fix the notation f_x to designate the translation of f by x. That is, $f_x(t) = f(t - x)$.

Definition. Let $\Lambda \subset \mathbb{R}$.

- We will say that Λ is a **discrete** set if it does not have finite accumulation points.
- We will say that Λ is a **uniformly discrete** or **separated** set if there is $\delta > 0$ such that $|\lambda - \gamma| > \delta$ for every $\lambda \neq \gamma \in \Lambda$.

The same definitions are valid in $\mathbb{R}^n$ or in any metric space with the suitable changes.

A discrete set Λ is a countable set such that for every $\lambda \in \Lambda$ there exists δ such that $(\lambda - \delta, \lambda + \delta) \cap \Lambda = \emptyset$. Separated sets are discrete sets where this δ is independent of λ.

Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}} \subseteq \mathbb{R}$ be a countable set (it can have finite accumulation points). Given a function φ, we define for this φ and this Λ the following set of functions:

$$T(\varphi,\Lambda) = \big\{\varphi(t-\lambda_n) = \varphi_\lambda(t) : \lambda_n \in \Lambda\big\}$$

We are interested in knowing for which sets (in principle discrete) Λ and for which functions φ the set $T(\varphi,\Lambda)$ generates $L^p(\mathbb{R})$. We formalize this concept.

Definition. Given a set $\Lambda \subseteq \mathbb{R}$ and a function $\varphi \in L^p(\mathbb{R})$ we say that the set $T(\varphi,\Lambda)$ **generates** $L^p(\mathbb{R})$ if $T(\varphi,\Lambda)$ is a generator system of $L^p(\mathbb{R})$.

Definition. We will say that a set $\Lambda \subseteq \mathbb{R}$ **accepts generators** for $L^p(\mathbb{R})$ if there exists a function $\varphi \in L^p(\mathbb{R})$ such that $T(\varphi,\Lambda)$ generates $L^p(\mathbb{R})$. In this case we say that φ is a **Λ-generator**.

Definition. We say that a function $\varphi \in L^p(\mathbb{R})$ is a **generator** if there is a discrete set Λ such that $T(\varphi,\Lambda)$ generates $L^p(\mathbb{R})$.

With these definitions we can already introduce the questions that we will treat in the first part of this work. On the one hand we are interested in finding out which sets Λ accept generators in the different spaces. This first question has been studied a lot and is solved in $L^1(\mathbb{R})$. We will be more interested in knowing for which functions φ there is a set Λ such that $T(\varphi,\Lambda)$ generates. In this case it is also interesting to attempt to describe all the sets Λ with this property. On the other hand, given a set Λ that accepts generators we will want to give conditions so that a function φ is a Λ-generator. In this work we will focus more on the description of the functions than in the sets, although both problems are intimately related.

These types of questions are a subject of classical study in the theory of the harmonic analysis. Wiener gave the first interesting results [Wie32], in a series of great importance theorems called *tauberians.* Wiener studied when a set $T(\varphi,\mathbb{R})$ generates $L^1(\mathbb{R})$ or $L^2(\mathbb{R})$. His work describes the set of functions such that all its translations (he was considering the case $\Lambda = \mathbb{R}$) generate the corresponding space, and arrived at a very clear complete description.

Theorem 1.2 (Wiener)**.** *Let $\varphi \in L^1(\mathbb{R})$. $T(\varphi,\mathbb{R})$ is a generator system of $L^1(\mathbb{R})$ if and only if $\widehat{\varphi}(\xi) \neq 0$ for every $\xi \in \mathbb{R}$.*

Let $\phi \in L^2(\mathbb{R})$. $T(\phi,\mathbb{R})$ is a generator system of $L^2(\mathbb{R})$ if and only if $\widehat{\phi}(\xi) \neq 0$ for every $\xi \in \mathbb{R}$ except a set of zero measure.

Proof. We will prove the case $L^2(\mathbb{R})$. Using 1.1, let $h \in L^2(\mathbb{R})$ such that:

$$\int_{\mathbb{R}} h(t)\overline{\phi(t-\lambda)}\,dt = 0 \qquad \forall\lambda \in \mathbb{R}$$

We take Fourier transform and we use Parseval theorem to see that it is equivalent to:

$$\int_{\mathbb{R}} \widehat{h}(\xi)\widehat{\phi}(-\xi)e^{2\pi i\lambda\xi}\, d\xi = 0 \qquad \forall\lambda \in \mathbb{R}$$

This is equivalent to $\widehat{h}(\xi)\widehat{\phi}(\xi) = 0$ almost for every ξ, which gives the double implication of the theorem.

The proof in elementary terms of the case $L^1(\mathbb{R})$ is quite more complex and long. A way to prove this theorem is to see that the Fourier transform of a function of $L^\infty(\mathbb{R})$ is a distribution that can be supported in discrete points and use the same idea as in $L^2(\mathbb{R})$. An argument based on Banach algebras can also be used. If we do not want to use any of these argument, we can find the direct proof in [Wie32], without using duality. ☐

Remark. The equivalent problem for $p \in (1,2)$ or $p > 2$ is not solved. The main obstacle is to describe the set of Fourier transform of the space $L^p(\mathbb{R})$. As a matter of fact, an equivalent problem is to characterize which sets are support of distributions that have Fourier transform in $L^p(\mathbb{R})$. Many people have studied this problem. For instance, in [Beu50] Beurling proved, for $p \in (1,2)$, that if the Hausforff dimension of the zero set of the Fourier transform of φ is less than $2 - \frac{2}{p}$, then $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$. In [RoS03] there are examples of functions such that $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$ for every $p > r$ and does not generate for $p \leqslant r$. For more information about this problem we recommend this last reference and the the articles quoted there.

Now, we can state a result that will be very useful throughout the first part of this work, and that is a consequence of this theorem.

Lemma 1.3. *Let $f \in L^\infty(\mathbb{R})$ and $h \in L^1(\mathbb{R})$ such that $\widehat{h}(\xi) \neq 0$ for every $\xi \in \mathbb{R}$. We suppose that:*

$$f * h = 0$$

Then $f = 0$.

Proof. As $f * h = 0$, also $f * h_x = 0$ for any translation of h (the convolution commutes with the translations). Therefore $f * g = 0$ for any g that is a linear combination of translations of h. As $\widehat{h}(\xi) \neq 0$, the Wiener theorem says that $f * g = 0$ for every $g \in L^1(\mathbb{R})$.

We take an identity approach ϕ_ε, for example

$$\phi_\varepsilon = \frac{1}{2\varepsilon}\chi_{[-\frac{1}{\varepsilon}, \frac{1}{\varepsilon}]}$$

Then $f * \phi_\varepsilon = 0$ for every ε. Lebesgue theorem says that $f * \phi_\varepsilon \to f$ when $\varepsilon \to 0$ almost everywhere. This implies that $f = 0$. ☐

Wiener theorem closes the problem in $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$ when the set that we consider is the straight line. But we are interested in discrete sets, where the problem is more complicate. We have, as we have commented, two parallel problems; to know which discrete sets Λ admit generators, and which can be these generators. Wiener theorem says that a necessary condition so that a function is a generator is that it transform does not cancel in any point if we are in $L^1(\mathbb{R})$ or almost everwhere in the $L^2(\mathbb{R})$ case.

We will focus especially on the cases $L^1(\mathbb{R})$ and $L^2(\mathbb{R})$, since they are the most interesting, in order to the theoretical study, as well as in order to the applications.

Focusing in the sets, for the time being there is not a description of the discrete sets that admit generators in $L^p(\mathbb{R})$, but there are partial results that gives light to the problem. As an example, in $L^1(\mathbb{R})$ we have that in [Bru06] and [BOU06] there are a complete description of the discreet sets Λ that accept generators in terms of the Beurling-Mallavin density. As a matter of fact, in these two works it is seen that they coincide with uniqueness sets of certain classes of functions that allow us to construct Λ-generators that belong to these classes.

In the rest of spaces $L^p(\mathbb{R})$ we do not have any description of these sets, but they are partial results. For example, in $L^2(\mathbb{R})$ it is easy to prove that a regular net ($\Lambda = \{a\mathbb{Z}\}$) does not accept generators, but in [Ole97] and [OlU04] there are constructed generators for small perturbations of these sets. However, for $L^p(\mathbb{R})$ with $p > 2$ $\mathbb{Z}$-generators can be constructed.

The general idea is that the number of sets that accept generators increases with p. We formalize this in the following statement.

Theorem 1.4. *Let Λ be a set that admits generators in $L^p(\mathbb{R})$. Then Λ accepts generators for $L^\alpha(\mathbb{R})$ for every $\alpha \geqslant p$.*

Moreover, if $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$ then $T(\varphi * h, \Lambda)$ generates $L^\alpha(\mathbb{R})$ for every $p \leqslant \alpha < \infty$, where h can be any function of $L^1(\mathbb{R})$ bounded and such that $\widehat{h}(\xi) \neq 0$ for every ξ.

Proof. As Λ accepts generators for $L^p(\mathbb{R})$ we know that there is a function $g \in L^p(\mathbb{R})$ such that $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$. This is equivalent to if for $f \in L^{p'}(\mathbb{R})$ (p' the dual exponent of p)

$$\int_{\mathbb{R}} f(t)\overline{\varphi(t-\lambda)}\, dt = 0$$

for every $\lambda \in \Lambda$ then $f = 0$. We take a function h as in the statement, for example $h(t) = e^{-t^2}$. We will construct a new generating function through convolution of φ whit this one:

$$\phi(t) = (h * \varphi)(t) = \int_{\mathbb{R}} h(x) g(t-x)\, dx$$

For every $f \in L^q(\mathbb{R})$ *Hausdorff-Young* theorem says that $h * f \in L^\beta(\mathbb{R})$ for every $\beta \geqslant q$, since $h \in L^r(\mathbb{R})$ for every $1 \leqslant r \leqslant \infty$.

Using this reasoning we see that $\phi \in L^\alpha(\mathbb{R})$ for every $\alpha \geqslant p$. To check that $T(\phi, \Lambda)$ generates $L^\alpha(\mathbb{R})$ we use duality. Thus, if we take $f \in L^{\alpha'}(\mathbb{R})$:

$$\begin{aligned}\int_{\mathbb{R}} f(t)\overline{\phi(t-\lambda)}\,dt &= \int_{\mathbb{R}} f(t)\overline{h * \varphi(t-\lambda)}\,dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(t)h(x)\overline{\varphi(t-x-\lambda)}\,dx\,dt \\ &= \int_{\mathbb{R}}\int_{\mathbb{R}} f(y+x)h(x)\overline{\varphi(y-\lambda)}\,dy\,dx \\ &= \int_{\mathbb{R}} \widetilde{f} * h(-y)\overline{\varphi(y-\lambda)}\,dy\end{aligned}$$

where $\widetilde{f}(t) = f(-t)$. Now we observe that $\widetilde{f} * h \in L^{p'}(\mathbb{R})$ since $p' \geqslant \alpha'$. If:

$$\int_{\mathbb{R}} f(t)\overline{\phi(t-\lambda)}\,dt = \int_{\mathbb{R}} \widetilde{f} * h(-y)\overline{\varphi(y-\lambda)}\,dt = 0 \quad \forall \lambda \in \Lambda$$

as $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$, $\widetilde{f} * h = 0$. We remember now that $\widehat{h}(\xi) \neq 0$ for every ξ and lemma 1.3 says that $f = 0$. Therefore $T(\phi, \Lambda)$ generates $L^\alpha(\mathbb{R})$. $\square$

Remark. We observe that if $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$ we can not say that it generates $L^\alpha(\mathbb{R})$.

1.3 Generators for $L^2(\mathbb{R})$.

The $L^2(\mathbb{R})$ case is interesting for several reasons. The first one is that, in order to the applications, this space corresponds with the finite energy signals. But beyond this fact, one of the particularities of this space is that it is a Hilbert space. This says that it is its own dual. Moreover, we have the concept of basis that allows us to generalize and to give more structure to the generator sets.

Another important property is that the Fourier transform is an isomorphism of $L^2(\mathbb{R})$ in itself. This fact makes easier the work, as we have seen in Wiener theorem. We can find another exemple in the translation of theorem 1.4 if we look only at the $L^2(\mathbb{R})$ case.

Theorem 1.5. *We suppose that $T(\varphi, \Lambda)$ generates $L^2(\mathbb{R})$. Let $\phi \in L^2(\mathbb{R})$ be such that $\widehat{\phi}(\xi) \neq 0$ almost for every ξ, and moreover it fulfills that:*

$$\left|\frac{\widehat{\phi}(\xi)}{\widehat{\varphi}(\xi)}\right| \leqslant C$$

Then $T(\phi, \Lambda)$ also generates $L^2(\mathbb{R})$

Proof. We suppose that there is $m \in L^2(\mathbb{R})$ such that:

$$\int_{\mathbb{R}} m(t)\overline{\phi(t-\lambda)}\, dt = 0 \quad \forall \lambda \in \Lambda$$

Then, by duality:

$$\int_{\mathbb{R}} e^{2\pi and\lambda\xi}\widehat{m}(\xi)\widehat{\phi}(-\xi)\, d\xi = \int_{\mathbb{R}} e^{2\pi and\lambda\xi}\widehat{m}(\xi)\frac{\widehat{\phi}(-\xi)}{\widehat{\varphi}(-\xi)}\widehat{\varphi}(-\xi)\, d\xi = 0 \quad \forall \lambda \in \Lambda$$

As $T(\varphi, \Lambda)$ generates and $\frac{\widehat{\phi}(\xi)}{\widehat{\varphi}(\xi)}$ is bounded we have that $m(\xi)\frac{\widehat{\phi}(\xi)}{\widehat{\varphi}(\xi)} \in L^2(\mathbb{R})$ and it is equal to 0. Taking into account that $\widehat{\phi} \neq 0$ we see that $m = 0$ and therefore $T(\phi, \Lambda)$ generates. ☐

In this theorem we can see that when we study the generator sets of a function we have to look at the module of its Fourier transform. As the Fourier transform is an isomorphism we can move the problem to the side of the transform and obtain an equivalent statement. If we have a weight $P(\xi) > 0$ almost for every ξ we can define $L^2(\mathbb{R}, P)$ as the set of functions such that:

$$\int_{\mathbb{R}} |f(\xi)|^2 |P(\xi)|\, d\xi < \infty$$

Proposition 1.6. *Let $\varphi \in L^2(\mathbb{R})$ be such that $\widehat{\varphi}(\xi) \neq 0$ almost for every ξ. The set $T(\varphi, \Lambda)$ generates $L^2(\mathbb{R})$ if and only if the set of exponentials $\{e^{2\pi i\lambda\xi}\}_{\lambda\in\Lambda}$ generate $L^2(\mathbb{R}, |\widehat{\varphi}|^2)$.*

Remark. Theorem 1.5 is a simple corollary of this result. We only have to compare the spaces that appear when we take Fourier transform. As a matter of fact both statements are saying that the capacity of generate of φ only depends of the module of its Fourier transform $|\widehat{\varphi}|$ and its decrease at infinite.

To prove this result we only need Parseval theorem and the definition of generator system.

We have commented in the former section that a regular net does not accept generators for $L^2(\mathbb{R})$. We can give now a very simple proof of this fact. If $T(\varphi, \mathbb{Z})$ is a generator system of $L^2(\mathbb{R})$ we have that all the functions of $L^2(\mathbb{R}, |\varphi|^2)$ are 2π-periodic and this is absurd.

The study of when a set of exponentials generates a certain space is a classical subject of the harmonic analysis. The case of the $L^2(I)$ space, where I is a closed interval, has been one of the most studied. This problem was solved by Beurling and Mallavin through a notion (the Beurling-Mallavin density) that is also useful to describe the sets that accept generators for $L^1(\mathbb{R})$.

1.4 Bases and frames.

Another important property of the Hilbert spaces is that they have much more structure. In these spaces we can define bases and frames. These are special cases of generator systems that, under some strong conditions, allow us to have stable reconstruction formulas. We go to describe this concepts and to give their properties.

Definition. Let H be a Hilbert space and $\{e_i\}_{i\in I} \subseteq H$. We say that $\{e_i\}_{i\in I}$ is an **orthonormal basis** if $\langle e_i, e_j\rangle = \delta_{i,j}\ \forall i,j \in I$ and for every $f \in H$ it fulfills that:

$$\|f\|^2 = \sum_{i\in I} |\langle e_i, e_j\rangle|^2 \tag{1.2}$$

The first part of the definition says that e_i are orthogonal. The second says that they generate H. That is, we have linear independence and completeness. Therefore the orthonormal bases are generator systems and the condition 1.2 gives an additional structure which allows us to reconstruct in a stable way. If $\{e_i\}_{i\in I}$ is an orthonormal basis, every element $f \in H$ can be expressed in an unique way as a linear combination of the elements of $\{e_i\}_{i\in I}$ by the following formula:

$$f = \sum_{i\in I} \langle f, e_i\rangle e_i$$

The fact that every element can be expressed as a linear combination of the elements of $\{e_i\}_{i\in I}$ implies that this set generates the space. The uniqueness of the expression comes from the linear independence, that, in this case, is a consequence of the orthogonality of the elements.

Frames are a more general concept of generator systems with this property (that every element can be expressed as an infinite linear combination). These will be the more general sets with this property.

Definition. We say that a set $\{e_k\}_{k\in\mathbb{N}}$ in a Hilbert space H is a **frame** if there are constants $A, B > 0$ such that:

$$A\|f\|^2 \leqslant \sum_{k=1}^{\infty} |\langle f, e_k\rangle|^2 \leqslant B\|f\|^2, \quad \forall f \in H$$

The constants A and B are called the bounds or constants of the frame.

In this definition we have not given importance to the concept of independence. When we work with frames the important property is that they can generate the space H in a stable way. The way to make it is very similar to the orthonormal bases.

Proposition 1.7. *Let $\{f_k\}_{k\in\mathbb{N}}$ be a frame for a Hilbert space H. There is a set $\{g_k\}_{k\in\mathbb{N}}$ such that:*

$$f=\sum_{k=1}^{\infty}\langle f,g_k\rangle f_k \quad \forall f\in H$$

The set $\{g_k\}_{k\in\mathbb{N}}$ is also a frame. As there is not independence, this does not have to be unique. There is, however, a privileged one, which is called the dual frame. This has the property of minimizing the norm of the coefficients of the representation, which obviously neither are unique. Moreover $\{f_k\}_{k\in\mathbb{N}}$ can be calculated in a explicit way from the original frame through the frame operator. The details can be find in [Dau92]. The bounds of the dual frame are $\frac{1}{B}$ and $\frac{1}{A}$ respectively. If $\{\widetilde{f_k}\}_{k\in\mathbb{N}}$ is the dual frame of $\{f_k\}_{k\in\mathbb{N}}$, the repconstruction formula for this remains:

$$f=\sum_{k=1}^{\infty}\langle f,f_k\rangle \widetilde{f_k} \quad \forall f\in H$$

that tells us that the values $\langle f,f_k\rangle$ completely determine the function.

One of the characteristics of the frames is that they can be redundant, that is, there are excess of vectors. In an informal way they would correspond, in finite dimension, with a set of vectors with cardinal greater than the dimension of the space and that generates.

There exists some special classes of frames:

Definition. Let $\{e_k\}_{k\in\mathbb{N}}$ be a frame for H with bounds A and B. We will say that the frame is **rigid** if $A=B$.

Remark. If a frame is rigid it can be seen that the dual frame is itself multiplied by $\frac{1}{A}$. The reconstruction formula comes off practically identical to that of the orthonormal bases. This fact makes these be the best frames in order to the applications in signal processing, since it is not necessary to calculate the dual frame to reconstruct the function.

Riesz bases are another class of frames that are of special importance. These sets can be characterized in many ways. Their main property is that the functions that shape it are linear independent. This makes that each function can be expressed in an unique way.

Definition. A set $\{e_k\}_{k\in\mathbb{N}}$ is a **Riesz basis** of H if every element $f\in H$ can be expressed in an unique way as a linear combination of elements of $\{e_k\}_{k\in\mathbb{N}}$:

$$f=\sum_{k\in\mathbb{N}}a_k e_k \qquad a_k\in\mathbb{R}(\mathbb{C})$$

so that there are $A, B > 0$ such that:

$$A\|f\|^2 \leqslant \sum_{k\in\mathbb{N}} |a_k|^2 \leqslant B\|f\|^2$$

The dual frame of a Riesz basis also has special properties.

Definition. Two sets of vectors $\{f_k\}_{k\in\mathbb{N}}$ and $\{g_k\}_{k\in\mathbb{N}}$ are called **bi-orthogonal** if:

$$\langle f_i, g_j\rangle = \begin{cases} 1 & \text{if } i = j \\ 0 & \text{if } i \neq j \end{cases}$$

Proposition 1.8. *If $\{f_k\}_{k\in\mathbb{N}}$ is a Riesz basis for H, there is a unique set $\{g_k\}_{k\in\mathbb{N}}$ in H such that:*

$$f = \sum_{k=1}^{\infty} \langle f, g_k\rangle f_k, \quad \forall f \in H$$

$\{g_k\}_{k\in\mathbb{N}}$ is also a Riesz basis that we call dual basis. $\{f_k\}_{k\in\mathbb{N}}$ and $\{g_k\}_{k\in\mathbb{N}}$ are bi-orthogonal and the series converges unconditionally.

The existence of a bi-orthogonal set practically assures that a set will be a Riesz basis. We give some equivalent definitions in order to be able to recognize Riesz bases.

Proposition 1.9. *Let H be a Hilbert space. Then they are equivalents:*

1. *The set $\{f_n\}_{n\in\mathbb{N}}$ is a Riesz basis.*
2. *The set $\{f_n\}_{n\in\mathbb{N}}$ is complete in H and there are positive constants A, B such that for any positive integer k and arbitrary scalars $c_1, \ldots, c_k$ it fulfills that:*

$$A\sum_{n=1}^{k} |c_n|^2 \leqslant \left\| \sum_{n=1}^{k} c_n f_n \right\|^2 \leqslant B\sum_{n=1}^{k} |c_n|^2$$

3. *The set $\{f_n\}_{n\in\mathbb{N}}$ is complete in H and it there is a complete bi-orthogonal set $\{g_n\}_{n\in\mathbb{N}}$ such that:*

$$\sum_{n\in\mathbb{N}} |\langle f, f_n\rangle|^2 < \infty \quad \textit{and} \quad \sum_{n\in\mathbb{N}} |\langle f, g_n\rangle|^2 < \infty$$

   for every $f \in H$.

4. *The set $\{f_n\}_{n\in\mathbb{N}}$ is of the form $\{Te_k\}_{k\in\mathbb{N}}$ where $\{e_k\}_{k\in\mathbb{N}}$ an orthonormal basis of H and $T : H \to H$ is a bounded and bijective linear operator.*

5. *The set* $\{f_n\}$ *is a frame and every element of* H *can be expressed in an only way as a linear combination of elements of* $\{f_n\}_{n\in\mathbb{N}}$.

We can find the proofs in [You01].

One of the advantages of Riesz bases with respect to orthonormal ones is that we have different stability results, as for example:

Proposition 1.10. *Let* H *be a Hilbert space and* $\{e_n\}_{n\in\mathbb{N}}$ *an orthonormal basis. If* $\{f_n\}_{n\in\mathbb{N}}$ *is a linearly independent set such that:*

$$\sum_{n\in\mathbb{N}} \|e_n - f_n\|^2 < \infty$$

Then $\{f_n\}_{n\in\mathbb{N}}$ *is a Riesz basis.*

This proposition says that bases closer to an orthonormal basis are Riesz bases, since the orthonormality is not a question of proximity.

But also we have results that allow us to say that sets next to Riesz bases are Riesz bases:

Theorem 1.11 (Paley)**.** *Let* $\{e_i\}_{i\in\mathbb{N}}$ *be a Riesz basis for a Hilbert space* H. *We suppose that* $\{f_i\}_{i\in\mathbb{N}}$ *is a set of elements of* H *that fulfills that:*

$$\left\|\sum_{i=1}^{n} c_i(e_i - f_i)\right\| \leqslant \lambda \left\|\sum_{i=1}^{n} c_i e_i\right\|$$

for some constant $0 \leqslant \lambda < 1$, *and for every selection of scalar* $c_1, \dots, c_n$ $(n \in \mathbb{N})$. *Then* $\{f_i\}_{i\in\mathbb{N}}$ *is a Riesz basis for* H.

Orthonormal bases, Riesz bases and frames are the three basic concepts that we will use in the second part of this work. The main difference with the general generator systems is the stability of the coefficients, which causes the reconstruction formulas.

When we consider generator systems we want that the linear combinations are dense in the space. This is fulfilled in the case of bases and frames. But now we can describe every function as a linear combination of the elements of the basis or the frame, which we could not insure in the case of the generator systems.

As it is deduced from the definitions, the orthonormal bases are also Riesz bases, Riesz bases are frames, a special class of frames called exact frames, and the orthonormal ones are rigid frames. We can think Riesz bases as linearly independent frames, without redundancy.

Orthonormal and Riesz bases as well as frames define operators (that we will name operators of synthesis) in the following way. If $\{e_k\}_{k\in\mathbb{N}}$ is a set in H, we define:

$$\begin{aligned} T : H &\longrightarrow l^2 \\ v &\longmapsto T(v) = \{\langle v, e_k\rangle\}_{k\in\mathbb{N}} \end{aligned}$$

We can recognize which class of set we are using observing how it operates this map. If $\{e_k\}_{k\in\mathbb{N}}$ is an orthonormal basis the operator T is an isometry. In Riesz bases the operator is bounded and invertible, but it does not preserve the norm. If the set is a frame the operator is also bounded, but it is not exhaustive, although it is invertible in its image.

Later on, when we will define the phase spaces, we will see that this operator corresponds in a direct way with the sampling operator. To understand the connection that we will establish is important a good understanding of this operator. For example, in orthonormal bases as well as in Riesz bases, this operator is bijective. This means two things. On the one hand the values of $T(v)$ determines completely v, and in other hand, for any set of values $a \in l^2$ there is a vector $v \in H$ such that $T(v) = H$. In the first case we have solved the problem of sampling (to determine v from its image) and in the other one we solve the interpolation problem (obtain v with a given image). This two problems are dual.

In the case of the frames the operator is not exhaustive and we can not solve the interpolation problem. In spite of that, v is totally determinate from its image.

The sets for which the problem of interpolation can be solved are the linearly independent sets (in a suitable sense). For these the operator is exhaustive but it can not be injective, since they do not have to be complete. In this case there is the possibility of not being able to solve the problem of sampling.

We have to observe that the operator T does not give the coefficients with which we represent the functions. These coefficients are useful to reconstruct the vectors but throughout the dual basis or the dual frame.

The main reason why we have not spoken about bases and frames until this moment is the following result.

Theorem 1.12. *A generator system by translations $T(\varphi, \Lambda)$ can never be a basis or a frame. That is, there are not frames or bases of translations in $L^2(\mathbb{R})$.*

Remark. This result is true in $L^2(\mathbb{R})$ but it is not in general. As an example, in the Paley-Wiener space (band limited functions) there are bases and frames of translations of a concrete function.

Lemma 1.13. *If $\{\varphi_n\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$, then:*

$$S(t) = \sum_{n\in\mathbb{Z}} |\varphi_n(t)|^2 = \infty$$

almost for every $t \in \mathbb{R}$.

Proof. If $S(t) < \infty$ for a set of positive measure I, then it exists $M > 0$ and another set of positive measure J such that

$$S(t) \leqslant M \qquad \forall t \in J$$

If $f(t) = \sum_{n\in\mathbb{Z}}\langle f, \widetilde{\varphi_n}\rangle \varphi_n(t)$ is the expression of $f \in L^2(\mathbb{R})$ where $\{\widetilde{\varphi_n}\}_{n\in\mathbb{Z}}$ is the dual frame of $\{\varphi_n\}_{n\in\mathbb{Z}}$, using the Schwart inequality we have that:

$$|f(t)|^2 = \Big|\sum_{n\in\mathbb{Z}}\langle f, \widetilde{\varphi_n}\rangle \varphi_n(t)\Big|^2 \leqslant S(t)\sum_{n\in\mathbb{Z}}\big|\langle f, \widetilde{\varphi_n}\rangle\big|^2 \leqslant \frac{M}{A}\|f\|^2$$

for every $t \in J$, where A is the lower bound of the frame $\{\varphi_n\}_{n\in\mathbb{Z}}$. This says that every function of $L^2(\mathbb{R})$ is bounded in J, which is absurd. $\square$

Proof of 1.12. We suppose that $\{\varphi(t-\lambda_n)\}_{n\in\mathbb{Z}}$ is a frame of $L^2(\mathbb{R})$. We define $h(\delta)$ the module of continuity in $L^2(\mathbb{R})$ of φ:

$$h(\delta) = \sup_{|s|\leqslant\delta}\big\|\varphi(t) - \varphi(t-s)\big\| \tag{1.3}$$

that is decreasing and with limit 0 when $\delta \to 0$. As the norm is an invariant for translations we have that:

$$\big\|\varphi(t-x) - \varphi(t-y)\big\| \leqslant h(|x-y|)$$

We suppose without losing generality that $\|\varphi\| = 1$. If $x \in \mathbb{R}$ we can deduce that $\big|\langle\varphi(t-x), \varphi(t-\lambda_n)\rangle\big| \geqslant 1 - \frac{1}{2}h(|x-\lambda_n|)$. As $\{\varphi(t-\lambda_n)\}_{n\in\mathbb{Z}}$ is a frame we have that:

$$\sum_{n\in\mathbb{Z}}\big|\langle\varphi(t-x), \varphi(t-\lambda_n)\rangle\big|^2 \leqslant B$$

From these two relations we can deduce that in any interval in the way $[a, a+1]$ there are as most N points of Λ since:

$$\sum_{\lambda_n\in[a,a+1]}\left|\left\langle\varphi\left(t-a+\frac{1}{2}\right), \varphi(t-\lambda_n)\right\rangle\right|^2 \geqslant \Big|\big\{\lambda_n \in [a, a+1]\big\}\Big|\left(1 - \frac{1}{2}h\left(\frac{1}{2}\right)\right)$$

Now we calculate:

$$\int_0^1 S(t)\,dt \leqslant \int_0^1\sum_{n\in\mathbb{Z}}|\varphi(t-\lambda_n)|^2\,dt = \int_{-\lambda_n}^{1-\lambda_n}|\varphi(t)|^2\,dt$$

As every interval of length 1 contains as most N points we have that each t will be as most in N intervals in the way $[-\lambda_n, 1-\lambda_n]$ and therefore:

$$\int_0^1 S(t)\,dt \leqslant N\int_{\mathbb{R}}|\varphi(t)|^2\,dt < \infty$$

and $S(t)$ is integrable in $[0,1]$. This implies that it is finite almost for every point in this interval, that is a contradiction with the former lemma. $\square$

1.5 Gabor and wavelets.

In order to use the Hilbert space structure of $L^2(\mathbb{R})$, and in view of the negative result that gives 1.12, one of the solutions that has been used along the last years is not just to restrict to translations of a given function, but to consider also other transformations.

In this work we will notice two alternatives that have been quite fruitful the last years. The first is to add modulations to obtain the Gabor transform. The other option is to take dilatation and translations. In this last case we will find the wavelet transform.

We will give first the definitions of continuous transform, that afterwards, after discretitzate, they will cause the respective bases and frames. We start with the Gabor transform [Gab46].

Definition. Fixed a function $g \in L^2(\mathbb{R})$ with $\|g\| = 1$, we define the **Gabor transform** of a function $f \in L^2(\mathbb{R})$ through g at the point $z = x + yi$ as:

$$Gf(z) = \big\langle f(t), e^{2\pi i y t} g(t-x)\big\rangle$$

We can write this transform in a more compact way like $Gf(z) = \langle f, g_z\rangle$. One of the main properties of this transform is that we can reconstruct the initial function f from the values of its transform in a stable way.

Theorem 1.14 (Gabor). *Let $g \in L^2(\mathbb{R})$ be such that $\|g\| = 1$ and we consider $Gf(z) = \langle f, g_z\rangle$ as we have said formerly. Then it is fulfilled that $Gf \in L^2(\mathbb{C}, dx\, dy)$ and $\|Gf\|_{\mathbb{C}} = \|f\|$. Moreover we have reconstruction formula given by:*

$$f(t) = \int_{\mathbb{R}}\int_{\mathbb{R}} Gf(z) g_z(t)\, dx\, dy \quad \forall f \in L^2(\mathbb{R}) \tag{1.4}$$

Remark. We have to understand the equality (1.4) as an equality of functions in $L^2(\mathbb{R})$, since it can not have sense punctually.

Proof. We will prove first the reconstruction formula. If we consider $Gf(z) = Gf(x,y) = F_y(x)$ as a function of a real variable (we think y as a parameter) we observe that:

$$F_y(x) = e^{-2\pi i x y}\int_{-\infty}^{\infty} f(t)\overline{g(t-x)}e^{2\pi i y(x-t)}\, dt = e^{-2\pi i x y}\left(f * g_y\right)(x)$$

where $g_y(t) = \overline{g(-t)}e^{2\pi i y t}$. Its Fourier transform with respect to x is:

$$\widehat{F_y}(\xi) = \widehat{f}(\xi + y)\widehat{g_y}(\xi + y) = \widehat{f}(\xi + y)\overline{\widehat{g}(\xi)}$$

(1.4) can be written as:

$$\int_{\mathbb{R}}\int_{\mathbb{R}} F_y(x)e^{2\pi iyt}g(t-x)\,dx\,dy = \int_{\mathbb{R}} e^{2\pi iyt}\int_{\mathbb{R}} F_y(x)\overline{\overline{g(t-x)}}\,dx\,dy \tag{1.5}$$

The integral in x can be thought as an scalar product. Taking into account that the Fourier transform of $\overline{g(t-x)}$ with respect to x is $\overline{\widehat{g}(\xi)}e^{-2\pi i\xi t}$, (1.5) remains:

$$\int_{\mathbb{R}} e^{2\pi iyt}\int_{\mathbb{R}} \widehat{f}(\xi+y)\overline{\widehat{g}(\xi)}\,\overline{\overline{\widehat{g}(\xi)}e^{-2\pi i\xi t}}\,d\xi\,dy =$$

$$\int_{\mathbb{R}} |\widehat{g}(\xi)|^2\int_{\mathbb{R}} \widehat{f}(\xi+y)e^{2\pi it(\xi+y)}\,dy\,d\xi = f(t)$$

In this last step we have used Fubini theorem. Therefore we have proved the reconstruction formula when $f \in L^1(\mathbb{R})$. By a density argument and by the continuity of the transform it can spread in every $L^2(\mathbb{R})$.

To prove that we have isometry we calculate:

$$\|Gf\|_{\mathbb{C}}^2 = \int_{\mathbb{R}}\int_{\mathbb{R}} |F_y(x)|^2\,dx\,dy = \int_{\mathbb{R}}\int_{\mathbb{R}} \left|\widehat{f}(\xi+y)\widehat{g}(\xi)\right|^2 d\xi\,dy = \|f\|_2^2$$

where we have used Parseval and Fubini theorems. ☐

The wavelet transform is analogous to this one, but considering dilatation instead of modulations. The parameter of dilatation only takes values in $\mathbb{R}^+$, and we will define it in the space $L^2(\mathbb{R})$ of functions that take real values.

Definition. Given a function $\psi \in L^2(\mathbb{R})$ we define the **wavelet transform** (for the wavelet ψ) of f at a point $z = x+iy \in \mathbb{R}\times\mathbb{R}^+$ as:

$$Wf(z) = \langle f, \psi_z\rangle = \int_{-\infty}^{\infty} f(t)\overline{y^{\frac{-1}{2}}\psi\left(\frac{t-x}{y}\right)}\,dt$$

As in the case of Gabor, for wavelets we also have continuous reconstruction formula. But this formula will be only valid when the wavelet belongs to a special class, which we will name admissible.

Definition. We will say that a wavelet $\psi \in L^2(\mathbb{R})$ is **admissible** if the following integral is convergent:

$$\int_0^{\infty} \frac{|\widehat{\psi}(\xi)|^2}{\xi}\,d\xi < \infty$$

In a general way we will consider normalized wavelets such that the norm as well as this last integral are 1. Under this hypothesis we can give the reconstruction formula.

Theorem 1.15 (Calderon, Grossmann, Morlet)**.** *We fix an admissible wavelet* ψ *in the former conditions. For every* $f \in L^2(\mathbb{R})$ *it is fulfilled that* $Wf \in L^2(\mathbb{R} \times \mathbb{R}^+, \frac{dx\,dy}{y^2})$, $\|Wf\|_{\mathbb{R}\times\mathbb{R}^+} = \|f\|_2$ *and we can reconstruct* f *from the values of its transform through the following formula:*

$$f(t) = \int_0^\infty \int_{-\infty}^\infty Wf(z)\psi_z(t)\,\frac{dx\,dy}{y^2} \tag{1.6}$$

Remark. As in the case of Gabor, (1.6) has to be understanded as an equality in $L^2(\mathbb{R})$.

Proof. We can write the integral that gives the reconstruction formula (1.6) as a convolution in the following way. We thought $z = (x, y)$ in two variables. We can write as a convolution $Wf(x, y) = (f * \psi_y^*)(x)$, where we use the notation $\psi_y^*(t) = y^{\frac{-1}{2}}\overline{\psi(\frac{-t}{y})}$. In this way the integral of (1.6) remains:

$$\int_0^\infty \big(Wf(\cdot, y) * \psi_y\big)(t)\,\frac{dy}{y^2} = \int_0^\infty \big(f * \psi_y^* * \psi_y\big)(t)\,\frac{dy}{y^2} \tag{1.7}$$

where '$\cdot$' indicates the variable with respect to what we are calculating the convolution. We observe that (1.7) is a function that depends on t. We will prove that it is equal to f seeing that they have the same Fourier transform. We calculate the transform of (1.7):

$$\int_0^\infty \widehat{f}(\xi)y^{\frac{1}{2}}\overline{\widehat{\psi}(y\xi)}y^{\frac{1}{2}}\widehat{\psi}(y\xi)\,\frac{dy}{y^2} = \widehat{f}(\xi)\int_0^\infty |\widehat{\psi}(y\xi)|^2\,\frac{dy}{y} \tag{1.8}$$

Making the change $\omega = y\xi$ we retrieve the condition of admissibility, so that (1.8) takes the form:

$$\widehat{f}(\xi)\int_0^\infty \frac{|\widehat{\psi}(\omega)|^2}{\omega}\,d\omega = \widehat{f}(\xi)$$

This proves the reconstruction formula. To prove the isometry we have to observe that the Fourier transform of $Wf(x, y)$ with respect to the variable x is $y^{\frac{1}{2}}\overline{\widehat{\psi}(y\xi)}\widehat{f}(\xi)$. Using Plancharel formula and Fubini theorem we can see that:

$$\begin{aligned}\|Wf(x,y)\|^2 &= \int_0^\infty \int_{-\infty}^\infty |Wf(x,y)|^2\,dx\,\frac{dy}{y^2} = \int_0^\infty \int_{-\infty}^\infty \left|y^{\frac{1}{2}}\overline{\widehat{\psi}(y\xi)}\widehat{f}(\xi)\right|^2 d\xi\frac{dy}{y^2}\\ &= \int_{-\infty}^\infty |\widehat{f}(\xi)|^2 \int_0^\infty \frac{|\widehat{\psi}(y\xi)|^2}{y}\,dy\,d\xi = \|\widehat{f}\|^2 = \|f\|^2\end{aligned}$$

That is what we want to prove. ☐

Remark. The equality (1.7) is known as the Calderon formula, that in [Cal64] find the same formula from another perspective. Years later, Grossmann and Morlet, who did not know the work of Calderon, proved it in [GrM84] from the point of view of the signal processing.

We have supposed, in the definition of wavelet transform ands in the proof of the theorem, that $\int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{\xi}\, d\xi = c_\psi = 1$. If this was not true, the wavelet transform has to be defined as:

$$Wf(z) = \frac{1}{\sqrt{c_\psi}} \langle f, \psi_z \rangle$$

In this way the reconstruction formula continues beeing valid. It is only necessary to take into account the constant c_ψ in the proof.

We compare the reconstruction formulas given here with the Wiener theorem 1.2. In all cases, under different conditions about the analyzing function, we can generate all the space $L^2(\mathbb{R})$ using all the action (all translations, the translations and modulations or translations and dilatation in each case). The main difference with respect to using only translations is that in the Gabor or wavelet transforms we have an explicit and rigid (there is conservation of the norm) reconstruction formula.

A way to interpret theorems 1.14 and 1.15 is to think that the isometry says that both transforms give a rigid frame of $L^2(\mathbb{R})$. The reconstruction formula is the continuous equivalent to the reconstruction formula of frames, with the dual frame equal to the original one.

Theorem 1.12 says that there are not discrete frames of translations, that is, that $T(\varphi, \Lambda)$ can not be a frame for any function $\varphi \in L^2(\mathbb{R})$ and any discrete set $\Lambda \subset \mathbb{R}$. This result is also true in the continuous version ($\Lambda = \mathbb{R}$), thinking in the concept of continuous frame.

Proposition 1.16. *There are not continuous frames of translations. That is, there do not exists $\varphi \in L^2(\mathbb{R})$ such that for every $f \in L^2(\mathbb{R})$ has sense the formula:*

$$f = \int_{\mathbb{R}} \langle f, \varphi_x \rangle \varphi_x \, dx$$

or in an equivalent way:

$$\|f\| = \big\|\langle f, \varphi_x\rangle\big\| = \left\| \int f(u)\overline{\varphi(u-x)}\, du \right\|$$

Proof. We calculate directly:

$$\int_{\mathbb{R}} \langle f, \varphi_x \rangle \varphi_x(t)\, dx = \int_{\mathbb{R}} \int_{\mathbb{R}} f(u)\overline{\varphi(u-x)}\varphi(t-x)\, dt\, dx$$

We can think the integral in x as an scalar product. We take Fourier transform and we apply Parseval. Then the former equation remains:

$$\int_{\mathbb{R}} f(u) \int_{\mathbb{R}} e^{-2\pi i\xi(t-u)} |\widehat{\varphi}(-\xi)|^2 \, d\xi \, du$$

The integral in U is the Fourier transform of f. In this way we obtain:

$$\int_{\mathbb{R}} |\widehat{\varphi}(-\xi)|^2 e^{-2\pi i\xi t} \widehat{f}(-\xi) \, d\xi = \int_{\mathbb{R}} |\widehat{\varphi}(\omega)|^2 \widehat{f}(\omega) e^{2\pi i\omega t} \, d\omega$$

It is clear that the last equation can not equal in general any $f \in L^2(\mathbb{R})$.

We can deduce that there can not be equality of norms in a similar way, or from the nonexistence of reconstruction formula. $\square$

In these two definitions we have used the notation $z = x + iy$. It has to be clear at all times that this is only a notation, that is, we are not thinking that we will have analytical functions of complex variable, but it is a more compact way to write the formulas. It is also necessary to take into account that we are using the same notation for two different things. The one that we mean is that g_z will mean different things when we work with Gabor or wavelet transforms. Even it changes the set where the z is defined. Any of both cases will be consistent with the definition of φ_x for translations. Normally there will not be confusion and it is simpler to see the similarities and to understand the theorems.

These two definitions are particular cases of actions of groups on $L^2(\mathbb{R})$. In the case of the Gabor transform the group that is acting is the Weil-Heisenberg group and in the case of wavelets is the affine group. The reconstruction formulas can be deduced directly from the theory of group actions in Hilbert spaces, as well as other properties that we will give later on.

As in the case of translations, we are interested in discrete sets of functions that can generate $L^2(\mathbb{R})$. We give therefore the definitions corresponding to these types of systems.

Definition. Let $\Lambda = \{z_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a discrete set and $g \in L^2(\mathbb{R})$ a function that we will call Gabor analyzing atom. We define the **Gabor system** $G(g, \Lambda)$ as:

$$G(g, \Lambda) = \left\{ g_z(t) = e^{2\pi iyt} g(t - x); z = x + iy \in \Lambda \right\}$$

Definition. Let $\Lambda = \{z_n\}_{n\in\mathbb{N}} \subset \mathbb{R} \times \mathbb{R}^+$ be a discrete set and $\psi \in L^2(\mathbb{R})$ an admissible function that we will call analyzing wavelet. We define the **wavelet system** $W(\psi, \Lambda)$ as:

$$W(\psi, \Lambda) = \left\{ \psi_z(t) = y^{\frac{-1}{2}} \psi\left(\frac{t - x}{y}\right); z = x + iy \in \Lambda \right\}$$

In a general way we will consider analyzing functions with norm 1 and normalized wavelets such that $\int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{\xi}\,d\xi = c_\psi = 1$.

As the continuous versions of these systems have much more structure than its equivalent in translations, we will ask for more properties to the discrete sets. Instead of asking when a system $G(g,\Lambda)$ or $W(\psi,\Lambda)$ generates $L^2(\mathbb{R})$, we will search more structure here. We will ask when it can be a basis or a frame. The questions are similar to those that we made in sets of translations. That is, to give conditions on the analyzing function and on the set in order to obtain a basis or a frame.

First of all we will consider sets Λ that are regular nets. In the case of Gabor they take the form $\Lambda = \{am + ibn; n, m \in \mathbb{Z}\} = \{a\mathbb{Z} \times b\mathbb{Z}\}$. The systems of the form $G(g, a\mathbb{Z} \times b\mathbb{Z})$ have been very studied in the last years. We can find several results that give necessary or sufficient conditions so that these sets are bases or frames. As an example, in [Dau92] we find:

Theorem 1.17 (Daubechies)**.** *Let $G(g, a\mathbb{Z} \times b\mathbb{Z})$ be a Gabor frame. Then:*

$$ab \leqslant 1 \tag{1.9}$$

The constants of the frame A, B have to fulfill:

$$A \leqslant \frac{1}{ab} \leqslant B$$
$$\forall t \in \mathbb{R}, A \leqslant \frac{1}{b}\sum_{n\in\mathbb{Z}} |g(t-na)|^2 \leqslant B$$
$$\forall \xi \in \mathbb{R}, A \leqslant \frac{1}{a}\sum_{m\in\mathbb{Z}} |\widehat{g}(\xi-mb)|^2 \leqslant B$$

The number $\frac{1}{ab}$ measures the density of the net. The condition (1.9) says that there will not be frames nor bases for density less than 1, and that we will have orthonormal bases when $ab = 1$. In the case of orthonormal bases we find a result that tells that the function g can not be well located and soft at the same time.

Theorem 1.18 (Balian-Low)**.** *If $G(g, a\mathbb{Z} \times b\mathbb{Z})$ is a frame with $ab = 1$, then:*

$$\int_{\mathbb{R}} t^2 |g(t)|^2\,dt = \infty \quad o \quad \int_{\mathbb{R}} \xi^2 |\widehat{g}(\xi)|^2\,d\xi = \infty$$

This result, proved in an independent way by Balian [Bal81] and Low [Low85] in the case of basis, and generalized later to frames by Daubechies and Janssen [DaJ93], is very important in order to the applications, since it tells that we can not find orthonormal bases well located in time and frequency at the same time. We will have to choose among good location or that there is not redundancy. We can find several generalizations of this theorem.

Regarding sufficint conditions we can also find as an example:

Theorem 1.19 (Daubechies)**.** *We define:*

$$\beta(u) = \sup_{0\leqslant t\leqslant a} \sum_{n\in\mathbb{Z}} |g(t-na)||g(t-na+u)|$$

and

$$\Delta = \sum_{k\in\mathbb{Z},k\neq 0} \left[\beta\Big(\frac{k}{b}\Big)\beta\Big(\frac{-k}{b}\Big)\right]^{\frac{1}{2}}$$

If a and b fulfill:

$$A_0 = \frac{1}{b}\left(\inf_{0\leqslant t\leqslant a} \sum_{n\in\mathbb{Z}} |g(t-na)|^2 - \Delta\right) > 0$$

and

$$B_0 = \frac{1}{b}\left(\inf_{0\leqslant t\leqslant a} \sum_{n\in\mathbb{Z}} |g(t-na)|^2 + \Delta\right) < \infty$$

then $G(g, a\mathbb{Z}\times b\mathbb{Z})$ is a frame. The constants A_0 and B_0 are respectively upper and lower bounds of the constants A and B of the frame.

The proof of these results as well as a much more accurate and deep study of the regular Gabor frames can be found in [Dau92], or we can also quote [Chr03] in the extensive bibliography about the subject.

If we search sufficient conditions for orthonormal bases in regular nets we can give a curious result that was not known until now. This result, tells that if the set $G(g, \mathbb{Z}\times\mathbb{Z})$ is orthonormal then it is a basis. It is interesting because it tells that the orthonormality (independence of the functions) already gives the capacity to generate when we use all the net $\mathbb{Z}\times\mathbb{Z}$. The key is the Poisson formula.

Theorem 1.20 (Poisson sumation formula)**.** *Let f be a function for which we can calculate its Fourier transform. Then:*

$$\sum_{m\in\mathbb{Z}^n} f(m) = \sum_{m\in\mathbb{Z}^n} \widehat{f}(m)$$

validates when the two sides of the equality have sense.

The idea is to apply this formula to the Gabor continuous transform of a function, which is defined in two variables. We first give a technical lemma.

Lemma 1.21. *Let $g\in L^2(\mathbb{R})$ be a Gabor atom and Gf the Gabor transform of $f\in L^2(\mathbb{R})$ with respect to g. The Fourier transform (in two variables) of Gf is:*

$$\mathcal{F}(Gf)(\xi_1,\xi_2) = \overline{\widehat{g}(\xi_1)}f(-\xi_2)e^{2\pi i\xi_1\xi_2}$$

Proof. We calculate directly:

$$\begin{aligned}
\mathcal{F}(Gf)(\xi_1,\xi_2) &= \int_{\mathbb{R}^2} Gf(x,y)e^{-2\pi i(x\xi_1+y\xi_2)}\,dx\,dy\\
&= \int_{\mathbb{R}^2}\int_{\mathbb{R}} f(t)e^{-2\pi iyt}\overline{g(t-x)}\,dt\,e^{-2\pi i(x\xi_1+y\xi_2)}\,dx\,dy\\
&= \int_{\mathbb{R}^2} f(t)e^{-2\pi iy(\xi_2+t)}\int_{\mathbb{R}}\overline{g(t-x)}e^{-2\pi ix\xi_1}\,dx\,dy\,dt\\
&= \int_{\mathbb{R}} e^{-2\pi iy\xi_2}\int_{\mathbb{R}} f(t)e^{-2\pi ity}e^{-2\pi it\xi_1}\overline{\widehat{g}(\xi_1)}\,dt\,dy\\
&=\overline{\widehat{g}(\xi_1)}\int_{\mathbb{R}} e^{-2\pi iy\xi_2}\widehat{f}(\xi_1+y)\,dy = \overline{\widehat{g}(\xi_1)}f(-\xi_2)e^{2\pi i\xi_1\xi_2}
\end{aligned}$$

that is that we want to prove. □

Theorem 1.22. *Let $g\in L^2(\mathbb{R})$ be a Gabor atom. If the system $G(g,\mathbb{Z}\times\mathbb{Z})$ is orthonormal then it is an orthonormal basis.*

Proof. We will apply the Poisson formula to the function $|Gf|^2$. Before calculating we observe that $|Gf|^2 = Gf\overline{Gf}$. Using lemma 1.21 we have that:

$$\mathcal{F}\left(\overline{Gf}\right)(\xi_1,\xi_2) = \widehat{g}(-\xi_1)\overline{f(\xi_2)}e^{-2\pi i\xi_1\xi_2}$$

Therefore:

$$\begin{aligned}
&\mathcal{F}(|Gf|^2)(\xi_1,\xi_2) =\\
&\qquad\int_{\mathbb{R}^2}\overline{\widehat{g}(\alpha_1)}\widehat{g}(\alpha_1-\xi_1)f(-\alpha_2)\overline{f(\xi_2-\alpha_2)}e^{-2\pi i\xi_1\xi_2}e^{2\pi i\xi_1\alpha_2}e^{2\pi i\xi_2\alpha_1}\,d\alpha_1\,d\alpha_2
\end{aligned}$$

Now we apply the Poisson formula to $|Gf|^2$:

$$\begin{aligned}
\sum_{m,n\in\mathbb{Z}}|Gf(m,n)|^2 &= \sum_{m,n\in\mathbb{Z}}\mathcal{F}\big(|Gf|^2\big)(m,n)\\
&= \int_{\mathbb{R}^2}\overline{\widehat{g}(\alpha_1)}\widehat{g}(\alpha_1-m)f(-\alpha_2)\overline{f(n-\alpha_2)}e^{-2\pi imn}e^{2\pi im\alpha_2}e^{2\pi in\alpha_1}\,d\alpha_1\,d\alpha_2\\
&= \sum_{m,n\in\mathbb{Z}}\left(\int_{\mathbb{R}}\overline{\widehat{g}(\alpha_1)}\widehat{g}(\alpha_1-m)e^{2\pi in\alpha_1}\,d\alpha_1\right)\left(\int_{\mathbb{R}}f(-\alpha_2)\overline{f(n-\alpha_2)}e^{2\pi im\alpha_2}\,d\alpha_2\right)\\
&= \sum_{m,n\in\mathbb{Z}}\langle e^{2\pi imt}g(t+n),g(t)\rangle\langle f(t),e^{2\pi imt}f(t+m)\rangle\\
&=\|f\|^2
\end{aligned}$$

where we have used the condition of orthonormality in the last step. We have to remember the definition of $Gf(m,n)$ to see:

$$\sum_{m,n\in\mathbb{Z}}|\langle f(t),e^{2\pi int}g(t+m)\rangle|^2 = \sum_{m,n\in\mathbb{Z}}|Gf(m,n)|^2 = \|f\|^2$$

that is the one that was necessary to prove that $G(g, \mathbb{Z}\times\mathbb{Z})$ is an orthonormal basis. □

In the case of wavelets the regular nets have the form $\{a^j nb + ia^j\}_{j,n\in\mathbb{Z}} \subset \mathbb{R}\times\mathbb{R}^+$, depending on parameters a, b. We will call this nets hyperbolically regular, since they have the property that the hyperbolic distance between the neighboring points of this net is constant. Also they correspond with the extremes of the partition in dyadic cubes of the half-plane, which have constant measure. In this case these have also been the more studied sets. We can give necessary conditions similiar to those that we have given in the case of Gabor transform. If we take an admissible wavelet ψ and we define $c_\psi = \int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{\xi}\, d\xi$ and $\Lambda = \{a^j nb + ia^j\}_{j,n\in\mathbb{Z}}$, we can state, as an example:

Theorem 1.23 (Daubechies)**.** *If* $W(\psi,\Lambda)$ *is a frame for* $L^2(\mathbb{R})$*, then the bounds of the frame fulfill:*

$$A \leqslant \frac{2\pi c_\psi}{b\log a} \leqslant B$$
$$\forall \xi \in \mathbb{R}\setminus\{0\}\ ,\ A \leqslant \frac{1}{b}\sum_{j\in\mathbb{Z}} |\widehat{\psi}(a^j\xi)|^2 \leqslant B$$

It is necessary to observe that, unlike those that it happend in Gabor, here there is not any restriction regarding the density of the net. In general we can justify this fact with the following argument; if $W(\psi,\Lambda)$ is a frame with Λ a regular net with parameters a, b, we define $\widetilde{\psi}(t) = \gamma^{\frac{1}{2}}\psi(\gamma t)$. Then $W\big(\widetilde{\psi}, \widetilde{\Lambda}\big)$ will be a frame with $\widetilde{\Lambda}$ a regular net with parameters $a, \frac{b}{\gamma}$. We can not do this reasoning when ψ is normalized so that $c_\psi = 1$, but either it is not know any result that tells that there are not frames under a determinate density.

Regarding sufficient conditions, we can also give examples:

Theorem 1.24 (Daubechies)**.** *We define:*

$$\beta(u) = \sup_{0\leqslant\xi\leqslant a} \sum_{j\in\mathbb{Z}} |\widehat{\psi}(a^j\xi)||\widehat{\psi}(a^j\xi + u)|$$

and

$$\Delta = \sum_{k\in\mathbb{Z}, k\neq 0} \left[\beta\Big(\frac{k}{b}\Big)\beta\Big(\frac{-k}{b}\Big)\right]^{\frac{1}{2}}$$

If a *and* b *fulfill:*

$$A_0 = \frac{1}{b}\left(\inf_{0\leqslant\xi\leqslant a} \sum_{j\in\mathbb{Z}} |\widehat{\psi}(a^j\xi)|^2 - \Delta\right) > 0$$

and

$$B_0 = \frac{1}{b}\left(\inf_{0\leqslant\xi\leqslant a}\sum_{j\in\mathbb{Z}}|\widehat{\psi}(a^j\xi)|^2 + \Delta\right) < \infty$$

Then $W(g,\Lambda)$ *is a frame for* Λ *a regular net with parameters* a, b. *The constants* A_0 *and* B_0 *are respectively upper and lower bounds of the constants* A *and* B *of the frame.*

Another time, we can find the proofs of these results and a deeper study of the frames of wavelets in [Dau92]. In the wavelet case the orthonormal bases have been much more studied. By one hand we do not have any restriction of the type *Balian-Low.* We can construct orthonormal bases of wavelets with good time-frequency location properties. Moreover we have some structures that allow us to create wavelet bases as good as we want in a simple way.

1.5.1 Multiresolution Analysis.

The multiresolution analysis are some structures introduced by Mallat [Mal98], formed by a series of subspaces with some properties that relate them. These structures allows us to generating orthonormal bases of wavelets in an explicit way. Most of orthonormal wavelets bases that are knew come from these structures that now we will define and comment briefly.

Definition. A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is a **multiresolution approach** if the following six properties are satisfied:

$$\forall(j,k)\in\mathbb{Z}^2,\ f(t)\in V_j \Leftrightarrow f(t-2^jk)\in V_j \tag{1.10}$$

$$\forall j\in\mathbb{Z},\ V_{j+1}\subseteq V_j \tag{1.11}$$

$$\forall j\in\mathbb{Z},\ f(t)\in V_j \Leftrightarrow f\left(\frac{t}{2}\right)\in V_{j+1} \tag{1.12}$$

$$\lim_{j\to\infty} V_j = \bigcap_{j=-\infty}^{\infty} V_j = \{0\} \tag{1.13}$$

$$\lim_{j\to-\infty} V_j = \overline{\left(\bigcup_{j=-\infty}^{\infty} V_j\right)} = L^2(\mathbb{R}) \tag{1.14}$$

$$\text{Exists } \phi \text{ such that } \{\phi(t-n)\}_{n\in\mathbb{Z}} \text{ is an orthonormal basis of } V_0 \tag{1.15}$$

These conditions are not independent among them nor are the minimum that can be demanded, but this is not important now

We will call ϕ the father wavelet or scalar function of the multiresolution analysis. We do not have to confuse ϕ with an admissible wavelet, since in general it will not be. We have to observe that ϕ characterize totally the multiresolution approach.

From this structure, using (1.11), we can affirm that there are closed subspaces $W_{j+1} \subseteq V_j$ such that $W_{j+1} \perp V_{j+1}$. This implies:

$$V_{j-1} = V_j \oplus W_j \tag{1.16}$$

Moreover we obtain that the W_j are orthogonal. Using (1.13), (1.14) and (1.16) we can affirm that:

$$L^2(\mathbb{R}) = \bigoplus_{j=-\infty}^{\infty} W_j \tag{1.17}$$

Using (1.12) we see also that:

$$\forall j, k \in \mathbb{Z},\ f(t) \in W_j \Leftrightarrow f\left(\frac{t}{2^k}\right) \in W_{j+k} \tag{1.18}$$

Using (1.15) we can see that there is $\psi(t)$ such that:

$$\{\psi(t-n)\}_{n\in\mathbb{Z}} \text{ it is an orthonormal basis of} W_0 \tag{1.19}$$

This ψ can be calculated explicitly from ϕ. We join (1.17), (1.18) and (1.19) to see that the set:

$$\mathcal{B} = \left\{ \frac{1}{2^{j/2}} \psi\left(\frac{t-2^j n}{2^j}\right) \right\}_{j,n\in\mathbb{Z}}$$

is an orthonormal basis for $L^2(\mathbb{R})$.

We will call multiresolution analysis to the structure formed by the W_j's and ψ. We can check that ψ is an admissible wavelet. There are many examples of wavelets coming from multiresolution analyses. They are a very useful tool and there is an extensive literature that studies them. We can find the details of this construction using the same notation in [Mal98].

The bases that give the multiresolution analysis correspond with the translations of the wavelet ψ for the regular net:

$$\Lambda = \{2^j n + i2^j\}_{j,n\in\mathbb{Z}} \subseteq \mathbb{R} \times \mathbb{R}^+$$

We have to comment that the wavelets that we construct by the multiresolution analysis are not normalized so that $c_\psi = 1$ and we do not have any contradiction with theorem 1.23.

1.6 Phase Spaces.

The classical tools of the harmonic analysis are only useful to study the regular sets, both in the case of translations or in the Gabor and wavelet transforms. To study the rest of cases we will have to introduce new concepts that allow us to use other tools, as the complex analysis or the theory of Hilbert spaces with reproductive kernel. The phase spaces are sets of functions that will be from great importance during all the work. In an informal way, these spaces are formed by, when an analyzing function is fixed, the continuous transform of all functions.

1.6.1 Translations.

Our goal is to describe a space of functions that is formed by the continuous transform of all functions. Under this so little precise idea the one that we search is a set of functions that allows us to transform proposition 1.1 in results about uniqueness and zeros sets of these spaces.

We think that we have a fixed (candidate to) generator $\varphi \in L^p(\mathbb{R})$. We want to attempt to know for which discrete sets Λ the system $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$. By duality this will be equivalent to look at the scalar product of the translations of φ with functions of the dual space $L^{p'}(\mathbb{R})$. The Hölder inequality says that for every x we can define the following function:

$$Tf(x) = \int_{\mathbb{R}} f(t)\overline{\varphi(t-x)}\,dt$$

when $f \in L^{p'}(\mathbb{R})$. Moreover it is bounded and it is continuous. In this way we can think that Tf is the continuous transform of f for the analyzing function φ. We have to notice that we define the continuous transform of the functions of the dual space of that which we want to generate. This is consistent with the one that we have made in the definition of the Gabor and wavelet continuous transform, since $L^2(\mathbb{R})$ is a Hilbert space and it is its own dual.

Definition. Given a function $\varphi \in L^p(\mathbb{R})$, $1 \leqslant p < \infty$ we define its **phase space** $H_\varphi = H$ as:

$$H_\varphi = \left\{ F : \exists f \in L^{p'}(\mathbb{R}) \text{ with } F(x) = \int_{\mathbb{R}} f(t)\overline{\varphi(t-x)}\,dt = \langle f(t), \varphi(t-x)\rangle \right\}$$

where p' is the dual exponent of p.

These spaces will be very useful to study our problem. There is a very direct relationship between the properties of generation of $T(\varphi, \Lambda)$ and the values that a function $F \in H$ can take in the points of Λ. They are necessary a pair of definitions to formalize these ideas.

Definition. Given a vectorial space of continuous functions H we will say that a discrete set Λ is a **set of uniqueness** for H if the values of any $F \in H$ in Λ determines F completely.

This is equivalent to $F(\lambda) = 0$ for every $\lambda \in \Lambda$ implying F identically 0.

Definition. Given a vectorial space of continuous functions H we will say that a discrete set Λ is a **zero set** of H if there is a function $F \in H$ such that $F(\lambda) = 0$ for every $\lambda \in \Lambda$ and $F(x) \neq 0$ if $x \notin \lambda$. That is, F cancels in Λ and in no more place.

The following results, although simple, will be from great utility, and they put of manifest the great importance that the phase spaces have in the study of generators by translations.

Proposition 1.25. *Let H be the phase space for translations of a function $\varphi \in L^p(\mathbb{R})$. $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$ if and only if Λ is a set of uniqueness for H.*

Proof. This is trivial if we observe that the equivalence of the end of the definition is the duality condition 1.1. ☐

Proposition 1.26. *Let H be the phase space by translations of a function $\varphi \in L^p(\mathbb{R})$. $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$ if and only if Λ is not included in any zero set of H.*

Proof. If Λ belongs to a zero set there is $f \in L^{p'}(\mathbb{R})$ such that $\langle f, \varphi_\lambda \rangle = 0$ for every $\lambda \in \Lambda$ and $f \neq 0$ (Tf can cancel in more points). The duality condition 1.1 says that in this case $T(\varphi, \Lambda)$ can not generate $L^p(\mathbb{R})$. This condition also says that if $T(\varphi, \Lambda)$ does not generate $L^p(\mathbb{R})$, Λ belongs to a zero set of H. ☐

These two results, although we have proved them in a different way, are equivalents. If we can know the sets of uniqueness or of zeros of the phase space we have solved the problem for a fixed φ. This will not be possible normally. Moreover it is necessary to comment that if Λ is included in a zero set, $T(\varphi, \Lambda)$ can then not generate. But his does not mean that Λ is a zero set. This will be anothe obstacle in our study.

The first step to characterize these sets is to describe the phase spaces, or more that to describe them, to recognize them. This will not be possible in general. But even in the best cases we will not have finished the work, since it will be necessary to know afterwards which are its sets of uniqueness (or of zeros). Characterizations of these will be simple almost never. The cases in that we will be able to solve this problem will be those that the phase space has very good properties. If, for example, this space is formed by analytical functions (or even it is a known space) we will be able to attempt it, but this is not any guarantee of exit. We will see in the next chapters that with the Poisson function as well as with the Gaussian one it appears spaces of analytic functions, and that in the first case we characterize the sets of uniqueness and in the second we just can give partial results.

When it is difficult always to describe the phase space we will have the option to include it in some more treatable space. Or also we will be able to see that it contains some set of functions that we know well. This way to work will not be able to solve completely the problem, but it will help to giving partial results. That is, we will often use the following results.

Proposition 1.27. *Let H be the phase space by translations of a function $\varphi \in L^p(\mathbb{R})$. We suppose that there is a space (set) of functions E such that $H \subseteq E$. If Λ is a set of uniqueness for E, then $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$.*

Proof. If Λ is of uniqueness for E, the values in Λ determines all the functions of E, and in particular all the functions of H. ☐

Proposition 1.28. *Let H be the phase space by translations of a function $\varphi \in L^p(\mathbb{R})$. We suppose that there is a space (set) of functions E such that $E \subseteq H$. If Λ is contained in a zero set for E, then $T(\varphi, \Lambda)$ does not generate $L^p(\mathbb{R})$.*

Proof. As Λ is in a zero set for E, there is $F \in E$ such that $F(\lambda) = 0$ for every $\lambda \in \Lambda$. This F also belongs to H, and therefore Λ is also in a zero set for H. ☐

1.6.2 Gabor.

In this case, as we are in a Hilbert space, it will be simpler to define the phase space.

Definition. Let $g \in L^2(\mathbb{R})$. We define the **phase space** of the Gabor atom g, H_g (or H) as:

$$H_g = \left\{F(z) \in L^2(\mathbb{C}) : \exists f \in L^2(\mathbb{R}) \text{ with } F(z) = Gf(z) = \langle f, g_z \rangle\right\}$$

This space has good properties. Theorem 1.14 says that all its functions are square integrable in $\mathbb{C}$. Moreover it is a Hilbert subspace and all its functions are continuous. Precisely it is a Hilbert space with reproductive kernel.

Proposition 1.29. *The phase space H_g of a Gabor atom $g \in L^2(\mathbb{R})$ is a Hilbert subspace of $L^2(\mathbb{C})$ that is characterized for the following reproductive kernel:*

$$k_g(z, z_0) = k(z, z_0) = k_{z_0}(z) = \langle g_{z_0}, g_z \rangle$$

That is, $F \in H_g$ if and only if:

$$F(z_0) = \int_{\mathbb{C}} F(z)\overline{k(z, z_0)}\, dx\, dy \tag{1.20}$$

The Gabor transform with respect to g is an isomorphism of $L^2(\mathbb{R})$ in H_g.

Proof. We introduce the reconstruction formula in the definition of the transform:

$$Gf(z_0) = \int_{\mathbb{R}} \left(\int_{\mathbb{C}} Gf(z) g_z(t)\, dx\, dy \right) \overline{g_{z_0}(t)}\, dt$$

Exchanging the integrals we have:

$$Gf(z_0) = \int_{\mathbb{C}} Gf(z) \left(\int_{\mathbb{R}} g_z(t)\overline{g_{z_0}(t)}\, dt \right) dx\, dy$$

Therefore k replays all the functions of the space. As it belongs to this space we can affirm that it is its reproductive kernel.

The isomorphism is deduced from theorem 1.14, since H_g inherits the norm from $L^2(\mathbb{C})$ and $\|f\| = \|Gf\|$. ☐

Remark. We can think in the reproduction formula (1.20) as a convolution. The argument is the following one, if we take the definition of k we have that:

$$k(z, z_0) = \overline{\langle g_z, g_{z_0}\rangle} = \overline{e^{2\pi i x(y-y_0)}\langle g, g_{z_0 - z}\rangle} = \overline{e^{2\pi i x(y-y_0)} k(z_0 - z)}$$

where we have defined $k(z) = \langle g, g_z\rangle = k(0, z)$. If we introduce this to the reproduction formula (1.20), this takes the form:

$$F(z_0) = \int_{\mathbb{C}} F(z) e^{2\pi i x(y-y_0)} k(z_0 - z)\, dx\, dy$$

that is practically a convolution, except for the exponential factor, that has module 1. This class of convolutions are called twisted convolutions. Writing the reconstruction formula in this way will be much more usefull.

We have to take into account that we are using k to designate the reproductive kernel thought in two variables as for the kernel of convolution, which only depends on a variable. We have to be a little attentive to avoid errors, although normally there will not be confusion.

As the space in that we will work is formed by continuous functions and it has reproductive kernel, we can sample the functions in any point. Given a set of points of $\mathbb{C}$ we can consider, for each $F \in H_g$, the succession of values that F takes in this set.

Definition. Given a discrete set $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{C}$ we will say that Λ is a **sampling set** for H_g if there are constants $A, B > 0$ such that:

$$A\|F\|^2 \leqslant \sum_{n\in\mathbb{Z}} |F(\lambda_n)|^2 \leqslant B\|F\|^2 \quad \forall F \in H_g$$

These sets are very important since they correspond with frames:

Proposition 1.30. *Let $\Lambda \subset \mathbb{C}$ be a discrete set and $g \in L^2(\mathbb{R})$ a Gabor atom. $G(g,\Lambda)$ is a frame of $L^2(\mathbb{R})$ if and only if Λ is a sampling set for H_g.*

Proof. This fact is easy to prove observing that $Gf(z) = \langle f, g_z\rangle$ and that $\|Gf\| = \|f\|$. As we have bijective correspondence among H_g and $L^2(\mathbb{R})$ by the Gabor transform, we can write:

$$\sum_{\lambda\in\Lambda} |Gf(\lambda)|^2 = \sum_{\lambda\in\Lambda} |\langle f, g_\lambda\rangle|^2$$

for every $f \in L^2(\mathbb{R})$ or for every $Gf \in H_g$. Therefore the frame condition and that of sampling are equivalents. ☐

With this result, to describe the sampling sets of H_g is equivalent to describe the sets Λ for which $G(g,\Lambda)$ is a frame. And if we can demonstrate that for a certain H_g there are sampling sets, there are then sets Λ for which $G(g,\Lambda)$ is a frame.

Related with the sampling sets we have the one that we could call the dual concept, which are the interpolation sets:

Definition. Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{C}$ be a discrete set. We will say that Λ is an **interpolation set** for H_g if for every succession $\{a_n\}_{n\in\mathbb{N}} \in l^2$ exists a function $F \in H_g$ such that:

$$F(\lambda_n) = a_n \qquad \forall n \in \mathbb{N}$$

We say that these two concepts are dual since if we look at the sampling operator in H_g for a set $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$:

$$\begin{aligned} M : H_g &\longrightarrow l^2 \\ F &\longmapsto M(F) = \{F(\lambda_n)\}_{n\in\mathbb{N}} \end{aligned}$$

This operator is injective if Λ is a sampling set, and exhaustive if it is an interpolation set. These operators correspond with the synthesis operators in $L^2(\mathbb{R})$ for the sets of functions $G(g,\Lambda)$. With this duality it is easy to see that Riesz bases correspond with the sets that are sampling and interpolation sets on time.

As we have isomorphism between $L^2(\mathbb{R})$ and H_g also we have the following equivalence, which is evident from the definitions:

Proposition 1.31. *Let $g \in L^2(\mathbb{R})$ be a discrete set and $\Lambda \subset \mathbb{C}$. $G(g, \Lambda)$ is a frame (or a Riesz basis) of $L^2(\mathbb{R})$ if and only if $\{k_\lambda\}_{\lambda \in \Lambda}$ is a frame (or a Riesz basis) of H_g.*

In view of these results, it is clear that a well knowing of the phase space of a Gabor atom will be very useful to prove the existence and to describe the sets that can cause a frame or a Riesz basis. Now we want to find atoms with a phase space with good properties. In general these phase spaces are already better than a general Hilbert space. We only find continuous functions and moreover they are characterized by a reproductive kernel, which gives a lot of structure. Informally speaking, we can say that all what it happens to the kernel inherits the whole space.

In spite of these good properties, we will not be able to improve know results for general Gabor atoms. What we will make is to ask to its reproductive kernel (that it is equivalent to put conditions about g) that it has good integration and discretization properties.

Definition. We define the **Feichtinger algebra** as the set of functions $g \in L^2(\mathbb{R})$ such that:

$$k(z) = Gg(z) = \langle g, g_z \rangle = \int_{\mathbb{R}} g(t) e^{-2\pi i t y} \overline{g(t-x)}\, dt \in L^1(\mathbb{C})$$

We will designate this set as $\mathcal{A}$.

These are the functions that we will use for analyzing, since they will cause phase spaces with the wished properties. This set was introduced by Feichtinger and Gröchenig in the theory of representations studied in [FeG89], [FeG89-2] and [Gro91]. Among the most interesting properties of this set we can declare that $\langle f, g_z \rangle \in L^1(\mathbb{C})$ if and only if $f, g \in \mathcal{A}$. This fact gives a simple way to recognize the algebra if we know any of its functions.

A trivial example of function that belongs to the Feichtinger algebra is the Gaussian function:

$$\phi(t) = 2^{\frac{1}{4}} e^{-\pi t^2}$$

This function will play a very important paper in this work. The advantage of this function is that we can calculate explicitly the reproductive kernel of its phase space, which is $k(z) e^{-\frac{\pi}{2}|z|^2} e^{-\pi i x y}$. Also we will be able to describe the phase space very well. It will appear the Fock space of entire functions, where we will be able to apply the techniques of the complex analysis. As a matter of fact in this particular space the sampling and interpolation sets had already been described, and therefore the problem that we consider is already solved.

For discretizate the reproduction and reconstruction formulas it is necessary to be able to control the punctual values by the continuous values of the functions. For making this we have to define the one that we will call the local maximal function, which gives us a greater control of the function:

Definition. Given a continuous function F defined in $\mathbb{C}$ we define its **local maximal function** as:

$$MF(z) = \sup_{w \in B(z,1)} |F(w)|$$

where $B(z,1)$ is the ball of center z and radius 1 in $\mathbb{C}$ using the Euclidean distance.

This function amplifies the exceptional points of the function F. Informally speaking, if F is big in a point MF is big in a set of positive measure.

The spaces for the that we will be able to discretize the reproduction formulas are those for which the maximal function of the reproductive kernel is integrable. Surprisingly this happens if $g \in \mathcal{A}$.

Proposition 1.32. *Let $g \in L^2(\mathbb{R})$ be a Gabor atom and k the reproductive kernel of its phase space. If $g \in \mathcal{A}$ then Mk is integrable in $\mathbb{C}$. That is, Mk is integrable if k is.*

This result confirms that the set $\mathcal{A}$ is the suitable one to work.

The ideal situation belongs if we can describe perfectly the phase space in any way that allows us to know its sets of sampling. These sets have been studied in the spaces of harmonic or holomorphic functions. In some of them complete characterizations of its interpolation and sampling sets have been given. But as we will see later on, we will only be in this case when the Gabor atom is the Gaussian function. In this case the phase space corresponds in direct way with the Fock space and they share the sampling and interpolation sets.

1.6.3 Wavelets.

Regarding wavelets we find a similar situation to that we have commented on here for Gabor. We have to take into account that now the phase spaces will be defined in $\mathbb{R} \times \mathbb{R}^+$ instead of $\mathbb{C}$.

Definition. Let $\psi \in L^2(\mathbb{R})$ be an admissible wavelet. We define the **phase space** of the wavelet ψ, H_ψ (or H) as:

$$H_\psi = \left\{ F(z) \in L^2(\mathbb{R} \times \mathbb{R}^+) : \exists f \in L^2(\mathbb{R}) \text{ with } F(z) = Wf(z) = \langle f, \psi_z \rangle \right\}$$

As in the Gabor case, this space is also formed by continuous functions, but now in $L^2\left(\mathbb{R} \times \mathbb{R}^+, \frac{dx\,dy}{y^2}\right)$. Also it is a space with reproductive kernel:

Proposition 1.33. *The phase space H_ψ of an admissible wavelet $\psi \in L^2(\mathbb{R})$ is a Hilbert subspace of $L^2(\mathbb{R} \times \mathbb{R}^+)$ that is characterized by the following reproductive kernel:*

$$k_\psi(z, z_0) = k(z, z_0) = k_{z_0}(z) = \langle \psi_{z_0}, \psi_z \rangle$$

That is, $F \in H_\psi$ if and only if:

$$F(z_0) = \int_{\mathbb{R}\times\mathbb{R}^+} F(z)\overline{k(z, z_0)}\frac{dx\,dy}{y^2} \tag{1.21}$$

and the wavelet transform with respect to ψ is an isomorphism from $L^2(\mathbb{R})$ to H_ψ.

The proof is identical to that of theorem 1.29 and we will not repeat it.

The first significant difference between both transforms is that in this case (1.21) is a true convolution, but with respect to the hyperbolic (or affine) group. If we define $z_0 \cdot z = y_0 z + x_0$ (where the juxtaposition is the usual product of $\mathbb{C}$) we can give a group structure to $\mathbb{R} \times \mathbb{R}^+$. In this group the identity will be $i = (0, 1)$ and $z_0^{-1} \cdot z = \frac{z - z_0}{y_0}$. This group is not commutative. Its invariant measure (by translations with respect to the group) is different for the right that for the left. We will work only with left translations. In this case the invariant measure will be $dm(z) = \frac{dx\,dy}{y^2}$. In this way we can write:

$$k(z, z_0) = \langle \psi_{z_0}, \psi_z \rangle = \langle \psi_{z^{-1}\cdot z_0}, \psi \rangle = \overline{k(z^{-1} \cdot z_0)}$$

where we have defined $k(z) = \langle \psi, \psi_z \rangle = k(0, z)$. As in the Gabor case, we introduce this into the reproduction formula of (1.21):

$$F(z_0) = \int_{\mathbb{R}\times\mathbb{R}^+} F(z)k(z^{-1} \cdot z_0)\frac{dx\,dy}{y^2}$$

where we find a formula of non commutative convolution because the group is not commutative, but that will have most of the properties of the usual convolutions.

When giving conditions of integrability or describing the maximal function it will be important that the structure of $\mathbb{R} \times \mathbb{R}^+$ is different of that of $\mathbb{C}$, but everything is until then identical.

Definition. Given a discrete set $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R} \times \mathbb{R}^+$ we will say that Λ is a **sampling set** for H_ψ if there are constants $A, B > 0$ such that:

$$A\|F\|^2 \leqslant \sum_{n\in\mathbb{Z}} |F(\lambda_n)|^2 \leqslant B\|F\|^2 \quad \forall F \in H_\psi$$

Proposition 1.34. *Let $\Lambda \subset \mathbb{R}\times\mathbb{R}^+$ be a discrete set and $\psi \in L^2(\mathbb{R})$ an admissible wavelet. $W(g, \Lambda)$ is a frame of $L^2(\mathbb{R})$ if and only if Λ is a sampling set for H_ψ.*

The proof is identical to the Gabor case. The definition of interpolation sets and the symmetry among the bases generated by ψ and the ones generated by the reproductive kernel also are identical.

Definition. Let $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}} \subset \mathbb{R}\times\mathbb{R}^+$ be a discrete set. We will say that Λ is an **interpolation set** for H_ψ if for every succession $\{a_n\}_{n\in\mathbb{N}} \in l^2$ there exists a function $F \in H_\psi$ such that:

$$F(\lambda_n) = a_n \qquad \forall n \in \mathbb{N}$$

Proposition 1.35. *Let $\psi \in L^2(\mathbb{R})$ be an admissible wavelet and $\Lambda \subset \mathbb{R}\times\mathbb{R}^+$ a discrete set. $W(g,\Lambda)$ is a frame (or a Riesz basis) of $L^2(\mathbb{R})$ if and only if $\{k_\lambda\}_{\lambda\in\Lambda}$ is a frame (or a Riesz basis) of H_ψ.*

When asking for conditions of integrability of the kernel, in the wavelet case we have an equivalent notion to the Feichtinger algebra considering those wavelets for which its kernel is an integrable function:

Definition. We define the set of **wavelets with integrable kernel** as those admissible wavelets $\psi \in L^2(\mathbb{R})$ such that:

$$k(z) = W_\psi(z) = \langle \psi, \psi_z\rangle = \int_{\mathbb{R}} \psi(t) y^{-\frac{1}{2}} \overline{\psi(\frac{t-x}{y})}\, dt \in L^1\big(\mathbb{R}\times\mathbb{R}^+, \frac{dx\,dy}{y^2}\big)$$

We will designate the set of all the wavelets with integrable kernel as $\mathcal{B}$.

This set preserves some of the properties of the Feichtinger algebra, but not all of them. For example, if $\langle f, \psi_z\rangle \in L^1(\mathbb{R}\times\mathbb{R}^+)$ for a $\psi \in \mathcal{B}$, then $\langle f, \phi_z\rangle \in L^1(\mathbb{R}\times\mathbb{R}^+)$ for any other $\phi \in \mathcal{B}$. But however we can not insure that $\langle \phi, \psi_z\rangle \in L^1(\mathbb{R}\times\mathbb{R}^+)$. This is because in the Gabor case $Gf(z)$ is integrable if and only if $Gf(-z)$ is it. However in wavelets $Wf(z)$ can be integrable and $Wf(z^{-1})$ not. This is one of the differences of the structure of the group.

Regarding discretizations, it is also necessary to define the local maximal function. We have to take into account that we have to use the hyperbolic distance.

Definition. Given a continuous function F defined in $\mathbb{R}\times\mathbb{R}^+$ we define its **local maximal function** as:

$$MF(z) = \sup_{w\in B(z,1)} |F(w)|$$

where $B(z,1)$ is the ball of center z and radius 1 in $\mathbb{R}\times\mathbb{R}^+$ using the hyperbolic distance.

The set of wavelets such that the local maximal function of its kernel is integrable differs from $\mathcal{B}$, and we will call it $\mathcal{MB}$.

As in the Gabor case, we will want to describe as well as we can the phase space of a wavelet, to be able to describe its sampling and interpolation sets. Casually we find a set of admissible wavelets for which the phase space corresponds with a space of holomorphic functions. The functions are the Pisson-type wavelets:

$$\psi(t) = \frac{1}{c_\alpha}(t+i)^{-\frac{\alpha+1}{2}}$$

and theirs phase spaces correspond with the Bergman spaces of the half-plane. These wavelets are not real, but this is only a technical question. We can always take their real part or their imaginary part and give the same results, since the real part of a holomorphic function determines directly its imaginary part.

Under certain conditions also we will have a result of uniqueness, although it will not be so good as in the Gabor case.

Part I

Generators by translations.

Chapter 2

Generators for $L^1(\mathbb{R})$.

We have commented in the preliminaries that when studying the generator systems in the way $T(\varphi, \Lambda)$ it appear different questions. Two first that we can consider are which sets Λ we will be able to use, and which functions φ will allow us to generate. These questions can be studied separately, but they are very related.

The characterization of the sets Λ that accept generators for $L^1(\mathbb{R})$ we can be found in [BOU06] and [Bru06]. In this pair of articles these sets are described in terms of the densities of Beurling-Malliavan, of the spectral radius of Λ or as sets of uniqueness of certain classes of functions. In the first section we will give the concrete result.

When we regard the characterization of the generators we do not have so good news. This problem has been less studied and it is not know any complete description. Wiener's theorem already gives a necessary condition that for the time being is the only one that is known. The examples and results that are known for the time being lead us to bringing up the following conjecture:

Conjecture. Let φ be a generator for $L^1(\mathbb{R})$. Then $\widehat{\varphi}(\xi) \neq 0$ for every $\xi \in \mathbb{R}$ and:

$$\int_{\mathbb{R}} \frac{\log|\widehat{\varphi}(\xi)|}{1+\xi^2} = -\infty$$

This last integral is known as the logarithmic integral of $\widehat{\varphi}$. It turns up in several fields of the mathematical analysi. For example, the monograph [Koo92] is dedicated to study this integral. There are several reasons that make us think that this conjecture is a certain. On a part we have two special cases (generators φ with decresing Fourier transform or sets Λ contained in a half–line) in that it is already known that the divergency of the integral is necessary. We devote a section to explaining these cases briefly.

Moreover, all the examples of generators for $L^1(\mathbb{R})$ that are known fulfill this condition. As a matter of fact they are part of a special class of generators, which we will call almost–analytical. Another section of this chapter is dedicated to studying these generators. For this set we will be capable of giving sufficient conditions closer to the necessary ones deals in [Bru06] (where already they were introduced) when constructing them.

We will also study another special class of generators, called analytical, and that they are a subset of the former ones. These are relevants because they are simple to construct and they have phase spaces with good properties.

Another interesting problem is to attempt to describe which sets Λ fulfill that $T(\varphi, \Lambda)$ is a generator system, when we fix φ. We will give some sufficient conditions when the generator with what we work is almost–analytical or analytical.

The two next chapters take care to study this problem in very special cases. In chapter 3 we explain the characterization of the sets Λ that it cause a generator system for the Poisson function that is obtained in [BrM07]. We will generalize afterwards this description to other functions closer to the Poisson function.

The last chapter of this part studies the same problem, but this time for the Gaussian function. Some partial results were already known for this function. We will tune more the necessary as well as sufficient conditions so that a set gives place to a generator system with the Gaussian, but we do not have any complete description.

2.1 Characterization of the sets of generators.

We have already commented that one of the first questions that we can consider is when, given a discrete set $\Lambda \subset \mathbb{R}$, there is a Λ-generador of $L^1(\mathbb{R})$. This question is solved in [Bru06] and [BOU06]. The theorem that they give is the following one:

Theorem 2.1 (Bruna, Olevskii, Ulanovskii)**.** *For a discrete set $\Lambda \subset \mathbb{R}$, the following conditions are equivalents:*

a) There is a Λ-generador φ for $L^1(\mathbb{R})$.

b) The spectral radius of Λ is $+\infty$.

c) Λ is a set of uniqueness for an almost–analytical class $C\{M_n\}$ with $M_n^{\frac{1}{n}} \to \infty$.

d) Λ is a set of uniqueness for a generalized Brenstein class.

Remark. The generalized Brenstein classes [BOU06] and the almost–analytics ones are very closed. In both cases, if Λ is a set of uniqueness for one of these classes we can then achieve a Λ-generador in the corresponding class.

The spectral radius of Λ is defined as the following supreme:

$$\sup\big\{T > 0 : \{e^{-i\lambda\xi}\}_{\lambda\in\Lambda} \text{ generates } L^2(0,T)\big\}$$

That is, it mesures the capacity of generation that has the set of exponentials with parameters in Λ. For example, the spectral radius of $\mathbb{Z}$ is 2π. This coincides with the density of Beurling-Mallavin. In this way we have a more geometric formula more to calculate the spectral radius. We will not enter in details about this subject.

We will be interested in describing the functions that can make the role of a generators of $L^1(\mathbb{R})$. This problem is difficult of solving directly. In this section we will limit ourselves to studying a special type of generators. The sets that accept generators are described in 2.1 in terms of uniqueness sets of certain classes of functions. The one that we can make is to search conditions on φ so that the phase space of this function is included in these classes. Afterwards we will be able to apply the proposition 1.27 to prove that there are discrete sets Λ such that $T(\varphi,\Lambda)$ generates $L^1(\mathbb{R})$. We will focus on the almost–analytical classes. Until now, all the generators of $L^1(\mathbb{R})$ that are known are contained in these classes. We will call them almost–analytical generators.

Wiener's theorem 1.2 gives the first condition so that a function φ is a generator of $L^1(\mathbb{R})$. In the case $L^1(\mathbb{R})$ this theorem tells us that the function φ has to have Fourier transform different of zero in every point. This condition is necessary and independent of which class of generator we search, therefore we will always have to have it present.

The rest of conditions that we will give will be conditions of decrease of the Fourier transform. We will find that as more decrease has the transform, more sets Λ will accept to be able to generate $L^1(\mathbb{R})$. It is important to comment that for a lot that decreases we will always have to have present that it can not take value zero. This says, for example, that we can not think of generator with Fourier transform with compact support.

The conditions of decrease of the transform correspond in an informal way with conditions of regularity of the function φ. This makes the generators that we will work to have very good conditions of derivation. As an example of this fact we will comment that the almost–analytical generators are always C^∞.

When we speak about necessary conditions here it is important to comment that these will not be necessary conditions for φ to be a Λ-generador, but they will be necessary so that it is a generator of a certain type (precisely almost–analytical). For the time being necessary conditions for a function to be a generator of $L^1(\mathbb{R})$ in general have not been found (except for the Wiener's theorem), and as we have commented formerly, all the generators that are known are from the same type that we will describe.

2.2 Divergent Logarithmic Integral.

Theorem 2.5 will say that if φ is an almost–analytical generator, then:

$$\int_{\mathbb{R}} \frac{\log|\widehat{\varphi}(\xi)|}{1+\xi^2}$$

In fact it will say more. Not just this integral is divergent, but there is a logarithmically convex majorant with the same property. This condition is necessary so that φ is an almost–analytical generator. We can propose also if it is necessary in general. This is for the time being an open problem, but there are two cases in that it is a certain. This makes us think that the conjecture will be a certain in general. Moreover one of the cases puts restrictions on the function and the other one about the set.

We go to comment on them now separately. The first case corresponds to generators φ such that $\widehat{\varphi}$ is even and decresing in the positive semiaxis. This case is already commented on in [Bru06], and we reproduce here the argumentation without important changes. The idea is the following one, we suppose that for any ε there is a function $f \in L^\infty(\mathbb{R})$ such that $f * \check{\varphi}$ is suported in $(-\varepsilon, \varepsilon)$. If this happens φ can not be a generator, since Λ should be a set dense everywhere and could not be discrete. Now it will be necessary to search necessary conditions so that this does not happen. The entity criterion is given by the theory of multipliers of *Beurling-Malliavin*. We need first a definition.

Definition. We say that a weight $\omega(\xi) \geqslant 1$ **accepts multipliers** if there is a entire function G_ε of arbitrary exponential type ε such that $\omega(\xi)G_\varepsilon(\xi)$ is bounded or belongs to $L^p(\mathbb{R})$.

Proposition 2.2 (Bruna-Ulanovski)**.** *If $|\widehat{\varphi}|^{-1}$ accepts multipliers, then φ is not a generator.*

Proof. For $\varepsilon > 0$, let G_ε be the function that gives the definition. Then $G_\varepsilon(\xi) = h(\xi)\widehat{\varphi}(\xi)$ with h bounded. Then:

$$G_\varepsilon(\xi)\left(\frac{\sin\varepsilon\xi}{\varepsilon\xi}\right)^2 = h(\xi)\left(\frac{\sin\varepsilon\xi}{\varepsilon\xi}\right)^2\widehat{\varphi}(\xi)$$

is also bounded. As $h(\xi)\left(\frac{\sin\varepsilon\xi}{\varepsilon\xi}\right)^2$ is in $L^1(\mathbb{R})$, we can write $f \in L^\infty(\mathbb{R})$ as like the Fourier transform of a function. This says that $f * \check{\varphi}$ is the Fourier transform of

$G_\varepsilon(\xi)\left(\frac{\sin\varepsilon\xi}{\varepsilon\xi}\right)^2$, that has exponential type 2ε. This makes $f*\check{\varphi}$ to be supported in $(-2\varepsilon,2\varepsilon)$, which proves the theorem. $\square$

So it is a necessary condition for a weight to accepts multipliers:

$$\int_{\mathbb{R}}\frac{\log\omega(\xi)}{1+\xi^2}<\infty$$

This does not help us, although it makes us believe that the conjecture is true. But it results that if ω is increasing and even this condition is also sufficient. The details can be found in [Koo92]. Therefore, if $|\widehat{\varphi}|$ is a decresing function in the positive semiaxis, the divergency of the logarithmic integral is a necessary condition so that φ is a generator. It is not difficult to see that it is not necessary to require that the transformed one is even.

The other case corresponds to sets Λ contained in the positive half–line. We suppose that we have a function $\varphi\in L^2(\mathbb{R})$ such that

$$\int_{\mathbb{R}}\frac{\log|\widehat{\varphi}(\xi)|}{1+\xi^2}<\infty$$

and $\widehat{\varphi}(\xi)\neq 0$ almost for every ξ. Then there is a function θ of $H^2(\mathbb{R})$ (the Hardy space of the half–plane restricted to the real line) such that $|\widehat{\varphi}|=|\theta|$. Form theorem 1.5 $T(\varphi,\Lambda)$ generates $L^2(\mathbb{R})$ if and only if $T(\widehat{\theta},\lambda)$ also generates $L^2(\mathbb{R})$. But $\widehat{\theta}(t)=0$ for every $t<0$, and therefore we can not generate every $L^2(\mathbb{R})$ only with positive translations, since any linear combination will cancel in the negatives.

For $L^1(\mathbb{R})$ we have to use the theorem 1.4 and to make convolution with $\widehat{P}$ (the Fourier transform of the Poisson function), which has finite logarithmic integral.

This is a particular case of the study of the invariant subspaces for translations of $H^p(\mathbb{R})$ and $H^p(\mathbb{D})$. This problem has been studied by many people, and today still continues being an important research field. The monograph [Nik86] is an excelent reference about this problem.

2.3 Almost–analytical Generators.

Definition. An **almost–analytical class** $C\{M_n\}$ associate to the positive numbers (M_n) consists in the set of functions $f\in C^\infty(\mathbb{R})$ such that:

$$\left|f^{n)}(x)\right|\leqslant C_f\beta_f^n M_n,\qquad n=0,1,2,\dots,x\in\mathbb{R}$$

and in more it is fulfilled than $f^{n)}(0)=0\,\forall n$ it implies $f=0$ for every $f\in C\{M_n\}$. This is the case if and only if:

$$\sum_{n=1}^{\infty}\frac{1}{M_n^{\frac{1}{n}}}=\infty$$

We can assume that the M_n are logarithmically convex. That is, $M_0 = 1$, $M_n^2 \leqslant M_{n-1}M_{n+1}$.

Remark. If we take the log–convex normalization we have that the succession $M_n^{\frac{1}{n}}$ is increasing. When $\left(\frac{M_n}{n!}\right)^{\frac{1}{n}}$ is a bounded succession we have the analytical classes.

When $f^{n)}(0) = 0\,\forall n$ does not imply $f = 0$ for the functions of the class we find the *Denjoy-Carleman* classes.

These classes have typically discrete sets of uniqueness. It is for this fact that they are interesting for us. There is not any characterization of these sets, however, for a fixed general almost–analytical class. Theorem 3 of [Bru06] gives a sufficient condition that we replay next.

Theorem 2.3 (Bruna). *An almost–analytical class is $C\{M_n\}$. We write:*

$$\overline{M}[k] = \sum_{n=1}^{k} \frac{M_{n-1}}{M_n}$$

Let Λ be a discrete set and $n_\Lambda(r) = |\Lambda \cap [-r, r]|$ its counting function. If Λ fulfills that:

$$\limsup_{r\to\infty} \frac{\overline{M}[n_\Lambda(r)]}{r} = \infty \tag{2.1}$$

then Λ is a set of uniqueness for $C\{M_n\}$

Moreover, if $f \in C\{M_n\}$ fulfills that $\|f^{n)}\|_\infty \leqslant C_f\beta_f^n M_n$. Then:

$$\overline{M}[n_f(r)] \leqslant 2e\beta_f r$$

where $n_f(r)$ is the counting function of the zeros of f.

Proof. The idea is to reclimb the *Bang's lemma* [Ban53]. As it is presented in [NSV04] this lemma says that if $f \in C^\infty[-1, 1]$ fulfills that $\|f^{n)}\|_\infty \leqslant M_n$, the cardinal of its zero set (counting multiplicity) can not overcome its *Bang's number*. This number is defined as the bigger N such that:

$$\sum_{\log\|f\|_\infty^{-1} < n \leqslant N} \frac{M_{n-1}}{M_n} < 2e$$

For $f \in C\{M_n\}$, it is fulfilled that $\|f^{n)}\|_\infty \leqslant C_f\beta_f^n M_n$. If $n_f(r)$ is the counting function of the zeros of f, reclimbing the lemma of Bang we have that:

$$\overline{M}[n_f(r)] \leqslant 2e\beta r$$

This already proves the second part of the theorem.

For the first part it is only necessary to observe that if Λ fulfills (2.1) it can not exist any function that cancels in this set and this demonstrates that it is of uniqueness. □

In [Hir50] they also study the zero sets, but focusing on analytical classes. The idea is to generalize a result that it says that if f fulfills:

$$|F^{n)}(x)| \leqslant C_F k^n n!$$

then:

$$\limsup_{r\to\infty} \frac{2\log n_\Lambda(r)}{\pi t} \leqslant k$$

The functions that fulfill this type of bound have an analytical extension on the band $|\Im z| < \frac{1}{k}$. These will be the class of functions that will appear in the following section. If we look at these classes, 2.3 is already giving a similar bound, since for $M_n = n!$ $\overline{M}[k] \sim \log k$, but we have to take care with the constants that can appear. The advantage of 2.3 lies in that it is applicable to general almost–analytical classes.

The way to see the connection among these classes and to construct generators is the following one. We start with a discrete set Λ that accepts generators. Theorem 2.1 says that Λ is of uniqueness for an almost–analytical class $C\{M_n\}$. The way to obtain the generator is to construct a function $\varphi \in L^1(\mathbb{R})$ such that:

$$L^\infty(\mathbb{R}) * \check{\varphi} \subset C\{M_n\}$$

where $\check{\varphi}(t) = \overline{\varphi(-t)}$. We suppose that this is fulfilled. Then the phase space of φ is included in $C\{M_n\}$. We can use now proposition 1.27 or directly we take a function $h \in L^\infty(\mathbb{R})$ such that:

$$\int_{\mathbb{R}} h(t)\overline{\varphi(t-\lambda)}\,dt = 0, \quad \lambda \in \Lambda$$

The function:

$$Th(x) = \int_{\mathbb{R}} h(t)\overline{\varphi(t-x)}\,dt$$

belongs to $C\{M_n\}$. As Λ is of uniqueness for this class this implies that $Th = 0$. We apply lemma 1.3 to see that $h = 0$.

Because of duality this argument tells us that φ is a Λ-generador.

Proposition 2.4. *Let $C\{M_n\}$ be an almost–analytics class and $\varphi \in L^1(\mathbb{R})$. It is equivalent*

$$L^\infty(\mathbb{R}) * \check{\varphi} \subset C\{M_n\} \tag{2.2}$$

to that

$$\int_{\mathbb{R}} |\varphi^{n)}(t)|\,dt \leqslant \beta^n M_n, \quad n = 0, 1, 2, \dots \tag{2.3}$$

where $\varphi^{n)}$ is the n'th derivative of φ.

Proof. We start with (2.3) implies (2.2). For $f \in L^\infty(\mathbb{R})$ we take

$$Tf(x) = \int_{\mathbb{R}} f(t)\overline{\varphi(t-x)}\,dt$$

To see

$$\left|(Tf)^{n)}(x)\right| \leqslant C_f \beta^n M_n$$

it is only necessary to derive under the integral and to bound f by its infinite norm.

For seeing the other implication we have to introduce a norm to $C\{M_n\}$ in the following way:

$$\|F\| = \sup_n \left(\frac{\sup|F|}{M_n}\right)^{\frac{1}{n}}$$

If we apply the closed graph theorem we see that:

$$\|\partial^n f * \check{\varphi}\|_\infty \leqslant A^n \|f\|_\infty M_n$$

As this is fulfilled for every $f \in L^\infty(\mathbb{R})$ and $\partial^n f * \check{\varphi} = f * (\partial^n \check{\varphi})$ we already have that φ has to fulfill (2.3). ☐

Remark. Let's notice that if (2.3) is fulfilled, for any function $f \in L^\infty(\mathbb{R})$ we have the bound:

$$|(Tf)^{n)}(x)| \leqslant \|f\|_\infty \beta^n M_n$$

This means that the same β is valid to all the functions of the phase space, while in the definition of almost–analytical class this constant depended on the function. In particular we see that we do not equal the class $C\{M_n\}$ and the inclusion in (2.2) is strict. As already we have commented in the former chapter, this provokes that there can be sets of uniqueness of the phase space of φ that are not of $C\{M_n\}$. As a matter of fact theorem 2.3 already gives examples of this type.

In [Bru06] is constructed a function φ fulfilling (2.3) in the following way. Given an almost–analytical class $C\{M_n\}$ we consider the *Ostrowski* decresing function

$$0 < \Theta(\xi) = \inf_n \frac{M_n}{|\xi|^n} \leqslant 1$$

From this we define

$$0 < \omega(\xi) = \int_\xi^\infty \Theta(s) e^{-s}\,ds$$

that fulfills that $\omega(\xi), |\omega'(\xi)| \leqslant \Theta(\xi)$. The generator that we are searching will have as Fourier transform $\widehat{\varphi}(\xi) = \omega(\xi) e^{-\xi^2}$. Defined in this way φ is C^∞ since

$\xi^n\widehat{\varphi}(\xi)$ is integrable for every n. In more $\widehat{\varphi^{n)}}(\xi) = (-2\pi i\xi)^n\widehat{\varphi}(\xi)$ fulfills

$$\begin{aligned}
\left|\widehat{\varphi^{n)}}(\xi)\right| &\leqslant e^{-\xi^2}|2\pi\xi|^n\Theta(\xi) \leqslant (2\pi)^n M_n e^{-\xi^2} \\
\left|\widehat{\varphi^{n)}}'(\xi)\right| &\leqslant \big(n|2\pi\xi|^{n-1}\omega(\xi) + |2\pi\xi|^n\omega'(\xi) + |2\pi\xi|^{n+1}\omega(\xi)\big)e^{-\xi^2} \\
&\leqslant \big(n(2\pi)^{n-1}M_{n-1} + (2\pi)^n M_n\big)e^{-\xi^2} + (2\pi)^n M_n|\xi|e^{-\xi^2}
\end{aligned}$$

This implies $\|\widehat{\varphi^{n)}}\|_1 \leqslant C(2\pi)^n M_n$ and $\|\widehat{\varphi^{n)}}\|_2 + \|\widehat{\varphi^{n)}}'\|_2 \leqslant C(2\pi)^n M_n$. We apply

$$\|\psi\|_1 \leqslant C\left(\|\widehat{\psi}\|_1 + \|(\widehat{\psi})'\|_2\right) \tag{2.4}$$

to $\psi = \varphi^{n)}$ and we see that it fulfills (2.3).

We want to study this class of generators, but in more general terms.

Definition. We will say that φ is an almost–analytical **generator** if there is an almost–analytical class $C\{M_n\}$ and $\beta > 0$ such that:

$$\int_{\mathbb{R}} |\varphi^{n)}(t)|\,dt \leqslant \beta^n M_n, \quad n = 0, 1, 2, \ldots \tag{2.5}$$

or in an equivalent way:

$$L^\infty(\mathbb{R}) * \check{\varphi} \subset C\{M_n\}$$

We have already commented that in [Bru06] is demonstrated that if Λ accepts generators for $L^1(\mathbb{R})$, then we can construct an almost–analytical Λ-generador. Now we want to find conditions on φ so that it is an almost–analytical generator. The first obvious condition is that $\widehat{\varphi}(\xi) \neq 0$ for every ξ by the Wiener's theorem 1.2. Apart from this, it is seen [Koo92][Bru06] that a necessary condition is the divergence of the logarithmic integral of the Fourier transform of φ:

$$\int_0^\infty \frac{\log|\widehat{\varphi}(\xi)|}{1+\xi^2} = -\infty$$

This is a condition of decrease, since it asks the logarithm to be big and negative, that it is equivalent to the Fourier transform of the function being very small ($\widehat{\varphi}$ is continuous and is in $L^2(\mathbb{R})$). But this condition is not sufficient, since we can find examples of continuous $L^1(\mathbb{R})$ functions f such that the logarithmic integral of $\widehat{f}$ is divergent but however the function is not C^∞. Our idea is to arrive at a necessary condition a little better than this, that is already obtained in [Bru06] although is not declared of explicit way, and a very next sufficient condition. The concrete statement is the following one:

Theorem 2.5. *Let $\varphi \in L^1(\mathbb{R})$. If φ is an almost–analytical generator then there exists a function $H(\xi) > 0$ such that $\frac{1}{H}$ is logarithmically convex and:*

$$\int_0^\infty \frac{\log H(\xi)}{1+\xi^2}\, d\xi = -\infty$$

so that:

$$0 < |\widehat{\varphi}(\xi)| \leqslant H(|\xi|)$$

It is a sufficient condition so that φ is an almost–analytical generator that there is a function $H(\xi)$ with the former properties such that:

$$\begin{aligned} 0 < |\widehat{\varphi}(\xi)| &\leqslant H(|\xi|) \\ |\widehat{\varphi}'(\xi)| &\leqslant H(|\xi|) \end{aligned}$$

To prove this theorem we will give a series of results that have interested in themself. We will begin following the outline of [HaJ94], who searched conditions so that a function did not have zeros of infinite order.

Lemma 2.6. *Given a function $\varphi \in L^1(\mathbb{R})$ such that $\widehat{\varphi}(\xi) \neq 0$ for every ξ it is a sufficient condition for $\varphi \in C^\infty$ with bounded derivatives that exists an increasing function $\omega(\xi)$ going to infinite such that:*

$$|\widehat{\varphi}(\xi)| \leqslant H(|\xi|)$$

with H the function defined as:

$$H(\xi) = e^{-p(\xi)} = e^{-A-\int_1^\xi \frac{\omega(u)}{u}\, du}$$

This condition is also necessary for $\varphi \in C^\infty$ with $\varphi^{n)} \in L^1(\mathbb{R})$ for every n.

Remark. The function H defined in the lemma has the property that $\frac{1}{H}$ is a logarithmically convex function. As we are interested in strictly positive functions with limit zero, any logarithmically convex function can be written in an analogous way as $\frac{1}{H}$.

Proof. If a function φ is C^∞ with integrable derivatives the Fourier transform fulfills that $|\xi^n \widehat{f}(\xi)|$ is bounded for every n. As the functions in that we are interested are always different of zero, we can take logarithms for obtaining:

$$\log|\widehat{\varphi}(\xi)| \leqslant K_n - n\log|\xi| \qquad \forall n$$

If we look at the second part of this inequality, we can define $p(\xi) = \sup_n(n\log|\xi| - K_n)$, which will be a convex function of $\log\xi$ (e^p logarithmically convex). This function can be written as:

$$p(\xi) = A + \int_1^\xi \frac{\omega(u)}{u}\, du \tag{2.6}$$

where $\omega(u)$ will be an increasing function with infinite limit (we are thinking of $\xi > 0$). Therefore the necessary condition for φ to be C^∞ with $\varphi^{n)} \in L^1(\mathbb{R})$ is that there exists a incresing function $\omega(u)$ going to infinite so that:

$$|\widehat{\varphi}(\xi)| \leqslant e^{-p(\xi)}$$

with p defined as in (2.6). We have to see that this condition is sufficient for φ to be C^∞. We observe that, given an n, there exists c_n such that $\omega(u) \geqslant n$ for $u \geqslant c_n$, since ω grows towards infinite. This says that if $\xi > c_n$:

$$\int_{c_n}^{\xi} \frac{\omega(u)}{u}\, du \geqslant n \int_{c_n}^{\xi} \frac{1}{u}\, du = n(\log \xi - \log c_n)$$

Taking this into account, if $\xi > c_n$ we have that:

$$|\xi^n \varphi(\xi)| \leqslant \xi^n e^{-A - \int_1^{\xi} \frac{\omega(u)}{u}\, du} \leqslant e^{-A} e^{-\int_1^{c_n} \frac{\omega(u)}{u}\, du} c_n^n \leqslant e^{-A} c_n^n$$

If $\xi < c_n$ (positive), then we bound directly:

$$|\xi^n \varphi(\xi)| \leqslant c_n^n e^{-A - \int_1^{\xi} \frac{\omega(u)}{u}\, du} \leqslant e^{-A} c_n^n$$

Joining both parts we have the bound that we searched:

$$|\xi^n \widehat{\varphi}(\xi)| \leqslant e^{-A} c_n^n$$

that says that φ is C^∞ with bounded derivatives. ☐

Remark. As the bounds in $L^1(\mathbb{R})$ norm give us bounds in $L^\infty(\mathbb{R})$ norm of the Fourier transform but the invers is not true, the result that we have given is not symmetrical, and as a matter of fact we can not improve it without supposing some extra condition of the function.

We need that the derivatives are in $L^1(\mathbb{R})$, and now we just have the existence of the derivatives. To obtain the $L^1(\mathbb{R})$ bound from these we need a bound of the same type for the derivative of $\widehat{\varphi}$ in order to use (2.4).

Lemma 2.7. *Let $\varphi \in L^1(\mathbb{R})$ for which there is a function H as in 2.6 such that:*

$$|\widehat{\varphi}(\xi)|, |\widehat{\varphi}'(\xi)| \leqslant H(|\xi|)$$

Then $\varphi \in C^\infty$ and $\varphi^{n)} \in L^1(\mathbb{R})$ for every n. Moreover in this case there is $K > 0$ such that:

$$\|\varphi^{n)}\|_1 \leqslant K^n c_{n+2}^{n+2}$$

where c_n fulfills that $\omega(u) \geqslant n$ if $u \geqslant c_n$.

Proof. We consider $\psi = \varphi^{n)}$. Using the bounds that we obtain in the proof of 2.6 we can give the following inequalities:

$$\|\widehat{\psi}\|_1 \leqslant e^{-A}(2\pi)^n\big(c_n^n + c_{n+2}^{n+2}\big)\Big\|\frac{1}{1+|\xi|^2}\Big\|_1$$

$$\|\widehat{\psi}'\|_2 \leqslant e^{-A}(2\pi)^n\big(nc_{n-1}^{n-1} + (n+1)c_n^n + c_{n+2}^{n+2}\big)\Big\|\frac{1}{1+|\xi|}\Big\|_2$$

and using (2.4) we have the bound that we searched:

$$\big\|\varphi^{n)}\big\|_1 \leqslant K^n c_{n+2}^{n+2}$$

□

Now it is necessary to remember that for φ to be an almost–analytical generator it has to be fulfilled (2.5) for an almost–analytical class $C\{M_n\}$. In an obvious way we will have to choose $\beta = K$ and $M_n \geqslant c_{n+2}^{n+2}$. But we remember that for $C\{M_n\}$to be almost–analytical we need that $\sum M_n^{-\frac{1}{n}}$ is divergent. We will be able to make this if and only if:

$$\sum_{n\geqslant 2} \frac{1}{c_{n+2}^{1+\frac{2}{n}}} = \infty \tag{2.7}$$

Lemma 2.8. *We suppose $a_n > 0$ for every n. Then the character of the series (a_n) and $\left(a_n^{1+\frac{2}{n}}\right)$ is the same one. That is:*

$$\sum_n a_n < \infty \Longleftrightarrow \sum_n a_n^{1+\frac{2}{n}} < \infty$$

Proof. We define $\widetilde{a}_n = a_n^{1+\frac{2}{n}}$. It is trivial that the convergence of (a_n) implies that of $\widetilde{a}_n$ since for n big enough $a_n < 1$. Now we suppose that $\sum a_n$ is divergent. Here we have to sort out the a_n such that $a_n \geqslant \frac{1}{2^n}$, which we call b_n, with $b_n = 0$ if $a_n < \frac{1}{2^n}$. In this way:

$$\sum_n b_n b_n^{\frac{2}{n}} \geqslant \sum_n \frac{b_n}{4} = \frac{1}{4}\sum_n b_n$$

For seeing that this last sum is divergent we define c_n in a complementary way to b_n. That is, $a_n = b_n + c_n$. It can be seen easily that $\sum c_n$ is convergent, and as $\sum a_n = \sum b_n + \sum c_n$ we already have that $\sum b_n$ is divergent. With this we have proved that $\sum \widetilde{a}_n$ is divergent, that completes the proof. □

Proof (of the theorem 2.5). We begin by proving the sufficiency. The former lemma is useful to study the character of the series $\sum \frac{1}{c_n}$ instead of (2.7). If we look to this series we have that:

$$\sum_{n\geqslant 1} \frac{1}{c_n} = \frac{1}{2}\int_1^\infty \frac{\sum_{n\geqslant 1} \chi_{[c_n,\infty)}(u)}{u^2}\,du$$

This last integral coincides by the comparision criterion with:

$$\int_1^\infty \frac{\sum_{n\geqslant 1} \chi_{[c_n,\infty)}(u)}{u^2}\,du \approx \int_1^\infty \frac{\omega(u)}{u^2}\,du$$

If we remember the definition of $H(\xi)$, by integration by parts, we see that the one that we are studying is the convergence of the logarithmic integral of H:

$$\int_0^\infty \frac{\log H(\xi)}{1+\xi^2}\,d\xi \approx -A\int_1^\infty \frac{\int_1^\xi \frac{\omega(u)}{u}\,du}{\xi^2}\,d\xi \approx \int_1^\infty \frac{\omega(\xi)}{\xi^2}\,d\xi$$

Therefore, $\sum \frac{1}{c_n}$ is divergent if and only if the logarithmic integral of $H(\xi)$ is divergent, and in this case we will be able to find an almost-analytic class $C\{M_n\}$ such that $L^\infty(\mathbb{R}) * \check{\varphi} \subset C\{M_n\}$.

To see the necessity we define $H(\xi) = \inf_n \frac{M_n}{\xi^n}$ (the called *Ostrowski* function). This function is already logarithmically convex, and the divergence of the logarithmic integral is equivalent to the condition of almost- analicity [Koo92]. ☐

Theorem 2.5 allows us to give examples of generators of $L^1(\mathbb{R})$in a simple way. But the reality is that the majority of generators that leave in a direct way are from a much better type, since, as we will see in the next section, its correspond with analytical classes. To give examples of non analytical almost–analytical generators we have to construct a $\omega(u) > 0$ such that:

$$\int_1^\infty \frac{\omega(u)}{u^2}\,du$$

is divergent, but $\omega(u) << u$. We can give as simple examples:

$$\widehat{\varphi}(\xi) = e^{-\int_1^{|\xi|} \frac{1}{\log u}\,du}$$

or

$$\widehat{\varphi}(\xi) = e^{-\int_1^{|\xi|} \frac{1}{\log^2 u}\,du}$$

These two examples fulfill the mentioned conditions. In the second case we can choose $C\{M_n\}$ with $M_n = (n\log n)^n$, which is the typical example of non analytical almost–analytical class. Another method of giving examples is to calculate the *Ostrowski* function directly for a given almost–analytical class. For example:

$$\widehat{\varphi}(\xi) = \inf_n \frac{(n\log n)^n}{\xi^n}$$

But either it does not seem that simple expresions of φ can be found in this way.

2.4 Analytical Generators.

A special case of almost–analytical classes are those that we will call analytical ones. These classes are characterized by the fact that its functions are not just $\mathcal{C}^\infty$ but are analytical functions.

We have given the description of these classes in the former section. We will say that an almost–analytical class $C\{M_n\}$ is analytical if $\left(\frac{M_n}{n!}\right)^{\frac{1}{n}}$ is a bounded succession.

Definition. We will say that φ is an **analytical generador** if there is an analytical class $C\{M_n\}$ and a $\beta > 0$ such that:

$$\int_{\mathbb{R}} |\varphi^{n)}(t)|\, dt \leqslant \beta^n M_n, \quad n = 0, 1, 2, \dots$$

or in an equivalent way:

$$L^\infty(\mathbb{R}) * \check{\varphi} \subset C\{M_n\}$$

The Poisson function is the first example of analytical generator:

$$P(t) = \frac{1}{\pi}\frac{1}{1+t^2}$$

The sets Λ for which $T(P, \Lambda)$ generate $L^1(\mathbb{R})$ are described in [BrM07]:

Theorem 2.9 (Bruna-Melnikov)**.** *A discret set $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ fulfills that $T(P, \Lambda)$ generates $L^1(\mathbb{R})$ for $P(t)$ the Poisson function if and only if:*

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{2}|\lambda_n|} = \infty \tag{2.8}$$

The same statement is true for $L^p(\mathbb{R})$.

We Will treat this example with much more detail in the following chapter.

We go now to study the analytical generators in general. Let's focus in how can be the majorant H in the case of analytical classes.

Theorem 2.10. *Given a function $\varphi \in L^1(\mathbb{R})$, if φ is an analytical generator then there are $A, C > 0$ such that:*

$$0 < |\widehat{\varphi}(\xi)| \leqslant Ae^{-C|\xi|}$$

for every $\xi \in \mathbb{R}$.

Proof. As φ is an analytical generator, there is an analytical class $C\{M_n\}$ and $\beta > 0$ such that:

$$\int_{\mathbb{R}} |\varphi^{n)}(t)|\, dt \leqslant \beta^n M_n$$

Taking Fourier transform this implies that:

$$|(2\pi\xi)^n \widehat{\varphi}(\xi)| \leqslant \beta^n M_n$$

We take logarithms on both sides of the inequality and we can affirm that:

$$\log|\widehat{\varphi}(\xi)| \leqslant n\log\beta + \log M_n - n\log 2\pi|\xi| \tag{2.9}$$

We remember that, by Wiener's theorem, $\widehat{\varphi}(\xi) \neq 0$ for every ξ and therefore the logarithm is always well defined, exept maybe of the case $\xi = 0$. As we are attempting to prove an assimptotic inequality this will not be a problem.

If $C\{M_n\}$ is an analytical class, $\left(\frac{M_n}{n!}\right)^{\frac{1}{n}}$ is bounded. Applying Stirling formula and taking logarithms we can give the following bound:

$$\log M_n \leqslant \frac{1}{2}\log 2\pi n + n\log n + n\log k_1$$

We introduce this to (2.9) to see that:

$$\begin{aligned}\log|\widehat{\varphi}(\xi)| \leqslant& n\log\beta + \frac{1}{2}\log 2\pi n + n\log n + n\log k_1 - n\log 2\pi|\xi|\\ \leqslant& \frac{1}{2}\log 2\pi - n\big(\log|\xi| - (k_2 + \log n)\big)\end{aligned}$$

We calculate $\sup_n n\log|\xi| - n(k_2 + \log n)$, which is taken in $n = \frac{|\xi|}{e^{1+k_2}}$. In this way we can bound:

$$\begin{aligned}\log|\widehat{\varphi}(\xi)| \leqslant& \frac{1}{2}\log 2\pi - \frac{|\xi|}{e^{1+k_2}}\left(\log|\xi| - \left(k_2 + \log\frac{|\xi|}{e^{1+k_2}}\right)\right)\\ =& \frac{1}{2}\log 2\pi - \frac{|\xi|}{e^{1+k_2}}\end{aligned}$$

We define $C = \frac{1}{e^{1+k_2}}$ and we take exponential. It is only necessary to add the constant A to solve problems when ξ is next to zero and we have proved the theorem. □

This result gives the necessary condition to obtain an analytical generator and it briefs us of how has to be the majorant in this case. From here it is simple to find the suffivient condition. It is just necessary to hve the same control for the derivative of $\widehat{\varphi}$.

Theorem 2.11. *Given a function* $\varphi \in L^1(\mathbb{R})$*, if there are* $A, C > 0$ *such that for every* $\xi \in \mathbb{R}$

$$\begin{aligned} 0 <&|\widehat{\varphi}(\xi)| \leqslant Ae^{-C|\xi|} \\ &|\widehat{\varphi}'(\xi)| \leqslant Ae^{-C|\xi|} \end{aligned} \tag{2.10}$$

then φ *is an analytical generator.*

Proof. The idea, as in the almost–analytical case, is to apply (2.4) with $\psi = \varphi^{n)}$. Using (2.10) and applying lemma 2.7 with $H(\xi) = Ae^{-C|\xi|}$ we have that:

$$\|\varphi^{n)}\|_1 \leqslant K^n c_{n+2}^{n+2}$$

where in this case $c_n = \frac{n}{C}$. We choose $M_n = \left(\frac{n}{C}\right)^{n+2}$ and we use the Stirling formula:

$$\left(\frac{M_n}{n!}\right)^{\frac{1}{n}} \leqslant k_1 \left(\frac{n^{n+2}e^n}{C^{n+2}n^n\sqrt{2\pi n}}\right)^{\frac{1}{n}} \leqslant k_1 \frac{n^{\frac{2}{n}}e}{C^{1+\frac{2}{n}}(2\pi n)^{\frac{1}{2n}}} \leqslant k_2$$

that it tells that we are in an analytical class. ☐

The phase spaces of the analytical generators have very good properties. In particular its elements are functions that have an holomorphic extension on a band $|\Im z| < C$. Moreover they are included in the phase space of a (concrete) dilatation of the Poisson function. For this concrete case (when the generator is a dilatation of the Poisson function) we know all the sets of uniqueness of the set of transforms, because they are described in [BrM07] by the condition that gives theorem 2.9. Λ is a discrete set of uniqueness for the phase space of one dilatated of the Poisson function if and only if:

$$\sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2C}|\lambda|} = \infty$$

where the constant C is the parameter of dilatation with respect to the Poisson function. Like this we obtain, by an argument of spaces inclusion, a sufficient condition for Λ in order to $T(\varphi, \Lambda)$ generates when φ is an analytical generator.

Corollary 2.12. *Let* $\varphi \in L^1(\mathbb{R})$ *be an analytical generator such that* $|\widehat{\varphi}(\xi)|, |\widehat{\varphi}'(\xi)| \leqslant e^{-C|\xi|}$*. Let* Λ *be a discret set such that:*

$$\sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2C}|\lambda|} = \infty$$

Then $T(\varphi, \Lambda)$ *is a generator system of* $L^1(\mathbb{R})$*.*

In the case of the Poisson function this condition was necessary and sufficient, but in general it will not be necessary. For example, in the case of the Gaussian function ($\phi(t) = e^{-\pi t^2}$) we are under these conditions and it has many more sets that generate. This happend because the set of functions with holomorphic extension to a band is the worst analytical class that we can find, in the sense that it is the one that has less sets of uniqueness.

This is a general fact in almost–analytical generators. The sets of uniqueness of $L^\infty(\mathbb{R}) * \check{\varphi}$ coincide exactly with the sets Λ such that $T(\varphi, \Lambda)$ is a generator system of $L^1(\mathbb{R})$. If we know that $L^\infty(\mathbb{R}) * \check{\varphi} \subseteq C\{M_n\}$ we can affirm that any set of uniqueness of $C\{M_n\}$ will be it of $L^\infty(\mathbb{R}) * \check{\varphi}$. But as the inclusion can be strict we do not have a bijective correspondence among the sets of uniqueness. It is for this reason that it is very hard to give necessary conditions on Λ for $T(\varphi, \Lambda)$ to be a generator system of $L^1(\mathbb{R})$ when we have fixed φ. Either we do not know any characterization of the sets of uniqueness of a general almost–analytical class, except for the case in that class is analytical, and in this last supposition just in some concrete cases.

It is for this reason that the result 2.9 of [BrM07] it is of special importance.

Chapter 3

Poisson type generators.

3.1 Poisson function.

We have commented in the former chapter that one of the first examples of analytical generators is the Poisson function:

$$P(t) = \frac{1}{\pi}\frac{1}{1+t^2}$$

We remember that the sets Λ for which $T(P,\Lambda)$ generate $L^1(\mathbb{R})$ are described in [BrM07]:

Theorem 3.1 (Bruna-Melnikov). *A discrete set $\Lambda = \{\lambda_n\}_{n\in\mathbb{Z}} \subset \mathbb{R}$ fulfills that $T(P,\Lambda)$ generates $L^1(\mathbb{R})$ for $P(t)$ the Poisson function if and only if:*

$$\sum_{n\in\mathbb{Z}} e^{-\frac{\pi}{2}|\lambda_n|} = \infty \tag{3.1}$$

The same is true for $L^p(\mathbb{R})$.

The Poisson function is one of the simpler analytical generators that is known. By convolution with $L^\infty(\mathbb{R})$ it will cause one of the more general analytical classes that we can find. Moreover it has the property that all sets that they generate have been able to be characterized. This result is attained thanks to the fact that its phase space has been very well described. The proof consists of two parts (as it is usual whenever a total description is obtained from the sets that cause generator systems). First it is searched a good description of the phase space. In this case we will see that this coincides with the restriction at a straight line of a space of holomorphic functions. Afterwards it have to be characterized the subclass of the sets of uniqueness of this space that interest us. We can solve this second part thanks to the fact that we find a Hardy type space. Translating the problem to the disk we can give a description of the uniqueness sets using the tools of complex analysis (mainly the Jensen formula and the Blaschke products).

We start with the description of the phase space. The way to make it is the following one. We remember that the phase space of P for $L^p(\mathbb{R})$ was:

$$H = \left\{ F : F(x) = \int_{\mathbb{R}} \frac{1}{\pi} \frac{f(t)}{(t-x)^2+1}\, dt, \text{ for one} f \in L^q(\mathbb{R}) \right\}$$

where q is the dual exponent of p. We can think without losing generality that f is real, since if we prove that $T(P,\Lambda)$ generates the real $L^p(\mathbb{R})$ also we will have the result for the complex one.

Let $h^q(\mathbb{R}^2_+)$ be the space of real harmonic functions of the upper half–plane such that:

$$\|u\|_q^q = \sup_{y>0} \int_{\mathbb{R}} |u(x+iy)|^q\, dx < \infty$$

and $h^\infty(\mathbb{R}^2_+)$ the space of real harmonic and bounded functions. A function of this space can be expressed as:

$$u(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{(t-x)^2+y^2}\, dt \qquad z = x+iy$$

with $f \in L^q(\mathbb{R})$ and real, and moreover it is a one to one correspondence. This proves that the restriction to the straight line $\Im z = 1$ of $h^q(\mathbb{R}^2_+)$ coincides with the phase space of the Poisson function. In an informal way we can think that they are the same one and we are searching the sets of uniqueness of $h^q(\mathbb{R}^2_+)$ contained in $\Im z = 1$.

To determine these sets we have to change again of space. The idea is to complexify the expression of a function of the phase space to find a holomorphic space of functions instead of harmonic ones:

$$F(z) = \frac{1}{\pi} \int_{\mathbb{R}} \frac{f(t)}{(t-z)^2+1}\, dt \tag{3.2}$$

Using the correspondence of the phase space with $h^q(\mathbb{R}^2_+)$, we can give a description of the set of functions that appear when complexify. We name B on the band $|\Im z| < 1$ and let $E^q(B)$ be the space of holomorphic functions in B such that satisfy:

$$\sup_{|y|<1} \int_{\mathbb{R}} |F(x+iy)|^q\, dx = \|F\|_q^q < \infty$$
$$F(\overline{z}) = \overline{F(z)}, \quad z \in B$$

For $E^\infty(B)$ we change the first condition for $\Re F$ bounded.

Theorem 3.2 (Bruna-Melnikov)**.** *The map* $u \mapsto F$ *for which* $u(x+i) = F(x)$, $x \in \mathbb{R}$ *is a one to one correspondence from* $h^q(\mathbb{R}^2_+)$ *onto* $E^q(B)$, $1 < q \leqslant \infty$.

Note. The proof of this theorem is new, and does not coincide with that of [BrM07]. The idea about this proof was proposed by Joaquim Ortega Cerdà,and we are thankfull to be allowed to reproduce it here.

Proof. The injectivity of this map is obvious. We go to see that an F defined as in (3.2) belongs to $E^q(B)$. F is holomorphic and is well defined for every $|y| < 1$ because $\Re\big(1 + (z-t)^2\big) = 1 + (x-t)^2 - y^2 > 0$. Also it is easy to see that $F(\overline{z}) = \overline{F(z)}$. We write F in the following way:

$$F(z) = \frac{1}{2\pi i}\int_{\mathbb{R}} f(t)\left\{\frac{1}{t-z-i} - \frac{1}{t-z+i}\right\} dt = Cf(z+i) - Cf(z-i)$$

where Cf is the Cauchy transform

$$Cf(w) = \frac{1}{2\pi i}\int_{\mathbb{R}} \frac{f(t)}{t-w}\, dt, \quad w \notin \mathbb{R}$$

If $f \in L^q(\mathbb{R})$, $1 < q < \infty$, Cf is in the Hardy space $H^q(\mathbb{R}^2_+)$ [Gar07] and it fulfills the bound:

$$\int_{\mathbb{R}} |Cf(x+iy)|^q\, dx \leqslant \int_{\mathbb{R}} |f(x)|^q\, dx$$

With this we have that F always belongs to $E^q(B)$ if $f \in L^q(\mathbb{R})$.

To see the other inclusion we take a function $F \in E^q(B)$. The idea is to write

$$F(z) = G(z) + \overline{G(\overline{z})} \tag{3.3}$$

with G a function of the Hardy space $H^q(\Pi)$ of the half–plane $\Pi = \{\Im z > -1\}$. A function of this Hardy space fulfills that it is holomorphic and moreover

$$\|G\|^q_{H^q} = \sup_{y > -1}\int_{\mathbb{R}} |G(x+iy)|^q\, dx < \infty$$

As $\overline{G(\overline{z})}$ belongs to $H^q(\Pi^-)$, $\Pi^- = \{\Im z < 1\}$, it is clear that an F as in (3.3) always belongs to $E^q(B)$. If we look at it when $x \in \mathbb{R}$

$$F(x) = G(x) + \overline{G(x)} = 2\Re G(x) = \frac{2}{\pi}\int_{\mathbb{R}} \frac{\Re G(t)}{(x-t)^2+1}\, dt$$

that is from the wished type, since $\Re G(z+i) \in h^q(\mathbb{R}^2_+)$. Therefore, to prove the inclusion it is only necessary to see that every $F \in E^q(B)$ can be written as in (3.3). We will achieve an expression of this type for the usual procedure. First we will make the decomposition in the category C^∞ and afterwards we will solve a $\overline{\partial}$

for achieving the analycity. We define an auxiliary decreasing function $\vartheta \in C^{\infty}(\mathbb{R})$ such that $\vartheta(t) = 1$ if $t \leqslant \frac{-1}{2}$, $\vartheta(t) = 0$ if $t \geqslant \frac{1}{2}$, and $\vartheta(-t) = 1 - \vartheta(t)$. In this way, if we take:

$$\widetilde{G}(z) = F(z)\vartheta(\Im z)$$

it is fulfilled:

$$\widetilde{G}(z) + \overline{\widetilde{G}(\overline{z})} = F(z)\vartheta(\Im z) + \overline{F(\overline{z})}\vartheta(-\Im z) = F(z)\vartheta(\Im z) + F(z)\big(1 - \vartheta(\Im z)\big) = F(z)$$

In more $\widetilde{G}$ fulfills the wished norm bounds, because it inherits them of F and ϑ cancels from $\frac{1}{2}$. The only obstacle is that it is not holomorphic. To solve this problem we will add a factor u to him by solving $\overline{\partial} u = \overline{\partial} F\vartheta$. We achieve this factor making convolution with the Cauchy kernel:

$$u(z) = \frac{1}{2\pi i}\int_{\mathbb{C}} \frac{F(w)\vartheta'(\Im w)}{z - w}\, dm(w)$$

where $dm(w)$ is the area measure of $\mathbb{C}$. There are not problems when calculating this integral since, although F is just defined on the band, ϑ' is 0 out of B. In this way the function G that we are searching will be:

$$G(z) = F(z)\vartheta(\Im z) - u(z)$$

We go to check out that all conditions are fulfilled. We look at first:

$$\begin{aligned}\overline{u(z)} =& \frac{-1}{2\pi i}\int_{\mathbb{C}} \frac{\overline{F(w)}\vartheta'(\Im w)}{\overline{z} - \overline{w}}\, dm(w) = \frac{-1}{2\pi i}\int_{\mathbb{C}} \frac{F(\overline{w})\vartheta'(\Im w)}{\overline{z} - \overline{w}}\, dm(w)\\ =& \frac{-1}{2\pi i}\int_{\mathbb{C}} \frac{F(s)\vartheta'(-\Im s)}{\overline{z} - s}\, dm(s) = \frac{-1}{2\pi i}\int_{\mathbb{C}} \frac{F(s)\vartheta'(\Im s)}{\overline{z} - s}\, dm(s)\\ =& - u(\overline{z})\end{aligned}$$

This implies that the decomposition continues being valid. It remains us to see that $G \in H^q(\Pi)$. We see first that is holomorphic:

$$\overline{\partial} G(z) = \frac{-1}{2i} F(z)\vartheta'(\Im z) - \overline{\partial} u(z) = 0$$

since we have defined u as the Cauchy integral of $F(z)\vartheta'(\Im z)$, which solves the $\overline{\partial}$. To see:

$$\|G\|_q = \sup_{y>-1}\left(\int_{\mathbb{R}} |G(x+iy)|^q\, dx\right)^{\frac{1}{q}} < \infty$$

we apply Minkovski. If each one of the addends fulfills this bound, G will also fulfill it. We already had the part corresponding to $F(z)\vartheta(\Im z)$. Then we have to bound $u(z)$.

$$\begin{aligned}\int_{\mathbb{R}} |u(x+iy)|^q\,dx &= \int_{\mathbb{R}} \left| \frac{-1}{2\pi i} \int_{\frac{-1}{2}}^{\frac{1}{2}} \int_{\mathbb{R}} \frac{F(a+ib)\vartheta'(b)}{x+iy-(a+ib)}\,da\,db \right|^q dx \\ &= \int_{\mathbb{R}} \left| \int_{\frac{-1}{2}}^{\frac{1}{2}} \vartheta'(b) \frac{1}{2\pi i} \int_{\mathbb{R}} \frac{F(a+ib)}{x+iy-(a+ib)}\,da\,db \right|^q dx\end{aligned}$$

The integral in a coincides with the Cauchy transform of F in the straight line $\Im z = b$. Making a change of notation we can say:

$$\int_{\mathbb{R}} |u(x+iy)|^q\,dx = \int_{\mathbb{R}} \left| \int_{\frac{-1}{2}}^{\frac{1}{2}} \vartheta'(b) CF_b(x+i(y-b))\,db \right|^q dx$$

We apply Hölder to the integral in b:

$$\int_{\mathbb{R}} |u(x+iy)|^q\,dx \leqslant \int_{\mathbb{R}} \left(\int_{\frac{-1}{2}}^{\frac{1}{2}} |\vartheta'(b)|^p\,db \right)^{\frac{q}{p}} \int_{\frac{-1}{2}}^{\frac{1}{2}} |CF_b(x+i(y-b))|^q\,db\,dx$$

As $\vartheta \in C^\infty$, its derivative is bounded and we can forget about the first integral. For the second we apply Fubini and we remember the Cauchy integral is bounded by its border values:

$$\int_{\mathbb{R}} \int_{\frac{-1}{2}}^{\frac{1}{2}} |CF_b(x+i(y-b))|^q|\,db\,dx \leqslant \int_{\frac{-1}{2}}^{\frac{1}{2}} \int_{\mathbb{R}} |F(x+ib)|^q\,dx\,db \leqslant \int_{\frac{-1}{2}}^{\frac{1}{2}} \|F\|_q^q\,db$$

since $b \in (\frac{-1}{2}, \frac{1}{2})$ and $F \in E^q(B)$. Moreover this bound is independent of y, and we will be able to bound the supreme in the same way.

With this we have proved that $G \in H^q(\Pi)$ and we have seen the double inclusion. For the case $E^\infty(B)$ is necessary to use the original proof of [BrM07]. ☐

Remark. The main advantage of this proof with respect to the original of [BrM07] is that it can become generalized easily to several variables. We have to observe that in this case we do not restrict to the real part of the function but we use the module of the function to define the norm and the space.

Proof of theorem 3.1. As $|F|^q$ has a harmonic majorant in B, we can deduce that if ϕ is the conformal map from B in to the disk $\mathbb{D} = \{z : |z| < 1\}$ defined by:

$$w = \phi(z) = \frac{e^{\frac{\pi}{2}z} - 1}{e^{\frac{\pi}{2}z} + 1}$$

then every function $H(w) = F(\phi^{-1}(w))$ with $F \in E^q(B)$ will be in the Hardy space of the disk $H^q(\mathbb{D})$ (the converse is not true). For $q = \infty$ we define $H(w) = \exp(F(\phi^{-1}(w))) - 1$.

We define $\Gamma = \phi(\Lambda) \subset \mathbb{D}$. In this way H cancels in Γ. We can also see:

$$\sum_{\gamma\in\Gamma} \log\frac{1}{|\gamma|} = \sum_{\lambda\in\Lambda} \frac{e^{\frac{\pi}{2}|\lambda|}+1}{|e^{\frac{\pi}{2}|\lambda|}-1|} \sim 2\sum_{\lambda\in\Lambda} e^{-\frac{\pi}{2}|\lambda|}$$

We apply now the Jensen formula:

$$\log|H(0)| + \sum_{\gamma\in\Gamma}\log\frac{1}{|\gamma|} \leqslant \frac{1}{2\pi}\int_0^{2\pi}\log|H(e^{i\theta})|\,d\theta \leqslant \log\|H\|_q$$

Without losing generality we can think that $H(0) \neq 0$ and we obtain the bound of (3.1) if Γ is contained in a zero set of any function of $E^q(B)$.

It remains to see that if the summation in (3.1) is bounded, then there is a function in $E^q(B)$ such that cancels in Λ.

The $\sum_{\gamma\in\Gamma}\log\frac{1}{|\gamma|} \sim \sum_{\gamma\in\Gamma} 1-|\gamma| < \infty$ condition is the Blaschke condition, that guarantees that the product:

$$\beta(w) = \prod_{\gamma\in\Gamma} \frac{-\gamma}{|\gamma|}\frac{w-\gamma}{1-\gamma w}$$

is convergent (it is necessary to multiply by w if $0 \in \Gamma$). Defined in this way, $\beta(w) = 0$ if and only if $w \in \Gamma$, $|\beta(w)| \leqslant 1$ for every $w \in \mathbb{D}$ and $|\beta(w)| = 1$ almost for every $w \in \partial\mathbb{D}$. As $\Im\gamma = 0$ for $\gamma \in \Gamma$ we have also that $\beta(\overline{w}) = \overline{\beta(w)}$.

We suppose that H is a holomorphic function of the disk. If we define $F(z) = H(\phi(z))$ then $g(s) = F(s \pm i) = H\left(\frac{ie^{\frac{\pi}{2}s}-1}{ie^{\frac{\pi}{2}s}+1}\right)$ will be in $L^q(\mathbb{R})$ if:

$$\int_{\mathbb{R}} |g(s)|^q\,ds = \frac{1}{\pi}\int_{|w|=1} |H(w)|^q \frac{|dw|}{|1-w^2|} < \infty$$

If we choose $H(w) = (1-w^2)\beta(w)$ we will be under these conditions and we have also the symmetry condition. With this we have proved the theorem. □

We can compare this result with the one which we would obtain just using that this phase space is contained in an analytical class. The class that we have to choose corresponds to take $M_n = n!$ or $M_n = n^n$. In addition we can see that every function F of the phase space fulfills:

$$|F^{n)}(x)| \leqslant C_F n! \quad \forall n$$

since they are restrictions of analytical functions in the band $|\Im z| < 1$. The results declared in [Hir50] say that the zero set of a function F in this class has to fulfill:

$$\limsup_{r\to\infty} \frac{2\log n_F(r)}{\pi t} \leqslant 1$$

Moreover this condition is very precise, since if the inequality is strict for a set there is a function that cancels in this.

This says that if there is $\varepsilon > 0$ such that $\log n_\Lambda(r) > (1+\varepsilon)\frac{\pi}{2}r$ for every $r > r_0$ big enough, Λ will be a set of uniqueness for this class. We sort the elements of Λ by increasing module. Looking at the inequality that Λ fulfills we can say that for k big enough it will have to be fulfilled:

$$\log k > (1+\varepsilon)\frac{\pi}{2}|\lambda_k|$$

or in an equivalent way:

$$k^{-\frac{1}{1+\varepsilon}} = e^{-\frac{\log k}{1+\varepsilon}} < e^{-\frac{\pi}{2}|\lambda_k|}$$

We observe that this condition, which will be sufficient so that $T(P,\Lambda)$ generates $L^1(\mathbb{R})$, implies that the summation in (3.1) is divergent. If we look at it well both conditions are very close but non identical. Moreover the phase space is included in the analytical class, but it can not equal it. This makes that the necessary conditions for uniqueness sets of the class do not transmit to the phase space.

3.2 Poisson type functions.

Our following step will be to generalize this result to a special class of functions that we can call Poisson type functions.

Theorem 3.3. *Let $\varphi \in L^1(\mathbb{R})$ be a function for which there exists constants $A, B > 0$ such that:*

$$Ae^{-2\pi|\xi|} \leqslant |\widehat{\varphi}(\xi)| \leqslant Be^{-2\pi|\xi|} \tag{3.4}$$

We also suppose that:

$$|\widehat{\varphi}'(\xi)| \leqslant Ce^{-2\pi|\xi|}$$

Then the set $T(\varphi,\Lambda)$ generates $L^1(\mathbb{R})$ if and only if:

$$\sum_{\lambda\in\Lambda} e^{-\frac{\pi}{2}|\lambda|} = \infty$$

Lemma 3.4. *$T(P,\Lambda)$ generates $L^2(\mathbb{R})$ if and only if $T(\psi,\Lambda)$ generates $L^2(\mathbb{R})$ with $\psi(t) = P * \widehat{P}(t)$.*

Remark. Theorem 3.1 says that the discrete sets Λ for which $T(P,\Lambda)$ generates $L^2(\mathbb{R})$ are the same than in $L^1(\mathbb{R})$.

Proof. As we have described the sets Λ for which $T(P,\Lambda)$ generates $L^2(\mathbb{R})$, we have to prove that $T(\psi,\Lambda)$ generates $L^2(\mathbb{R})$ if and only if:

$$\sum_{\lambda\in\Lambda} e^{-\frac{\pi}{2}|\lambda|} = \infty$$

It is clear that this condition is sufficient, since ψ is a convolution of P with a function of $L^1(\mathbb{R})$ and we can apply theorem 1.4. For the necessity we will revise the proof of theorem 3.1. The idea is to use duality, and to see that if $\sum_{\lambda\in\Lambda} e^{-\frac{\pi}{2}|\lambda|} < \infty$ then we can find $f \in L^2(\mathbb{R})$ such that $\langle f(t), \psi(t-\lambda)\rangle = 0$ for every $\lambda \in \Lambda$ with $f \neq 0$.

We start again with $h^2(\mathbb{R}^2_+)$. We remember that a function of this space can be expressed as:

$$u(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{(x-t)^2+y^2}\,dt \qquad z = x+iy$$

with $f \in L^2(\mathbb{R})$ and real, and moreover it is a one to one correspondence.

The system $T(\psi,\Lambda)$ will not generate $L^2(\mathbb{R})$ if and only if there is $g \in L^2(\mathbb{R})$ such that:

$$\int_{\mathbb{R}} g(t)\psi(t-\lambda)\,dt = 0 \qquad \forall\lambda\in\Lambda$$

where we can think that g is real. If we develop this integral we have that:

$$\begin{aligned}\int_{\mathbb{R}} g(t)\psi(t-\lambda)\,dt &= \int_{\mathbb{R}} g(t)\big(P*\widehat{P}\big)(t-\lambda)\,dt\\ &=\frac{1}{\pi}\int_{\mathbb{R}} g(t)\int_{\mathbb{R}} \frac{e^{-2\pi|s|}}{1+(t-\lambda-s)^2}\,ds\,dt\\ &=\frac{1}{\pi}\int_{\mathbb{R}}\int_{\mathbb{R}} \frac{g(t)e^{-2\pi|t-w|}}{1+(w-\lambda)^2}\,dw\,dt\\ &=\frac{1}{\pi}\int_{\mathbb{R}} \big(g*\widehat{P}\big)(t)\frac{1}{1+(w-\lambda)^2}\,dw\end{aligned}$$

We have to find a harmonic function

$$u(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{(x-t)^2+y^2}\,dt$$

that cancels in $\lambda+i$ for $\lambda\in\Lambda$ and so that $f = g*\widehat{P}$. In [BrM07] it is proved that there is $f \in L^2(\mathbb{R})$ such that $u(\lambda+i) = 0$ for $\lambda\in\Lambda$. We have to prove that we can take this f as $g*\widehat{P}$ with $g\in L^2(\mathbb{R})$.

We remember that if:

$$F(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{f(t)}{(t-z)^2+1}\,dt \tag{3.5}$$

Theorem 3.2 says that the map $u \mapsto F$ when $u(x+i) = F(x)$ is a one to one correspondence among $E^2(B)$ and $h^2(\mathbb{R}^2_+)$.

Then we want to find $F \in E^2(B)$ such that $F(\lambda) = 0$ for every $\lambda \in \Lambda$ and that it can be written as (3.5) with $f = g * \widehat{P}$ for some $g \in L^2(\mathbb{R})$. We can think (3.5) as a scalar product in $L^2(\mathbb{R})$. We use the Fourier transform and we apply Parseval theorem to see that:

$$F(x) = \int_{\mathbb{R}} \widehat{f}(\xi) e^{-2\pi|\xi|} e^{2\pi i x \xi}\, d\xi$$

By analytical continuation we obtain:

$$F(z) = \int_{\mathbb{R}} \widehat{f} e^{-2\pi|\xi|} e^{2\pi i z \xi}\, d\xi$$

where we want that $\widehat{f}(\xi) = \frac{1}{\pi}\frac{\widehat{g}(\xi)}{1+\xi^2}$ with $g \in L^2(\mathbb{R})$. That is, we search $F \in E^2(B)$ that can be written as:

$$F(z) = \frac{1}{\pi}\int_{\mathbb{R}} \frac{\widehat{g}(\xi)}{1+\xi^2} e^{-2\pi|\xi|} e^{2\pi i z \xi}\, d\xi$$

with $g \in L^2(\mathbb{R})$. If $F'' \in E^2(B)$ then we can write it as:

$$F''(z) = \int_{\mathbb{R}} \widetilde{f}(\xi) e^{-2\pi|\xi|} e^{2\pi i z \xi}\, d\xi$$

with $\widetilde{f} \in L^2(\mathbb{R})$. But on the other hand we see that:

$$F''(z) = \int_{\mathbb{R}} \widehat{f}(\xi)(2\pi i \xi)^2 e^{-2\pi|\xi|} e^{2\pi i z \xi}\, d\xi$$

and therefore $\widehat{f}(\xi)(2\pi i\xi)^2 = \widetilde{f}(\xi) \in L^2(\mathbb{R})$. This allows us to write:

$$\widehat{f}(\xi) = \frac{1}{4\pi^2}\frac{4\pi^2\widehat{f}(\xi) - \widetilde{f}(\xi)}{1+\xi^2}$$

with $4\pi^2\widehat{f} - \widetilde{f} \in L^2(\mathbb{R})$. In this way we have reduced the problem to find $F \in E^2(B)$ such that $F(\lambda) = 0$ for $\lambda \in \Lambda$ and such that $F'' \in E^2(B)$.

Now the idea, as in the proof of 3.1, is to translate the problem to the disk. We remember the definition of ϕ, the comformal map of B in the disk:

$$w = \phi(z) = \frac{e^{\frac{\pi}{2}z} - 1}{e^{\frac{\pi}{2}z} + 1}$$

We define $\Gamma = \phi(\Lambda) \subset \mathbb{D}$ as before, and we remember that:

$$\sum_{\gamma\in\Gamma} \log\frac{1}{|\gamma|} \sim 2\sum_{\lambda\in\Lambda} e^{-\frac{\pi}{2}|\lambda|} < \infty$$

that is the Blaschke condition. This guarantees that the product:

$$\beta(w) = \prod_{\gamma\in\Gamma} \frac{-\gamma}{|\gamma|}\frac{w-\gamma}{1-\gamma w}$$

is convergent (it is necessary to multiply by w if $0 \in \Gamma$).

We suppose that H is a holomorphic function of the disk. If we define $F(z) = H(\phi(z))$ then $g(s) = F(s \pm i) = H\left(\frac{ie^{\frac{\pi}{2}s}-1}{ie^{\frac{\pi}{2}s}+1}\right)$ will be in $L^2(\mathbb{R})$ if:

$$\int_{\mathbb{R}} |g(s)|^2\, ds = \frac{1}{\pi}\int_{|z|=1} |H(z)|^2 \frac{|dz|}{|1-z^2|} < \infty \tag{3.6}$$

This condition also had to be fulfilled in 3.1, but now we need to ask more to the function. We calculate the second derivative of F:

$$F''(z) = H''\big(\phi(z)\big)\left(\frac{\pi e^{\frac{\pi}{2}z}}{(e^{\frac{\pi}{2}z}+1)^2}\right)^2 + H'\big(\phi(z)\big)\frac{\pi^2}{2}\frac{e^{\frac{\pi}{2}z}(1-e^{\frac{\pi}{2}z})}{(e^{\frac{\pi}{2}z}+1)^3}$$

In an equivalent way, for $h(s) = \Re F''(s \pm i)$ to be in $L^2(\mathbb{R})$ we will have enough with:

$$\int_{|z|=1} \left| H''(z)\frac{\pi i(z+1)(z-1)}{\big(i(z+1)+(z-1)\big)^2}\right|^2 \frac{|dz|}{|1-z^2|} < \infty \tag{3.7}$$

$$\int_{|z|=1} \left| H'(z)\frac{\pi^2}{2}\frac{i(z+1)(z-1)\big((z-1)-i(z+1)\big)}{\big(i(z+1)+(z+1)\big)^3}\right|^2 \frac{|dz|}{|1-z^2|} < \infty \tag{3.8}$$

Therefore we have to find a holomorphic function H in the disk with $H(\gamma) = 0$ for $\gamma \in \Gamma$, $H(\overline{z}) = \overline{H(z)}$ and so that (3.6), (3.7) and (3.8) are fulfilled. What we will make is to choose $H(z) = (1-z^2)^n\beta(z)$ with n big enough. We see first the bound of the derivatives of β:

$$\beta'(z) = \sum_{\gamma\in\Gamma} \frac{-\gamma}{|\gamma|}\frac{1-\gamma^2}{(1-\gamma z)^2}\prod_{\lambda\in\Gamma,\lambda\neq\gamma}\frac{-\lambda}{|\lambda|}\frac{z-\lambda}{1-\lambda z}$$

The product is bounded by 1 independently of γ almost for every z. In more, for $|z| = 1$ we have that $|1-\gamma z| \geqslant \frac{1}{2}|1-z^2|$ and $|1-\gamma^2| \leqslant 2(1-|\gamma|)$. Therefore:

$$|\beta'(z)| \leqslant \frac{2}{|1-z^2|^2}\sum_{\gamma\in\Gamma}|1-\gamma^2| \leqslant \frac{4}{|1-z^2|^2}\sum_{\gamma\in\Gamma} 1-|\gamma| \leqslant \frac{2K}{|1-z^2|^2}$$

where we have found again the Blaschke condition. For the second derivative we have:

$$\beta''(z) = 2\sum_{\gamma_1\neq\gamma_2\in\Gamma} \frac{-\gamma_1}{|\gamma_1|}\frac{1-\gamma_1^2}{(1-\gamma_1 z)^2}\frac{-\gamma_2}{|\gamma_2|}\frac{1-\gamma_2^2}{(1-\gamma_2 z)^2}\prod_{\lambda\in\Gamma,\lambda\neq\gamma_1,\gamma_2}\frac{-\lambda}{|\lambda|}\frac{z-\lambda}{1-\lambda z}+$$

$$+2\sum_{\gamma\in\Gamma}\frac{\gamma^2}{|\gamma|}\frac{1-\gamma^2}{(1-\gamma z)^3}\prod_{\lambda\in\Gamma,\lambda\neq\gamma}\frac{-\lambda}{|\lambda|}\frac{z-\lambda}{1-\lambda z}$$

that we can bound using the same ideas as in the case of the first derivative:

$$|\beta''(z)| \leqslant \frac{12K^2}{|1-z^2|^4}$$

If we choose $H(z) = (1-z^2)^4\beta(z)$ we will be in the mentioned conditions. $F(z) = H\big(\phi(z)\big)$ and $F''(z)$ will be in $E^2(B)$ with $F(\lambda) = 0$ for every $\lambda \in \Lambda$. In this way we have proved that $T(\psi, \Lambda)$ can not generate $L^2(\mathbb{R})$ if $\sum_{\lambda\in\Lambda} e^{\frac{\pi}{2}|\lambda|} < \infty$ and we obtain the necessity. $\square$

Proof of theorem 3.3. The sufficient condition is deduced from theorem 2.11.

For seeing that it is necessary we suppose that $T(\varphi, \Lambda)$ generates $L^1(\mathbb{R})$. Using theorem 1.4 we have that $T(\varphi * \widehat{P}, \Lambda)$ generates $L^2(\mathbb{R})$. If we calculate the Fourier transform of $\varphi * \widehat{P}$ we see that:

$$\left|\frac{\widehat{\varphi}(\xi)}{1+\xi^2}\right| \approx \frac{e^{-2\pi|\xi|}}{1+\xi^2}$$

Using 1.5, $\varphi * \widehat{P}$ and $P * \widehat{P}$ have the same sets of generators in $L^2(\mathbb{R})$. By the former lemma these coincide with those of P and $T(P, \Lambda)$ also generates $L^2(\mathbb{R})$. This tells that the summation has to be infinite. $\square$

3.3 Rational functions.

A special case in that we can use 3.3 are some of the rational functions of $L^1(\mathbb{R})$. The case of functions of the type:

$$\varphi(t) = \frac{1}{k^2 + (t-x)^2}$$

is solved by dilatation and shifting of P. But the same one we can say of functions of the type:

$$\varphi(t) = \sum_{n=1}^{N} \frac{c_n}{k_n^2 - (t-x_n)^2}$$

whenever $k_i \neq k_j$ if $i \neq j$. If we calculate Fourier transform we see that:

$$\widehat{\varphi}(\xi) = \sum_{n=1}^{N} c_n \frac{\pi}{k_n} e^{-2\pi k_n |\xi|} e^{-2\pi i x_n \xi}$$

For $|\xi|$ big enough we will have that $|\widehat{\varphi}(\xi)| \approx e^{-2\pi k|\xi|}$ if $k = \inf k_n$. If $\widehat{\varphi}(\xi) \neq 0$ for every ξ we can apply theorem 3.3 and we obtain:

Theorem 3.5. *Let*

$$\varphi(t) = \sum_{n=1}^{N} \frac{c_n}{k_n^2 - (t - x_n)^2}$$

with $k_i \neq k_j$ *if* $i \neq j$, $k_i > 0$ *for every* i.

We suppose that $\widehat{\varphi}(t) \neq 0$ *for every* $\xi \in \mathbb{R}$. *Then* $T(\varphi, \Lambda)$ *generates* $L^1(\mathbb{R})$ *if and only if:*

$$\sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2k}|\lambda|} = \infty$$

with $k = \inf k_n$.

As a matter of fact in this theorem the condition $k_i \neq k_j$ is only necessary for the smallest k_j, since this will be the dominant factor. Also when this is not isolated we can find cases in that we will be able to describe the sets that they generate. We consider:

$$\varphi(t) = \sum_{n=1}^{N} \frac{c_n}{1 + (t - x_n)^2}$$

The Fourier transform of this function will be:

$$\widehat{\varphi}(\xi) = \pi e^{-2\pi|\xi|} \sum_{n=1}^{N} c_n e^{-2\pi x_n \xi} = e^{-2\pi|\xi|} P_{\varphi}(\xi)$$

If all x_n are like $x_n = i\alpha$ for $i \in \mathbb{Z}$, P_φ is then a periodic trigonometric polynomial, and therefore we will only to ask that if it is different of zero in every point of the period. If this is not the case we will have an almost–periodic function that in general we will not be able to bound inferiorly, although sometimes it will be possible. The most general result that we can give is the following one:

Theorem 3.6. *Let*

$$\varphi(t) = \sum_{n=1}^{N} \sum_{m=1}^{M} \frac{c_{n,m}}{k_n^2 - (t - x_{n,m})^2}$$

with $0 < k_i < k_j$ *if* $i \neq j$.

We suppose that $\widehat{\varphi}(t) \neq 0$ *and* $|P_1(\xi)| = |\sum_{m=1}^{M} c_{n,m} e^{-2\pi i x_{1,m}\xi}| > C > 0$ *for every* $\xi \in \mathbb{R}$*. Then* $T(\varphi, \Lambda)$ *generates* $L^1(\mathbb{R})$ *if and only if:*

$$\sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2k_1}|\lambda|} = \infty$$

In the rest of cases it will be difficult to give results, and each particular case will have to be looked separately. In a general way whenever we see that $|\widehat{\varphi}(\xi)| \leqslant Ae^{-k_1|\xi|}$ we will be able to give sufficient conditions about Λ and when we have $|\widehat{\varphi}(\xi)| \geqslant Be^{-k_2|\xi|}$ we can give necessary conditions. If $k_1 \neq k_2$ we will not be able to give a complete characterization, but we will obtain partial results.

An example of the same style than these but where we will be able to give a complete result is the following one. We take $\varphi = P - P''$ as analyzing function. The Fourier transform of this function is:

$$\widehat{\varphi}(\xi) = (1 + 4\pi^2\xi^2)\widehat{\varphi}(\xi) = (1 + 4\pi^2\xi^2)e^{-\pi|\xi|}$$

that is different of zero in every point. The results mentioned before say that for $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$ it is needed that:

$$\sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2C}|\lambda|} = \infty$$

with $C = 1$, and it is a sufficient condition that it happens for $C > 1$. We will improve now this last condition in order to give a complete characterization for this case. The way for obtaining this result is the same that we are usually using; we will see that the phase space is contained in a bigger space but with the same sets of uniqueness than $E^q(B)$, and we will obtain the sufficiency. We already have the necessity, but also we will be able to obtain it seeing that $E^q(B)$ is contained in the phase space.

Lemma 3.7. *Let* $\varphi(t) = P(t) - P''(t)$ *with* P *the Poisson function. The phase space of* φ *for* $L^p(\mathbb{R})$ *is the restriction to the straight line of:*

$$E^q(B) - E^q(B)'' = \Big\{G : G(z) = F(z) - F''(z) \text{ with } F \in E^q(B)\Big\}$$

with q *the dual exponent of* p *and* $E^q(B)$ *the space defined in 3.2.*

Proof. This is clear if we take into account the definition of the phase space of P. □

Theorem 3.8. *Let* $\varphi(t) = P(t) - P''(t)$ *with* P *the Poisson function. Then* $T(\varphi, \Lambda)$ *generates* $L^2(\mathbb{R})$ *if and only if:*

$$\sum_{\lambda \in \Lambda} e^{-\frac{\pi}{2}|\lambda|} = \infty$$

Proof. We call H the phase space of φ for $L^2(\mathbb{R})$. If $F \in H$, then there is $f \in L^2(\mathbb{R})$ such that:

$$F(z) = \langle f, \varphi \rangle = \langle \widehat{f}, \widehat{\varphi} \rangle = \int_{\mathbb{R}} \widehat{f}(\xi)(1 + 4\pi^2\xi^2)e^{-2\pi|\xi|}$$

If $G \in E^2(B)$ then it is clear that G belongs to H looking at the former equation. It is just necessary to take $\widehat{f}(\xi) = \frac{\widehat{g}(\xi)}{(1+4\pi^2\xi^2)}$ with the g that gives place to G using as analyzing function the Poisson function. This is valid when $z \in \mathbb{R}$, but for analytical extension we have it on all the band. From here we can deduce the necessary condition. If a set Λ is of zeros for $E^2(B)$ also it will be it for H. Therefore, if the summation converges, Λ can not be of uniqueness for H.

For the sufficiency we go to see to which space belongs the second derivative of a function of $E^2(B)$.

First of everything we observe that for a holomorphic function in a disk of center 0 and radius R it is fulfilled that:

$$\int_{|z|\leqslant R} |f(z)|^2\, dA(z) = \int_0^R r \int_0^{2\pi} |f(re^{i\theta})|^2\, d\theta\, dr = \int_0^R r \sum_{n=0}^{\infty} |c_n|^2 r^{2n}$$

where c_n are the Taylor coefficients of f and we have used Parseval. We exchange the summation for the integral and we can arrive to:

$$\int_{|z|\leqslant R} |f(z)|^2\, dm(z) = \sum_{n=0}^{\infty} |c_n|^2 \int_0^R r^{2n+1} \geqslant |c_2|^2 \frac{R^6}{6}$$

This argument works in any disk, even if it is not centered on the zero, and c_2 is the second derivative of f in the center of the disk. This gives a punctual bound of f'' that we will be able to make uniform immediately in our case.

We take $F \in E^2(B)$. For any $z \in B$ the ball of center z and radius $\frac{1-|y|}{2}$ belongs to B and F is holomorphic in this ball. Applying the former inequality we can say that:

$$\int_B (1 - |y|)^4 |F''(z)|^2\, dm(z) \leqslant 2^4 \int_B \frac{4}{(1-|y|)^2} \int_{B\left(z, \frac{1-|y|}{2}\right)} |F(w)|^2\, dm(w)\, dm(z)$$

We apply Fubini and the former equation goes to being:

$$\int_B |F(w)|^2 \int_{\left\{z : w \in B\left(z, \frac{1-|y|}{2}\right)\right\}} \frac{4\, dm(z)}{(1-|y|)^2}$$

As $\left\{z : w \in B\big(z, \frac{1-|y|}{2}\big)\right\} \subset B(w, 1 - |\Im w|)$ and in this set

$$\frac{1}{\pi}\frac{1}{(1-|\Im w|)^2} \geqslant \frac{1}{4\pi}\frac{1}{(1-|y|)^2}$$

we can bound:

$$\int_{\left\{z:w\in B\left(z,\frac{1-|y|}{2}\right)\right\}} \frac{4\,dm(z)}{(1-|y|)^2} \leqslant \int_{B(w,1-|\Im w|)} \frac{16\pi\,dm(z)}{\pi(1-|\Im w|)^2} \leqslant 16\pi$$

Therefore we can affirm:

$$\int_B (1-|y|)^4 |F''(z)|^2\,dm(z) \leqslant 2^8\pi \int_B |F(w)|^2\,dm(w)$$

This last integral is bounded, since:

$$\int_B |F(z)|^2\,dm(z) = \int_{-1}^{1}\int_{\mathbb{R}} |F(x+iy)|^2\,dx\,dy \leqslant \int_{-1}^{1} \|F\|^2\,dy = 2\|F\|^2$$

(we remember that $\|F\|^2$ were the supreme of the norms in straight lines $y = const.$). This tells that F'' is in a Bergman type space. We go to translate this integral to the disk to give the results.

We can check out that if $w = \phi(z) = \frac{e^{\frac{\pi}{2}z}-1}{e^{\frac{\pi}{2}z}+1}$ then $1-|y| \geqslant 1-|w|$. We define $H(w) = F''(\phi^{-1}(w))$ and we can say that:

$$\begin{aligned}\int_{\mathbb{D}} |H(w)|^2(1-|w|)^3\,dm(w) &\leqslant \int_D (1-|w|)^4|H(w)|^2\frac{dm(w)}{|1-w^2|}\\ &\leqslant \int_B (1-|y|)^4|F''(z)|^2\,dm(z)\end{aligned}$$

Therefore H is in the Bergman space of the disk with weight $(1-|w|)^3$. A subset of zeros contained in a diameter of a function of this space has to fulfill the Blaschke condition [Kor75]. We remember that $\Gamma = \phi(\Lambda)$ fulfill the Blaschke condition in the disk if and only if:

$$\sum_{\lambda\in\Lambda} e^{-\frac{\pi}{2}|\lambda|} < \infty$$

It is easy to see that F is also in the Bergman type space of the band. As the summation diverges Λ can not belong to any set of zeros of this space, and therefore neither to $E^2(b) - E^2(B)''$. Then we have completed the proof of the theorem. ☐

Remark. We can give the same statement for $L^1(\mathbb{R})$. As we already had the necessity, we only need to see the sufficiency, and for this we can use theorem 1.4.

Chapter 4

Generator systems by translations with the Gaussian function.

In this chapter we will study for which sets Λ the set $T(\varphi, \Lambda)$ generates $L^p(\mathbb{R})$ when we restrict to the Gaussian function $\phi(t) = 2^{\frac{1}{4}} e^{-\pi t^2}$. This case can be treated because we will find a space of entire functions, and we can use complex analysis, mainly the Jensen formula, the Hadamard representation and the Lindelöf theorems.

4.1 Growth and zeros of entire functions.

In this section we will revise a series of concepts and classical results about entire functions. These results relate the growth at infinite of these functions with its zeros, precisely with the amount of these. This will be the theoretical basis that we will use to study the sets of translations of the Gaussian function that can generate in $L^p(\mathbb{R})$.

For this summary we will follow [Lev96]. We will attempt to sustain the notation. In this reference we can find all the proofs of the results that we present here without it. We start with some definitions.

Definition. Let f be a entire function. We define:

$$M_f(r) = \sup_{|z|=r} |f(z)|$$

Definition. Let f be a entire function. We define the **order** (of growth) of f as:

$$\rho = \limsup_{r\to\infty} \frac{\log\log M_f(r)}{\log r}$$

Intuitively it means that f grows like e^{r^ρ}.

Definition. Let f be a entire function of order ρ. We define the **type** of f as:

$$\sigma = \limsup_{r\to\infty} \frac{\log M_f(r)}{r^\rho}$$

Intuitively it means that f grows like $e^{\sigma r^\rho}$.

Definition. Given a succession $a_1, a_2, \dots, a_n, \dots \in \mathbb{C}$ with $\lim_{n\to\infty} |a_n| = \infty$, $a_n \neq 0\ \forall n$, the **convergence exponent** of $\{a_n\}$ (ρ_1) is the greatest lower bound of λ's such that:

$$\sum_n \frac{1}{|a_n|^\lambda} < \infty$$

If $n(r)$ is the counting function of $\{a_n\}$, the convergence exponent of $\{a_n\}$ then coincides with the order of $n(r)$:

$$\rho_1 = \limsup_{r\to\infty} \frac{\log n(r)}{\log r}$$

Definition. Given a discrete set $\Lambda \subset \mathbb{R}$ with counting function $n(r)$, we define the **upper density** of Λ with respect to ρ as:

$$\Delta^+(\Lambda) = \limsup_{r\to\infty} \frac{n(r)}{r^\rho}$$

and the **lower density** as:

$$\Delta^-(\Lambda) = \liminf_{r\to\infty} \frac{n(r)}{r^\rho}$$

$\Delta_f^\pm$ will be the (upper or lower) density of the zero set of f.

Remark. If the convergence exponent of Λ is less than ρ, automatically both densities are infinite, and if it is greater they are equal to 0. For this motive ρ does not turn up in the notation. It is easy to prove that even if the convergence exponent is equal to ρ, if $\sum 1/|\lambda_n|^\rho < \infty$, then both densities are also equal to 0.

The fundamental result that relates the growth of a function with its zero set is the **Jensen formula**:

Theorem 4.1 (Jensen)**.** *Let f be a entire function such that $f(0) \neq 0$. Let $\{a_m\}_{m\in\mathbb{N}}$ be its zero set and $n(r)$ the counting function of this set. Then:*

$$\int_0^R \frac{n(t)}{t}\,dt = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})|\,d\psi - \log|f(0)|$$

or in an equivalent way:

$$\log|f(0)| = \frac{1}{2\pi}\int_0^{2\pi} \log|f(Re^{i\psi})|\,d\psi + \sum_{|a_m|<R} \log\frac{|a_m|}{R}$$

The first corollary that is deduced in trivial way of this result says, assuming $f(0) = 1$, that:

$$\log M_f(er) \geqslant \int_0^{er} \frac{n(t)}{t}\,dt \geqslant \int_r^{er} \frac{n(t)}{t} \geqslant n(r)$$

and we obtain:

$$n(r) \leqslant \log M_f(er)$$

This inequality gives a direct relationship between the zero set of a function and its growth. We can deduce now the Hadamard theorem:

Theorem 4.2 (Hadamard)**.** *The convergence exponent of the zero set of a entire function does not exceed its order of growth.*

For improving this result we have to construct functions with a prescribed zero set. We can make this with the Weierstrass products:

Definition. We define

$$G(u,p) = \begin{cases} 1-u, & p = 0 \\ (1-u)\exp\bigl(u + \frac{u^2}{2} + \cdots + \frac{u^p}{p}\bigr), & p > 0 \end{cases}$$

We name **Weierstrass primary factors** to this functions.

For a set $\{a_n\} \subseteq \mathbb{C}$ such that $\sum |a_n|^{-p-1} < \infty$ with p a non negative integer we define the **Weierstrass canonical product** associated to $\{a_n\}$ as:

$$\Pi(z) = \prod_n G\Bigl(\frac{z}{a_n}, p\Bigr)$$

Under these conditions the product is uniformly convergent in closed disks and therefore defines a entire function in $\mathbb{C}$ that only cancels in $\{a_n\}$.

The Borel theorem gives the order of these products:

Theorem 4.3 (Borel)**.** *The order of convergence of a canonical product is equal to the convergence exponent of its zero set.*

Practically we only are interested in this type of functions. We see this thanks to the Hadamard decomposition:

Theorem 4.4 (Hadamard)**.** *A entire function of finite order* ρ *can be represented as:*

$$f(z) = z^m e^{P_q(z)} \prod_{n=1}^{\infty} G\Big(\frac{z}{a_n}, p\Big)$$

where $a_1, a_2, \dots$ *are the roots of* f *different from zero,* $p \leqslant \rho$*,* $P_q(z)$ *is a polynomial in* z *of degree* $q \leqslant \rho$*, and* m *is the multiplicity of the root at the origin.*

If we multiply two entire functions f and g, the order of the resulting product will be the maximum of the orders of f and g. The Borel theorem calculates the order of a Weierstrass canonical product, and that of the exponential of a polynomial is trivial. In this way it is not very difficult to calculate the order of a function when we have the Hadamard decomposition.

To calculate the type will be more difficult in general. If in the Hadamard decomposition the degree of P_q is bigger that the order of the zero set then is trivial to calculate the type, which is the coefficient of the maximum degree term of the polynomial. If we are not in these conditions we will need the Lindelöf theorems:

Theorem 4.5 (Lindelöf)**.** *Let* f *be a entire function with Hadamard decomposition*

$$f(z) = z^m e^{P_q(z)} \prod_{n\in\mathbb{N}} G\Big(\frac{z}{a_n}, p\Big)$$

with P_q *a polynomial of degree* q*. If the degree of* P_q *is less or equal to* $p+1$ *(integer), the order of convergence of the zero set of* f *(*$\Lambda = \{a_n\}_{n\in\mathbb{N}}$*) is also* $p+1$ *and* $\sum |a_n|^{-(p+1)} < \infty$*, then* f *is a function of order* $p+1$ *and minimal type if* $b_{p+1} = 0$ *or of mean type if* $b_{p+1} \neq 0$*, where* $P_q(z) = b_0 + b_1 z + \cdots + b_q z^q$*.*

Theorem 4.6 (Lindelöf)**.** *Let* $f(z)$ *be a entire function of entire order* p *with Hadamard decomposition:*

$$f(z) = z^m e^{b_0 + b_1 z + \cdots + b_p z^p} \prod G\Big(\frac{z}{a_n}, p\Big)$$

such that $\sum \frac{1}{|a_n|^p} = \infty$*. Let:*

$$\delta_f(r) = \left| b_p + \frac{1}{p} \sum_{|a_n|<r} a_n^{-p} \right|, \qquad \overline{\delta}_f = \limsup_{r\to\infty} \delta_f(r) \tag{4.1}$$

and $\gamma_f = \max(\Delta_f^+, \overline{\delta}_f)$*. Then* γ_f *and the type of* f *are simultaneously zero, infinite or positive numbers.*

The second Lindelöf theorem relates the type of f with the density of its zero set and the compensation condition (4.1). But we obtain that the type will be finite and positive if the density and (4.1) also they are it. In [Lev96] we can find explicit bounds looking at the proof of the result, but these are not very good. The following definition will be very useful to study the functions of mean type (finite and different of zero).

Definition. Let f be a entire function of finite order ρ. We define the **indicator function** of f (with respect to the order ρ) as:

$$h_f(\theta) = \limsup_{r\to\infty} \frac{\log|f(re^{i\theta})|}{r^\rho}$$

It is not very difficult to see that $h_{fg}(\theta) \leqslant h_f(\theta) + h_g(\theta)$, and we can also give the bound $h_{f+g}(\theta) \leqslant \max\{h_f(\theta), h_g(\theta)\}$. For a function of mean type [Lev80] can also be proved that this function is bounded from up and below. We can also see that:

$$\sup_\theta h_f(\theta) = \sigma_f$$

This tells that if we know h_f we have the type of f. Moreover this bounds the indicator function. We define the auxiliary function:

$$h_{f,r}(\theta) = \frac{\log|f(re^{i\theta})|}{r^\rho}$$

Definition. Let f be a entire function of finite order ρ. We will say that f has **completely regular growth** if almost for every θ there exists the following limit:

$$\lim_{r\to\infty} h_{f,r}(\theta) = h_f(\theta)$$

or in an equivalent way, if:

$$\log|f(re^{i\theta})| = h_f(\theta)r^\rho + o(r^\rho)$$

The Jensen formula 4.1 says us that:

$$\frac{N_f(r)}{r^\rho} = \int_0^{2\pi} h_{f,r}(\theta)\, d\theta - \frac{\log|f(0)|}{r^\rho}$$

where $N_f(r) = \int_0^r \frac{n_f(t)}{t}\, dt$, with n_f the counting function of the zero set of f. Using this we can give [Lev80] the following results to bound the density of the zero set of a function of mean type.

Theorem 4.7 (Levin). *Let f be a not identically zero entire function of order $\rho > 0$. Then:*

$$\limsup_{r\to\infty} \frac{n_f(r)}{r^\rho} \leqslant \frac{e\rho}{2\pi} \int_0^{2\pi} h_f(\theta)\, d\theta$$

Theorem 4.8 (Levin)**.** *Let f be a not identically zero entire function of order $\rho > 0$. Then:*

$$\liminf_{r\to\infty} \frac{n_f(r)}{r^\rho} \leqslant \frac{\rho}{2\pi} \int_0^{2\pi} h_f(\theta)\, d\theta$$

We have equality for functions of completely regular growth and only in this case.

Definition. Let Λ be a discrete set of $\mathbb{C}$ of finite order ρ. Let $n(r; \psi_1, \psi_2)$ be the number of points of Λ in the sector $\{z : |z| \leqslant r, \psi_1 \leqslant \arg z < \psi_2\}$. We define the **angular density** of Λ:

$$\Delta_\Lambda(\psi_1, \psi_2) = \lim_{r\to\infty} \frac{n(r; \psi_1, \psi_2)}{r^\rho}$$

when this limit exists almost for every $\psi_1, \psi_2 \in [0, 2\pi)$. In this case we will say that Λ has angular density.

The equation:

$$\Delta(\psi_1) - \Delta(\psi_2) = \Delta_\Lambda(\psi_1, \psi_2)$$

defines a function $\Delta(\theta)$ (except a constant) which we will use as angular density.

It can be proved that a function of non entire order has completely regular growth if and only if its zero set has angular density. For functions of entire order (those that will interest us) the result is a little more complicated, since it is necessary the convergence of (4.1).

Theorem 4.9 (Levin,Pfluger)**.** *Let f be a entire function of integer order p with Hadamard decomposition*

$$f(z) = z^m e^{b_0 + b_1 z + \cdots + b_p z^p} \prod G\Big(\frac{z}{a_n}, p\Big)$$

We suppose that its zero set $\{a_n\}_{n\in\mathbb{N}}$ has angular density and there exists the following limit:

$$\delta_f = \lim_{r\to\infty} \delta_f(r), \qquad \delta_f(r) = b_p + \frac{1}{p} \sum_{|a_n|<r} a_n^{-p} \tag{4.2}$$

Then f has completely regular growth and we can calculate its indicator function:

$$h_f(\theta) = \int_{\theta-2\pi}^{\theta} (\theta - \psi) \sin p(\psi - \theta)\, d\Delta(\psi) + \tau_f \cos p(\theta - \theta_f)$$

with $b_p = \tau_f e^{i\theta_f}$. In these conditions $\limsup_{r\to\infty} h_{f,r}(\theta)$ converges uniformly in $h_f(\theta)$ almost in every $[0, 2\pi]$.

It is more, if f has completely regular growth then its zero set fulfills the conditions mentioned here.

4.2 The Fock space and the $L^2(\mathbb{R})$ case.

The tools that we have commented in the former section will be useful for studying the zero sets of the Fock space. These sets are related with the sets Λ for which $T(\phi, \Lambda)$ is a generator system for $L^2(\mathbb{R})$, which is the first case that we will treat. We begin defining this space.

Definition. We define the **Fock space** of entire functions as the set of functions:

$$\mathcal{F} = \left\{ f : f \text{ is entire and} \|f\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |f(z)|^2 e^{-\pi|z|^2} \, dm(z) < \infty \right\}$$

This is one of the classical spaces of holomorphic functions. It has been studied from different points of view. For example its sets of sampling and of interpolation are characterized, which will be very useful when we study the Gabor transform. But now we are interested in its zero sets.

Definition. We will say that $\Lambda = \{a_n\}_{n\in\mathbb{N}}$ is a **zero set** of the Fock space if $\exists f \in \mathcal{F}$ such that:

$$\begin{aligned} f(z) &= 0 \qquad z \in \Lambda \\ f(z) &\neq 0 \qquad z \notin \Lambda \end{aligned}$$

That is, there is $f \in \mathcal{F}$ such that f cancels in all Λ and nowhere more.

The Bargmann transform gives the relationship among the Fock space and the systems of translations of the Gaussian functions:

Definition. Given $f \in L^2(\mathbb{R})$, we define the **Bargmann transform** of f as:

$$Bf(z) = 2^{\frac{1}{4}} \int_{\mathbb{R}} f(t) e^{2\pi tz - \pi t^2 - \frac{\pi}{2} z^2} \, dt$$

Theorem 4.10 (Bargmann). *The Bargmann transform is an isomorphism between $L^2(\mathbb{R})$ and the Fock space:*

$$\|Bf\|_{\mathcal{F}} = \|f\|_2$$

The proof can be found in [Fol89]. If now we calculate this transform for a value $x \in \mathbb{R}$ we see that:

$$Bf(x) = 2^{\frac{1}{4}} \int_{\mathbb{R}} f(t) e^{2\pi tx - \pi t^2 - \frac{\pi}{2} x^2} \, dt = 2^{\frac{1}{4}} \int_{\mathbb{R}} f(t) e^{-\pi\left((t-x)^2 - \frac{x^2}{2}\right)} \, dt$$

Therefore the Bargmann transform of a function f in the real axis is the scalar product of this function with a translation the Gaussian function ϕ, multiplied by a constant that is always different of zero:

$$Bf(x) = 2^{\frac{1}{4}} e^{\frac{\pi}{2}x^2} \langle f, \phi_x \rangle$$

This says that $Bf(x) = 0$ if and only if $\langle f, \phi_x \rangle = 0$. As $L^2(\mathbb{R})$ is a Hilbert space, using duality we can prove the following statement:

Proposition 4.11. *The set $T(\varphi, \Lambda)$ generates $L^2(\mathbb{R})$ if and only if Λ is not included in a zero set for the Fock space.*

Proof. Using duality we have that $T(\varphi, \Lambda)$ generates $L^2(\mathbb{R})$ if and only if $\langle f, \varphi_\lambda \rangle = 0$ for every λ implies that $f = 0$. As the Bargmann transform is an isomorphism, this last condition is equivalent to $F(\lambda) = 0$ for every λ implying that $F = 0$ for every $F \in \mathcal{F}$. But if Λ is in a zero set we have that there is $F \in \mathcal{F}$, $F \neq 0$, with $F(\lambda) = 0$ for every $\lambda \in \Lambda$. $\square$

This relation justifies that we become interested in the zero sets of this space. Zhu already studied them in [Zhu93], where it proved the first results that we will give here. He also gave examples that proves that a subset of a zero set does not have to continue being it. Later, in [Tun05] is proved that a set of interpolation either is necessary that it is a zero set for the Fock space.

The Jensen formula and the rest of theoretical results that we have remembered in the former section related the order of growth of a function with the convergence exponent and other properties of its zero set. The following result gives the way to use these theorems.

Proposition 4.12 (Zhu). *If $f \in \mathcal{F}$ then f has order less or equal to 2 and if it has order 2 has type less or equal to $\frac{\pi}{2}$.*

Proof. We define:

$$f_w(z) = f(z+w)e^{-\frac{\pi}{2}|w|^2 - \pi z\overline{w}}$$

With this definition we have $f_w \in \mathcal{F}$ and in more $\|f_w\| = \|f\|$. As f_w is a holomorphic function we have that $|f_w|^2$ is subharmonic and therefore:

$$\begin{aligned}
|f_w(0)|^2 = \left| f(w)e^{-\frac{\pi}{2}|w|^2} \right|^2 &\leqslant \int_0^{2\pi} \left| f(re^{i\psi}+w)e^{-\frac{\pi}{2}|w|^2 - \pi re^{i\psi}\overline{w}} \right|^2 d\psi \\
&= \frac{1}{\pi}\int_0^\infty re^{-\pi r^2} \int_0^{2\pi} \left| f(re^{i\psi}+w)e^{-\frac{\pi}{2}|w|^2 - \pi re^{i\psi}\overline{w}} \right|^2 d\psi \\
&= \frac{1}{\pi}\int_{\mathbb{C}} \left| f(z+w)e^{-\frac{\pi}{2}|w|^2 - \pi z\overline{w}} \right|^2 e^{-\pi|z|^2}\, dz \\
&= \frac{1}{\pi}\int_{\mathbb{C}} |f(z+w)|^2 e^{-\pi|z+w|^2}\, dz = \frac{1}{\pi}\|f\|^2
\end{aligned}$$

Therefore:

$$|f(w)| \leqslant \frac{1}{\pi^{1/2}} \|f\| e^{\frac{\pi}{2}|w|^2}$$

that proves the result. □

Remark. It is easy to see that if a function f has order strictly less than 2 then $f \in \mathcal{F}$, and if f has order 2 and type strictly less than $\frac{\pi}{2}$ then also belongs to $\mathcal{F}$. However, among the functions of order 2 and type $\frac{\pi}{2}$ there is that they are in the Fock space and other that not.

Theorem 4.13 (Zhu)**.** *If $\Lambda = \{a_n\}_{n\in\mathbb{N}}$ is a zero set for the Fock space, the convergence exponent of Λ is then less or equal to 2. That is:*

$$\sum_{n\in\mathbb{N}} \frac{1}{|a_n|^{2+\varepsilon}} < \infty \qquad \forall \varepsilon > 0$$

Proof. If the convergence exponent of $\{a_n\}$ is greater than 2, its Weierstass product will have also order greater than 2. Therefore any function of finite order that cancels in $\{a_n\}$ will have order greater than 2 and will not be able to belong to the Fock space. □

This gives a necessary condition for being a zero set for the Fock space. We can also give sufficient conditions.

Theorem 4.14. *Let $\Lambda = \{a_n\}_{n\in\mathbb{N}}$. If*

$$\sum_{n\in\mathbb{N}} \frac{1}{|a_n|^{2}} < \infty$$

then Λ is a zero set for the Fock space.

Proof. We take directly the canonical product of the set Λ. If the order of convergence is strictly less than 2 then the Borel theorem 4.3 says that the order of this function is strictly less than 2. Therefore it is in the Fock space. However, if the order of convergence of Λ is exactly 2, as it is an integer, we can use the first theorem of Lindelöf 4.5 that says that the function will have order 2 and type 0 and therefore also will be in the Fock space. □

The translation from these results by the Bargmann transform give automatically necessary and sufficient conditions for a set Λ to generates with the Gaussian function.

Theorem 4.15 (Zalik)**.** *We consider a discrete set $\Lambda = \{\lambda_n\}_{n\in\mathbb{N}}$. It is a necessary condition so that a set of functions $T(\varphi, \Lambda)$ generates that:*

$$\sum_{n\in\mathbb{N}} \frac{1}{|\lambda_n|^2} = \infty$$

It is a sufficient condition that for some $\varepsilon > 0$:

$$\sum_{n\in\mathbb{N}} \frac{1}{|\lambda_n|^{2+\varepsilon}} = \infty$$

Proof. We take the necessary and sufficient conditions for a set to be of zeros for the Fock space and we use 4.11. ☐

We can find this result in [Zal78], where is proved using techniques close to these, but without using the isomorphism with the Fock space.

It rest to study the case in that $\sum \frac{1}{|\lambda_n|^{2+\varepsilon}} < \infty$ for every $\varepsilon > 0$ but $\sum \frac{1}{|\lambda_n|^2} = \infty$. This case has to be treated more carefully. These last results tell that if the convergence exponent of Λ is greater than 2, then $T(\phi, \Lambda)$ generates and can not be contained in any zero set of the Fock space. And if it is less, then it does not generate and it can be a zero set. It left the case equal to 2, which is the critic one and had not been studied until now.

4.3 Sets with order of convergence 2.

For studying the critical case we have to introduce a concept that allows us to differentiate discrete sets. The densities defined in the first section will play this role.

Theorems 4.13, 4.14 and 4.15 classify all the cases except convergence exponent equal to 2 and density greater than 0. Using the Hadamard factorization we can see that given a set Λ with convergence exponent 2 we will always be able to construct a function of order 2 such that its zero set is Λ. We are interested in the type of this function, since this will help us to know if it is or not in the Fock space.

In this sense the Fock space is entailed in a very particular way and that complicates the study of this problem a lot. The main obstacle comes from the fact that a subset of a zero set for the Fock space does not have that to be necessarily a zero set of the Fock space [Zhu93]. However, we can give restrictions about the density that can have a zero set.

Theorem 4.16. *If Λ is a zero set for the Fock space, then $\Delta^+(\Lambda) \leqslant e\pi$ and $\Delta^-(\Lambda) \leqslant \pi$.*

Proof. The indicator function of a function of the Fock space is bounded by $\frac{\pi}{2}$. If we introduce this in theorems 4.7 and 4.8 we obtain the bounds for the upper and lower density respectively. $\square$

This tells that if a set Λ has densities above these values it will not be able to be a zero set nor be able to be contained in any of them. We can also state the following result referring to generators:

Theorem 4.17. *Let $\Lambda \subset \mathbb{R}$ be a discrete set. If $\Delta^-(\Lambda) > \pi$ or $\Delta^+(\Lambda) > e\pi$ then $T(\phi, \Lambda)$ is a generator system for $L^2(\mathbb{R})$.*

To obtain necessary conditions we will look at sets with angular density, since these allow us to calculate the indicator function of any function that cancels in these set.

If we have a discrete set $\Lambda \in \mathbb{R}$ such that $\Delta^+(\Lambda) = \Delta^-(\Lambda) \neq 0$ and we calculate:

$$\sum_{\lambda\in\Lambda} \frac{1}{\lambda^2} = \sum_{\lambda\in\Lambda} \frac{1}{|\lambda|^2} = \infty$$

The second theorem of Lindelöf 4.6 says that this set will ever not be able to be a zero set of a function of the Fock space. The way to act in this case will be to construct a function with a zero set Γ that includes this first and so that the condition (4.2) is fulfilled (as a matter of fact we will make $\delta_f = 0$). In this way we will be able to calculate the indicator function and to know if the function is in the Fock space or not. This will be useful to give necessary conditions so that $T(\phi, \Lambda)$ generates $L^2(\mathbb{R})$.

Lemma 4.18. *Let $\Lambda \subseteq \mathbb{R}$ a discreet set such that:*

$$\lim_{r\to\infty} \frac{n^+(r)}{r^2} = \lim_{r\to\infty} \frac{n^-(r)}{r^2} = \Delta$$

where n^+ and n^- are the counting functions of the positive and negative part of Λ respectively, and we are asking for the existence of the limit as well as the coincidence.

Then there is a entire function f of order 2 and type $\pi\Delta$ such that $f(\lambda) = 0$ for every $\lambda \in \Lambda$.

Proof. We construct the set $\Gamma = \Lambda \cup e^{\frac{\pi}{2}i}\Lambda$. In the conditions of the statement this set has angular density

$$\Delta(\theta) = \Delta \sum_{k=0}^{3} \chi_{[\frac{k\pi}{2}, 2\pi)}(\theta)$$

Moreover,

$$\sum_{\gamma\in\Gamma,|\gamma|<r} \frac{1}{\gamma^2} = 0 \qquad \forall r$$

since in the summation we find each term in the way λ^{-2} with $\lambda \in \Lambda$ with positive and negative sign and we have cancellation. We are therefore in conditions of using theorem 4.9 with the function

$$f(z) = \prod_{\gamma\in\Gamma} G\Big(\frac{z}{\gamma}, 2\Big)$$

For this function $d\Delta(\theta) = \sum_{k=0}^{3} \delta_{\frac{k\pi}{2}}$ (spreading in a 2π-periodic way) and we can calculate its indicator function:

$$h_f(\theta) = 2\pi\Delta \sin\theta \cos\theta$$

when $\theta \in [0, \frac{\pi}{2}]$, and it spreads in a $\frac{\pi}{2}$-periodic way to the rest of $[0, 2\pi]$. As

$$\sup_{\theta\in[0,2\pi]} h_f(\theta) = \pi\Delta$$

we see that f fulfills the conditions of the statement and it cancels in Λ. ☐

Theorem 4.19. *Let $\Lambda \subseteq \mathbb{R}$ be a discrete set such that $T(\phi, \Lambda)$ generates $L^2(\mathbb{R})$. Then:*

$$\Delta^+(\Lambda^+) \geqslant \frac{1}{2} \quad \acute{o} \quad \Delta^+(\Lambda^-) \geqslant \frac{1}{2}$$

Proof. We suppose that both $\Delta^+(\Lambda^+)$ and $\Delta^+(\Lambda^-)$ are less than $\frac{1}{2} - \varepsilon$. Adding points we can construct $\widetilde{\Lambda}$ such that:

$$\Delta^+(\widetilde{\Lambda}^+) = \Delta^+(\widetilde{\Lambda}^-) = \frac{1}{2} - \varepsilon$$

The former lemma tells that there is a function in the Fock space (it has type less than $\frac{\pi}{2}$) that cancels in Λ. The isomorphism of the Bargmann transform says then that there is a function $f \in L^2(\mathbb{R})$ orthogonal to $T(\phi, \Lambda)$ and this can not be a generator system. ☐

4.4 The $L^1(\mathbb{R})$ case.

The isomorphism given by the Bargmann transform only is useful to study the generator systems of translations of the Gaussian function when the space in that we work is $L^2(\mathbb{R})$. If we want to generalize the results obtained to the other $L^p(\mathbb{R})$ spaces we will find with two clearly different cases depending on if we search necessary or sufficient conditions.

We can define the Bargmann transform of any function of $L^p(\mathbb{R})$, $1 < p \leqslant \infty$, and see that the functions that we obtain are entire, of order as much 2 and type bounded by $\frac{\pi}{2}$. As the only property of the Fock space that we have used for giving conditions was that we were under these assumptions, this reasoning will allow us to generalize the sufficient conditions for all $L^p(\mathbb{R})$.

If we are interested in the necessary conditions the situation is not so good. To prove these results what we made was to construct a function in the Fock space that canceled in a set that does not fullfill these properties. The isomorphism of the Bargmann transform said then that there was an orthonormal function to the set of translations of the Gaussian function. We will just be able to make this in $L^2(\mathbb{R})$. We will be able to generalize the necessary conditions to the $L^1(\mathbb{R})$ case using theorem 1.4 and observing that the convolution of a Gaussian function with itself is another Gaussina function.

Lemma 4.20. *Let $f \in L^p(\mathbb{R})$, $1 < p \leqslant \infty$. Then the function*

$$Bf(z) = 2^{\frac{1}{4}} \int_{\mathbb{R}} f(t) e^{2\pi tz - \pi t^2 - \frac{\pi}{2} z^2}\, dt$$

is an entire function and

$$|Bf(z)| \leqslant \|f\|_p \|\phi\|_q e^{\frac{\pi}{2}|z|^2}$$

with $\frac{1}{p} + \frac{1}{q} = 1$.

Proof. If we look at the Bargmann transform of f we have that:

$$Bf(z) = e^{\frac{\pi}{2} z^2} \int_{\mathbb{R}} f(t) e^{-\pi(t-z)^2}\, dt$$

that is a convolution of f with an entire function. As the integrals will always be well defined this convolution defines an entire function in z for every f.

For the bound we observe that:

$$e^{2\pi tz - \pi t^2 - \frac{\pi}{2} z^2} = e^{\frac{\pi}{2}|z|^2} e^{-\pi(t-x)^2} e^{2\pi andty - \pi ixy}$$

and therefore:

$$|Bf(z)| \leqslant e^{\frac{\pi}{2}|z|^2} \int_{\mathbb{R}} |f(t)| \left| e^{-\pi(t-x)^2} \right| dt$$

Here we apply Hölder and we have the bound of the statement. □

Theorem 4.21. *Let $\Lambda \subset \mathbb{R}$ be a discrete set. If $\Delta^-(\Lambda) > \pi$ or $\Delta^+(\Lambda) > e\pi$ then $T(\phi, \Lambda)$ is a generator system for $L^p(\mathbb{R})$, $1 \leqslant p < \infty$.*

Proof. We fix $1 \leqslant p < \infty$ and suppose that there is $f \in L^q(\mathbb{R})$ $(\frac{1}{p} + \frac{1}{q} = 1)$ such that:

$$\int_{\mathbb{R}} f(t)\phi(t-\lambda)\,dt = 0 \qquad \forall\lambda \in \Lambda$$

This is equivalent to the existence of an entire function of order 2 and type $\frac{\pi}{2}$ (or less) that cancels in Λ. But using theorems 4.7 and 4.8 we see that under the conditions of the statement this function has to be null and $T(\phi, \Lambda)$ generates $L^p(\mathbb{R})$. □

We only will be able to generalize the necessary conditions to the $L^1(\mathbb{R})$ case. We observe first that if we define:

$$\phi_a(t) = 2^{\frac{1}{4}} e^{-a\pi t^2}$$

when $\frac{1}{a} + \frac{1}{b} = 1$, using the Fourier transform, we can see that:

$$\phi_a * \phi_b(t) = \frac{(1-a)^{\frac{1}{2}}}{a}\phi(t)$$

Theorem 4.22. *Let $\Lambda \subseteq \mathbb{R}$ be a discrete set such that $T(\phi, \Lambda)$ generates $L^1(\mathbb{R})$. Then:*

$$\Delta^+(\Lambda^+) \geqslant \frac{1}{2} \quad \acute{o} \quad \Delta^+(\Lambda^-) \geqslant \frac{1}{2}$$

Proof. If $T(\phi, \Lambda)$ generates $L^1(\mathbb{R})$, $T(\phi_a, \frac{1}{a}\Lambda)$ also generates for any $a > 1$. Making convolution with ϕ_b theorem 1.4 says that $T(\phi, \frac{1}{a}\Lambda)$ generates any $L^p(\mathbb{R})$ $(1 \leqslant p < \infty)$. In particular it generates $L^2(\mathbb{R})$ and theorem 4.19 gives the condition:

$$\Delta^+(\frac{1}{a}\Lambda^+) \geqslant \frac{1}{2} \quad \text{ó} \quad \Delta^+(\frac{1}{a}\Lambda^-) \geqslant \frac{1}{2}$$

But as $\Delta^+(\frac{1}{a}\Lambda^+) = \frac{1}{a^2}\Delta^+(\Lambda^+)$ and the same one happens with Λ^-. As this has to be fulfilled for every $a > 1$ and we have the proof of the statement. □

As we see, we arrive to the same conditions, both necessary and sufficient, for Λ than in the $L^2(\mathbb{R})$ case. It seems difficult to arrive to a complete description of the sets which they allow to generate due to the complicated structure of the zero sets of the Fock space. This description should depend on the density of the set. We can try a partial description restricting just to sets where both densities (the upper and the lower ones) coincide, but this does not help us very much, since it can be contained in a zero set where these do not coincide.

A problem that seems to be more reachable is the following one:

Question. $T(\phi, \Lambda)$ generates $L^p(\mathbb{R})$ independently of p?

That is, the sets Λ for which $T(\phi, \Lambda)$ generates $L^p(\mathbb{R})$ are the same than in $L^q(\mathbb{R})$ for any $1 \leqslant p, q < \infty$?

When we look to the Poisson function this result was true, but here, for the time being, we can not give an answer. The Bargmann transform, which allows us to know the phase space, only is a one to one correspondence in $L^2(\mathbb{R})$. Even so it seems that it can give a valuable information about the rest of phase spaces, when we work in $L^p(\mathbb{R})$ with $p \neq 2$.

Part II

Frames in $L^2(\mathbb{R})$.

Chapter 5

Gabor transform.

5.1 Discretization in Gabor phase spaces.

We introduce the Gabor transform in order to use the $L^2(\mathbb{R})$ Hilbert space structure. In this case we will be interested in finding bases and frames instead of sets of generators, since for these last it is not necessary to introduce modulations, because with translations it is enough.

We comment first on the notations that we will use in this chapter, and that we have defined in the preliminaries. When we work with Gabor transform we will take $z = x + iy \in \mathbb{C}$ and we will write

$$\rho(z)f(t) = f_z(t) = e^{2\pi iyt} f(t-x)$$

We will also use the notation $dm(z) = dx\,dy$ to designate the area measure of $\mathbb{C}$. In this way we define the Gabor transform of a function $f \in L^2(\mathbb{R})$ for an analyzing Gabor atom $g \in L^2(\mathbb{R})$ as:

$$Gf(z) = \langle f, g_z \rangle = \int_{\mathbb{R}} f(t) e^{-2\pi ity} \overline{g(t-x)}\, dt$$

Theorem 1.14 says that $Gf \in L^2(\mathbb{C})$ and $\|Gf\| = \|f\|$, with the following reconstruction formula:

$$f(t) = \int_{\mathbb{R}} \int_{\mathbb{R}} Gf(z) g_z(t)\, dm(z) \quad \forall f \in L^2(\mathbb{R})$$

When we fix the analyzing atom g, theorem 1.29 says that the set of functions of $L^2(\mathbb{C})$ that are transformed of some $f \in L^2(\mathbb{R})$ form a Hilbert subspace characterized by a reproductive kernel. This says that if

$$H = \left\{ F(z) \in L^2(\mathbb{C}) : \exists f \in L^2(\mathbb{R}) \text{ with } F(z) = Gf(z) = \langle f, g_z \rangle \right\}$$

then:

$$F(z_0) = \int_{\mathbb{C}} F(z)e^{2\pi i x(y-y_0)}k(z_0 - z)\,dm(z)$$

with $k(z) = \langle g, g_z\rangle$. This formula is special type of convolution called twisted convolution.

Using these notations we can deduce that:

$$Gf_{z_0}(z) = \langle f_{z_0}, g_z\rangle = e^{2\pi i x_0(y_0-y)}\langle f, g_{z-z_0}\rangle = e^{2\pi i x_0(y_0-y)}Gf(z-z_0)$$

In this way, to be consistent with the notation and the definition of the transform, we have to define the translations in $\mathbb{C}$ of a function $F \in H$ (or in $L^2(\mathbb{C})$ in a general way) as:

$$F_{z_0}(z) = e^{2\pi i x_0(y_0-y)}F(z-z_0)$$

since in general the function $F(z-z_0)$ can not belong to H. Taking this into account we can write the reproduction formula in a bit more compact way:

$$F(z_0) = \int_{\mathbb{C}} F(z)k_z(z_0)\,dm(z)$$

and moreover we have invariance of the norms:

$$\|F\| = \|F_{z_0}\| = \|f\| = \|f_{z_0}\|$$

if $F = Gf$.

It is necessary to observe that these translations do not coincide in general with the usual translation of $\mathbb{C}$. But if we look at the module of the function (with what we will work usually), then we have an equality:

$$|F_{z_0}(z)| = |F(z-z_0)|$$

The phase space has good properties of continuity. Since we have invariance by translations, the functions of the space will be uniformly continuous, with uniformity not just with respect to the distance of the points, but also with respect to the function. We formalize this idea with the following result:

Proposition 5.1. *The module of the Gabor transform is uniformly continuous. In other words, given* ε *there exists* δ *such that if* $|z_1 - z_2| < \delta$*, then for every* $F \in H$*:*

$$F(z) = \langle f, g_z\rangle$$

it is fulfilled that

$$\big||F(z_1)| - |F(z_2)|\big| < \|F\|\varepsilon = \|f\|\varepsilon$$

Proof. Calculating directly and using the properties of the scalar product we see that:

$$\begin{aligned} \big||F(z_1)| - |F(z_2)|\big| =& \big||F_{z_2}(z_1 - z_2)| - |F_{z_2}(0)|\big| \\ \leqslant& |F_{z_2}(z_1 - z_2) - F_{z_2}(z)| \\ =& |\langle f_{z_2}, g_{z_1 - z_2}\rangle - \langle f_{z_2}, g\rangle| \\ \leqslant& \|f_{z_2}\|\|g_{z_1 - z_2} - g\| = \|f\|\|g_{z_1 - z_2} - g\| \end{aligned}$$

and the result is deduced from the continuity of the translation and modulation operators in $L^2(\mathbb{R})$. ☐

Remark. Modifying a bit the proof it can be checked out that F is continuous, and as $F(z) \to 0$ when $|z| \to \infty$ (this can be deduced directly from the definition of the transform, or in a simpler way, from the fact that $|F| \in L^2(\mathbb{C})$ and it is uniformly continuous) we will also be able to prove that F is uniformly continuous, but δ will depend on F and will not obtain a result as good as the heading here.

A curious property of these spaces is that they do not contain functions with compact support.

Proposition 5.2. *Let $g \in L^2(\mathbb{R})$ be a Gabor atom and $f \in L^2(\mathbb{R})$ a function different of zero. Then $Gf(z)$ can never have compact support.*

Proof. We suppose that $Gf(t)$ has compact support contained, for example, in $[0,1] \times [0,1]$. If we look at the formula

$$Gf(z) = \int_{\mathbb{R}} f(t) e^{-2\pi andty} \overline{g(t - x)\, dt}$$

as $Gf(z) = 0$ for every y if $x \notin [0,1]$, we can affirm that:

$$f(t)\overline{g(t - x)} = 0$$

as a function, for every $x \notin [0,1]$, since its Fourier transform is zero. This says that both f and g have compact support. By Parseval theorem:

$$Gf(z) = e^{2\pi ixy} \int_{\mathbb{R}} \widehat{f}(\xi) e^{2\pi ix\xi} \widehat{g}(-\xi - y)\, d\xi$$

We can deduce that both $\widehat{f}$ and $\widehat{g}$ also have compact support and this is a contradiction. ☐

We will not use this result during our study, but it helps us to understanding a bit more the structure of these spaces. Moreover it tells that taking separate sets has possibilities of exit. If it exists in the phase space some function with compact support and we take a separate set $\Lambda \subset \mathbb{C}$, just a finite number of translations of g_λ would not be orthogonal to a function $f \in L^2(\mathbb{R})$. This makes very difficult to generate f from the translations of g_λ with $\lambda \in \Lambda$.

Even though no function can have compact support, we can find atoms for which there are functions in the phase space with support in a band. For example, if both the atom g and f have compact support, Gf will be supported in a vertical band. The same happens with Fourier transform with compact support. And we can find atoms with compact support and very good localization in time and frequency (those that we are interested to).

The zero set of a function of the phase space will usually be a union of curves. That is, we can think that it has dimension 1. In the former examples, in the band where the function Gf is supported the zeros are usually also entailed in this way. For example, we take the expression of the reproductive kernel of the phase space of the Poisson function $P(t) = \frac{1}{\pi}\frac{1}{1+t^2}$, that can be calculated in a explicit way:

$$k(z) = \begin{cases} \frac{\pi}{x} e^{-2\pi|y|}\left(\frac{1}{x+2i} + \frac{e^{-2\pi ixy}}{x-2i}\right) & \text{if } y \geqslant 0 \\ \frac{\pi}{x} e^{-2\pi|y|}\left(\frac{1}{x-2i} + \frac{e^{-2\pi ixy}}{x+2i}\right) & \text{if } y < 0 \end{cases}$$

We observe that the zero set of the kernel is a joint of curves closer to hyperboles. There is a case where the zero set is discrete, the phase space of the Gaussian function, which we will treat in the next section. We can think in an informal way that all its elements are entire functions (we will formalize this idea later on). But this case seems to be an accident. There is not know any other case with this property. In the last section of this chapter we will see that for a class of very good atoms we can prove that the zero sets of the phase space have as much dimension 1 and it does not seem that we can improve this result except for a very particular examples (which are not known yet and it is not clear that they exist).

In this part of the work we will look for frames of $L^2(\mathbb{R})$, which are a special class of generator systems. Therefore the discrete sets Λ that we will construct avoid the zero sets (they are never contained in them). This will be difficult because the zero sets of the Gabor transforms are very big with respect to the zero sets that we found in the case of translations.

We are interested on studying when a system $G(g, \Lambda)$ is a basis or a frame of $L^2(\mathbb{R})$ for a Gabor atom $g \in L^2(\mathbb{R})$ and a set Λ. In this case the set will be uniformly discrete (or separated) and it will be contained in $\mathbb{C}$. Later on we will see why we restrict to this class of sets. We have commented in the first chapter that, fixed g, $G(g, \Lambda)$ is a frame if and only if Λ is a sampling set for the phase space of g. In this chapter we will study this type of sets. We will give necessary and sufficient conditions for a set Λ to be a sampling set. We will study at the same time the interpolation sets of the phase spaces. Sampling and interpolation are dual concepts. We study interpolation sets because it helps to understand better the structure of the spaces as well as the particularities of the problem.

If we look at the condition of sampling we see that we have to bound the summation

$$\sum_{\lambda\in\Lambda} |F(\lambda)|^2 = \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{C}} F(z) k_\lambda(z)\, dm(z) \right|^2$$

where we have introduced the reproduction formula, since the idea is to bound this summation of independent way of F. The following step will be introduce the absolute values inside of the integral. But this will give us an inequality that just will be useful in one of both bounds. Big part of the work of this chapter will be to attempt to control the loss of information on making this step.

The general idea will be to obtain an equivalence of the type:

$$\|F\|^2 = \int_{\mathbb{C}} \left| \int_{\mathbb{C}} F(z) k_\lambda(z)\, dm(z) \right|^2 dm(\lambda) \approx \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{C}} F(z) k_\lambda(z)\, dm(z) \right|^2$$

where we see again that the problem is to know how much information do we lose when we discretize the reproductive kernel (that is, to sample it in a discrete set). We will search conditions on the reproductive kernel k (or in equivalent way, on g) to be able to assure the existence of sampling sets. We will have to restrict to atoms for which the kernel is an integrable function. That is, those atoms that belong to the Feichtinger algebra $\mathcal{A}$.

After fixing an analyzing atom $g \in \mathcal{A}$, we will attempt to give necessary and sufficient conditions for a set Λ to be a sampling set in the corresponding phase space. As a matter of fact, when we prove the existence of sampling sets already we will give enough conditions so that a set is it.

Regarding necessary conditions we will only reproduce the ones already obtained by Ramanathan and Steger in [RaS95]. These characterize the sets that can be sampling sets for some phase space through their uniform densities (which we will define later on). This is a very strong result, since the necessary condition does not depend on the atom that we use for analyzing, and is necessary for all the atoms of $L^2(\mathbb{R})$ and not just for the Feichtinger algebra.

5.2 Another time the Fock space.

We will give in this section the best example of Gabor atom that is know when we look for phase spaces with good properties of sampling and interpolation. As we have said before, the study of these concepts corresponds with the study of frames and bases former by translations and modulation of the atom.

Sampling and interpolation sets have been studied in different spaces of holomorphic functions. In this section we will study the particular case of an atom where the phase space corresponds with the Fock space of entire functions (that we have defined in the former chapter). In this space there are described the sets of sampling and of interpolation. We will remember the definitions and some of the properties declared formerly to improve the understanding.

We will work again with the normalized Gaussian function:

$$\phi(t) = 2^{\frac{1}{4}} e^{-\pi t^2}$$

Calculating directly we see that:

$$\phi_z(t) = 2^{\frac{1}{4}} e^{2\pi i y t} \phi(t-x) = 2^{\frac{1}{4}} e^{2\pi i t y - \pi (t-x)^2}$$

We introduce this calculation in the definition of Gabor transform. In this way we obtain for every $f \in L^2(\mathbb{R})$:

$$Gf(z) = \langle f, \phi_z \rangle = 2^{\frac{1}{4}} \int_{\mathbb{R}} f(t) e^{-2\pi i t y - \pi (t-x)^2} \, dt \tag{5.1}$$

We can think that this functions are in the Fock space. We remember the definition of this space:

Definition. We define the **Fock space** as the following set of functions:

$$\mathcal{F} = \left\{ F : F \text{ is entire and } \|F\|_{\mathcal{F}}^2 = \int_{\mathbb{C}} |F(z)|^2 e^{-\pi |z|^2} \, dm(z) < \infty \right\}$$

The Bargmann transform gives the relationship betwen this space and $L^2(\mathbb{R})$.

Definition. Given $f \in L^2(\mathbb{R})$, we define the **Bargmann transform** of f as:

$$Bf(z) = 2^{\frac{1}{4}} \int_{\mathbb{R}} f(t) e^{2\pi t z - \pi t^2 - \frac{\pi}{2} z^2} \, dt$$

This transform is an isomorphism between $L^2(\mathbb{R})$ and $\mathcal{F}$ [Fol89]. For this reason this space is called also the Bargmann-Fock space. If now we look again at (5.1) we see that:

$$Gf(x+iy) = e^{-\frac{\pi}{2}|x+iy|^2} e^{-\pi i x y} Bf(x-iy)$$

or in a equivalent way:

$$Gf(\overline{z}) = e^{-\frac{\pi}{2}|z|^2} e^{\pi i x y} Bf(z)$$

Therefore we have that $e^{\frac{\pi}{2}|z|^2}e^{-\pi ixy}Gf(\overline{z}) \in \mathcal{F}$, and for every function $F \in \mathcal{F}$ exists $f \in L^2(\mathbb{R})$ such that $Gf(\overline{z}) = e^{-\frac{\pi}{2}|z|^2}e^{\pi ixy}F(z)$. Moreover we can prove:

$$\begin{aligned}
\|Gf(\overline{z})\|^2 &= \int_{\mathbb{C}} |Gf(z)|^2\, dm(z) \\
&= \int_{\mathbb{C}} \left|e^{-\frac{\pi}{2}|z|^2}e^{\pi ixy}F(z)\right|^2 dm(z) \\
&= \int_{\mathbb{C}} |F(z)|^2 e^{-\pi|z|^2} dm(z) = \|F\|^2_{\mathcal{F}}
\end{aligned}$$

The former correspondence is an isometry. We can think in an informal way that the phase space of the Gaussian function is the Fock space. It is important to look at the correspondence among reproductive kernels. The reproductive kernel of the Fock space is:

$$k_{\mathcal{F}}(z, z_0) = e^{\pi z\overline{z_0}}$$

and the reproductive kernel of the phase space of ϕ is:

$$k_\phi(z, z_0) = e^{-\frac{\pi}{2}(|z|^2+|z_0|^2)}e^{\pi i(x_0y_0-xy)}e^{-\pi\overline{z}z_0}$$

We can write it in a one variable function:

$$k_\phi(z) = e^{-\frac{\pi}{2}|z|^2}e^{-\pi ixy}$$

and in this way we obtain that:

$$k_\phi(z, z_0) = e^{2\pi ix_0(y_0-y)}k_\phi(z)$$

In the Fock space we have a complete characterization of the sampling and interpolation sets. This characterization has been proved in an independent way in [Sei92], [SeW92] and [Lyu92]. The characterization is given though a density that we will define now.

The way to define these densities is the following one: given a uniformly discrete set $\Gamma = \{z_j\}_{j\in\mathbb{N}}$, we fix a set Q of measure 1 and border of measure 0 (we can think in a ball). We denote for $n^-(r)$ and $n^+(r)$ respectively the minimum and the maximum number of points of Γ that we can find in any translation of rQ.

Definition. We define the **lower and upper uniform densities** as:

$$D^-(\Gamma) = \liminf_{r\to\infty} \frac{n^-(r)}{\pi r^2} \qquad D^+(\Gamma) = \limsup_{r\to\infty} \frac{n^+(r)}{\pi r^2}$$

In [Lan67] it is proved that this definition is independent of the choice of Q. The difference between these densities and those which that we used in the study of the sets of uniqueness of this space in the former chapter is that now we take supreme and infimum in balls centered on any point of $\mathbb{C}$, and not just in the origin. The idea is to make more uniforms the arguments for proving that a set is of uniqueness. In this way we can prove that a uniformly discreet set Γ is a sampling set if and only if $D^-(\Gamma) > \pi$ and it is an interpolation set if and only if $D^+(\Gamma) < \pi$.

As a direct consequence of these results we have that there are not sets that are sampling and interpolation sets on time. Or in an equivalent way, there are not Riesz bases using translations and modulations of the Gaussian function. This fact is not very good if we take into account that bases are minimum representations. We find that any stable representation of $L^2(\mathbb{R})$ through the Gaussian function will always be redundant. We will comment later on that if we want a characterization only in terms of density it seems that there must not be bases.

In this particular case we can use the results of the former chapter, when we studied the generator systems for translations in the way $T(\phi, \Lambda)$. If we look at the conditions that we obtained about the zero sets of a function of the Fock space we can state:

Theorem 5.3. *Let ϕ be the Gaussian function and Λ a discrete set (not necessarily in a uniform way). Then, if:*

$$\Delta^-(\Lambda) > \pi$$

or

$$\Delta^+(\Lambda) > e\pi$$

$G(\phi, \Lambda)$ generates $L^2(\mathbb{R})$.

These results are in consonance with those that we have given about sampling and interpolation (we think in sets in which the upper and lower density coincide), but they are not so good. In the case of sampling there is a complete characterization, that we can not attain when we study the sets of uniqueness. One of the reasons is that the condition of compensation of the zeros can not be avoided when we search sets of uniqueness.

5.3 Analytical Gabor phase spaces.

The case of the Gaussian function is exceptional not just because its phase space is formed by holomorphic functions. Also we can say that it is the only atom with this property.

In this section we will use the conjugate space for convenience. This space is the phase space, but evaluated in $\overline{z}$ and using to analyze $\overline{g}$ instead of g. If we look at the case of the Gaussian function, the functions of the phase space were anti–holomorphic. Working in this way the statement will be consistent with what we have said in the former section.

This change will not introduce any substantial difference with respect to the usual case.

Theorem 5.4. *We consider the conjugate of the phase space of a Gabor atom* $g \in L^2(\mathbb{R})$*:*

$$\overline{H} = \{F(z) = \int_{\mathbb{R}} f(t) e^{2\pi i t y} g(t-x)\, dt,\ f \in L^2(\mathbb{R})\}$$

Then this space is a space of holomorphic functions with respect to a weight if and only if g *is time-frequency translation of the Gaussian function.*

Proof. We suppose that there is a $M(z) = M(x,y)$ such that:

$$M(z)F(z) \in Hol(\mathbb{C}) \qquad \forall F \in \overline{H}$$

Then

$$\overline{\partial} M F(z) = \int_{\mathbb{R}} f(t) \overline{\partial}\big(M(z) e^{2\pi i t y} g(t-x)\big)\, dt = 0 \qquad \forall f \in L^2(\mathbb{R})$$

This fact is equivalent to:

$$\overline{\partial}\big(M(z) e^{2\pi i t y} g(t-x)\big) = 0 \qquad \forall t \in \mathbb{R}$$

We can calculate this derivative,

$$\begin{aligned}\overline{\partial}\big(M(z) e^{2\pi i t y} g(t-x)\big) = \overline{\partial}\big(M(x,y)\big) e^{2\pi i t y} g(t-x) - \frac{1}{2} M(x,y) e^{2\pi i t y} g'(t-x) \\ - \pi t M(x,y) e^{2\pi i t y} g(t-x)\end{aligned}$$

This tells:

$$2\big(\overline{\partial} M(x,y) - M(x,y)\pi t\big) g(t-x) = M(x,y) g'(t-x)$$

where we have a differential equation in g that is not difficult to solve. If we make the change $t - x = w$ we obtain:

$$\frac{g'(w)}{g(w)} = 2\frac{\overline{\partial} M(x,y)}{M(x,y)} - 2\pi(w+x)$$

Then the solution is of the form

$$g(w) = C e^{-\pi (w+x)^2 + 2\frac{\overline{\partial} M}{M} w} = C e^{-\pi \left[(w+x-\frac{1}{\pi}\frac{\overline{\partial} M}{M})^2 - (\frac{1}{\pi}\frac{\overline{\partial} M}{M})^2 + \frac{2x}{\pi}\frac{\overline{\partial} M}{M}\right]}$$

that is a translation of the Gaussian function. □

Remark. We also have conditions on the weight. The solution of the differential equation says that:

$$x - \frac{1}{\pi}\frac{\overline{\partial} M}{M} = C_1 \qquad -\left(\frac{1}{\pi}\frac{\overline{\partial} M}{M}\right)^2 + \frac{2x}{\pi}\frac{\overline{\partial} M}{M} - \pi \log C = C_2$$

since the function g can not depend on x nor on y. In the case of the Gaussian function centered in zero the weight is:

$$M(z) = e^{\frac{\pi}{2}|z|^2} e^{-\pi ixy}$$

where $\overline{\partial} M(z) = \pi x M(z)$ and both factors are canceled taking $C = e^{-\pi x}$.

When we say that the solution is a translation of the Gaussian function we have to comment that we admit complex translations, or in an equivalent way, that they can be translations and modulations with real parameter.

Note. This result is also true if we ask for almost–regular functions instead of holomorphic ones.

This result says that we will be able to apply the techniques of the complex analysis in any other case. For this reason the results that we will obtain will not be as good as the ones that exists for the Gaussian function.

5.4 Sampling results.

From now on H will be the phase space of a fixed Gabor atom g normalized so that $\|g\| = 1$. We will take $z, w \in \mathbb{C}$, with $z = x + iy$ and $w = a + ib$, with $x, y, a, b \in \mathbb{R}$. We will use the equivalent notation when we have subscripts. We remember the definition of uniformly discrete sets, which we will usually call separate.

Definition. Let $\Gamma = \{z_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$. We will say that Γ is an **uniformly discrete** or **separate set** if there is $\varepsilon > 0$ such that $|z_i - z_j| > \varepsilon \ \forall i \neq j$.

We will refer to ε as the **separation constant** of Γ.

The following results prove that these are the only sets that we have to consider.

Lemma 5.5. *Let $F \in H$ such that $F(w) = 0$. Let $z \in \mathbb{C}$, then:*

$$|F(w)|^2 \leqslant \|F\|^2\big(1 - |k_w(z)|^2\big) = \|F\|^2\big(1 - |\langle g_w, g_z\rangle|^2\big)$$

Proof. We consider H_w the set of functions of H that cancel in w. This is a Hilbert subspace with the following reproductive kernel:

$$\Phi_{z_0}(z) = k_{z_0}(z) - k_{z_0}(w) k_w(z)$$

To see this we have to observe first that $\Phi_{z_0} \in H$ since it is a linear combination of functions of H and $\Phi_{z_0}(w) = 0$ for every $z_0 \in \mathbb{C}$. Therefore $\Phi_{z_0} \in H_w$.

If we take $h \in H_w$ we see that:

$$\langle h, \Phi_{z_0}\rangle = \langle h, k_{z_0}\rangle - k_{z_0}(w)\langle h, k_w\rangle = h(z_0)$$

That is, Φ_{z_0} reproduce the functions of H_w. As it belongs to the space, it is its reproductive kernel.

As F belongs to H_w we apply the reproduction formula in this space.

$$\begin{aligned} |F(z)|^2 =& |\langle F, \Phi_z\rangle|^2 \leqslant \|F\|^2 \langle \Phi_z, \Phi_z\rangle = \|F\|^2 \Phi_z(z) \\ =& \|F\|^2 \left(k_z(z) - \frac{k_z(w)}{k_w(w)} k_w(z) \right) = \|F\|^2 \big(1 - |k_w(z)|^2\big) \end{aligned}$$

and we obtain the result. □

Theorem 5.6. *Let* $\Gamma = \{z_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$. *If* Γ *is an interpolation set for* H *it is fulfilled that* Γ *is a separate set.*

Proof. By the closed graph theorem, if Γ is an interpolation set, there is $M > 0$ such that $\forall \{a_n\}_{n\in\mathbb{N}} \in l^2$ exists $F \in H$ with $F(z_n) = a_n$ and $\|F\|^2 \leqslant M^2 \sum_{n\in\mathbb{N}} |a_n|^2$. That is, we can carry out the interpolation with bounded norm.

Let $z_i \neq z_j$. There is $F \in H$ such that $F(z_i) = 1$ and $F(z_j) = 0$ with $\|F\| \leqslant M$. Applying 5.5 we have:

$$\begin{aligned} & 1 = |F(z_i)|^2 \leqslant \|F\|^2 (1 - |k_{z_i}(z_j)|^2) \leqslant M^2 (1 - |k_{z_i}(z_j)|^2) \\ & \Rightarrow 1 - |k_{z_i}(z_j)|^2 \geqslant \frac{1}{M^2} \Rightarrow |k_{z_i}(z_j)|^2 \leq 1 - \frac{1}{M^2} = C < 1 \end{aligned}$$

This means that if $i \neq j$, $|\langle g_{z_i}, g_{z_j}\rangle|^2 \leqslant C < 1$. As the module of the kernel is uniformly continuous, this fact implies the condition of separation, since C does not depend on z_i and z_j. □

Now we know that all interpolation sets are separate sets, and it only remains to analyze the case of the sampling sets. Here the situation is a bit different.

Theorem 5.7. *Let* $\Gamma = \{z_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$. *If* Γ *is a sampling set for* H *then* Γ *is a finite union of separate sets.*

Proof. As Γ is a sampling set, $\exists C > 0$ such that:

$$\sum_{\Gamma} |F(z_j)|^2 \leqslant C\|F\|^2 \tag{5.2}$$

If Γ is not a finite union of separate sets then for every N and every δ we can find a ball B of radius δ such that $|\Gamma \cap B| > N$.

As $|k_{z_0}(z)|$ is uniformly continuous, $\exists\delta$ such that:

$$\big||k_{z_0}(z_1)| - |k_{z_0}(z_2)|\big| < 1/2$$

if $|z_1 - z_2| < \delta$, where this δ does not depend on z_0.

We take this δ and $N > 4C$. Let B be the corresponding ball of radius δ that contains N points of the set. If w is the center of B, (5.2) has to be fulfilled for the function $k_w(z) \in H$:

$$\sum_{\Gamma} |k_w(z_j)|^2 \leqslant C\|k_w\|^2 = C$$

But on the other hand:

$$\sum_{\Gamma} |k_w(z_j)|^2 = \sum_{\Gamma\cap B^c} |k_w(z_j)|^2 + \sum_{\Gamma\cap B} |k_w(z_j)|^2 \geqslant \sum_{\Gamma\cap B} (1-1/2)^2 \geqslant N\frac{1}{4} > C$$

and we have a contradiction. □

This result is the best that we can achieve in the case of sampling sets since there are sampling sets that are not separate. For example we can join two regular nets with different parameters, each one of them forming a frame, so that we do not have separation.

Until now we have not asked the analyzing atom g to fullfill any special condition. To prove the existence of sampling sets and obtain sufficient conditions we will have to suppose that $g \in \mathcal{A}$. That is, that $k \in L^1(\mathbb{C})$. We remember that in [FeG89] and [FeG89-2] it is proved that if $g \in \mathcal{A}$ then that the local maximal function of k:

$$Mk(z) = \sup_{w\in B(z,1)} |k(w)|$$

also belongs to $L^1(\mathbb{C})$. Therefore we have an extra of integration that allows us to discretize.

Lemma 5.8. *Let k such that $Mk \in L^1(\mathbb{C})$ and $\Lambda \subset \mathbb{C}$ a separate set with separation constant ε. Then:*

$$\sum_{\lambda\in\Lambda} |k(\lambda)| < \frac{\varepsilon^{-2}}{4\pi}\|Mk\|_1$$

Proof. We suppose without losing generality that $\frac{\varepsilon}{2} < 1$. If we take two different points $\gamma, \lambda \in \Lambda$, as $B(\gamma, \frac{\varepsilon}{2}) \cap B(\lambda, \frac{\varepsilon}{2}) = \emptyset$ we can bound:

$$\sum_{\lambda\in\Lambda} |k(\lambda)| \leqslant \sum_{\lambda\in\Lambda} \frac{1}{|B(0,\frac{\varepsilon}{2})|}\int_{B(\lambda,\frac{\varepsilon}{2})} Mk(z)\,dm(z) \leqslant \frac{\varepsilon^{-2}}{4\pi}\int_{\mathbb{C}} Mk(z)\,dm(z)$$

that is what we want to prove. □

Remark. In an informal way we can think that the bound that we obtain is independent of Λ since it only depends on its separation constant.

The following results are used prove that a set next to a sampling set will also be a sampling set. This result can be obtained using the theory of representations of Feichtinger and Groshenig [FeG89][FeG89-2], but here we will use other techniques, adapting the ideas of [OlS92] to of Gabor case. In this way we can give explicit bounds only depending on the separation constants and on $\|Mk\|_1$. This improvement will allow us to prove that every sampling set contains a separate subset that is also a sampling set. This problem has also been studied in [SuZ01], [SuZ02], and [FeS06], where similar results are obtained, but using other technical, and in the two first, changing the set of analyzing atoms.

We give the following result as an example. It tells that a separate set fullfill the upper bound of the sampling operator. The proof is very simple but it illustrates in a very good way the techniques that we will use throughout this section.

Proposition 5.9. *Given an analyzing Gabor atom* $g \in \mathcal{A}$ *and* Γ *a separate set it exists* $B > 0$ *such that:*

$$\sum_{\gamma\in\Gamma} |F(\gamma)|^2 \leqslant B\|F\|^2 \quad \forall F \in H$$

Proof. Calculating directly we have that:

$$\begin{aligned}\sum_{\gamma\in\Gamma} |F(\gamma)|^2 &= \sum_{\gamma\in\Gamma} \left| \int_{\mathbb{C}} F(z)\overline{k_\gamma(z)}\, dm(z) \right|^2 \\ &\leqslant \sum_{\gamma\in\Gamma} \left(\int_{\mathbb{C}} |F(z)|^2 |k_\gamma(z)|\, dm(z) \right) \left(\int_{\mathbb{C}} |k_\gamma(z)|\, dm(z) \right) \\ &= \int_{\mathbb{C}} |F(z)|^2 \sum_{j\in\mathbb{N}} |k(z-\gamma)|\, dm(z) \int_{\mathbb{C}} |k(z)|\, dm(z) \leqslant B\|F\|^2\end{aligned}$$

since $\int_{\mathbb{C}} |k(z)|\, dm(z) < \infty$ because the kernel is integrable and

$$\sum_{\gamma\in\Gamma} |k(z-\gamma)| = \sum_{\lambda\in(z-\Gamma} |k(\lambda)|$$

since $z - \Gamma$ has the same separation constant than Γ and we can apply 5.8 to bound independently of z. ☐

Remark. As we see in the proof, the bound constant depends on k in terms of its $L^1(\mathbb{R})$–norm and not in the $L^2(\mathbb{R})$–norm. That is, it depends on $\|k\|_1$, discrete as well as continuous. It is for this reason that we have to restrict to the study of atoms of the Feichtinger algebra.

Using the same ideas we can prove the result that we have announced formerly. The only innovation here consists in the way to prove it.

Theorem 5.10. *Let* $g \in \mathcal{A}$ *be a Gabor atom of the Feichtinger algebra. Given* $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ *a sampling set for* H *there exists* $\delta > 0$ *such that if* $\Gamma = \{w_j\}_{j\in\mathbb{N}}$ *fulfills that* $|z_j - w_j| < \delta\ \forall j$*, then* Γ *is also a sampling set.*

This result is deduced in a trivial way from the following pair of lemmas.

Lemma 5.11. *Let* $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ *and* $\Gamma = \{w_j\}_{j\in\mathbb{N}}$ *be two discrete sets in* $\mathbb{C}$*. Then it is fulfilled that:*

$$\left|\left(\sum_{j\in\mathbb{N}} |F(z_j)|^2\right)^{1/2} - \left(\sum_{j\in\mathbb{N}} |F(w_j)|^2\right)^{1/2}\right| \leqslant d_1 d_2 \|F\| \quad \forall F \in H$$

where d_1 *and* d_2 *are defined by:*

- $d_1^2 = \sup_j \int_{\mathbb{C}} |k_{z_j - w_j}(z) - k(z)|\, dm(z)$
- $d_2^2 = \sup_z \sum_{j\in\mathbb{N}} |k_{z_j}(z) - k_{w_j}(z)|$

Proof. Calculating directly we have that:

$$\left|\left(\sum_{j\in\mathbb{N}} |F(z_j)|^2\right)^{1/2} - \left(\sum_{j\in\mathbb{N}} |F(w_j)|^2\right)^{1/2}\right| = \big|\|(F(z_j))_j\|_2 - \|(F(w_j))_j\|_2\big|$$

where $\|\cdot\|_2$ means the l^2–norm, thinking $\big(F(z_j)\big)_j$ as a succession.

$$\begin{aligned}\big|\|(F(z_j))_j\|_2 - \|(F(w_j))_j\|_2\big| \leqslant& \big\|(|F(z_j)| - |F(w_j)|)_j\big\|_2 \\ =& \big\|(|F_{-w_j}(z_j - w_j)| - |F_{-w_j}(0)|)_j\big\|_2\end{aligned}$$

due to the invariance from translations of $|F|$. We introduce the reproduction formula,

$$\begin{aligned}&\big\|(|F_{-w_j}(z_j - w_j)| - |F_{-w_j}(0)|)_j\big\|_2 \\ &\quad= \left\|\left(\left|\int_{\mathbb{C}} F_{-w_j}(z)\overline{k_{z_j - w_j}(z)}\, dm(z)\right| - \left|\int_{\mathbb{C}} F_{-w_j}(z)\overline{k(z)}\, dm(z)\right|\right)_j\right\|_2 \\ &\quad\leqslant \left\|\left(\int_{\mathbb{C}} F_{-w_j}(z)[\overline{k_{z_j - w_j}(z)} - \overline{k(z)}]\, dm(z)\right)_j\right\|_2 \\ &\quad\leqslant \left[\sum_{j\in\mathbb{N}}\left(\int_{\mathbb{C}} |F_{-w_j}(z)||k_{z_j - w_j}(z) - k(z)|\, dm(z)\right)^2\right]^{1/2}\end{aligned}$$

and in this way we have transported the problem to look at points nearby to 0 and to the reproductive kernel. We can use the Schwartz inequality,

$$\left[\sum_{j\in\mathbb{N}}\left(\int_{\mathbb{C}}|F_{-w_j}(z)||k_{z_j-w_j}(z)-k(z)|\,dm(z)\right)^2\right]^{1/2}$$
$$\leqslant\left[\sum_{j\in\mathbb{N}}\int_{\mathbb{C}}|F_{-w_j}(z)|^2|k_{z_j-w_j}(z)-k(z)|\,dm(z)\int_{\mathbb{C}}|k_{z_j-w_j}(z)-k(z)|\,dm(z)\right]^{1/2}$$

Now we bound separately,

$$\int_{\mathbb{C}}|k_{z_j-w_j}(z)-k(z)|\,dm(z)\leqslant d_1^2$$

and on the other hand

$$\begin{aligned}\sum_{j\in\mathbb{N}}&\int_{\mathbb{C}}|F(z+w_j)|^2|k_{z_j-w_j}(z)-k(z)|\,dm(z)\\&\leqslant\sum_{j\in\mathbb{N}}\int_{\mathbb{C}}|F(z)|^2|k_{z_j-w_j}(z-w_j)-k(z-w_j)|\,dm(z)\\&\leqslant\int_{\mathbb{C}}|F(z)|^2\sum_{j\in\mathbb{N}}|k_{z_j}(z)-k_{w_j}(z)|\,dm(z)\leqslant d_2^2\|F\|^2\end{aligned}$$

and joining both bounds we obtain the statement. □

Lemma 5.12. *Let $\Lambda=\{z_j\}_{j\in\mathbb{N}}$ be a separate, and we suppose that $g\in\mathcal{A}$.*

For every ε $\exists\delta$ such that if $\Gamma=\{w_j\}_{j\in\mathbb{N}}$ fulfills that $|z_j-w_j|\leqslant\delta$ $\forall j$ then $d_1d_2\leqslant\varepsilon$, with d_1 and d_2 defined as in 5.11.

Proof. First we will see that we can make d_1 so small as we want if Γ is close enough to Λ. To see this we write:

$$\int_{\mathbb{C}}|k_{z_j-w_j}(z)-k(z)|\,dm(z)=\int_{\mathbb{C}}\left|e^{2\pi i(x_j-a_j)(y_j-b_j-y)}k\big(z-(z_j-w_j)\big)-k(z)\right|dm(z)$$

As k is integrable, there is $R>0$ such that, if $|\alpha|<1$,

$$\int_{\mathbb{C}\setminus B(0,R)}|k(z-\alpha)|dm(z)<\frac{\varepsilon}{4}$$

In this way, if we assume $\delta < 1$, we have that:

$$\begin{aligned}
&\int_{\mathbb{C}} \left| e^{2\pi i(x_j-a_j)(y_j-b_j-y)} k\big(z-(z_j-w_j)\big) - k(z) \right| dm(z) \\
&= \int_{B(0,R)} \left| e^{2\pi i(x_j-a_j)(y_j-b_j-y)} k\big(z-(z_j-w_j)\big) - k(z) \right| dm(z) + \\
&\qquad + \int_{\mathbb{C}\setminus B(0,R)} \left| e^{2\pi i(x_j-a_j)(y_j-b_j-y)} k\big(z-(z_j-w_j)\big) - k(z) \right| dm(z) \\
&\leqslant \int_{B(0,R)} \left| e^{2\pi i(x_j-a_j)(y_j-b_j-y)} - 1 \right| \left| k\big(z-(z_j-w_j)\big) \right| dm(z) + \\
&\qquad + \int_{B(0,R)} \left| k\big(z-(z_j-w_j)\big) - k(z) \right| dm(z) + \frac{\varepsilon}{2}
\end{aligned}$$

For the first integral, we use that $|x_j - a_j|, |y_j - b_j| \leqslant |z_j - w_j|$. As $|y| < R$, choosing δ small enough we can achieve $\left| e^{2\pi i(x_j-a_j)(y_j-b_j-y)} - 1 \right| < \frac{\varepsilon}{4\|k\|_1}$ for every $|y| < R$. For the second integral it is only necessary to use that the translation operator is a continuous in $L^1(\mathbb{C})$.

We go now to bound d_2. If $\alpha = \sup_{i\neq j} |z_i - z_j|$ is the separation constant of Λ, we assume $\delta < \frac{\alpha}{3}$, and it is fulfilled that $|w_i - w_j| > \frac{\alpha}{3}$ $\forall i \neq j$. That is, $\frac{\alpha}{3}$ can be used as separation constant for Λ as well as for Γ. As a matter of fact, $\Lambda_z = \{z - z_j\}_{j\in\mathbb{N}}$ and $\Gamma_z = \{z - w_j\}_{j\in\mathbb{N}}$ also have the same separation constant. Here we can apply 5.8 to prove that there is $C = C(\frac{\alpha}{3})$ such that $d_2 \leqslant 2C$.

If we join both parts we see that d_2 is bounded by C when we take δ small, and d_1 can be made as much small one as we wish taking δ small enough. Therefore $\exists\delta$ such that if $|z_j - w_j| \leqslant \delta$ $\forall j$, it is fulfilled that $d_1 d_2 < \varepsilon$. ☐

We can compare theorem 5.10 with the stability results given for general Riesz bases. If we look at 1.10 and 1.11 we see that we can change the system, so that the sum of changes is not big. In 5.10 we change a little each vector, but we can make it in an uniform way. This is much better, since the sum can diverge and we however continue having a frame. In this result we do not ask for independence because we work with frames instead of bases.

In theorem 5.7 we saw that everything sampling set was a finite union of separate sets, but the former lemmas allow us to improve this result. The idea is that a sampling set is a union of some very close sets, and lemmas 5.12 and 5.11 say that close sets give close information about the functions. That is, in a non separate sampling sets we have repeated information.

Theorem 5.13. *We suppose that $g \in \mathcal{A}$ and a let $\Gamma = \{z_j\}_{j\in\mathbb{N}}$ be a sampling set for H.*

Then Γ contains a subset $\widetilde{\Gamma} \subseteq \Gamma$ such that $\widetilde{\Gamma}$ is a sampling and separate set.

Proof. As Γ is sampling set we know by 5.7 that it is a finite union of separate sets, and that there are $A, B > 0$ such that for every $F \in H$ it is fulfilled that:

$$A\|F\|^2 \leqslant \sum_{j\in\mathbb{N}} |F(z_j)|^2 \leqslant B\|F\|^2$$

For every δ (we think in a very small δ) we can define $\widetilde{\Gamma} \subseteq \Gamma$ so that if $\widetilde{\Gamma} = \{w_i\}_{i\in\mathbb{N}}$, it is fulfilled that:

- $B(w_i, \delta) \cap \widetilde{\Gamma} = \{w_i\}$
- $\cup_{i\in\mathbb{N}}\big(B(w_i,\delta) \cap \Gamma\big) = \Lambda$
- $|B(w_i, \delta) \cap \Gamma| \leqslant N$

where N is a constant that can be bounded by the number of separate sets that form Γ. We can make all this thanks to the fact that Γ is a finite union of separate sets. That is, the one that we make is to define $\widetilde{\Gamma}$ so that $B(w_i, \delta)$ only contains w_i of the points of $\widetilde{\Gamma}$, every $z_j \in \Gamma$ is contained in a $B(w_i, \delta)$ for some $w_i \in \widetilde{\Gamma}$, and each one of these balls only contains a finite (bounded) number of points of Γ. We can think that each z_j belongs only to a $B(w_i, \delta)$. This is not true, but we can assign an only w_i to each z_j when working.

Taking this into account we can write $\Gamma = \cup_{i\in\mathbb{N}}\{w_i^1, \dots, w_i^{N_i}\}$, so that:

- $w_i^1 = w_i \in \widetilde{\Gamma}$
- $w_i^k \notin \widetilde{\Gamma}$ if $k \neq 1$
- $w_i^k \in B(w_i, \delta)$
- $N_i \leqslant N$ $\forall i$
- The sets $\{w_i^k\}_{k=1}^{N_i}$ are disjoint.

With this notation we calculate:

$$\begin{aligned}
\sum_{z_j\in\Gamma} |F(z_j)|^2 &= \sum_{w_i\in\widetilde{\Gamma}} \sum_{k=1}^{N_i} |F(w_i^k)|^2 \\
&= \sum_{w_i\in\widetilde{\Gamma}} \left[\sum_{k=1}^{N_i} \Big(|F(w_i^k)|^2 - |F(w_i^1)|^2\Big) + N_i|F(w_i^1)|^2\right] \\
&\leqslant N \sum_{w_i\in\widetilde{\Gamma}} |F(w_i)|^2 + \sum_{w_i\in\widetilde{\Gamma}} \sum_{k=1}^{N_i} \Big(|F(w_i^k)|^2 - |F(w_i^1)|^2\Big)
\end{aligned}$$

If we define $w_i^k = w_i^1$ for $N_i < k \leqslant N$ we can write:

$$\begin{aligned}
&\sum_{w_i\in\widetilde{\Gamma}}\sum_{k=1}^{N_i}\Big(|F(w_i^k)|^2 - |F(w_i^1)|^2\Big)\\
&\qquad=\sum_{w_i\in\widetilde{\Gamma}}\sum_{k=1}^{N}\Big(|F(w_i^k)|^2 - |F(w_i^1)|^2\Big)\\
&\qquad=\sum_{k=1}^{N}\Big[\sum_{w_i\in\widetilde{\Gamma}}|F(w_i^k)|^2 - \sum_{w_i\in\widetilde{\Gamma}}|F(w_i^1)|^2\Big]\\
&\qquad=\sum_{k=1}^{N}\Bigg[\Big(\sum_{w_i\in\widetilde{\Gamma}}|F(w_i^k)|^2\Big)^{1/2} + \Big(\sum_{w_i\in\widetilde{\Gamma}}|F(w_i^1)|^2\Big)^{1/2}\Bigg]\\
&\qquad\qquad\qquad\qquad\Bigg[\Big(\sum_{w_i\in\widetilde{\Gamma}}|F(w_i^k)|^2\Big)^{1/2} - \Big(\sum_{w_i\in\widetilde{\Gamma}}|F(w_i^1)|^2\Big)^{1/2}\Bigg]
\end{aligned}$$

The first parenthesis we can be bounded by $2B^{1/2}\|F\|$, since they are partial summations of Γ. To bound the second parenthesis, in absolute value, what we will make is to apply 5.11 to the sets $\{w_i^k\}_{i\in\mathbb{N}}$ and $\{w_i^1\}_{i\in\mathbb{N}}$. This says that:

$$\left|\Big(\sum_{i\in\mathbb{N}}|F(w_i^k)|^2\Big)^{1/2} - \Big(\sum_{i\in\mathbb{N}}|F(w_i^1)|^2\Big)^{1/2}\right| \leqslant d_1^k d_2^k\|F\|$$

where k is not an exponent but an index, and d_1^k, d_2^k are defined by:

- $(d_1^k)^2 = \sup_{i\in\mathbb{N}}\int_{\mathbb{C}}|k_{w_i^k - w_i^1}(z) - k(z)|\,d\mu(z)$
- $(d_2^k)^2 = \sup_{z\in\mathbb{C}}\sum_{i\in\mathbb{N}}|k_{w_i^k}(z) - k_{w_i^1}(z)|$

We define now:

- $d_1 = \sup_k d_1^k$
- $d_2 = \sup_k d_2^k$

We can bound independently of k:

$$\left|\Big(\sum_{i\in\mathbb{N}}|F(w_i^k)|^2\Big)^{1/2} - \Big(\sum_{i\in\mathbb{N}}|F(w_i^1)|^2\Big)^{1/2}\right| \leqslant d_1 d_2\|F\|$$

If we join all this bounds we see that:

$$
\begin{aligned}
A\|F\|^2 \leqslant \sum_{j\in\mathbb{N}} |F(z_j)|^2 \leqslant N \sum_{i\in\mathbb{N}} |F(w_i)|^2 + \left| \sum_{i\in\mathbb{N}} \sum_{k=1}^{N} \left(|F(w_i^k)|^2 - |F(w_i^1)|^2 \right) \right| \\
\leqslant N \sum_{i\in\mathbb{N}} |F(w_i)|^2 + 2NB^{1/2} d_1 d_2 \|F\|^2
\end{aligned}
$$

This implies

$$
\frac{A - 2NB^{1/2} d_1 d_2}{N} \|F\|^2 \leqslant \sum_{i\in\mathbb{N}} |F(w_i)|^2
$$

As N, A and B are fixed, applying 5.9 and 5.12 we have that there is δ small enough such that $\widetilde{\Gamma}$, previously defined, fulfills that there are A', $B' > 0$ with:

$$
A'\|F\|^2 \leqslant \sum_{i\in\mathbb{N}} |F(w_i)|^2 \leqslant B'\|F\|^2
$$

that is, $\widetilde{\Gamma}$ is a sampling set. ☐

Remark. This result tells us that when we consider atoms of Gabor in the Feichtinger algebra, the only discrete sets that interest us are the separate ones.

The results that we have obtained until now allow us to compare some sets with other. But they are not useful when we ask if some set is a sampling set or when we want to prove the existence of sampling sets.

The following step is to compare the discrete information with the continuous one. Our goal i to prove that sets that, speaking in an informal way, have points in all the plan are sampling sets.

Lemma 5.14. *Let $\Gamma = \{z_j\}_{j\in\mathbb{N}}$ be such that it is fulfilled that $\forall j\ \exists V_j \subseteq \mathbb{C}$ (open) so that $\mathbb{C} = \cup_{j\in\mathbb{N}} \overline{V_j}$, with $z_j \in V_j$ and $\cup_{j\in\mathbb{N}} (V_j - z_j) \subseteq V$ compact, $V_j \cap V_k = \emptyset$. Then it is fulfilled that:*

$$
\left| \|F\| - \left(\sum_{j\in\mathbb{N}} c_j |F(z_j)|^2 \right)^{1/2} \right| \leqslant d_1' d_2' \|F\|
$$

where:

- $d_1' = \left(\sup_{z\in V} \int_{\mathbb{C}} |k_z(w) - k(w)|\, dm(w) \right)^{1/2}$
- $d_2' = \left(\sup_{w\in\mathbb{C}} \sum_{j\in\mathbb{N}} \int_{V_j} |k_w(z) - k_w(z_j)|\, dm(z) \right)^{1/2}$

and c_j is the area of V_j.

Proof. Calculating directly:

$$\begin{aligned}
&\left|\|F\| - \Big(\sum_{j\in\mathbb{N}} c_j |F(z_j)|^2\Big)^{1/2}\right|^2 = \left|\|F\| - \Big\|\sum_{j\in\mathbb{N}} \chi_{V_j}(z)|F(z_j)|\Big\|\right|^2 \\
&\leqslant \Big\||F(z)| - \sum_{j\in\mathbb{N}} \chi_{V_j}(z)|F(z_j)|\Big\|^2 = \Big\|\sum_{j\in\mathbb{N}} (|F(z)| - |F(z_j)|)\chi_{V_j}(z)\Big\|^2 \\
&= \int_{\mathbb{C}} \Big|\sum_{j\in\mathbb{N}} (|F(z)| - |F(z_j)|)\chi_{V_j}(z)\Big|^2 \, dm(z) = \sum_{j\in\mathbb{N}} \int_{V_j} \big||F(z)| - |F(z_j)|\big|^2 \, dm(z)
\end{aligned}$$

Now we have to take into account that V_j is a neighbourhood of z_j, and therefore $V_j - z_j$ is a neighbourhood of 0, that moreover we have to remember that it is contained in V. We write $F_{-z_j}(z) = \langle f_{-z_j}, g_z\rangle$. It is fulfilled that $|F(z)| = |F_{-z_j}(z - z_j)|$ and $\|F\| = \|F_{-z_j}\|$. If we take this into account, we can write:

$$\begin{aligned}
&\sum_{j\in\mathbb{N}} \int_{V_j} \big||F(z)| - |F(z_j)|\big|^2 dm(z) = \sum_{j\in\mathbb{N}} \int_{V_j - z_j} \big||F_{-z_j}(z)| - |F_{-z_j}(0)|\big|^2 dm(z) \\
&\leqslant \sum_{j\in\mathbb{N}} \int_{V_j - z_j} \big|F_{-z_j}(z) - F_{-z_j}(0)\big|^2 dm(z) \\
&\leqslant \sum_{j\in\mathbb{N}} \int_{V_j - z_j} \left(\int_{\mathbb{C}} |F_{-z_j}(w)||k_z(w) - k(w)| \, dm(w)\right)^2 dm(z) \\
&= \sum_{j\in\mathbb{N}} \int_{V_j - z_j} \left[\int_{\mathbb{C}} |F_{-z_j}(w)|^2 |k_z(w) - k(w)| dm(w) \int_{\mathbb{C}} |k_z(w) - k(w)| dm(w)\right] dm(z)
\end{aligned}$$

We bound each part separately. If we take into account that $z \in V_j - z_j$ and that $V_j - z_j \subseteq V$, we can write:

$$\int_{\mathbb{C}} |k_z(w) - k(w)| \, dm(w) \leqslant \sup_{z\in V} \int_{\mathbb{C}} |k_z(w) - k(w)| \, dm(w) = (d_1')^2$$

It is only missing the second part:

$$\begin{aligned}
\sum_{j\in\mathbb{N}}\int_{V_j-z_j}\int_{\mathbb{C}}&|F_{-z_j}(w)|^2|k_z(w)-k(w)|\,dm(w)\,dm(z)\\
=&\sum_{j\in\mathbb{N}}\int_{V_j-z_j}\int_{\mathbb{C}}|F(w+z_j)|^2|k_z(w)-k(w)|\,dm(w)\,dm(z)\\
=&\sum_{j\in\mathbb{N}}\int_{V_j-z_j}\int_{\mathbb{C}}|F(w)|^2|k_z(w-z_j)-k(w-z_j)|\,dm(w)\,dm(z)\\
=&\sum_{j\in\mathbb{N}}\int_{V_j-z_j}\int_{\mathbb{C}}|F(w)|^2\left|e^{2\pi i x_j b}k_{z+z_j}(w)-e^{2\pi i x_j b}k_{z_j}(w)\right|\,dm(w)\,dm(z)\\
=&\sum_{j\in\mathbb{N}}\int_{V_j}\int_{\mathbb{C}}|F(w)|^2|k_z(w)-k_{z_j}(w)|\,dm(w)\,dm(z)\\
=&\int_{\mathbb{C}}|F(w)|^2\sum_{j\in\mathbb{N}}\int_{V_j}|k_w(z)-k_w(z_j)|\,dm(z)\,dm(w)\leqslant(d_2')^2\|F\|^2
\end{aligned}$$

as we want to prove. □

In this lemma we can think that V is a closed ball of center 0. If the radius of V is small enough, which means that the set is dense enough, we are in conditions of proving that Γ is a sampling set when $g\in\mathcal{A}$.

Theorem 5.15. *Let H be a phase space of a Gabor atom $g\in\mathcal{A}$. There is δ such that if $\Gamma=\{z_j\}_{j\in\mathbb{N}}$ is a separate set fulfilling the conditions of 5.14 with V contained in $B(0,\delta)$, then Γ is a sampling set for H.*

Proof. What we will make is to bound d_2' and to make d_1' as small as we need in 5.14. For d_2', we fix $w\in\mathbb{C}$ and we have that:

$$\begin{aligned}
\sum_{j\in\mathbb{N}}\int_{V_j}|k_w(z)-k_w(z_j)|\,dm(z)\leqslant&\sum_{j\in\mathbb{N}}\int_{V_j}|k_w(z)|\,dm(z)+\sum_{j\in\mathbb{N}}|V_j||k_w(z_j)|\\
\leqslant&\|k\|_1+\sum_{j\in\mathbb{N}}|V_j|\frac{1}{|V_j|}\int_{V_j}|Mk(z-w)|\,dm(z)\\
=&\|k\|_1+\|Mk\|_1
\end{aligned}$$

if $\delta\leqslant 1$. Therefore $d_2'\leqslant\left(\|k\|_1+\|Mk\|_1\right)^{1/2}$.

For d_1' we fix $z\in V$ and we calculate:

$$\int_{\mathbb{C}}|k_z(w)-k(w)|\,dm(w)=\int_{\mathbb{C}}\left|e^{2\pi i b x}k(w-z)-k(w)\right|\,dm(w)$$

As $k \in L^1(\mathbb{C})$, there is R big enough so that, for every $z \in V$:

$$\int_{\mathbb{C}\setminus B(0,R)} |k(w-z)|\, dm(w) \leqslant \frac{\varepsilon^2}{4} \tag{5.3}$$

In this way we have that:

$$\begin{aligned}\int_{\mathbb{C}} |k_z(w)-k(w)|\, dm(w) &\leqslant \int_{B(0,R)} \left|e^{2\pi i x(y-b)} k(w-z) - k(w)\right| dm(w) + \frac{\varepsilon^2}{2}\\ &\leqslant \int_{B(0,R)} \left|e^{2\pi i x(y-b)} - 1\right| |k(w-z)|\, dm(w)\\ &\quad + \int_{B(0,R)} |k(w-z)-k(w)|\, dm(w) + \frac{\varepsilon^2}{2}\end{aligned}$$

Now, as $w \in B(0,R)$ we have that $|b| < R$, and in an equivalent way $|x|, |y| < \delta$. Therefore, if we take δ small enough, we have that $\left|e^{2\pi i x(y-b)} - 1\right| < \frac{\|k\|_1 \varepsilon^2}{4}$, which bounds the first integral. We can make the second integral smaller than $\frac{\varepsilon^2}{4}$ choosing δ small by the continuity of the translation operator of in $L^1(\mathbb{C})$. If we join all this it is deduced that if δ is small enough we can achieve $d_1' < \varepsilon$ for any $\varepsilon > 0$. Therefore, applying 5.14 we see that:

$$\left| \|F\| - \left(\sum_{j\in\mathbb{N}} c_j |F(z_j)|^2\right)^{1/2} \right| < \varepsilon(\|k\|_1 + \|Mk\|_1)^{1/2} \|F\|$$

We take δ such that $\varepsilon\big(\|k\|_1 + \|Mk\|_1\big)^{1/2} < 1$ and we obtain:

$$\begin{aligned}\left(1 - \varepsilon(\|k\|_1 + \|Mk\|_1)^{1/2}\right) \|F\| &\leqslant \left(\sum_{j\in\mathbb{N}} c_j |F(z_j)|^2\right)^{1/2}\\ &\leqslant \left(1 + \varepsilon(\|k\|_1 + \|Mk\|_1)^{1/2}\right) \|F\|\end{aligned}$$

Now it is only necessary to define A and B so that

$$A = \frac{\left(1 - \varepsilon(\|k\|_1 + \|Mk\|_1)^{1/2}\right)^2}{\sup_j\{c_j\}}$$

and

$$B = \frac{\left(1 + \varepsilon(\|k\|_1 + \|Mk\|_1)^{1/2}\right)^2}{\inf_j\{c_j\}}$$

and we obtain:

$$A\|F\|^2 \leqslant \sum_{j\in\mathbb{N}} |F(z_j)|^2 \leqslant B\|F\|^2$$

Both $\sup_j\{c_j\}$ and $\inf_j\{c_j\}$ are bounded and are bigger than 0. □

Remark. We have not used 5.9 for the right inequality.

We will give now a result that gives a simpler way to recognize the sets that are under the conditions of 5.14.

Lemma 5.16. *Let* $\Gamma = \{z_j\}_{j\in\mathbb{N}}$ *be a separate set such that* $\exists\delta$ *with the property that* $\forall z \in \mathbb{C}$, $B(z,\delta)\cap\Gamma \neq \emptyset$.

Then Γ *fulfills the conditions of 5.14*

Proof. As Γ is a separate set it is only necessary to see that $\exists V_j$ (open), with $V_j \cap V_k = \emptyset$ if $j \neq k$, $\mathbb{C} = \cup_{j\in\mathbb{N}}\overline{V_j}$, $(V_j - z_j) \subseteq V$ compact and $|\cap_{j\in\mathbb{N}}(V_j - z_j)| > 0$.

Let ε be the separation constant of Γ, $b_j = \overline{B(z_j, \frac{\varepsilon}{2})}$. We define $B_j = B(z_j,\delta)$. We define the V_j now:

- $V_1 = B_1 \setminus \cup_{j\neq 1} b_j$
- $V_k = B_i \setminus (\cup_{j\neq k} b_j) \bigcup \overline{\left(\cup_{j=1}^{k-1} B_i\right)}$

We observe first that in each compact of $\mathbb{C}$ all these unions and intersections are finite, and therefore the union is closed. This tells that the V_j are all open sets, and for construction they are disjointed.

As $B_j \subseteq \cup_{k=1}^{j}\overline{V_k}$, and every z is in some B_j, we already have that $\cup_{j\in\mathbb{N}}\overline{V_j} = \mathbb{C}$. As $V_j \subseteq B_j \Rightarrow (V_j - z_j) \subseteq (B_j - z_j) = B(0,\delta) \subseteq \overline{B(0,\delta)}$ that is compact. For construction, $B(0,\frac{\varepsilon}{2}) \subseteq (V_j - z_j)$ and we obtain the last condition. ☐

Joining the two last results we obtain a totally geometric sufficient condition for sampling sets.

Theorem 5.17. *Let* H *be the phase space of a Gabor atom* $g \in \mathcal{A}$. *There is* δ *such that if* Γ *is a separate set such that* $B(z,\delta)\cap\Gamma \neq \emptyset$ $\forall z$ *then* Γ *is a sampling set for* H.

5.5 The necessary condition of Ramanathan-Steger.

The last result of the former section says that a sufficiently dense set, where any small ball always contains some point of the set, is a sampling set. The next result, which can be found in [RaS95], goes in the other direction. Ramanathan and Steger obtain a result of comparison that says that every complete set of functions (with some restrictions) comes from a set of points with density bigger than any set that generates a Riesz basis.

One of the particularities of this result is that it uses the same notion of density that we used in the Fock space, and that is useful for classifying the sampling and interpolation sets into that space.

We give here the definitions and concrete results, omitting the proofs because of the great importance that these have in the study of the problem that we treat in this chapter.

We remember the definition of uniform density of a set in $\mathbb{C}$. Given a discrete set $\Gamma = \{z_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$, we denote for $n^+(r)$ the maximum number of points of Γ in a ball of radius $r\pi^{\frac{1}{2}}$ with center in any point of $\mathbb{C}$, and $n^-(r)$ the minimum number (the factor $\pi^{\frac{1}{2}}$ is to sustain the same normalization as in [RaS95] but it is not consistent with the one that we have given to the section 5.2).

Definition. We define the **uniform upper density** of Γ as:

$$D^+(\Gamma) = \limsup_{r\to\infty} \frac{n^+(r)}{r^2}$$

Definition. We define the **uniform lower density** of Γ as:

$$D^-(\Gamma) = \liminf_{r\to\infty} \frac{n^-(r)}{r^2}$$

If both densities coincide we will say that Γ has uniform density. It is easy to see that a regular net in the way $\{am + ibn\}_{m,n\in\mathbb{Z}}$ has uniform density $\frac{1}{ab}$.

For this type of nets we know that they can not cause frames (to be sets of sampling) if $D^+(\Gamma) = D^-(\Gamma) < 1$. The one that we will see is that this result can become generalized to general sets.

Definition. We suppose that the set $G(\varphi, \Gamma)$ is complete for a Gabor atom $\varphi \in L^2(\mathbb{R})$ and a discrete set $\Gamma \subseteq \mathbb{C}$. We will say that $G(\varphi, \Gamma)$ has the **homogeneous approximation property** if for any fixed f and any $\varepsilon > 0$ there is a $R > 0$ such that for any $z_0 \in \mathbb{C}$, f_{z_0} can be approximate with error less than ε in L^2-norm by a vector of the space generated by:

$$\{f_z : z \in \Gamma \cap B(z_0, R)\}$$

Theorem 5.18 (Ramanathan-Steger)**.** *Let Γ and Λ be discrete sets. We suppose that there are two functions $\phi, \varphi \in L^2(\mathbb{R})$ such that $G(\phi, \Gamma)$ is a Riesz basis and $G(\varphi, \Lambda)$ is complete and it has the uniform approximation property.*

i) If Γ and Λ have uniform upper densities given by $D^+(\Gamma)$ and $D^+(\Lambda)$ then:

$$D^+(\Gamma) \leqslant D^+(\Lambda)$$

ii) If Γ *and* Λ *have uniform lower densities given by* $D^-(\Gamma)$ *and* $D^-(\Lambda)$ *then:*

$$D^-(\Gamma) \leqslant D^-(\Lambda)$$

This result is very interesting because it compares Gabor systems generated by two different functions and two different discrete sets. We observe that we do not ask for any special condition for the Gabor atoms. The proof that we find in [RaS95] only uses linear algebra and for this reason this the result is easily exportable to similar fields.

It is only necessary to know which systems fulfill the uniform approximation property. In [RaS95] they give some results in this direction that we next replay.

Theorem 5.19 (Ramanathan-Steger). *We suppose that* $G(\varphi, \Lambda)$ *is complete for* $\varphi \in L^2(\mathbb{R})$ *and* Λ *a regular net. Then* $G(\varphi, \Lambda)$ *has the uniform approximation property.*

This result, together with the fact that we know Riesz bases generated by sets with density 1, allows to improve the existing results. We can affirm that not just there are not frames coming from a net with $ab > 1$, but either that we will not find complete systems.

Theorem 5.20 (Ramanathan-Steger). *Let* $\Gamma \subseteq \mathbb{C}$ *be a separate set. We suppose that, for* $\varphi \in L^2(\mathbb{R})$, $G(\varphi, \Gamma)$ *is a frame. Then* $G(\varphi, \Gamma)$ *has the uniform approximation property.*

This result says that all sampling sets have density bigger than 1. We obtain also the following corollary:

Corollary 5.21. *Let* Γ *be a separate set and* $\varphi \in L^2(\mathbb{R})$ *be such that* $G(\varphi, \Gamma)$ *is a Riesz basis. Then* Γ *has uniform density:*

$$D^+(\Gamma) = D^-(\Gamma) = 1$$

We can not affirm that, if $D^+(\Gamma) = D^-(\Gamma) = 1$, Γ only generates Riesz bases, since if we add one point to a set these densities do not change.

In [RaS95] it is conjectured that $D^+(\Gamma) < 1$ implies that $G(\varphi, \Gamma)$ is incomplete for every $\varphi \in L^2(\mathbb{R})$, but this is false, since the generators by translations are particular cases of Gabor systems and we can find examples of complete sets with $D^+(\Gamma) = D^-(\Gamma) = 0$.

These results are generalized in [CDH99] to several dimensions and with unions of nets that use different atoms. Also we have extensions of the concept of uniform approximation to other types of frames (for example the dual frame of a Gabor frame). We can find a good summary of this class of concepts and its applications in [BCHL06].

5.6 An interpolation result.

In this section we will give an interpolation in phase spaces result, inspired in [Dya94]. We will prove that a set with enough separation is an interpolation set.

Theorem 5.22. *Let g be a Gabor atom of the Feichtinger algebra. Then there is R_0 such that every set $\Gamma = \{z_k\}_{k\in\mathbb{N}}$ with separation constant bigger than $2R_0$ is an interpolation set for H, the phase space of g.*

This R_0 is the minimum R such that:

$$\int_{B(O,R)} |k(z)|\, dm(z) \geqslant \frac{1}{2}\|k\|_1$$

If $R = R_0$ the former integral is $\frac{1}{2}\|k\|_1$ by continuity.

Proof. We remember that a set $\Gamma = \{z_j\}_{j\in\mathbb{N}} \subseteq \mathbb{C}$ is an interpolation set for H if for any succession of values $(a_j)_{j\in\mathbb{N}} \in l^2$ there is a function $F \in H$ such that $F(z_j) = a_j\ \forall j \in \mathbb{N}$. This is equivalent to see that there is $F \in H$ such that:

$$\int_{\mathbb{C}} F(z)k(z, z_j)\, dm(z) = \langle F, k_{z_j}\rangle = a_j \qquad \forall j \in \mathbb{N} \tag{5.4}$$

If we take $\phi \in L^2(\mathbb{C})$ it works out that:

$$\widetilde{\phi}(z) = \int_{\mathbb{C}} \phi(w)\overline{k(z, w)}\, dm(w) \in H$$

As a matter of fact $\widetilde{\phi}$ is the orthogonal projection of ϕ in H. We observe that:

$$\langle \phi, k_{z_j}\rangle = \langle \widetilde{\phi}, k_{z_j}\rangle \qquad \forall j \in \mathbb{N}$$

and it works out that to solve the problem (5.4) for a function $F \in H$ is equivalent to find a $\phi \in L^2(\mathbb{C})$ such that:

$$\int_{\mathbb{C}} \phi(z)\overline{k(z_j, z)}\, dm(z) = \langle \phi, k_{z_j}\rangle = a_j \qquad \forall j \in \mathbb{N} \tag{5.5}$$

This is not strange, since the fact that $\Gamma = \{z_j\}_{j\in\mathbb{N}}$ is an interpolation set is equivalent to the linear independence of the set of functions $\{k_{z_j}\}_{j\in\mathbb{N}}$, and this does not depend on if we work in H or in all $L^2(\mathbb{C})$.

Therefore we want to see that, if Γ fulfills the conditions of the statement, we can assure that for every succession $(a_j)_{j\in\mathbb{N}}$ exists $\phi \in L^2(\mathbb{C})$ that solves (5.5).

To study this problem we call:

$$\alpha = \int_{\mathbb{C}} |k(z)|\, dm(z) = \int_{\mathbb{C}} |k_w(z)|\, dm(z) = \|k\|_1$$

Given a succession $(c_k)_{k\in\mathbb{N}} \in l^2$, we consider the following function:

$$f(z) = \sum_{k=1}^{\infty} \frac{c_k}{\alpha} \frac{|k_{z_k}(z)|}{\overline{k_{z_k}(z)}} \chi_{B_k}(z)$$

where B_k will be balls of center z_k and radius $R > R_0$ (half of the separation constant of Γ). We define $\frac{|k_{z_k}(z)|}{\overline{k_{z_k}(z)}} = 1$ if $k_{z_k}(z) = 0$. We have chosen R so that B_k are disjoint. In this way it is clear that $f \in L^2(\mathbb{C})$. Now we calculate and we have that:

$$\langle f, k_{z_j} \rangle = \sum_{k=1}^{\infty} b_{jk} c_k$$

with b_{jk} defined as:

$$b_{jk} = \int_{B_k} \frac{1}{\alpha} \frac{|k_{z_k}(z)|}{\overline{k_{z_k}(z)}} \overline{k_{z_j}(z)}\, dm(z)$$

Now we define the following operator:

$$\begin{aligned} T : l^2 &\longrightarrow l^2 \\ (c_k)_k &\longmapsto \left(\sum_{k=1}^{\infty} b_{jk} c_k \right)_j \end{aligned}$$

to see that this is an operator is bounded we can calculate:

$$\left\| T\big((c_k)_k\big) \right\|^2 = \sum_{j=1}^{\infty} \Big| \sum_{k=1}^{\infty} b_{jk} c_k \Big|^2 \leqslant \sum_{j=1}^{\infty} \Big(\sum_{k=1}^{\infty} |b_{jk}| \sum_{k=1}^{\infty} |b_{jk}| |c_k|^2 \Big)$$

On one hand we have that:

$$\sum_{k=1}^{\infty} |b_{jk}| = \sum_{k=1}^{\infty} \Big| \int_{B_k} \frac{|k_{z_k}(z)|}{\overline{k_{z_k}(z)}} \frac{\overline{k_{z_j}(z)}}{\alpha}\, dm(z) \Big| \leqslant 1$$

On the other hand we see:

$$\begin{aligned} \sum_{j=1}^{\infty} \sum_{k=1}^{\infty} |b_{jk}| |c_k|^2 &= \sum_{k=1}^{\infty} |c_k|^2 \sum_{j=1}^{\infty} |b_{jk}| = \sum_{k=1}^{\infty} |c_k|^2 \sum_{j=1}^{\infty} \Big| \int_{B_k} \frac{|k_{z_k}(z)|}{\overline{k_{z_k}(z)}} \frac{\overline{k_{z_j}(z)}}{\alpha}\, dm(z) \Big| \\ &\leqslant \sum_{k=1}^{\infty} |c_k|^2 \sum_{j=1}^{\infty} \int_{B_k} \frac{|k(z - z_j)|}{\alpha}\, dm(z) \\ &\leqslant \sum_{k=1}^{\infty} |c_k|^2 \sum_{j=1}^{\infty} \int_{B_k - z_j} \frac{|k(w)|}{\alpha}\, dm(w) \leqslant \|(c_k)_k\|^2 \end{aligned}$$

Therefore the operator T has norm less or equal to 1. The following step is to check out that this operator is invertible. We will make this seeing that $\|Id-T\|<1$.

$$\begin{aligned}\|Id-T\|^2 &= \sup_{\|(c_k)_k\|=1}\sum_{j=1}^{\infty}\Big|\sum_{k=1}^{\infty}(\delta_{jk}-b_{jk})c_k\Big|^2 \\ &\leqslant \sup_{\|(c_k)_k\|=1}\sum_{j=1}^{\infty}\Big(\sum_{k=1}^{\infty}|\delta_{jk}-b_{jk}|\sum_{k=1}^{\infty}|\delta_{jk}-b_{jk}||c_k|^2\Big)\end{aligned}$$

We have to look at this expression in parts. We observe first that $b_{jj}\geqslant 0$ and it does not depend on j and that $\sum_j|b_{jk}|\leqslant 1$ and $\sum_k|b_{jk}|\leqslant 1$. Taking this into account we see that, fixed j:

$$\sum_{k=1}^{\infty}|\delta_{jk}-b_{jk}| = 1-2b_{jj}+\sum_{k=1}^{\infty}|b_{jk}|\leqslant 2(1-b_{jj})$$

On the other hand we have, fixed k:

$$\sum_{j=1}^{\infty}|\delta_{jk}-b_{jk}| = 1-2b_{kk}+\sum_{j=1}^{\infty}|b_{jk}|\leqslant 2(1-b_{kk})$$

we call $\beta=b_{jj}$, and we remember that it does not depend on j. Now we add this bound to arrive to:

$$\|Id-T\|\leqslant 2(1-\beta)$$

We need that $\beta>\frac{1}{2}$. After making a change of variable, the definition of β is the following one:

$$\beta=\frac{1}{\|k\|_1}\int_{B(0,R)}|k(z)|\,dm(z)$$

It is clear that if $R>R_0$ it is fulfilled that $\beta=\beta(R)>\frac{1}{2}$. If we are in these conditions the operator T is invertible. That is, for every succession $(a_j)_{j\in\mathbb{N}}\in l^2$ there exists another succession $(c_k)_{k\in\mathbb{N}}\in l^2$ such that $T((c_k)_{k\in\mathbb{N}})=(a_j)_{j\in\mathbb{N}}$. This says that for every succession $(a_j)_{j\in\mathbb{N}}\in l^2$ there exists a function $f\in L^2(\mathbb{C})$ such that:

$$\langle f(z),k_{z_j}(z)\rangle=a_j \qquad \forall j\in\mathbb{N}$$

For that we have commented on formerly we can think that this f belongs to H. □

These sets correspond with Riesz bases of the subspace that they generate, since they can not be complete.

5.7 Atoms with convex bound.

There is a class of Gabor atoms for which we can give some generator results. Those are the atoms for which we can extend the phase space to an space of two complex variables. This is not the same that we made in section 5.2. In that example the space was composed by entire functions in one variable. Here the goal is more modest since we have proved that this example was unique. The idea is to achieve a similar structure for being able to use the tools of the complex analysis.

The atoms that we will consider have to have very good location in time as well as in frequency. The way to obtain this is working with convex functions and its conjugates.

Definition. Given a convex function $\phi(t)$ we define its **Legendre transform** as:

$$\widetilde{\phi}(x) = \sup_t xt - \phi(t)$$

$\widetilde{\phi}$ is also a convex function and we will say that ϕ and $\widetilde{\phi}$ are conjugated convex functions.

Using this concept we can give a result that says which class of functions we will use.

Theorem 5.23. *Let $g \in L^2(\mathbb{R})$ be a function such that both g and $\widehat{g}$ have an holomorphic extension in the complex plane fulfilling:*

$$|g(z)| \leqslant Ce^{a\phi(\Im z)}e^{-b\phi(\Re z)}$$
$$|\widehat{g}(w)| \leqslant Ce^{a\widetilde{\phi}(\Im z)}e^{-b\widetilde{\phi}(\Re z)}$$

with $a, b > 0$ and ϕ, $\widetilde{\phi}$ conjugated convex functions. Then the Gabor transform of any function $f \in L^2(\mathbb{R})$ with respect to g accepts an holomorphic extension in $\mathbb{C}^2$ and it fulfills the bound:

$$|Gf(z,w)|^2 \leqslant \|f\|^2 \|e^{-b\phi}\|_1 Ce^{2a\phi(\Im z)}e^{b\widetilde{\phi}(\frac{4\pi}{b}\Im w)}e^{4\pi\Im w\Re z}$$

Proof. First we see how we can complexify the first variable. We write directly:

$$Gf(z,u) = \int_{\mathbb{R}} f(t)e^{-2\pi itu}g(t-z)\,dt$$

with $u \in \mathbb{R}$ and $z \in \mathbb{C}$. This expression is well defined since $g \in L^2(\mathbb{R})$ in any line $\Im z = const.$. Deriving under the integral we have that it is holomorphic in z. We can see the extension in the other variable by a symmetrical argument.

Now we will obtain the bound:

$$\begin{aligned}|Gf(z,w)|^2 =& \left|\int_{\mathbb{R}} f(t)e^{-2\pi itw}g(t-z)\,dt\right|^2 \leqslant \|f\|^2\int_{\mathbb{R}}\left|e^{-2\pi itw}g(t-z)\right|^2 dt\\ \leqslant&\|f\|^2\int_{\mathbb{R}} e^{4\pi t\Im w}Ce^{2a\phi(\Im z)}e^{-2b\phi(t-\Re z)}\,dt\\ \leqslant&\|f\|^2Ce^{2a\phi(\Im z)}\int_{\mathbb{R}} e^{4\pi(s+\Re z)\Im w}e^{-2b\phi(s)}\,ds\end{aligned}$$

First of all we put out of the integral the term $e^{4\pi\Re z\Im w}$. Afterwards we remember that:

$$\widetilde{\phi}(t) = \sup_x xt - \phi(x)$$

We will use $e^{-b\phi(s)}$ for integrating and we can bound:

$$e^{4\pi s\Im w - b\phi(s)} \leqslant e^{b\widetilde{\phi}(\frac{4\pi}{b}\Im w)}$$

that completes the proof. □

If we compare this case with the one studied in 5.2, where we used the Gaussian function, we can see that:

$$Gf(z,w) = Bf(z-iw)e^{-\frac{\pi}{2}(z^2+w^2)}e^{-\pi izw}$$

where Bf was the Bargmann transform of f. In this way we see that in this concrete case our entire function in two variables is in fact a entire function in a variable (modified a bit) evaluated in $z-iw$. The bounds come in a fast way of this formula. Moreover also we can observe that:

$$Gf(x,-u)e^{-\frac{\pi}{2}(x^2+u^2)}e^{\pi ixu}$$

will be an entire function in $x+iu$.

The way to obtain functions fulfilling the conditions from the former theorem is to take $g(t) = e^{-p(t)}$ with p a convex polynomial. To obtain the bound we inspire in [BNW88], where some of these results are already given. In this way we take a positive and convex polynomial vanishing at the origin $p(t) = \sum_{n=1}^{N} a_n t^n$. A convex polynomial vanishing at the origin is positive if and only if the first derivative at 0 also cancels. Therefore we can think that the two first coefficients of the polynomial are 0. Moreover a polynomial of this type is practically symmetrical, as it can be deduced from the following result:

Lemma 5.24. *Fixed a maximum degree N there exists C (only depending on N) such that:*

$$p(t) \leqslant \sum_{n=2}^{N} |a_n||t|^n \leqslant Cp(t)$$

Proof. The first inequality is trivial. For the second we define

$$\Psi = \left\{ p\text{convex} : \text{degree of} p \leqslant N, p(0) = p'(0) = 0, \sum_{n=2}^{N} |a_n| = 1 \right\}$$

This set is compact (bounded and with finite dimension). Now, we define the map:

$$\begin{aligned} \Psi &\xrightarrow{\Phi} \mathbb{R} \\ p &\longmapsto p(1) \end{aligned}$$

As p is convex and positive, $p(1) = 0$ implies that $p = 0$. In this way we can affirm that:

$$m = \inf_{p \in \Psi} p(1) = \min \Phi(\Psi) > 0$$

Then we can deduce that:

$$p(1) \geqslant m = m \sum_{n=2}^{N} |a_n|$$

for every $p \in \Psi$. Now, we fix t and we define $q(x) = p(tx) = \sum_{n=2}^{N} a_n t^n x^n$. Then:

$$p(t) = q(1) \geqslant \sum_{n=2}^{N} |a_n||t|^n$$

and we have proved the lemma. ☐

Lemma 5.25. *Let p be positive and convex polynomial vanishing at the origin and of degree N. Then:*

$$|\Re p(z) - p(\Re z)| \leqslant \varepsilon p(\Re z) + C_\varepsilon p(\Im z)$$

for every $\varepsilon > 0$, and C_ε a constant that just depend of ε and of the degree of the polynomial.

Proof. We begin writing:

$$\left|\Re p(z) - p(\Re z)\right| \leqslant \left| \Re \sum_{n=2}^{N} a_n (x+iy)^n - \sum_{n=2}^{N} a_n x \right|$$

We go to develop this sum:

$$\left|\sum_{n=2}^{N} a_n\left(x^n + \Re\sum_{k=1}^{n-1}\begin{pmatrix} n \\ k \end{pmatrix}x^k(iy)^{n-k} + \Re(iy)^n\right) - \sum_{n=2}^{N} a_n x^n\right|$$
$$\leqslant C\sum_{n=2}^{N}|a_n|\left(\sum_{k=1}^{n-1}|x|^k|y|^{n-k} + |y|^k\right)$$

We take $\varepsilon > 0$ (very small). If $|y| \leqslant \varepsilon|x|$ we see that:

$$|x|^k|y|^{n-k} \leqslant |x|^n\varepsilon^{n-k} \leqslant \varepsilon|x|^n$$

in this way we can affirm that in this case:

$$\sum_{k=1}^{n-1}|x|^k|y|^{n-k} \leqslant N\varepsilon|x|^n$$

However, if $|y| \geqslant \varepsilon|x|$ we bound

$$\sum_{k=1}^{n-1}|x|^k|y|^{n-k} \leqslant \frac{N}{\varepsilon}|y|^n$$

We take into account all this and use the former lemma to obtain:

$$\left|\Re p(z) - p(\Re z)\right| =\leqslant CN\varepsilon p(\Re z) + C\left(\frac{N}{\varepsilon} + 1\right)p(\Im z)$$

and we have proved the lemma. □

Corollary 5.26. *Let p be a convex polynomial vanishing at the origin. Then there are $a, b > 0$ such that:*

$$-\Re p(z) \leqslant -ap(\Re z) + bp(\Im z)$$

Theorem 5.27. *Let p be a convex and positive polynomial vanishing at the origin. Let be $g = e^{-p}$. Then there are $a, b > 0$ such that:*

$$|g(z)| \leqslant e^{-a_1 p(a_2\Re z)}e^{b_1 p(b_2\Im z)}$$
$$|\widehat{g}(w)| \leqslant e^{-a_1\widetilde{p}(a_2\Re z)}e^{b_1\widetilde{p}(b_2\Im w)}$$

Proof. The bound of g is the former corollary. For the bound of $\widehat{g}$ we first take the function:

$$g_\xi(z) = e^{-p(z)}e^{-2\pi i\xi z}$$

We integrate this function into the square of vertexes t, $-t$, $t+ih$ and $-t+ih$ in counter-clock wise. As g_ξ is an holomorphic function this integral is 0. Now we observe that:

$$\left|\int_0^h e^{-p(t+is)} e^{-2\pi i\xi(t+si)}\,ds\right| \leqslant e^{-ap(t)}\int_0^h e^{bp(s)} e^{2\pi\xi s} \to 0$$

when $t \to \infty$ for every ξ. For the segment from $-t$ to $-t+ih$ we can make the same argument and in this way we see the coincidence of the integrals:

$$\widehat{g}(\xi) = \int_{\mathbb{R}} e^{-p(t)} e^{-2\pi andt}\,dt = \int_{\mathbb{R}} e^{-p(t+hi)} e^{-2\pi i\xi(t+hi)}\,dt$$

We will work with this second expression to compleixify the Fourier transform of g. Therefore:

$$\widehat{g}(w) = \int_{\mathbb{R}} e^{-p(t+hi)} e^{-2\pi iw(t+hi)}\,dt$$

We observe that this formula is true for every h. Then we can bound:

$$|\widehat{g}(w)| \leqslant \int_{\mathbb{R}} e^{-ap(t)} e^{bp(h)} e^{2\pi(t\Im w + h\Re w)}\,dt = e^{bp(h)+h\Re w}\int_{\mathbb{R}} e^{-ap(t)+t\Im w}\,dt$$

for every h. we bound the first factor:

$$\inf_h e^{bp(h)+h\Re w} = e^{-b\sup_h \frac{-\Re w}{b}h - p(h)} = e^{-b\widetilde{p}(\frac{-\Re w}{b})}$$

Now we have one of the terms. To obtain the other one the method is similar:

$$\int_{\mathbb{R}} e^{-ap(t)+t\Im w}\,dt \leqslant \int_{\mathbb{R}} e^{\sup_t t\Im w - \frac{a}{2}p(t)} e^{-\frac{a}{2}p(t)}\,dt \leqslant e^{-\frac{a}{2}\widetilde{p}(\frac{2}{a}\Im w)}\|e^{-\frac{a}{2}p}\|_1$$

And we have proved the theorem. ☐

Now we have the existence of functions fullfilling the conditions of 5.23. This result said that we can think that all the functions of the phase space are entire functions in two variables. We want to use this properties to give sufficient conditions for uniqueness sets.

We remember how this is made in the case of functions holomorphic in one variable. The zero set of a entire function is discrete and each zero has assigned a multiplicity. If we define the measure $\mu = \sum \lambda \in \Lambda m_\lambda \lambda$, it is a known result that if f has zeros in Λ with the given multiplicity, in the language of the distributions we can say that:

$$\Delta \log|f| = 2\pi\mu$$

If now we take the second Green identity:

$$\int_\Omega u\Delta v - v\Delta u = \int_{\partial\Omega} u\frac{\partial v}{\partial n} - v\frac{\partial u}{\partial n}$$

taking $u = \log|f|$ and $v = 1 - |z|^2$ in the unit disk we obtain:

$$4\int_{\mathbb{D}} \log|f| + 2\pi \sum_{\lambda\in\Lambda} (1 - |\lambda|^2) m_\lambda = 2\int_{|z|=1} \log|f|$$

This is the equation that allows us to give conditions about the zero set if we have the growth of f bounded.

What we will make is a similar reasoning in two variables. The zero sets of entire functions of two variables is an analytical variety of dimension 1 that has irreducible components V_k with multiplicity m_k.

In this way, if $\{V_k, m_k\}$ is the variety where f cancels with its multiplicity it is fulfilled that:

$$\partial\overline{\partial} \log|f| = \theta$$

where θ is a closed and positive current intrinsically associated to the variety that has as expression

$$\theta = \sum_{j,k} \theta_{jk} i dz_j \wedge d\overline{z_k}$$

so that $\sum \theta_{kk}$ is the planar measure of integration in the variety. We have to understand these formulas in the sense of the currents, which are the generalization of the distributions. $\partial\overline{\partial} \log|f| = \theta$ is the equivalent in two variables to the formula $\Delta \log|f| = 2\pi\mu$.

Whit this language we can deduce the Jensen formula that will be useful for us. We define the auxiliary $(1,1)$-form:

$$\beta = i\partial\overline{\partial}\big(|z|^2 + |w|^2\big) = i(dz \wedge d\overline{z} + dw \wedge d\overline{w})$$

This form fulfills:

$$\theta \wedge \beta = \Big(\sum \theta_{kk}\Big)\ dM$$

and it completes θ to a volume form of $\mathbb{C}^2$. We take R (big) and δ (small) and we will describe the set in that we will integrate. We put:

$$\rho = \frac{x^2}{R^2} + \frac{x^2}{\delta^2} + \frac{u^2}{R^2} + \frac{v^2}{\delta^2}$$

where $z = x + iy$ and $w = u + iv$. In this way $\rho = 1$ defines an ellipsoid in $\mathbb{C}^2$ that have radius R over the real variables and δ over the imaginary ones.

It is also necessary to observe that:

$$\partial\overline{\partial}\rho = \Big(\frac{1}{R^2} + \frac{1}{\delta^2}\Big)(dz \wedge d\overline{z} + dw \wedge d\overline{w})$$

And also we can complete this form for obtaining a volume element:

$$\beta \wedge i\partial\overline{\partial}\rho = \Big(\frac{1}{R^2} + \frac{1}{\delta^2}\Big) dM$$

Moreover it is fulfilled [GrL86]:

$$\beta \wedge i\overline{\partial}\rho_{|\rho=1} = dA$$

the area element (in three real dimensions) of the surface. With these notations Stokes theorem says that:

$$\int_{\rho=1} G\beta \wedge i\overline{\partial}\rho = \int_{\rho<1} \partial G \wedge \beta \wedge i\overline{\partial}\rho + \int_{\rho<1} G\beta \wedge i\partial\overline{\partial}\rho$$

We apply again this theorem to see that:

$$\int_{\rho<1} \partial G \wedge \beta \wedge i\overline{\partial}\rho = \int_{\rho<1} \partial G \wedge \beta \wedge i\overline{\partial}(\rho-1) = -\int_{\rho<1} (\rho-1) i\partial\overline{\partial} G \wedge \beta$$

since $\rho - 1$ cancels at the border. With this we arrive to:

$$\int_{\rho=1} G\beta \wedge i\overline{\partial}\rho = \int_{\rho<1} G\beta \wedge i\partial\overline{\partial}\rho + \int_{\rho<1} (1-\rho) i\partial\overline{\partial} G \wedge \beta \tag{5.6}$$

This is the equation that will allow us to deduce a Jensen formula in two variables.

Theorem 5.28. *We suppose that we have a Gabor atom fulfilling the bound of theorem 5.23 (for example $g = e^{-p}$ with p a polynomial like those that we have treated). Let $F(z,w)$ be the entire extension in two variables of a function of the phase space. Let V be the variety of zeros of this function. Then:*

$$\limsup_{R\to\infty} \frac{H^1\big(V \cap \mathbb{R}^2 \cap B(0,R)\big)}{R^2} \leqslant C_f$$

where H^1 is the 1–dimensional Haussdorf measure*, and we are making intersection of V with the plane $\Im z = \Im w = 0$.*

Remark. This result bounds the 1–dimensional measure of the variety of zeros. This is the type of result that we can wait. We have commented in the first section that it is usual that the zero–variety of a function of the phase space is a set of curves (sometimes it can be bigger, and in the case of the Gaussian function it is a set of points). This theorem tells that the amount of curves (the sum of its lengths) can not be arbitrarily big. We remember that the Jensen formula gives a bound of the same type, but with the 0-dimensional measure, that is, bounds the number of points.

Proof. We will bound the different integrals that turn up in (5.6) for $G = \log|F|$ with F in the phase space. We begin by observing that:

$$G(z,w) \leqslant 2a\phi(\Im z) + b\widetilde{\phi}\Big(\frac{4\pi}{b}\Im w\Big) + 4\pi \Im w \Re z$$

We first bound the integral on $\rho = 1$. If $\rho = 1$ then $|\Re z|, |\Re w| \leqslant R$ and $|\Im z|, |\Im w| \leqslant \delta$. Therefore:

$$G(z,w) \leqslant C_F + 2a\phi(\delta) + b_1\widetilde{\phi}\Big(\frac{4\pi}{b}\delta\Big) + 4\pi\Im w\Re z$$

and we obtain, using that $\beta \wedge i\overline{\partial}\rho$ is positive, that:

$$\int_{\rho=1} G\beta \wedge i\overline{\partial}\rho \leqslant \Big(2a\phi(\delta) + b\widetilde{\phi}\Big(\frac{4\pi}{b}\delta\Big)\Big) \int_{\rho=1} \beta \wedge i\overline{\partial}\rho$$

since the part corresponding to $4\pi\Im w\Re z$ is zero because we are integrating on a symmetrical set. We apply Stokes again and we arrive to:

$$\int_{\rho=1} G\beta \wedge i\overline{\partial}\rho \leqslant \Big(2a\phi(\delta) + b\widetilde{\phi}\Big(\frac{4\pi}{b}\delta\Big)\Big) \int_{\rho<1} \beta \wedge i\partial\overline{\partial}\rho$$

This bound is useful also for the integral on $\rho < 1$. It is evident that:

$$\int_{\rho<1} \beta \wedge i\partial\overline{\partial}\rho \leqslant R^2\delta^2\Big(\frac{1}{R^2} + \frac{1}{\delta^2}\Big)$$

And we obtain the inequality:

$$\int_{\rho<1} (1-\rho) i\partial\overline{\partial}G \wedge \beta \leqslant 2\Big(C + 2a\phi(\delta) + b\widetilde{\phi}\Big(\frac{4\pi}{b}\delta\Big)\Big) R^2\delta^2\Big(\frac{1}{R^2} + \frac{1}{\delta^2}\Big) \tag{5.7}$$

We can find the inferior bound of this integral in [Ber78], that says that:

$$\delta H^1\big(V \cap \mathbb{R}^2 \cap B(0,R)\big) \leqslant \int_{\rho<1} i\partial\overline{\partial}G \wedge \beta$$

Therefore, if we bound the integral in $\rho < 1$ by the same integral but in $\rho < \frac{1}{2}$ we have that:

$$\frac{\delta}{2} H^1\Big(V \cap \mathbb{R}^2 \cap B\Big(0, \frac{R}{2}\Big)\Big) \leqslant \int_{\rho<1} i(1-\rho)\partial\overline{\partial}G \wedge \beta \tag{5.8}$$

We join (5.8) with (5.7) to obtain the inequality:

$$\delta H^1\big(V\cap\mathbb{R}^2\cap B(0,R)\big) \leqslant 2\Big(C+2a\phi(2\delta)+b\widetilde{\phi}\Big(\frac{4\pi}{b}2\delta\Big)\Big)(2R)^2(2\delta)^2\Big(\frac{1}{(2R)^2}+\frac{1}{(2\delta)^2}\Big)$$

From here we arrive to:

$$H^1\big(V \cap \mathbb{R}^2 \cap B(0,R)\big) \leqslant \inf_{\delta} \frac{C + 2a\phi(2\delta) + b\widetilde{\phi}(\frac{4\pi}{b}2\delta)}{\delta} 8R^2$$

that proves the theorem. □

Now we can deduce the following result:

Theorem 5.29. *Let g be a Gabor atom fulfilling the conditions of 5.23. If Λ is a separate set such that:*

$$\limsup_{R\to\infty} \frac{|\Lambda \cap B(0,R)|}{R^2} = \infty$$

Then $G(g,\Lambda)$ generates $L^2(\mathbb{R})$.

For the proof it is only necessary to use theorem 5.23 and the bound of the number of zeros which can have a function of this space that we have given formerly.

This result is not as good as the one that we can give in the Fock space, where the functions were analytic in one variable. The advantage is that it is an asymptotic result that works in a more general class of atoms.

Chapter 6

Wavelets.

6.1 Discretitzation in wavelet phase spaces.

When we studied the Gabor transform we considered translations and modulations of a fixed function. In the case of wavelets we change the modulations for dilations. This modification, in a historical way, was introduced to save the problem that supposed the Balian–Low theorem 1.18, and that did not allow us to find well localized bases in time and frequency at time. The good properties of analysis that presents this transform and the utility of the AMR have converted it into an important field of work (of study as well as of application).

The main differences that we will find with respect to the Gabor transform come from this modification. The first important difference was the admissibility condition. The reconstruction formula is only valid for wavelets that fulfill:

$$c_\psi = \int_0^\infty \frac{|\widehat{\psi}(\xi)|^2}{\xi}\, d\xi < \infty$$

We call $\mathbb{H} = \mathbb{R} \times \mathbb{R}^+$. We remember the notations that we used for the wavelet transform. If $x + iy \in \mathbb{H}$, we define for $f \in L^2(\mathbb{R})$:

$$f_z(t) = y^{-\frac{1}{2}} f\Big(\frac{t-x}{y}\Big)$$

We will use $dm(z) = \frac{dx\,dy}{y^2}$ to designate the hyperbolic measure in $\mathbb{H}$. If ψ is an admissible wavelet and f any function of $L^2(\mathbb{R})$, using these notations, we can define the transform of f with respect to the wavelet ψ as:

$$Wf(z) = \langle f, \psi_z \rangle = \int_{\mathbb{R}} f(t) \overline{y^{-\frac{1}{2}} \psi\Big(\frac{t-x}{y}\Big)}\, dt$$

If the wavelet is normalized so that $\|\psi\| = c_\psi = 1$, theorem 1.15 says that $Wf \in L^2(\mathbb{H})$ and $\|Wf\| = \|f\|$. Moreover we can reconstruct f from the values of the transform:

$$f(t) = \int_{\mathbb{H}} Wf(z)\psi_z(t)\,dm(z) \qquad \forall f$$

Here it is necessary to make some comments about the validity of this formula. We have said that the admissibility condition is necessary, but it is not enough. If $\psi \in H^2(\mathbb{R})$ ($\widehat{\psi}(\xi) = 0$ for $\xi < 0$), then the reconstruction formula is only valid for functions $f \in H^2(\mathbb{R})$. If we want a reconstruction formula for $L^2(\mathbb{R})$ we have to take ψ such that $\psi(t) \in \mathbb{R}$. In this way we can define the transform of a function $f \in L^2(\mathbb{R})$ that only takes real values, and the reconstruction formula is valid in this subspace. From here it is not very difficult to see that we can extend the transform to all $L^2(\mathbb{R})$ and everything works perfectly.

Taking this into account, we can define the wavelet transform in $H^2(\mathbb{R})$, $L^2(\mathbb{R})$ to real values or $L^2(\mathbb{R})$ to complex values, taking the appropriate precautions. The results that we will give during this chapter do not depend in general of in which case we are, and therefore it will not be necessary to indicate it. The only differences will be that the phase spaces do not coincide, although structurally they will be identical. From now on we will think that $\psi \in L^2(\mathbb{R})$ is real valued, and we will define the transformed for $f \in L^2(\mathbb{R})$ complex valuated, which is the most general case. When ψ is real, k is also real and we can forget of the conjugated in all the of reproduction and reconstruction formulas.

The notations that we are using are the same ones that we used in the former chapter for the Gabor transform. This can bring in confusion sometimes, however it helps a lot to seeing the similarities and differences between both cases. Also it is more comfortable when working.

As in Gabor case, we fix an admissible (normalized) wavelet. Theorem 1.33 says that the set of transforms of all the functions of $L^2(\mathbb{R})$ form a Hilbert subspace of $L^2(\mathbb{H})$:

$$H = \Big\{ F(z) \in L^2(\mathbb{H}) : \exists f \in L^2(\mathbb{R}) \text{ with} F(z) = Wf(z) = \langle f, \psi_z \rangle \Big\}$$

This is a Hilbert space with reproductive kernel $k(z) = \langle \psi, \psi_z \rangle$, so that:

$$F(z_0) = \int_{\mathbb{H}} F(z) k(z_0^{-1} \cdot z)\,dm(z)$$

if and only if $F \in H$, with $z_0^{-1} \cdot z = \frac{z - z_0}{y_0}$, as we have commented in the preliminaries. If we define the translations in $\mathbb{H}$ in this way ($\tau_{z_0}(z) = z_0^{-1} \cdot z$), this formula is a convolution of F with k, with the particularity that the group of translations is not commutative. We can use the same notation for translations of a function of the space:

$$F_{z_0}(z) = F(z_0^{-1} \cdot z) = Wf_{z_0}(z)$$

if $F = Wf$. This is another of the important differences with the Gabor transform. In the Gabor case we did not have a true convolution, since we have an exponential factor. But however the translations were commutative. These differences will make some steps be simpler. At the same time they will provoke that the results obtained are not so good as in the former chapter.

There is some counterexample. In the case of uniform continuity, the fact that the exponential factor does not appear makes the result be better. In $\mathbb{H}$ we will use the hyperbolic distance $d(z_1, z_2)$, which is invariant for left–translations. We can also use the pseudohyperbolic one, which is simpler to calculate:

$$\overline{d}(z_1, z_2) = \left| \frac{z_1 - z_2}{z_1 - \overline{z_2}} \right|$$

The relationship between these two distances is:

$$d(z_1, z_2) = \frac{1}{2} \log \frac{1 + \overline{d}(z_1, z_2)}{1 + \overline{d}(z_1, z_2)}$$

that allows us to calculate the hyperbolic distance in a simple way.

After the definition of the distance, we can give the continuity result.

Proposition 6.1. *The wavelet transform is uniformly continuous. That is, given* $\varepsilon > 0$ *exists* $\delta > 0$ *such that if* $d(z_1, z_2) < \delta$*, for every* $F \in H$*,*

$$F(z) = \langle f, \psi_z \rangle$$

it is fulfilled that $|F(z_1) - F(z_2)| \leqslant \|F\|\varepsilon = \|f\|\varepsilon$*.*

Proof. We calculate directly:

$$\begin{aligned} |F(z_1) - F(z_2)| =& \big|\langle f, \psi_{z_1} \rangle - \langle f, \psi_{z_2} \rangle\big| = \big|\langle f, \psi_{z_1} - \psi_{z_2} \rangle\big| \\ \leqslant& \|f\| \big\|\psi_{z_1} - \psi_{z_2}\big\| = \|f\| \big\|\psi - \psi_{z_1^{-1} \cdot z_2}\big\| \end{aligned}$$

and the result is deduced of the continuity of the translation and dilatation operators in $L^2(\mathbb{R})$. ☐

As we see, the result is a bit better than in the Gabor case. In that case we could just give this result for the module of the functions, due to the exponential factor. This will be, however, the only case in that we will have better behavior in the case of wavelets. Sometimes the proofs will be simpler, but the results will not improve those of the former chapter.

The way to act is identical to the Gabor case. We want to study when a system $W(\psi,\Lambda)$ is a frame for $L^2(\mathbb{R})$. We know that this is equivalent to search the sampling sets of H. The way to make this is to discretitzate the reproduction formula:

$$\int_{\mathbb{H}}\left|\int_{\mathbb{H}} F(z)k(\lambda^{-1}\cdot z)\,dm(z)\right|^2 dm(\lambda)\approx\sum_{\lambda\in\Lambda}\left|\int_{\mathbb{H}} F(z)k(\lambda^{-1}\cdot z)\,dm(z)\right|^2$$

In this case, the equivalent to the Feichtinger algebra will be the set of wavelets with integrable or strongly integrable kernel, depending on what we need, because in this case this two sets do not coincide.

We will obtain identical sampling results than in the Gabor case. We prove that a set that has points in any translation of a small ball is a sampling set and therefore causes a frame. We give also explicit bounds of the constants of the frame that depend on the $L^1(\mathbb{H})$–norm of the kernel and its local maximal function. These results can also be achieved, without explicit bounds, using theory of representations, although the wavelet case (which corresponds to a non–unimodular group) is not so studied. On the other hand, Sun and Zhou give sufficient conditions close to those that we will give in this work, also with explicit bounds, but asking to the analyzing wavelet to belong to certain Sobolev spaces. The bounds depend on the norms in these spaces and the methods that they use in [SuZ03] and [SuZ04] are different from the ones that we use.

Regarding necessary conditions, for the time being it seems unattainable to find a result similar to 5.20. On the one hand the definition of density is not clear in $\mathbb{H}$. Seip defines in [Sei93] a density for the Bergman space of the disk that by conformal mapping can be translated to the half–plain. This allows to describe sampling and interpolation sets in this space, which is related with the wavelet transform, as we will see later on.

Heil and Kutyniok also study in [HeK03] and [HeK06] the irregular frames of wavelets using a equivalent definition but with different structural properties. The results that they achieve are also close to the ones that we give here, but without explicit bounds. In their works they deduce a theorem of the type 5.18. Introducing a notion of homogeneous approximation they prove that every set that causes a frame has to have positive density (with certain restrictions on the analyzing wavelet). We also give a result close to this, using the same ideas. The main difference is the wavelet representation that we use. Even though this seems that it should be a minor question, since in principle both representations are equivalents, when we work we have that the set of admissible wavelets that we can use is different in each case.

Sun and Zhou, in the articles cited before, also introduce a density different to the former ones, which they use for giving necessary conditions that do not defer a lot of the rest of works.

6.2 Rigidity of the Haar basis.

In this section we will give a curious result that gives a concrete example of the rigidity of orthogonal bases. This was one of the reasons that brought us to studying Riesz bases and frames instead of these.

We will fix in the Haar wavelet, that due to its simplicity it facilitates a lot the calculations. We can think that this wavelet come from the Multirresolution Analysis generated by the scale function $\phi(t) = \chi_{[0,1)}(t)$, even though we will not need it. For comfort we will work in the real valued $L^2(\mathbb{R})$. In this way it is not necessary to take into account the conjugate in the scalar product.

Definition. We define the **Haar wavelet** as:

$$\psi(t) = \chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},1)}(t)$$

In this way we have that:

$$\psi_z(t) = y^{\frac{-1}{2}} \left(\chi_{[x,x+\frac{y}{2})}(t) - \chi_{[x+\frac{y}{2},x+y)}(t)\right)$$

As we are interested in the orthogonality, we want to find the places where the scalar product of the wavelet with a translation is 0. In other words, the places where the reproductive kernel cancels. That is, we are interested on knowing when:

$$\langle \psi_z, \psi \rangle = \int_{-\infty}^{\infty} y^{\frac{-1}{2}} \left(\chi_{[x,x+\frac{y}{2})} - \chi_{[x+\frac{y}{2},x+y)}\right) \left(\chi_{[0,\frac{1}{2})}(t) - \chi_{[\frac{1}{2},1)}(t)\right) dt = 0 \quad (6.1)$$

There are some trivial cases where this is fulfilled:

- $x + y \leqslant 0$
- $x \geqslant 1$
- $x \geqslant 0,\ x + y \leqslant \frac{1}{2}$
- $x \geqslant \frac{1}{2},\ x + y \leqslant 1$
- $x \leqslant 0,\ x + \frac{y}{2} \geqslant 1$
- $x + \frac{y}{2} \leqslant 0,\ x + y \geqslant 1$

The two first cases correspond to the case in that both wavelets have disjoint support. The other four are owed to the fact that one of the wavelets is totally contained in one of the intervals where the other one is constant. But these are not the only cases where the former scalar product is canceled. We have it left to find two more cases:

- If $x < 0$, $0 < x + \frac{y}{2} < \frac{1}{2}$ and $\frac{1}{2} < x + y < 1$ we have that the scalar product (6.1) is:

$$y^{\frac{-1}{2}} \left(\int_0^{x+\frac{y}{2}} dt - \int_{x+\frac{y}{2}}^{\frac{1}{2}} dt + \int_{\frac{1}{2}}^{x+y} dt \right) = y^{\frac{-1}{2}} (3x + 2y - 1) \tag{6.2}$$

  In this way, if $y = \frac{1-3x}{2}$ with $x \in [-1, 0]$, (6.2) is 0.

- If $0 < x < \frac{1}{2}$, $\frac{1}{2} < x + \frac{y}{2} < 1$ and $1 < x + y$ we have that (6.1) remains as this:

$$y^{\frac{-1}{2}} \left(\int_x^{\frac{1}{2}} dt - \int_{\frac{1}{2}}^{x+\frac{y}{2}} dt + \int_{x+\frac{y}{2}}^{1} dt \right) = y^{\frac{-1}{2}} (2 - 3x - y) \tag{6.3}$$

  Therefore, when $y = 2 - 3x$ with $x \in [0, \frac{1}{2}]$, (6.3) is also 0.

Now we have obtained all cases in that (6.1) cancels (it is easy to check that there is not any more). Calculating a little more we can see that the graph of signs of the scalar product is the represented in the **Figure 1**.

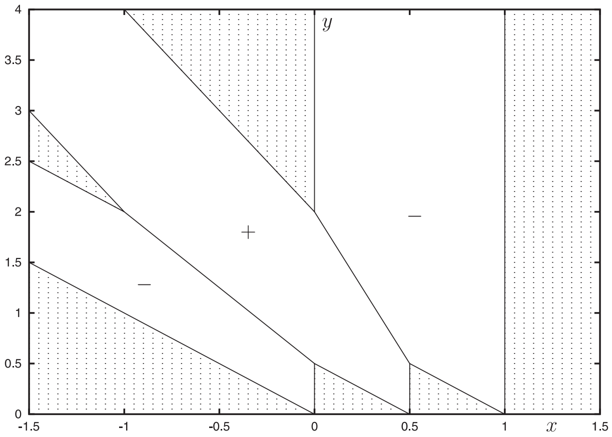

Figure 1. Reproductive kernel of the Haar wavelet.

We can observe that in the first cases the regions are described by inequalities, and they have infinite hyperbolic area. The two last come given by equations and correspond to two segments. Informally we will name triangles to the firsts and diagonal to the seconds.

Lemma 6.2. *Let $\mathcal{B} = \{\psi_{z_n}(t)\}_{z_n} \in \Lambda$ be an orthonormal basis of $L^2(\mathbb{R})$ such that $\psi_i = \psi \in \mathcal{B}$. Then $\psi_z \notin \mathcal{B}$ if z belongs to any of the diagonals.*

Proof. We will make it by contradiction. We suppose that there is ψ_z with $z \in D$ the diagonal corresponding to $y = \frac{1-3x}{2}$ with x among -1 and 0. For the other diagonal the reasoning is the same. We take the function:

$$f(t) = c_1 \chi_{[0,x+\frac{y}{2})}(t) - c_2 \chi_{[x+\frac{y}{2},\frac{1}{2})}(t)$$

taking c_1, $c_2 > 0$ so that $\int f(t)\,dt = 0$ and $\|f\| = 1$. We go to calculate $\langle f, \psi_{z_n} \rangle$ now. As $z \in D$ we can affirm by orthogonality that it does not exist other $z_n \in D$. If there is any other z_n in the diagonal, its support will have to be disjoint with that of ψ_z, and therefore also with f. Also we see that the ψ_{z_n} that have disjoint support to ψ also have it disjoint to f, and those that are constants in all the support of ψ also are it in all the support of f. Therefore, in all this set, it is fulfilled that $\langle f, \psi_{z_n} \rangle = 0$ in a trivial way. On another part, $\langle f, \psi_z \rangle = 0$ by construction. It only remains to look at the ψ_{z_n} that have support contained in one of the two halves where ψ is constant, that is, those that z_n is in one of the inferior triangles. But as $\psi_z \in \mathcal{B}$ we can say that a ψ_{z_n} with z_n in one of this two triangles also has to have its support contained in one and only one of the four following intervals:

- $I_1 = [0, x + \frac{y}{2}]$
- $I_2 = [x + \frac{y}{2}, \frac{1}{2}]$
- $I_3 = [\frac{1}{2}, x + y]$
- $I_4 = [x + y, 1]$

since if not, they can not be orthogonal to ψ_z. Now, those that have support in I_1 or I_4 are automatically orthogonal to f because have disjoint support, and those that are in I_2 or I_3 are it because they have support in intervals where f is constant. Therefore the only element of the basis that is not orthogonal to f is ψ, which is different of f, and therefore $|\langle f, \psi \rangle| < 1$. This is a contradiction with the fact that $\mathcal{B}$ is an orthonormal basis. ☐

Taking into account that $\langle \psi_z, \psi_{z_0} \rangle = \langle \psi_{z_0^{-1} \cdot z}, \psi \rangle$, this result can become generalized to any element of the basis, not exclusively to ψ, and either it is necessary that this belongs to the basis.

Lemma 6.3. *Let $\mathcal{B} = \{\psi_{z_n}\}_{z_n \in \Lambda}$ be an orthonormal basis such that $\psi \in \mathcal{B}$. Then its two songs $\psi_{i/2}$ and $\psi_{1/2+i/2}$ also belong to $\mathcal{B}$.*

Proof. We will see if only the left song belongs to it. We consider the ψ_{z_n} that have support contained in the left part of ψ, the interval $[0, \frac{1}{2}]$ (it is obvious that there is one). We take one of those whit minimum resolution, or the one that is the same, one of those of support with maximum length. We name it ψ_z. If this is the left song we have finished. If not, we will suppose that $x > 0$ (the same can be made with $x + y < \frac{1}{2}$). We take the function:

$$f(t) = c_1 \chi_{[0,x)}(t) - c_2 \chi_{[x,x+\frac{y}{2})}(t)$$

with c_1, $c_2 > 0$ so that $\int f\,dt = 0$ and $\|f\| = 1$, as in the former lemma. Using this lemma and with a reasoning similar to that of before we can see that $\langle f, \psi_{z_n} \rangle = 0$ for all the z_n of Λ by the condition of orthonormality, but this contradicts the fact that $\mathcal{B}$ is a basis. $\square$

As before, we can generalize this to any element of the basis. That is, this result is saying that if ψ_z belongs to an orthonormal basis, its two songs $\psi_{x+i\frac{y}{2}}$ and $\psi_{x+\frac{y}{2}+i\frac{y}{2}}$ also have to belong to the basis. As a matter of fact, we have proved the following theorem:

Theorem 6.4. *Let $\mathcal{B} = \{\psi_{z_n}\}_{z_n \in \Lambda}$ be an orthonormal basis of translations of the Haar wavelet. It is fulfilled that Λ is a hyperbolically regularly net. That is, it is a hyperbolic translation of the dyadic net.*

This example is useful to check out the rigidity of the concept of orthonormal basis.

6.3 The Bergman space.

In this section we will work with the Poisson wavelets. This is a family of admissible wavelets that have the property that its phase spaces correspond with the Bergman spaces of the half–plane. These are a family of spaces of holomorphic functions that will play the role corresponding to the Fock spaces in the Gabor case.

As it happend in the Fock space, in the Bergman spaces we have characterizations of the sampling and of interpolation sets. Even though the tools that are used are typical of the complex analysis and are not applicable directly to other wavelets, they give an excellent example to point the problem in the general case.

In this section we will work in an exceptional way with the wavelet transform defined in $H^2(\mathbb{R})$.

Definition. For $\alpha > 1$, we define the **Poisson wavelet** $\psi^\alpha(t)$ as:

$$\psi^\alpha(t) = \frac{1}{c_\alpha}(t+i)^{-\frac{\alpha+1}{2}}$$

where $c_\alpha = \left\|(t+i)^{-\frac{\alpha+1}{2}}\right\|$.

When we work we will think only in odd α, since they are the cases in that they can be made the calculations. As an example we can say that for $\alpha = 3, 5, 7$, $c_\alpha = \sqrt{\frac{\pi}{2}}, \sqrt{\frac{3\pi}{8}}, \sqrt{\frac{5\pi}{16}}$ respectively. We have to take into account that these wavelets are not normalized. That is, $\int_0^\infty \frac{|\widehat{\psi}^\alpha(\xi)|^2}{\xi}\,d\xi = \widetilde{c}_\alpha^{\,2} \neq 1$

Definition. For $\alpha > 1$, we define the **Bergman space** of the half–plane $A_\alpha(\mathbb{H})$ as:

$$A_\alpha(\mathbb{H}) = \left\{ F \text{ analytical in } \mathbb{H} : \int_{\mathbb{H}} |F(z)|^2 y^\alpha \, dm(z) < \infty \right\}$$

The norm in these spaces is:

$$|F\|_\alpha^2 = \int_{\mathbb{H}} |F(z)|^2 y^\alpha \, dm(z)$$

Now, we will see that the phase spaces of the Poisson wavelets are contained in these spaces. That is, we can think that the transform of a function $f \in H^2(\mathbb{R})$ for a Poisson wavelet ψ^α is in A_α.

If we look at the wavelet transform for the Poisson wavelet ψ^α of a function $f \in H^2(\mathbb{R})$ we see that:

$$\begin{aligned}
Wf(z) =& \frac{1}{\widetilde{c}_\alpha c_\alpha} \int_{-\infty}^{\infty} f(t) y^{\frac{-1}{2}} \Big(\frac{t-x}{y} - i\Big)^{-\frac{\alpha+1}{2}} dt \\
=& \frac{1}{\widetilde{c}_\alpha c_\alpha} \int_{-\infty}^{\infty} f(t) y^{\frac{-1}{2}} (t - x - iy)^{-\frac{\alpha+1}{2}} y^{\frac{\alpha+1}{2}} \, dt \\
=& \frac{1}{\widetilde{c}_\alpha c_\alpha} \int_{-\infty}^{\infty} f(t) y^{\frac{\alpha}{2}} (t-z)^{-\frac{\alpha+1}{2}} \, dt = \frac{y^{\frac{\alpha}{2}}}{\widetilde{c}_\alpha c_\alpha} \int_{-\infty}^{\infty} \frac{f(t)}{(t-z)^{\frac{\alpha+1}{2}}} \, dt
\end{aligned}$$

We define:

$$F(z) = \frac{1}{\widetilde{c}_\alpha c_\alpha} \int_{-\infty}^{\infty} \frac{f(t)}{(t-z)^{\frac{\alpha+1}{2}}} \, dt$$

(F is not the transform of f). The function $F(z)$ is holomorphic in a trivial way. Using that $Wf \in L^2(\mathbb{H})$ we obtain:

$$\int_{\mathbb{H}} |F(z)|^2 y^\alpha \, dm(z) = \int_{\mathbb{H}} \left| y^{\frac{\alpha}{2}} F(z) \right|^2 dm(z) = \int_{\mathbb{H}} |Wf(z)|^2 \, dm(z)$$

and therefore $F(z) \in A_\alpha$ and $\|F\|_\alpha = \|Wf\| = \|f\|$. With the identification $Wf(z) = y^{\frac{\alpha}{2}} F(z)$ we can identify the phase space of a Poisson wavelet in the Bergman space.

To see that we have equality we have to check that the reproductive kernels are the same. We will make this only for odd α. We calculate first the reproductive kernel of the phase spaces.

$$\begin{aligned}
k_\alpha(z,z_0)=\langle\psi^\alpha_{z_0},\psi^\alpha_z\rangle=&\int_{-\infty}^{\infty} y_0^{\frac{-1}{2}}\Big(\frac{t-x_0}{y_0}+i\Big)^{-\frac{\alpha+1}{2}}\overline{y^{\frac{-1}{2}}\Big(\frac{t-x}{y}+i\Big)^{-\frac{\alpha+1}{2}}}\,dt\\
=&\int_{-\infty}^{\infty} y_0^{\frac{\alpha}{2}}(t-x_0+y_0i)^{-\frac{\alpha+1}{2}}y^{\frac{\alpha}{2}}(t-x-yi)^{-\frac{\alpha+1}{2}}\,dt\\
=&\int_{-\infty}^{\infty} y_0^{\frac{\alpha}{2}}y^{\frac{\alpha}{2}}(t-\overline{z_0})^{-\frac{\alpha+1}{2}}(t-z)^{-\frac{\alpha+1}{2}}\,dt\\
=&y_0^{\frac{\alpha}{2}}y^{\frac{\alpha}{2}}\int_{-\infty}^{\infty}(t-\overline{z_0})^{-\frac{\alpha+1}{2}}(t-z)^{-\frac{\alpha+1}{2}}\,dt
\end{aligned}$$

This integral can be calculated when α is odd, and we obtain:

$$\langle\psi^\alpha_{z_0},\psi^\alpha_z\rangle=y_0^{\frac{\alpha}{2}}y^{\frac{\alpha}{2}}c'_\alpha 2\pi i(z-\overline{z_0})^{-\alpha}$$

where $c_{\alpha'}$ is a integer constant, that for $\alpha=3,5,7$ is $-2,6,-20$ respectively. The reproductive kernel says that, fixed α:

$$\begin{aligned}
Wf(z_0)=&\int_{\mathbb{H}} y_0^{\frac{\alpha}{2}}y^{\frac{\alpha}{2}}c'_\alpha 2\pi i(z_0-\overline{z})^{-\alpha}Wf(z)\,dm(z)\\
=&y_0^{\frac{\alpha}{2}}\int_{\mathbb{H}} c'_\alpha 2\pi i(z_0-\overline{z})^{-\alpha}y^{\frac{\alpha}{2}}Wf(z)\,dm(z)\\
=&y_0^{\frac{\alpha}{2}}\int_{\mathbb{H}} c'_\alpha 2\pi i(z_0-\overline{z})^{-\alpha}y^{\frac{-\alpha}{2}}Wf(z)y^\alpha\,dm(z)
\end{aligned}$$

If we take into account that $y^{\frac{-\alpha}{2}}Wf(z)=F(z)\in A_\alpha$, we see that the functions of A_α that are transform of a function of $H^2(\mathbb{R})$ are characterized by the equation:

$$F(z_0)=\int_{\mathbb{H}} c'_\alpha 2\pi i(z_0-\overline{z})^{-\alpha}F(z)y^\alpha\,dm(z)$$

But this is exactly the relation of the reproductive kernel of the Bergman spaces. Therefore we can affirm that, under the correspondence $y^{\frac{-\alpha}{2}}W_f(z)=F(z)$, both spaces are the same.

Normally people work with the version in the disk of the Bergman spaces, that for conformal mapping is equivalent to work in the half–plane. In this way we define the Bergman spaces of the disk $A_\alpha(\mathbb{D})$ (we sustain the notation because we are speaking about the same) as:

$$A_\alpha(\mathbb{D})=\left\{F(z)\text{ analytical en }\mathbb{D}:\int_{\mathbb{D}}|F(z)|^2(1-|z|^2)^\alpha\,d\mu(z)<\infty\right\}$$

where

$$d\mu(z)=\frac{dx\,dy}{(1-|z|^2)^2}$$

that is the measure that works in this case. As we have commented, Seip characterizes in [Sei93] the sampling and interpolation sets of these spaces. In his work he consider the general case (L^p or L^∞ norms), but here we will comment briefly his results in the particular case of L^2. That is, of the Bergman spaces that we have described.

In the disk we can work with the hyperbolic distance or the pseudohyperbolic one, which is more practical. The second is defined as:

$$\widetilde{d}(z,w) = \left|\frac{z-w}{1-\overline{z}w}\right|$$

The notion of separation of a set is exactly the same as in the plane or in the half–plane:

Definition. We will say that a set $\gamma = \{z_j\}$ is **uniformly discrete** if:

$$\inf_{j\neq k} \widetilde{d}(z_j, z_k) > 0$$

These are the only sets that have interest in the classification. We define now the upper and lower densities introduced by Seip for classifying the different sets.

For a uniformly discrete set $\Gamma = \{z_j\}$ and $\frac{1}{2} < r < 1$, let be:

$$D(\Gamma, r) = \frac{\sum\limits_{\frac{1}{2}<|z_j|<r} \log \frac{1}{|z_j|}}{\log \frac{1}{1-r}}$$

For each $z \in \mathbb{D}$, we form new sets by translation:

$$\Gamma_z = \left\{ \frac{z_j - z}{1 - \overline{z} z_j} \right\}$$

Definition. Given a set $\Gamma = \{z_j\}$ we define its upper and lower **uniform densities** as:

$$D^-(\Gamma) = \liminf_{r\to 1} \inf_{z\in\mathbb{D}} D(\Gamma_z, r)$$
$$D^+(\Gamma) = \limsup_{r\to 1} \sup_{z\in\mathbb{D}} D(\Gamma_z, r)$$

If $D^-(\Gamma) = D^+(\Gamma)$ we will say that Γ has uniform density. In [Sei93] it is proved the following statements:

Theorem 6.5 (Seip). *A set Γ of different points in $\mathbb{D}$ is a sampling set for A_α if and only if it can be expressed as a finite union of uniformly discrete sets and if it contain a uniformly discrete set Γ' such that $D^-(\Gamma') > \frac{\alpha-1}{2}$.*

Theorem 6.6 (Seip). *A set Γ of different points in $\mathbb{D}$ is an interpolation set for A_α if and only if it is uniformly discrete and $D^+(\Gamma) < \frac{\alpha-1}{2}$.*

As in the Fock space, in these spaces there are not sets of sampling and interpolation on time. Therefore we will not be able to find Riesz bases generated by the Poisson wavelets.

6.4 Harmonic phase spaces of wavelets.

In this section we will characterize the set of admissible wavelets for which its phase space is composed by harmonic functions in z.

In this section we will work in the real valued $L^2(\mathbb{R})$ space. If we take an admissible wavelet ψ, we want to know when there is a weight $\omega(y)$ such that $\omega(y)Wf(x,y)$ is a harmonic function for every real valued $f \in L^2(\mathbb{R})$. This restriction in the weight is related with the invariance by translations of the wavelet transform, but it is not clear that there are not other weights and harmonic wavelets in the general case.

Theorem 6.7. *Let ψ be an admissible and real valued wavelet. Then there is a weight $\omega(y)$ such that the set of functions $\omega(y)Wf(x,y)$ for $f \in L^2(\mathbb{R})$ is formed by harmonic functions if and only if ψ is a linear combination of $\Re(t+i)^\alpha$ and $\Im(t+i)^\alpha$ with $\alpha < -1$.*

Proof. We have to study when $\Delta\omega(y)Wf(x,y) = 0$:

$$\begin{aligned}\Delta\omega(y)Wf(x,y) =& \Delta\int_{-\infty}^{\infty} f(t)y^{\frac{-1}{2}}\omega(y)\psi\Big(\frac{t-x}{y}\Big)\,dt\\ =& \int_{-\infty}^{\infty} f(t)\left(\Delta y^{\frac{-1}{2}}\omega(y)\psi\Big(\frac{t-x}{y}\Big)\right)\,dt = 0\end{aligned}$$

for every $f \in L^2(\mathbb{R})$. This is equivalent to:

$$\Delta y^{\frac{-1}{2}}\omega(y)\psi\Big(\frac{t-x}{y}\Big) = 0$$

and as it has to be valid for every t. By change of variable we arrive to:

$$\Delta y^{\frac{-1}{2}}\omega(y)\psi\Big(\frac{x}{y}\Big) = 0$$

If we name $\beta(y) = y^{\frac{-1}{2}}\omega(y)$, the one that we want is to find for which wavelets ψ there is β such that $\beta(y)\psi(\frac{x}{y})$ is a harmonic function. If we calculate we have that:

$$\Delta\left(\beta(y)\psi\Big(\frac{x}{y}\Big)\right) = 2\frac{x}{y^3}\beta(y)\psi'\Big(\frac{x}{y}\Big) + \beta''(y)\psi\Big(\frac{x}{y}\Big) \\ - 2\frac{x}{y^2}\beta'(y)\psi'\Big(\frac{x}{y}\Big) + \Big(\frac{x^2}{y^2}+1\Big)\frac{1}{y^2}\beta(y)\psi''\Big(\frac{x}{y}\Big)$$

Therefore we want:

$$2\frac{x}{y^3}\beta(y)\psi'\Big(\frac{x}{y}\Big) + \beta''(y)\psi\Big(\frac{x}{y}\Big) - 2\frac{x}{y^2}\beta'(y)\psi'\Big(\frac{x}{y}\Big) \\ + \Big(\frac{x^2}{y^2}+1\Big)\frac{1}{y^2}\beta(y)\psi''\Big(\frac{x}{y}\Big) = 0 \quad (6.4)$$

We make the change $t = \frac{x}{y}$ and we multiply (6.4) by y^2 so that we obtain:

$$2\beta(y)t\psi'(t) + y^2\beta''(y)\psi(t) - 2y\beta'(y)t\psi'(t) + \beta(y)(t^2+1)\psi''(t) = 0 \qquad (6.5)$$

We introduce the change $y = e^u$ (we remember that $y > 0$) and $\gamma(u) = \beta(e^u)$. In this way we have that:

- $\gamma'(u) = e^u\beta'(e^u) = y\beta'(y)$
- $\gamma''(u) = e^u\beta'(e^u) + (e^u)^2\beta''(e^u) = y\beta'(y) + y^2\beta''(y)$

and this makes that (6.5) remains:

$$\gamma'(u)\psi(t) - \gamma'(u)\big(\psi(t) + 2t\psi'(t)\big) + \gamma(u)\big((t^2+1)\psi''(t) + 2t\psi'(t)\big) = 0$$

We can think this last equation as a scalar product in $\mathbb{R}^3$:

$$\langle\big(\gamma''(u), \gamma'(u), \gamma(u)\big), \big(\psi(t), -\psi(t) - 2t\psi'(t), (t^2+1)\psi''(t) + 2t\psi'(t)\big)\rangle = 0 \quad (6.6)$$

where this equality is independent of t and u.

We call E to the subspace generated by $\big(\gamma''(u), \gamma'(u), \gamma(u)\big)$ when we vary u, and F to the one generated by $\big(\psi(t), -\psi(t) - 2t\psi'(t), (t^2)\psi''(t) + 2t\psi'(t)\big)$ when t varying. Equation (6.6) tells that $E \perp F$. As both spaces are contained in $\mathbb{R}^3$ we can say that $\dim E + \dim F \leqslant 3$.

This inequality gives a very small number of options. To start we have the trivial cases, which do not have sense:

- $\dim F = 0$ corresponds with $\psi(t) = 0$.
- $\dim E = 0$ corresponds with $\gamma(u) = 0 \Rightarrow \omega(y) = 0$.

Therefore it only remains to consider two cases, $\dim F = 1$ ó $\dim E = 1$:

- **Case** $\dim F = 1$**.**

  This means that there exist a vector (c_1, c_2, c_3) and a function $b(t)$ such that:

  $$\big(\psi(t), -\psi(t) - 2t\psi'(t), (t^2+1)\psi''(t) + 2t\psi'(t)\big) = b(t)(c_1, c_2, c_3)$$

  That is:

  $$\left.\begin{aligned} \psi(t) &= c_1 b(t) \\ -\psi(t) - 2t\psi'(t) &= c_2 b(t) \\ (t^2+1)\psi''(t) + 2t\psi'(t) &= c_3 b(t) \end{aligned}\right\} \Rightarrow \frac{\psi(t)}{-\psi(t) - 2t\psi'(t)} = \frac{c_1}{c_2}$$

  The cases in that the denominator cancels are trivial. The former equality says that:

  $$\begin{aligned} \Big(1 + \frac{c_1}{c_2}\Big)\psi(t) + \frac{c_1}{c_2} t\psi'(t) = 0 \Rightarrow \frac{\psi}{\psi'} = \frac{-2\frac{c_1}{c_2}t}{1 + \frac{c_1}{c_2}} = \alpha t \\ \Rightarrow \log\psi = \frac{\alpha}{2}t^2 + c \Rightarrow \psi(t) = \mathrm{C}e^{\frac{\alpha}{2}t^2} \end{aligned}$$

  But this case does not have interest, since it will not be an admissible wavelet for real values of α.

- **Case** $\dim E = 1$**:**

  As in the former case, this means that there exist a vector (c_1, c_2, c_3) and a function $a(u)$ such that:

  $$\big(\gamma''(u), \gamma'(u), \gamma(u)\big) = a(u)(c_1, c_2, c_3)$$

  That is:

  $$\left.\begin{aligned} \gamma''(u) &= a(u)c_1 \\ \gamma'(u) &= a(u)c_2 \\ \gamma(u) &= a(u)c_3 \end{aligned}\right\} \Rightarrow \frac{\gamma(u)}{\gamma'(u)} = \frac{c_3}{c_2}$$

  $$\Rightarrow \log\gamma(u) = \frac{c_3}{c_2}u + c \Rightarrow \mathrm{C}e^{\frac{c_3}{c_2}u} = (e^u)^{\frac{c_3}{c_2}}\,\mathrm{C}$$

  If we undo the change of variable that we have made in (6.5) and we forget about the constant (it is not important), we have that $\beta(y) = y^{\frac{c_3}{c_2}} = y^{\alpha}$ ($c_2 \neq 0$ always).

  For this β, using (6.5) and that:

  - $\beta'(y) = \alpha y^{\alpha-1}$
  - $\beta''(y) = \alpha(\alpha-1)y^{\alpha-2}$

we see that ψ has to fulfill:

$$2y^{\alpha}t\psi'(t)+\alpha(\alpha-1)y^{\alpha}\psi(t)-2\alpha y^{\alpha}t\psi'(t)+y^{\alpha}\left(t^2+1\right)\psi''(t)=0$$

Simplifying y^{α} and taking $\psi(t)=(t+i)^{\alpha}$ we see that:

$$\begin{aligned}(t+i)^{\alpha}\alpha(\alpha-1)+(t+i)^{\alpha-1}(2t\alpha-2\alpha^2t)+(t+i)^{\alpha-2}\alpha(\alpha-1)(t^2+1)\\ =(t+i)^{\alpha-2}\alpha(\alpha-1)(t^2-1+2ti-2t^2-2it+t^2+1)=0\end{aligned}$$

Therefore $(t+i)^{\alpha}$ is a solution of the equation. However we search real solutions. Then we have to take $\Re(t+i)^{\alpha}$ and $\Im(t+i)^{\alpha}$, that, by the linearity of the Laplacian, will also be solutions of the equation. As this equation has order 2 and we have two solutions, we can say that the rest of solutions are a linear combination of these.

As we are only interested in admissible wavelets of $L^2(\mathbb{R})$, we have to restrict to the case $\alpha<-1$. ☐

Remark. The phase spaces of these wavelets will be related with the Bergman spaces, and we will be able to describe its sampling and interpolation sets from the description in that space.

6.5 Sampling results.

The results that we will give in this section are practically identical to the sampling results that we have given in the Gabor case, and the ideas are the same. The differences between the proofs are due to the different structure of both transforms and the respective phase spaces. Even if it is redundant, we will repeat the proofs and we will put emphasis in the differences that can turn up.

As in the Gabor case, we establish first some notation that will simplify the study. H will always be the phase space of an admissible wavelet ψ normalized so that $\|\psi\|=1$ and $\int_0^{\infty}\frac{|\widehat{\psi}(\xi)|^2}{\xi}\,d\xi=1$. In general, $z,w\in\mathbb{H}$ with $z=x+iy$, $w=a+bi$, fulfilling $x,a\in\mathbb{R}$, $y,b\in\mathbb{R}^+$. We shall use the same notation when sub–indices appear. We remember that when we speak about balls and about separation conditions we have to use the hyperbolic distance, which we can calculate by the formula:

$$d(z_1,z_2)=\frac{1}{2}\log\frac{1+\left|\frac{z_1-z_2}{z_1-\overline{z_2}}\right|}{1+\left|\frac{z_1-z_2}{z_1-\overline{z_2}}\right|}$$

With this distance the definition of separation remains:

Definition. Let $\Lambda = \{z_j\}_{j\in\mathbb{N}} \subseteq \mathbb{H}$. We will say that Λ is an **uniformly discrete** or **separate** set if there is $\varepsilon > 0$ such that $d(z_i, z_j) > \varepsilon \ \forall i \neq j$.

ε will be the **separation constant** of Λ.

As in the Gabor case, we need that the sets that we use have to be separated sets. The first results prove that these are the only ones that have interest.

Lemma 6.8. *Let $F \in H$ be a function such that $F(w) = 0$. Let $z \in \mathbb{H}$, then:*

$$|F(w)|^2 \leqslant \|F\|^2\big(1 - |k_w(z)|^2\big) = \|F\|^2\big(1 - |\langle\psi_w, \psi_z\rangle|^2\big)$$

Proof. We consider H_w the set of functions of H that cancel in w. This is a Hilbert subspace with the following reproductive kernel:

$$\Phi_{z_0}(z) = k_{z_0}(z) - k_{z_0}(w)k_w(z)$$

To see it we have to observe first that $\Phi_{z_0} \in H$ since it is a linear combination of functions of H and $\Phi_{z_0}(w) = 0$ for every $z_0 \in \mathbb{C}$. Therefore $\Phi_{z_0} \in H_w$.

If we take $h \in H_w$ we see that:

$$\langle h, \Phi_{z_0}\rangle = \langle h, k_{z_0}\rangle - k_{z_0}(w)\langle h, k_w\rangle = h(z_0)$$

That is, Φ_{z_0} reproduces the functions of H_w. As it belongs to the space we can affirm that it is its reproductive kernel.

As F belongs to H_w we apply the reproduction formula in this space.

$$\begin{aligned}|F(z)|^2 =&|\langle F, \Phi_z\rangle|^2 \leqslant \|F\|^2\langle\Phi_z, \Phi_z\rangle = \|F\|^2\Phi_z(z)\\ =&\|F\|^2\left(k_z(z) - \frac{k_z(w)}{k_w(w)}k_w(z)\right) = \|F\|^2\big(1 - |k_w(z)|^2\big)\end{aligned}$$

and we obtain the result. □

Theorem 6.9. *Let $\Lambda = \{z_j\}_{j\in\mathbb{N}} \subseteq \mathbb{H}$. If Λ is an interpolation set for H it is fulfilled that Λ is a separate set.*

Proof. By the closed graph theorem, if Λ is an interpolation set there is $M > 0$ such that $\forall\{a_n\}_{n\in\mathbb{N}} \in l^2$ there exists $F \in H$ such that $f(z_n) = a_n$ and $\|F\|^2 \leqslant M^2\sum_{n\in\mathbb{N}}|a_n|^2$. That is, we can carry out the interpolation with bounded norm.

Let $z_i \neq z_j$. There is $F \in H$ such that $F(z_i) = 1$ and $F(z_j) = 0$ with $\|F\| \leqslant M$. Applying 6.8 we have that:

$$\begin{aligned}&1 = |F(z_i)|^2 \leqslant \|F\|^2(1 - |k_{z_i}(z_j)|^2) \leqslant M^2\big(1 - |k_{z_i}(z_j)|^2\big)\\ &\Rightarrow 1 - |k_{z_i}(z_j)|^2 \geqslant \frac{1}{M^2} \Rightarrow |k_{z_i}(z_j)|^2 \leq 1 - \frac{1}{M^2} = C < 1\end{aligned}$$

This means that if $i \neq j$, $|\langle\psi_{z_i}, \psi_{z_j}\rangle|^2 \leqslant C < 1$, and as the kernel is uniformly continuous, this implies the separation condition, since C does not depend on z_i and z_j. □

The sampling sets also have analogous properties than in the Gabor case.

Theorem 6.10. *Let* $\Lambda = \{z_j\}_{j\in\mathbb{N}} \subseteq \mathbb{H}$*. If* Λ *is a sampling set for* H *it is fulfilled that* Λ *is a finite union of separated sets.*

Proof. As Λ is a sampling set, $\exists C > 0$ such that:

$$\sum_{\Lambda} |F(z_j)|^2 \leqslant C\|F\|^2 \tag{6.7}$$

We will make it by contradiction. If Λ is not a finite union of uniformly discrete sets is equivalent to that for every N and every δ there exists a hyperbolic ball of radius δ, we call B to it , such that $|\Lambda \cap B| > N$.

As $k_{z_0}(z)$ is an uniformly continuous function, $\exists\delta$ such that $|k_{z_0}(z_1)-k_{z_0}(z_2)| < \frac{1}{2}$ if $d(z_1, z_2) < \delta$, where this δ does not depend on z_0.

We take this δ and $N > 4C$. Let B be the corresponding ball of radius δ that contains N points of the succession. Let a be the center of B and consider the function $k_a(z) \in H$. This function has to fulfill (6.7).:

$$\sum_{\Lambda} |k_a(z_j)|^2 \leqslant C\|k_a\|^2 = C$$

But on the other hand:

$$\sum_{\Lambda} |k_a(z_j)|^2 = \sum_{\Lambda\cap B^c} |k_a(z_j)|^2 + \sum_{\Lambda\cap B} |k_a(z_j)|^2 \geqslant \sum_{\Lambda\cap B} \Big(1-\frac{1}{2}\Big)^2 \geqslant N\frac{1}{4} > C$$

and we have the contradiction. ☐

Remark. Here, we have used that we can make small $|k_z(z_1)-k_z(z_2)|$ independently of z, while in Gabor we had to consider the absolute value of the kernel.

We find again that this result is not improvable, since there are sampling sets that are not separated set.

Now we have to restrict the set of wavelets that we use. Here we find one of the main differences with the Gabor case.

The bounds that we will give when we will bound the sums will depend on the $L^1(\mathbb{H})$–norm of the reproductive kernel k and of its local maximal function:

$$Mk(z) = \sup_{w\in B(z,1)} |k(w)| \in L^1(\mathbb{H})$$

where $B(z, 1)$ is the hyperbolic ball of center z and radius 1. The area of this ball for the left invariant measure is $4\pi\sinh^2(\frac{1}{2})$.

Unlike it happend in the Gabor case, the set of admissible wavelets with integrable kernel does not coincide with those that its kernel has integrable local maximal function (we will call them **wavelets with strongly integrable kernel**). We will be able to discretizate the phase space in this last case.

We remember that the function $k(z^{-1}) \in L^1(z)$ and moreover its integral coincides with the integral of $k(z)$. As a matter of fact $k(z^{-1}) = k(z)$ since $\langle \psi, \psi_z \rangle = \langle \psi_z, \psi \rangle$. This does not happen in general with other functions of H, where $F(z)$ can be in $L^1(\mathbb{H})$ or $L^2(\mathbb{H})$ and $F(z^{-1})$ not.

Lemma 6.11. *Let k be such that $Mk \in L^1(\mathbb{H})$ and $\Lambda \subset \mathbb{C}$ be a separate set with separation constant ε. Then:*

$$\sum_{\lambda\in\Lambda} |k(\lambda)| < \frac{(\sinh\frac{\varepsilon}{4})^{-2}}{4\pi}\|Mk\|_1$$

Proof. We suppose without losing generality that $\frac{\varepsilon}{2} < 1$. If we take two different points $\gamma, \lambda \in \Lambda$, as $B(\gamma, \frac{\varepsilon}{2}) \cap B(\lambda, \frac{\varepsilon}{2}) = \emptyset$ we can bound:

$$\sum_{\lambda\in\Lambda} |k(\lambda)| \leqslant \sum_{\lambda\in\Lambda} \frac{1}{|B(0,\frac{\varepsilon}{2})|}\int_{B(\lambda,\frac{\varepsilon}{2})} Mk(z)\,dm(z) \leqslant \frac{(\sinh\frac{\varepsilon}{4})^{-2}}{4\pi}\int_{\mathbb{C}} Mk(z)\,dm(z)$$

that is what we wanted to prove. □

Remark. We see another time that the bound only depends on the separation constant of Λ.

As in the Gabor case, We will give a series of results directed to prove that a set close to a sampling set also is a sampling set. We will use the ideas of [OlS92], that will allow us to know the bounds in an enough explicit way so that we will be able to prove that every sampling set contains a subset that is both a sampling set and a separated set on time.

Proposition 6.12. *Given a wavelet ψ with strongly integrable kernel and Λ a separated set there exists $B > 0$ such that:*

$$\sum_{\lambda\in\Lambda} |F(\lambda)|^2 \leqslant B\|F\|^2 \quad \forall F \in H$$

Proof. Calculating directly we have that:

$$\begin{aligned}
\sum_{\lambda\in\Lambda} |F(\lambda)|^2 &= \sum_{\lambda\in\Lambda} \left| \int_{\mathbb{C}} F(z)k(z^{-1}\cdot\lambda)\,dm(z)\right|^2 \\
&\leqslant \sum_{\lambda\in\Lambda} \left(\int_{\mathbb{C}} |F(z)|^2|k(z^{-1}\cdot\lambda)|\,dm(z)\right)\left(\int_{\mathbb{C}} |k(z^{-1}\cdot\lambda)|\,dm(z)\right) \\
&= \int_{\mathbb{C}} |F(z)|^2 \sum_{\lambda\in\Lambda} |k(z^{-1}\cdot\lambda)|\,dm(z)\int_{\mathbb{C}} |k(z)|\,dm(z) \leqslant B\|F\|^2
\end{aligned}$$

since $\int_{\mathbb{C}} |k(z)|\, dm(z) < \infty$ because the kernel is integrable and

$$\sum_{\lambda\in\Lambda} |k(z^{-1}\cdot\lambda)| = \sum_{\gamma\in\Gamma} |k(\gamma)|$$

with $\Gamma = z^{-1}\cdot\Lambda$, that has the same separation constant than Λ. Now, we apply 6.11 to bound independently of z. ☐

Remark. As we see in the proof, the bounding constant depends on k in terms of its absolute value and not in the square. That is, it depends on $\|k\|_1$, discrete as well as continuous. It is for this reason that we have to limit the study to wavelets with strongly integrable kernel.

Theorem 6.13. *We consider an admissible wavelet with strongly integrable kernel. Given $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ a sampling set for H there exists $\delta > 0$ such that if $\Gamma = \{w_j\}_{j\in\mathbb{N}}$ fulfills that $|z_j - w_j| < \delta\ \forall j$, Γ is also a sampling set.*

This result is deduced in a trivial way from the following pair of lemmas.

Lemma 6.14. *Let $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ and $\Gamma = \{w_j\}_{j\in\mathbb{N}}$ be two sets in $\mathbb{H}$, then it is fulfilled that:*

$$\left| \left(\sum_{j\in\mathbb{N}} |F(z_j)|^2\right)^{\frac{1}{2}} - \left(\sum_{j\in\mathbb{N}} |F(w_j)|^2\right)^{\frac{1}{2}} \right| \leqslant d_1 d_2 \|F\| \quad \forall F\in H$$

where d_1 and d_2 are defined as:

- $d_1^2 = \sup_j \int_{\mathbb{H}} |k(z^{-1}\cdot z_j) - k(z^{-1}\cdot w_j)|\, dm(z)$
- $d_2^2 = \sup_z \sum_{j\in\mathbb{N}} |k(z^{-1}\cdot z_j) - k(z^{-1}\cdot w_j)|$

Proof. Calculating directly we have that:

$$\left| \left(\sum_{j\in\mathbb{N}} |F(z_j)|^2\right)^{\frac{1}{2}} - \left(\sum_{j\in\mathbb{N}} |F(w_j)|^2\right)^{\frac{1}{2}} \right| = \left| \|(F(z_j))_j\|_2 - \|(F(w_j))_j\|_2 \right|$$

where $\|\cdot\|_2$ means the l^2–norm, thinking $\big(F(z_j)\big)_j$ as a succession.

$$\begin{aligned}
\big|\|(F(z_j))_j\|_2 - \|(F(w_j))_j\|_2\big| &\leqslant \|(F(z_j) - F(w_j))_j\|_2 \\
&= \left\| \left(\int_{\mathbb{H}} F(z)k(z^{-1}\cdot z_j)\,dm(z) - \int_{\mathbb{H}} F(z)k(z^{-1}\cdot w_j)\,dm(z) \right)_j \right\|_2 \\
&= \left\| \left(\int_{\mathbb{H}} F(z)\big[k(z^{-1}\cdot z_j) - k(z^{-1}\cdot w_j)\big]\,dm(z) \right)_j \right\|_2 \\
&\leqslant \left[\sum_{j\in\mathbb{N}} \left(\int_{\mathbb{H}} |F(z)|\big|k(z^{-1}\cdot z_j) - k(z^{-1}\cdot w_j)\big|\,dm(z) \right)^2 \right]^{\frac{1}{2}}
\end{aligned}$$

where we have introduced the absolute values inside the integral because we will use Schwartz inequality. We can bound the former equation by:

$$\left[\sum_{j\in\mathbb{N}} \int_{\mathbb{H}} |F(z)|^2 \big|k(z^{-1}\cdot z_j) - k(z^{-1}\cdot w_j)\big|\,dm(z) \int_{\mathbb{H}} \big|k(z^{-1}\cdot z_j) - k(z^{-1}\cdot w_j)\big|\,dm(z) \right]^{\frac{1}{2}}$$

We bound separately,

$$\int_{\mathbb{H}} \big|k(z^{-1}\cdot z_j) - k(z^{-1}\cdot w_j)\big|\,dm(z) \leqslant d_1^2$$

On the other hand,

$$\int_{\mathbb{H}} |F(z)|^2 \sum_{j\in\mathbb{N}} \big|k(z^{-1}\cdot z_j) - k(z^{-1}\cdot w_j)\big|\,dm(z) \leqslant d_2^2\|F\|^2$$

Joining both bounds we obtain the statement. ☐

Lemma 6.15. *Let $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ be a separated set, and we suppose that ψ has strongly integrable kernel.*

For every ε $\exists\delta$ such that if $\Gamma = \{w_j\}_{j\in\mathbb{N}}$ fulfills that $d(z_j, w_j) \leqslant \delta$ $\forall j$ then $d_1 d_2 \leqslant \varepsilon$, with d_1 and d_2 defined as in 6.14.

Proof. First, we will see that we can make d_1 as small as we want if Γ is close enough to Λ. But this is trivial, since we only have to observe that if $|k(z)|$ is integrable $|k(z^{-1})|$ also is, and that the translation operator is uniformly continuous in $L^1(\mathbb{H})$.

We go now to bound d_2. If $\alpha = \sup_{i\neq j} d(z_i, z_j)$ is the separation constant of Λ, we assume $\delta < \frac{\alpha}{3}$, and it is fulfilled that $d(w_i, w_j) > \frac{\alpha}{3}$ $\forall i \neq j$. That is, $\frac{\alpha}{3}$ can be used as the separation constant for Λ as well as for Γ. As a matter of fact

$\Lambda_z = \{z^{-1} \cdot z_j\}_{j\in\mathbb{N}}$ and $\Gamma_z = \{z^{-1} \cdot w_j\}_{j\in\mathbb{N}}$ also have the same separation constant. Here we can apply 6.11 to prove that there is $C = C(\frac{\alpha}{3})$ such that $d_2 \leqslant 2C$.

If we join both parts we see that d_2 is bounded by C when we take small δ, and d_1 can be made as small as we wish taking δ small enough. Therefore $\exists\delta$ such that if $d(z_j, w_j) \leqslant \delta\ \forall j$, it is fulfilled that $d_1 d_2 < \varepsilon$. ☐

Remark. In these proofs we see that some of the calculations are simplifier due to translations in $\mathbb{H}$ coinciding exactly with those that we apply to the functions. But on the other hand they are more confusing because these are not commutative and the structure of $\mathbb{H}$ is not as simple as that of $\mathbb{C}$.

In theorem 6.10 we saw that every sampling set was a finite union of separate sets. Using the former results we can improve this statement. As in the Gabor case, the idea is that a sampling set is an union of some very close sets. Lemmas 6.15 and 6.14 say that closed sets give closed information about the functions. That is, in a sampling non separated set there is repeated information.

Theorem 6.16. *Let $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ be a sampling set for the phase space of a wavelet with strongly integrable kernel.*

Then Λ contains a subset $\widetilde{\Lambda} \subseteq \Lambda$ such that $\widetilde{\Lambda}$ is a sampling and separated set.

Proof. As Λ is a sampling set we know by 6.10 that it is a finite union of separated sets, and that there are $A, B > 0$ such that for every $F \in H$ it is fulfilled that:

$$A\|F\|^2 \leqslant \sum_{j\in\mathbb{N}} |F(z_j)|^2 \leqslant B\|F\|^2$$

For every δ (we think in a very small δ) we can define $\widetilde{\Lambda} \subseteq \Lambda$ so that if $\widetilde{\Lambda} = \{w_i\}_{i\in\mathbb{N}}$, it is fulfilled that:

- $B(w_i, \delta) \cap \widetilde{\Lambda} = \{w_i\}$
- $\cup_{i\in\mathbb{N}} (B(w_i, \delta) \cap \Lambda) = \Lambda$
- $|B(w_i, \delta) \cap \Lambda| \leqslant N$

where N is a constant that as a matter of fact can be bounded by the number of separate sets that form Λ. We can make all this thanks to the fact that Λ is a finite union of separated sets. That is, the one that we make is to define $\widetilde{\Lambda}$ so that $B(w_i, \delta)$ only contains w_i of the points of $\widetilde{\Lambda}$, every $z_j \in \Lambda$ is contained in some $B(w_i, \delta)$ for some $w_i \in \widetilde{\Lambda}$, and each one of these balls only contains a finite bounded number of points of Λ. We can think that each z_j belongs only to a $B(w_i, \delta)$. This is not true, but we can assign an only w_i to each z_j when we work.

Taking this into account we can write $\Lambda = \cup_{i\in\mathbb{N}}\{w_i^1, \dots, w_i^{N_i}\}$, so that:

- $w_i^1 = w_i \in \widetilde{\Lambda}$
- $w_i^k \notin \widetilde{\Lambda}$ if $k \neq 1$
- $w_i^k \in B(w_i, \delta)$
- $N_i \leqslant N \ \forall i$
- The sets $\{w_i^k\}_{k=1}^{N_i}$ are disjoint sets when we change i.

With this notation we calculate:

$$\begin{aligned}
\sum_{z_j \in \Lambda} |F(z_j)|^2 &= \sum_{w_i \in \widetilde{\Lambda}} \sum_{k=1}^{N_i} |F(w_i^k)|^2 \\
&= \sum_{w_i \in \widetilde{\Lambda}} \left[\sum_{k=1}^{N_i} \Big(|F(w_i^k)|^2 - |F(w_i^1)|^2 \Big) + N_i |F(w_i^1)|^2 \right] \\
&\leqslant N \sum_{w_i \in \widetilde{\Lambda}} |F(w_i)|^2 + \sum_{w_i \in \widetilde{\Lambda}} \sum_{k=1}^{N_i} \Big(|F(w_i^k)|^2 - |F(w_i^1)|^2 \Big)
\end{aligned}$$

If we define $w_i^k = w_i^1$ for $N_i < k \leqslant N$ we can write:

$$\begin{aligned}
&\sum_{w_i \in \widetilde{\Lambda}} \sum_{k=1}^{N_i} \Big(|F(w_i^k)|^2 - |F(w_i^1)|^2 \Big) = \sum_{w_i \in \widetilde{\Lambda}} \sum_{k=1}^{N} \Big(|F(w_i^k)|^2 - |F(w_i^1)|^2 \Big) \\
&= \sum_{k=1}^{N} \left[\sum_{w_i \in \widetilde{\Lambda}} |F(w_i^k)|^2 - \sum_{w_i \in \widetilde{\Lambda}} |F(w_i^1)|^2 \right] \\
&= \sum_{k=1}^{N} \left[\Big(\sum_{w_i \in \widetilde{\Lambda}} |F(w_i^k)|^2 \Big)^{\frac{1}{2}} + \Big(\sum_{w_i \in \widetilde{\Lambda}} |F(w_i^1)|^2 \Big)^{\frac{1}{2}} \right] \\
&\qquad \left[\Big(\sum_{w_i \in \widetilde{\Lambda}} |F(w_i^k)|^2 \Big)^{\frac{1}{2}} - \Big(\sum_{w_i \in \widetilde{\Lambda}} |F(w_i^1)|^2 \Big)^{\frac{1}{2}} \right]
\end{aligned}$$

We can bound the first bracket by $2B^{\frac{1}{2}}\|F\|$, since they are partial sums of Λ. To bound the second bracket in absolute value, what we will make is to apply 6.14 to the sets $\{w_i^k\}_{i \in \mathbb{N}}$ and $\{w_i^1\}_{i \in \mathbb{N}}$. This says that:

$$\left| \Big(\sum_{i \in \mathbb{N}} |F(w_i^k)|^2 \Big)^{\frac{1}{2}} - \Big(\sum_{i \in \mathbb{N}} |F(w_i^1)|^2 \Big)^{\frac{1}{2}} \right| \leqslant d_1^k d_2^k \|F\|$$

where k is not an exponent, it is an index, and d_1^k, d_2^k come defined by:

- $(d_1^k)^2 = \sup_{i\in\mathbb{N}} \int_{\mathbb{H}} |k(z^{-1}\cdot w_i^k) - k(z^{-1}\cdot w_i^1)|\, d\mu(z)$
- $(d_2^k)^2 = \sup_{z\in\mathbb{H}} \sum_{i\in\mathbb{N}} |k(z^{-1}\cdot w_i^k) - k(z^{-1}\cdot w_i^1)|$

We define now:

- $d_1 = \sup_k d_1^k$
- $d_2 = \sup_k d_2^k$

and we can bound independently of k:

$$\left|\left(\sum_{i\in\mathbb{N}} |F(w_i^k)|^2\right)^{\frac{1}{2}} - \left(\sum_{i\in\mathbb{N}} |F(w_i^1)|^2\right)^{\frac{1}{2}}\right| \leqslant d_1 d_2 \|F\|$$

Therefore, if we join all bounds we obtain:

$$\begin{aligned} A\|F\|^2 \leqslant \sum_{j\in\mathbb{N}} |F(z_j)|^2 \leqslant N\sum_{i\in\mathbb{N}} |F(w_i)|^2 + \left|\sum_{i\in\mathbb{N}}\sum_{k=1}^{N} \left(|F(w_i^k)|^2 - |F(w_i^1)|^2\right)\right| \\ \leqslant N\sum_{i\in\mathbb{N}} |F(w_i)|^2 + 2NB^{1/2} d_1 d_2 \|F\|^2 \end{aligned}$$

This implies

$$\frac{A - 2NB^{1/2}d_1 d_2}{N}\|F\|^2 \leqslant \sum_{i\in\mathbb{N}} |F(w_i)|^2$$

As N, A and B are fixed, applying 6.12 and 6.15 we see that there is δ small enough such that $\widetilde{\Lambda}$, defined at the beginning, fulfills that there are A', B' with:

$$A'\|F\|^2 \leqslant \sum_{i\in\mathbb{N}} |F(w_i)|^2 \leqslant B'\|F\|^2$$

That is, $\widetilde{\Lambda}$ is a sampling set. ☐

In this way we arrive to the same result than we had obtained in the Gabor case. The following step is to compare the discrete measure with the continuum one in order to demonstrate the existence of sampling sets and give sufficient conditions.

Lemma 6.17. *Let $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ be a set that fulfills that $\forall j\ \exists V_j \subseteq \mathbb{H}$ (open) so that $\mathbb{H} = \cup_{j\in\mathbb{N}}\overline{V_j}$, with $z_j \in V_j$ and $\cup_{j\in\mathbb{N}} z_j^{-1}\cdot V_j \subseteq V$ compact and symmetrical (around i), $V_j \cap V_k = \emptyset$. Then it fulfills that:*

$$\left|\|F\| - \left(\sum_{j\in\mathbb{N}} c_j |F(z_j)|^2\right)^{\frac{1}{2}}\right| \leqslant d_1' d_2' \|F\|$$

where:

- $d_1' = \big(\sup_{\xi\in V} \int_{\mathbb{H}} |k(z\cdot\xi) - k(z)|\, d\mu(z)\big)^{1/2}$
- $d_2' = \Big(\sup_{w\in\mathbb{H}} \sum_{j\in\mathbb{N}} \int_{V_j} |k(w\cdot z) - k(w\cdot z_j)|\, d\mu(z)\Big)^{1/2}$

and c_j *is the (hyperbolic) area of* V_j*.*

Proof. Calculating directly:

$$\begin{aligned}
&\left|\|F\| - \Big(\sum_{j\in\mathbb{N}} c_j |F(z_j)|^2\Big)^{\frac{1}{2}}\right|^2 = \left|\|F\| - \Big\|\sum_{j\in\mathbb{N}} \chi_{V_j}(z)|F(z_j)|\Big\|\right|^2 \\
\leqslant &\Big\|F(z) - \sum_{j\in\mathbb{N}} \chi_{V_j}(z)|F(z_j)|\Big\|^2 = \Big\|\sum_{j\in\mathbb{N}} \big(F(z) - F(z_j)\big)\chi_{V_j}(z)\Big\|^2 \\
= &\int_{\mathbb{H}} \Big|\sum_{j\in\mathbb{N}} \big(F(z) - F(z_j)\big)\chi_{V_j}(z)\Big|^2 \, dm(z)
\end{aligned}$$

We introduce here the reproduction formula for writing this last term as:

$$\int_{\mathbb{H}} \left|\sum_{j\in\mathbb{N}} \left[\int_{\mathbb{H}} F(w)\big(k(w^{-1}\cdot z) - k(w^{-1}\cdot z_j)\big)\, dm(w)\right] \chi_{V_j}(z)\right|^2 dm(z)$$

We apply Schwartz inequality to bound:

$$\begin{aligned}
\int_{\mathbb{H}} \Bigg[\sum_{j\in\mathbb{N}} &\Big(\int_{\mathbb{H}} |F(w)|^2 \big|k(w^{-1}\cdot z) - k(w^{-1}\cdot z_j)\big|\, dm(w)\Big)^{\frac{1}{2}} \\
&\Big(\int_{\mathbb{H}} \big|k(w^{-1}\cdot z) - k(w^{-1}\cdot z_j)\big|\, dm(w)\Big)^{\frac{1}{2}} \chi_{V_j}(z)\Bigg]^2 dm(z)
\end{aligned}$$

As the V_j are disjointed sets, we can introduce the square in the integral and we simplify the expression a bit:

$$\begin{aligned}
\int_{\mathbb{H}} \sum_{j\in\mathbb{N}} \Bigg[&\int_{\mathbb{H}} |F(w)|^2 \big|k(w^{-1}\cdot z) - k(w^{-1}\cdot z_j)\big|\chi_{V_j}(z)\, dm(w) \\
&\int_{\mathbb{H}} \big|k(w^{-1}\cdot z) - k(w^{-1}\cdot z_j)\big|\chi_{V_j}(z)\, dm(w)\Bigg]\, dm(z)
\end{aligned}$$

We sort out in parts. The second integral can be bonded by:

$$\begin{aligned}
\int_{\mathbb{H}} \big|k(w^{-1}\cdot z) - k(w^{-1}\cdot z_j)\big|\, dm(w) &= \int_{\mathbb{H}} \big|k(t^{-1}\cdot z_j\cdot z) - k(t^{-1})\big|\, dm(t) \\
&\leqslant \sup_{\xi\in V} \int_{\mathbb{H}} |k(t\cdot\xi) - k(t)|\, dm(t) = (d_1')^2
\end{aligned}$$

For the first one we make

$$\begin{aligned}
&\int_{\mathbb{H}} \sum_{j\in\mathbb{N}} |F(w)|^2 \big|k(w^{-1}\cdot z) - k(w^{-1}\cdot z_j)\big| \chi_{V_j}(z)\, dm(w)\, dm(z) \\
&\quad = \int_{\mathbb{H}} |F(w)|^2 \sum_{j\in\mathbb{N}} \int_{V_j} \big|k(w^{-1}\cdot z) - k(w^{-1}\cdot z_j)\big|\, dm(z)\, dm(w) \\
&\quad \leqslant \int_{\mathbb{H}} |F(w)|^2 \sup_{w\in\mathbb{H}} \left[\sum_{j\in\mathbb{N}} \int_{V_j} \big|k(w\cdot z) - k(w\cdot z_j)\big|\, dm(z)\right] dm(w) \\
&\quad \leqslant (d_2')^2 \|F\|^2
\end{aligned}$$

that is what we want to prove. □

Theorem 6.18. *Let H be a phase space of a wavelet ψ with strongly integrable kernel. There is δ such that if $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ is a separate set fulfilling the conditions of 6.17 with V contained in $B(0,\delta)$, then Λ is a sampling set for H.*

Proof. What we will make is to bound d_2' and to make d_1' as small as we want in 6.17. For d_2', we fix $w \in \mathbb{H}$ and we have that:

$$\begin{aligned}
\sum_{j\in\mathbb{N}} \int_{V_j} \big|k(w\cdot z) - k(w\cdot z_j)\big|\, dm(z) &\leqslant \sum_{j\in\mathbb{N}} \int_{V_j} |k(w\cdot z)|\, dm(z) + \sum_{j\in\mathbb{N}} |V_j||k(w\cdot z_j)| \\
&\leqslant \|k\|_1 + \sum_{j\in\mathbb{N}} |V_j| \frac{1}{|V_j|} \int_{w^{-1}\cdot V_j} |Mk(z)|\, dm(z) \\
&= \|k\|_1 + \|Mk\|_1
\end{aligned}$$

if $\delta \leqslant 1$. Therefore $d_2' \leqslant (\|k\|_1 + \|Mk\|_1)^{1/2}$.

By the continuity of the translation operator in $L^1(\mathbb{H})$ we can deduce that if δ is small enough we can achieve $d_1' < \varepsilon$ for any $\varepsilon > 0$. Therefore applying 6.17 we have that:

$$\left| \|F\| - \left(\sum_{j\in\mathbb{N}} c_j |F(z_j)|^2\right)^{\frac{1}{2}} \right| < \varepsilon\big(\|k\|_1 + \|Mk\|_1\big)^{\frac{1}{2}} \|F\|$$

We take δ such that $\varepsilon(\|k\|_1 + \|Mk\|_1)^{1/2} < 1$ and we obtain:

$$\begin{aligned}
\left(1 - \varepsilon\big(\|k\|_1 + \|Mk\|_1\big)^{\frac{1}{2}}\right) \|F\| &\leqslant \left(\sum_{j\in\mathbb{N}} c_j |F(z_j)|^2\right)^{\frac{1}{2}} \\
&\leqslant \left(1 + \varepsilon\big(\|k\|_1 + \|Mk\|_1\big)^{\frac{1}{2}}\right) \|F\|
\end{aligned}$$

Now we only have to define A and B so that

$$A = \frac{\Big(1-\varepsilon\big(\|k\|_1+\|Mk\|_1\big)^{\frac{1}{2}}\Big)^2}{\sup_j\{c_j\}}$$
$$B = \frac{\Big(1+\varepsilon\big(\|k\|_1+\|Mk\|_1\big)^{\frac{1}{2}}\Big)^2}{\inf_j\{c_j\}}$$

to obtain:

$$A\|F\|^2 \leqslant \sum_{j\in\mathbb{N}} |F(z_j)|^2 \leqslant B\|F\|^2$$

Both $\sup_j\{c_j\}$ and $\inf_j\{c_j\}$ are bounded and greater that 0. ☐

Remark. We have not used 6.12 for the right inequality.

Now we will give a simpler way to recognize the sets that are under the conditions of 5.14.

Lemma 6.19. *Let $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ be a separate set such that $\exists\delta$ with the property that $\forall z \in \mathbb{C}$, $B(z,\delta)\cap\Lambda \neq \emptyset$.*

Then Γ fulfills the conditions of 6.17.

Proof. As Λ is a separated set we only have to see that $\exists V_j$ open, with $V_j \cap V_k = \emptyset$ if $j \neq k$, $\mathbb{H} = \cup_{j\in\mathbb{N}}\overline{V_j}$, $z_j^{-1}\cdot V_j \subseteq V$ compact and symmetric, with $|\cap_{j\in\mathbb{N}} z_j^{-1}\cdot V_j| > 0$.

We define, if ε is the separation constant of Λ, $b_j = \overline{B(z_j,\frac{\varepsilon}{2})}$, $B_j = B(z_j,\delta)$. We now define the V_j:

- $V_1 = B_1 \setminus \cup_{j\neq 1} b_j$
- $V_k = B_i \setminus (\cup_{j\neq k} b_j) \bigcup \overline{\left(\cup_{j=1}^{k-1} B_i\right)}$

We observe that in each compact of $\mathbb{H}$ all these unions and intersections are finite, and therefore the union of closed sets is closed. This says that the V_j are all open sets, and for construction they are disjoint.

As $B_j \subseteq \cup_{k=1}^{j}\overline{V_k}$, and every z is in some B_j, we have that $\cup_{j\in\mathbb{N}}\overline{V_j} = \mathbb{H}$, and as $V_j \subseteq B_j \Rightarrow z_j^{-1}\cdot V_j \subseteq z_j^{-1}\cdot B_j = B(i,j) \subseteq \overline{B(i,\delta)}$ compact and symmetric, and also for construction we have that $B(i\varepsilon/2) \subseteq z_j^{-1}\cdot V_j$ and we obtain the last condition. ☐

Joining these two last results we obtain a sufficient totally geometric condition for sampling sets, identical to the Gabor case.

Theorem 6.20. *Let H be the phase space of a wavelet ψ with strongly integrable kernel. There is δ such that if Λ is a separate set such that $B(z,\delta)\cap\Lambda \neq \emptyset$ $\forall z$ then Λ is a sampling set for H.*

We have proved that a set dense enough (with points all over the plane) is a sampling set, but we do not have any notion about what kind of sets (of which density) can be sampling sets. That is, we know that a very dense set generates a frame, but we ask if a set that generates a frame has to be bit dense.

From the point of view of regular nets, we have seen that we can find frames for any selection of parameters, which makes us thinking that we will not have a result of minimum density, but although this, we can say something.

We will see a theorem analogous to 5.18, that allowed us to compare Gabor systems coming from different functions and sets. The situation, however, is not so good in the wavelet case.

We have to observe that, even though we are using the notion of density in an informal way, we have not given any formal definition of this concept (only in the case of the Poisson wavelet). As a matter of fact we will not give it, so that the results that we will have will not be directly of comparison of densities.

We will not be concerned with the homogeneous approximation condition, since we will compare systems coming from the same function, and it will not be necessary.

Theorem 6.21. *Let ψ be a wavelet with strongly integrable kernel and we suppose that $W(\psi,\Gamma)$ is a Riesz basis and $W(\psi\Lambda)$ a frame of $L^2(\mathbb{R})$ with $\Gamma=\{z_j\}_{j\in\mathbb{N}}$ and $\Lambda=\{w_j\}_{j\in\mathbb{N}}$ separated sets of $\mathbb{H}$.*

Then for every a, r, ε there is R such that:

$$(1-\varepsilon)|B(a,r)\cap\Gamma| \leqslant |B(a,r+R)\cap\Lambda| \tag{6.8}$$

Proof. Following the proof of 5.18 we consider $a\in\mathbb{H}$. We define $V_r(a)$ as the space generated by:

$$\{\psi_{z_j} : z_j\in B(a,r)\cap\Gamma\}$$

and we define $W_r(a)$ as the space generated by:

$$\{\psi_{w_k} : w_k\in B(a,r)\cap\Lambda\}$$

The balls and the radius that we consider are hyperbolic.

We consider the selfadjoint and positive operator $T: V_r(a)\to V_r(a)$ defined as the composition of projections:

$$T = P_{V_r(a)}\circ P_{W_{r+R}(a)}$$

As $W(\psi,\Gamma)$ is a Riesz basis, there are functions $\phi_j\in\mathcal{H}$ such that:

$$\langle\psi_{z_j},\phi_l\rangle=\delta_{j,l}$$

These ϕ_j also form a Riesz basis. Therefore they are uniformly bounded in norm. The former arguments imply that:

$$\begin{aligned} Tr(T) = \sum_{z_j \in B(a,r)\cap\Gamma} \langle T(\psi_{z_j}), \phi_j\rangle &\geqslant \sum_{z_j \in B(a,r)\cap\Gamma} \langle \psi_{z_j}, \phi_j\rangle - |\langle T(\psi_{z_j}) - \psi_{z_j}, \phi_j\rangle| \\ &\geqslant \big(1 - C \sup_j \|T(\psi_{z_j}) - \psi_{z_j}\|\big) |B(a,r)\cap\Gamma| \end{aligned}$$

where C is the upper bound of the norms of the ϕ_j and $|A|$ is the cardinal of A if A is finite.

We observe that the rank of T can not be greater than the dimension of $W_{r+R}(a)$ and we have the estimation:

$$Tr(T) \leqslant \dim W_{r+R}(a) \leqslant |B(a, r + R| \cap \Lambda)$$

Therefore we arrive to the inequality:

$$\big(1 - C \sup_j \|T(\psi_{z_j}) - \psi_{z_j}\|\big) |B(a,r)\cap\Gamma| \leqslant |B(a, r + R) \cap \Lambda|$$

We have to see that if we choose R big enough we can make $\sup_j \|T(\psi_{z_j}) - \psi_{z_j}\|$ as small as we want. To see this we observe that:

$$\|T(\psi_{z_j}) - \psi_{z_j}\| = \left\| P_{V_r(a)}\big(P_{W_{r+R}(a)}(\psi_{z_j})\big) - P_{V_r(a)}(\psi_{z_j}) \right\| \leqslant \|P_{W_{r+R}(a)}(\psi_{z_j}) - \psi_{z_j}\|$$

That is, we have to make $\|P_{(W_{r+R}(a))^\perp}(\psi_{z_j})\|$ small. If we use that $\{\psi_{w_k}\}_{k\in\mathbb{N}}$ is a frame, we can see that, using the dual frame, $P_{(W_{r+R}(a))^\perp}(\psi_{z_j})$ can be written as:

$$P_{(W_{r+R}(a))^\perp}(\psi_{z_j}) = \sum_{k\in\mathbb{N}} \langle P_{(W_{r+R}(a))^\perp}(\psi_{z_j}), \psi_{w_k}\rangle \widetilde{\psi}_k$$

where the $\widetilde{\psi}_k$'s are the elements of the dual frame. Now we observe that:

$$\langle P_{(W_{r+R}(a))^\perp}(\psi_{z_j}), \psi_{w_k}\rangle = 0$$

if $w_k \in B(a, r + R)$. In this way we have the former sum left:

$$\sum_{w_k\in\Lambda, w_k\notin B(a,r+R)} \langle P_{(W_{r+R}(a))^\perp}(\psi_{z_j}), \psi_{w_k}\rangle \widetilde{\psi}_k$$

The $\widetilde{\psi}_k$'s are bounded in norm, therefore we have the bound:

$$\|P_{(W_{r+R}(a))^\perp}(\psi_{z_j})\| \leqslant \sum_{w_k\in\Lambda, w_k\notin B(a,r+R)} |\langle \psi_{z_j}, \psi_{w_k}\rangle| = \sum_{w_k\in\Lambda, w_k\notin B(a,r+R)} |k(z_j^{-1} \cdot w_k)|$$

As ψ has strongly integrable kernel we obtain the following inequality:

$$\sum_{w_k\in\Lambda, w_k\notin B(a,r+R)} |k(z_j^{-1}\cdot w_k)| \leqslant \widetilde{C}\left\|Mk_{|(B(i,R))^c}\right\|_1$$

where $\widetilde{C}$ only depends on Λ and on ψ, but not on z_j. We have also used that $z_j \in B(a,r)$. Now, if we take R big enough we have the bound that we wanted.

Therefore for every a, r, ε there is R such that:

$$(1-\varepsilon)|B(a,r)\cap\Gamma| \leqslant |B(a,r+R)\cap\Lambda|$$

that it what we want to prove. □

If now we attempt to deduce a result of density as in 5.18 we have the problem that the hyperbolic area of a ball of radius r is $4\pi(\sinh\frac{r}{2})^2$ and we see that:

$$\lim_{r\to\infty}\frac{(\sinh\frac{r}{2})^2}{(\sinh\frac{r+R}{2})^2} \neq 1$$

This does not allow us to transform (6.8) in an inequality among densities as in the Gabor case. However, (6.8) allows us to say that we will always have a minimum of points in balls big enough.

6.6 An interpolation result.

In this section we will give an interpolation result in phase spaces as in 5.22, inspired also in [Dya94].

Theorem 6.22. *Let ψ be an admissible wavelet with integrable reproductive kernel k. Then there is R_0 such that every set $\Lambda = \{z_k\}_{k\in\mathbb{N}}$ with separation constant greater than $2R_0$ is an interpolation set for H, the phase space of ψ.*

This R_0 is the minimum R such that:

$$\int_{B(i,R)} |k(z)|\,d\mu(z) \geqslant \frac{1}{2}\|k\|_1$$

and if $R = R_0$ the former integral is equal to $\frac{1}{2}\|k\|_1$ by continuity.

Proof. We remember that a set $\Lambda = \{z_j\}_{j\in\mathbb{N}} \subseteq \mathbb{H}$ is an interpolation set for $H_\psi = H$ if for any succession of values $(a_j)_{j\in\mathbb{N}} \in l^2$ there is a function $F \in H$ such that $F(z_j) = a_j$ $\forall j \in \mathbb{N}$. This is equivalent to see that there is $F \in H$ such that:

$$\int_{\mathbb{H}} F(z)k(z^{-1}\cdot z_j)\,dm(z) = \langle F, k_{z_j}\rangle = a_j \qquad \forall j\in\mathbb{N} \tag{6.9}$$

If we have a function $\phi \in L^2(\mathbb{H})$ we have that:

$$\widetilde{\phi}(z) = \int_{\mathbb{H}} \phi(w) k(w^{-1} \cdot z)\, dm(w) \in H$$

As a matter of fact $\widetilde{\phi}$ is the orthogonal projection of ϕ in H. If we observe that:

$$\langle \phi, k_{z_j} \rangle = \langle \widetilde{\phi}, k_{z_j} \rangle \qquad \forall j \in \mathbb{N}$$

we see that to solve the problem (6.9) for a function $F \in H$ is equivalent to find a $\phi \in L^2(\mathbb{H})$ such that:

$$\int_{\mathbb{H}} \phi(z) k(z^{-1} \cdot z_j)\, dm(z) = \langle \phi, k_{z_j} \rangle = a_j \qquad \forall j \in \mathbb{N} \tag{6.10}$$

This is not strange, since the fact that $\Lambda = \{z_j\}_{j\in\mathbb{N}}$ is an interpolation set is equivalent to the set of functions $\{k_{z_j}\}_{j\in\mathbb{N}}$ being linearly independent. This fact does not depend on if our looking at it in H or in every $L^2(\mathbb{H})$.

Therefore we have to check out that if Λ fulfills the conditions of the statement we can assure that for every succession $(a_j)_{j\in\mathbb{N}}$ there exists $\phi \in L^2(\mathbb{H})$ that solves (6.10).

To study this problem we call:

$$\alpha = \int_{\mathbb{H}} |k(z)|\, dm(z) = \|k\|_1$$

Given a succession $(c_k)_{k\in\mathbb{N}} \in l^2$, we define the following function:

$$f(z) = \sum_{k=1}^{\infty} \frac{c_k}{\alpha} \frac{|k_{z_k}(z)|}{\overline{k_{z_k}(z)}} \chi_{B_k}(z)$$

where the B_k will be balls of center z_k and radius $R > R_0$, half of the separation constant of Λ. We define $\frac{|k_{z_k}(z)|}{\overline{k_{z_k}(z)}} = 1$ if $k_{z_k}(z) = 0$. As the B_k are disjoint we have that $f \in L^2(\mathbb{H})$. If now we calculate we see that:

$$\langle f, k_{z_j} \rangle = \sum_{k=1}^{\infty} b_{jk} c_k$$

with the b_{jk} defined as:

$$b_{jk} = \int_{B_k} \frac{1}{\alpha} \frac{|k_{z_k}(z)|}{\overline{k_{z_k}(z)}} \overline{k_{z_j}(z)}\, dm(z)$$

We define the following operator:

$$\begin{aligned} T : l^2 &\longrightarrow l^2 \\ (c_k)_k &\longmapsto \left(\sum_{k=1}^{\infty} b_{jk} c_k \right)_j \end{aligned}$$

To see that this operator is bounded we only have to calculate:

$$\|T((c_k)_k)\|^2 = \sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty} b_{jk}c_k\right|^2 \leqslant \sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|b_{jk}|\sum_{k=1}^{\infty}|b_{jk}||c_k|^2\right)$$

Now, on one hand we see that:

$$\sum_{k=1}^{\infty}|b_{jk}| = \sum_{k=1}^{\infty}\left|\int_{B_k}\frac{|k(z_k^{-1}\cdot z)|}{\overline{k(z_k^{-1}\cdot z)}}\,\frac{\overline{k(z_j^{-1}\cdot z)}}{\alpha}\,dm(z)\right| \leqslant 1$$

On the other hand we have that:

$$\begin{aligned}\sum_{j=1}^{\infty}\sum_{k=1}^{\infty}|b_{jk}||c_k|^2 &= \sum_{k=1}^{\infty}|c_k|^2\sum_{j=1}^{\infty}|b_{jk}|\\ &= \sum_{k=1}^{\infty}|c_k|^2\sum_{j=1}^{\infty}\left|\int_{B_k}\frac{|k(z_k^{-1}\cdot z)|}{\overline{k(z_k^{-1}\cdot z)}}\,\frac{\overline{k(z_j^{-1}\cdot z)}}{\alpha}\,dm(z)\right|\\ &\leqslant \sum_{k=1}^{\infty}|c_k|^2\sum_{j=1}^{\infty}\int_{B_k}\frac{|k(z_j^{-1}\cdot z)|}{\alpha}\,dm(z)\\ &\leqslant \sum_{k=1}^{\infty}|c_k|^2\sum_{j=1}^{\infty}\int_{z_j^{-1}\cdot B_k}\frac{|k(w)|}{\alpha}\,dm(w) \leqslant \|(c_k)_k\|^2\end{aligned}$$

Therefore the operator T has norm less or equal to 1. The following step is to check out that this operator is invertible. We will see this checking that $\|Id - T\| < 1$.

$$\begin{aligned}\|Id - T\|^2 &= \sup_{\|(c_k)_k\|=1}\sum_{j=1}^{\infty}\left|\sum_{k=1}^{\infty}(\delta_{jk} - b_{jk})c_k\right|^2\\ &\leqslant \sup_{\|(c_k)_k\|=1}\sum_{j=1}^{\infty}\left(\sum_{k=1}^{\infty}|\delta_{jk} - b_{jk}|\sum_{k=1}^{\infty}|\delta_{jk} - b_{jk}||c_k|^2\right)\end{aligned}$$

We have to look at this expression in parts. We observe first that $b_{jj} \geqslant 0$ and it does not depend on j and that $\sum_j |b_{jk}| \leqslant 1$ and $\sum_k |b_{jk}| \leqslant 1$. Taking this into account we see that when j is fixed:

$$\sum_{k=1}^{\infty}|\delta_{jk} - b_{jk}| = 1 - 2b_{jj} + \sum_{k=1}^{\infty}|b_{jk}| \leqslant 2(1 - b_{jj})$$

On another part, when k is fixed, we can bound:

$$\sum_{j=1}^{\infty}|\delta_{jk} - b_{jk}| = 1 - 2b_{kk} + \sum_{j=1}^{\infty}|b_{jk}| \leqslant 2(1 - b_{kk})$$

We call $\beta = b_{jj}$, which we remember that it does not depend on j. Now we add up this bound to arrive to:

$$\|Id - T\| \leqslant 2(1-\beta)$$

Therefore we need that $\beta > \frac{1}{2}$. After making a change of variable, the definition of β is the following one:

$$\beta = \frac{1}{\|k\|_1} \int_{B(i,R)} |k(z)|\, dm(z)$$

As we had chosen $R > R_0$ it is fulfilled that $\beta = \beta(R) > \frac{1}{2}$. If we are in these conditions $(a_j)_{j\in\mathbb{N}} \in l^2$ we have that the operator T is invertible. That is, for every succession it exists another succession $(c_k)_{k\in\mathbb{N}} \in l^2$ such that $T((c_k)_{k\in\mathbb{N}}) = (a_j)_{j\in\mathbb{N}}$. This says that for every succession $(a_j)_{j\in\mathbb{N}} \in l^2$ there exists a function $f \in L^2(\mathbb{H})$ such that:

$$\langle f(z), k(z_j^{-1} \cdot z)\rangle = a_j \qquad \forall j \in \mathbb{N}$$

For that we have commented on formerly we can think that f belongs to H. $\square$

As in the Gabor case, these sets correspond with Riesz bases of the subspace that they generate.

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