HILBERT SPACE GEOMETRY
**Definition:** A **vector space** over $\mathbb{R}$ is a set $V$ (whose elements are called **vectors**) together with a binary operation

$$+: V \times V \to V,$$

which is called **vector addition**, and an external binary operation

$$\cdot: \mathbb{R} \times V \to V,$$

which is called **scalar multiplication**, such that

(i) $(V, +)$ is a commutative group

(whose neutral element is called **zero vector**) and

(ii) for all $\lambda, \mu \in \mathbb{R}, x, y \in V$: $\lambda (\mu x) = (\lambda \mu) x,$

$1 \cdot x = x,$

$\lambda (x + y) = (\lambda x) + (\lambda y),$

$(\lambda + \mu) x = (\lambda x) + (\mu x),$

where the image of $(x, y) \in V \times V$ under $+$ is written as $x + y$ and the image of $(\lambda, x) \in \mathbb{R} \times V$ under $\cdot$ is written as $\lambda x$ or as $\lambda \cdot x$.

**Exercise:** Show that the set $\mathbb{R}^2$ together with vector addition and scalar multiplication defined by

$$
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} + 
\begin{pmatrix}
  y_1 \\
  y_2
\end{pmatrix} =
\begin{pmatrix}
  x_1 + y_1 \\
  x_2 + y_2
\end{pmatrix}
$$

and

$$
\lambda 
\begin{pmatrix}
  x_1 \\
  x_2
\end{pmatrix} =
\begin{pmatrix}
  \lambda x_1 \\
  \lambda x_2
\end{pmatrix},
$$

respectively, is a vector space.
Remark: Usually we do not distinguish strictly between a vector space \((V,+,\cdot)\) and the set of its vectors \(V\). For example, in the next definition \(V\) will first denote the vector space and then the set of its vectors.

**Definition:** If \(V\) is a vector space and \(M \subseteq V\), then the set of all linear combinations of elements of \(M\) is called **linear hull** or **linear span** of \(M\). It is denoted by \(\text{span}(M)\). By convention, \(\text{span}(\emptyset) = \{0\}\).

**Proposition:** If \(V\) is a vector space, then the linear hull of any subset \(M\) of \(V\) (together with the restriction of the vector addition to \(M \times M\) and the restriction of the scalar multiplication to \(\mathbb{R} \times M\)) is also a vector space.

**Proof:** We only need to prove that \(\text{span}(M)\) contains the zero vector and that it is closed under vector addition and scalar multiplication:

\[
\begin{align*}
M = \emptyset & \Rightarrow \text{span}(M) = \{0\} \Rightarrow 0 \in \text{span}(M) \\
M \neq \emptyset & \Rightarrow \exists x \in M: 0 \cdot x = 0 \in \text{span}(M)
\end{align*}
\]

\[
\begin{align*}
x, y \in \text{span}(M) & \Rightarrow x + y = 1 \cdot x + 1 \cdot y \in \text{span}(M) \\
x \in \text{span}(M), \lambda \in \mathbb{R} & \Rightarrow \lambda \cdot x \in \text{span}(M)
\end{align*}
\]

The other properties of a vector space are satisfied for all elements of \(V\) and therefore also for all elements of \(M \subseteq V\).

**Definition:** If a subset \(M\) of a vector space \(V\) is also a vector space, it is called a **linear subspace** of \(V\).
**Definition:** An inner product space is a vector space $V$ together with a function

$$\langle \cdot, \cdot \rangle : V \times V \to \mathbb{R}$$

(called **inner product**) satisfying the following axioms:

For all $x, y, z \in V$, $\lambda \in \mathbb{R}$

(i) $\langle x, y \rangle = \langle y, x \rangle$,

(ii) $\langle x + y, z \rangle = \langle x, z \rangle + \langle y, z \rangle$,

(iii) $\langle \lambda x, y \rangle = \lambda \langle x, y \rangle$,

(iv) $\langle x, x \rangle \geq 0$,

(v) $\langle x, x \rangle = 0 \iff x = 0$.

A **semi-inner product** satisfies (i) – (iv), but $\langle x, x \rangle$ can be zero if $x \neq 0$.

**Exercise:** Show that the inner product axioms (i)-(iii) imply that for all $x, y, z, u \in V$, $\lambda, \mu, \nu, \xi \in \mathbb{R}$

$$\langle \lambda x + \mu y, \nu z + \xi u \rangle = \lambda \nu \langle x, z \rangle + \lambda \xi \langle x, u \rangle + \mu \nu \langle y, z \rangle + \mu \xi \langle y, u \rangle.$$ 

**Exercise:** Show that the vector space $\mathbb{R}^2$ together with the function $\langle \cdot, \cdot \rangle$ defined by

$$\langle \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \end{pmatrix} \rangle = x_1 y_1 + x_2 y_2$$

is an inner product space.
Definition: The norm (seminorm) of an element $x$ of an inner product space (semi-inner product space) is defined by

$$\|x\| = \sqrt{x, x}.$$ 

Cauchy-Schwarz Inequality: If $x$ and $y$ are elements of an inner product space, then

$$\langle x, y \rangle \leq \|x\| \|y\|.$$ 

Proof: 

$$0 \leq \langle \|y\| x \pm \|x\| y, \|y\| x \pm \|x\| y \rangle$$

$$= \|y\|^2 \langle x, x \rangle \pm 2 \|x\| \|y\| \langle x, y \rangle + \|x\|^2 \langle y, y \rangle$$

$$= 2 \|x\|^2 \|y\|^2 \pm 2 \|x\| \|y\| \langle x, y \rangle$$

$$= 2 \|x\| \|y\| (\|x\| \|y\| \pm \langle x, y \rangle)$$

$$\Rightarrow 0 \leq \|x\| \|y\| \pm \langle x, y \rangle \Rightarrow \pm \langle x, y \rangle \leq \|x\| \|y\|$$

Exercise: Let $V$ be a semi-inner product space. Show that for all $x, y, z \in V$, $\lambda \in \mathbb{R}$

(i) $\|x + y\| \leq \|x\| + \|y\|,$

(ii) $\|\lambda x\| = |\lambda| \|x\|,$

(iii) $\|x\| \geq 0,$

and, if $V$ is an inner product space, also

(iv) $\|x\| = 0 \iff x = 0.$
**Lemma:** The triangle inequality \( \|x + y\| \leq \|x\| + \|y\| \) implies that for all \( x \) and \( y \)

\[
\|x - y\| \geq \|x\| - \|y\|.
\]

**Proof:** \( \|x\| = \|(x - y) + y\| \leq \|x - y\| + \|y\| \Rightarrow \|x - y\| \geq \|x\| - \|y\| \)

\[
\|y\| = \|(y - x) + x\| \leq \|y - x\| + \|x\| \Rightarrow \|y - x\| \geq \|y\| - \|x\|
\]

**Continuity of the Norm:** If the sequence \((x_n)\) of elements of an inner product space \(V\) converges in norm to \(x \in V\), then the sequence \(\|x_n\|\) converges to \(\|x\|\), i.e.,

\[
\|x_n - x\| \to 0 \Rightarrow \|x_n\| \to \|x\|.
\]

**Proof:** \( 0 \leq \|x_n\| - \|x\| \leq \|x_n - x\| \to 0 \)

**Continuity of the Inner Product:** If the sequences \((x_n)\) and \((y_n)\) of elements of an inner product space \(V\) converge in norm to \(x \in V\) and \(y \in V\), respectively, then the sequence \(\langle x_n, y_n \rangle\) converges to \(\langle x, y \rangle\), i.e.,

\[
\|x_n - x\| \to 0, \quad \|y_n - y\| \to 0 \Rightarrow \langle x_n, y_n \rangle \to \langle x, y \rangle.
\]

**Proof:** \( 0 \leq \langle x_n, y_n \rangle - \langle x, y \rangle = \langle x_n, y_n - y \rangle + \langle x_n - x, y \rangle \)

\[
\leq \langle x_n, y_n - y \rangle + \langle x_n - x, y \rangle
\]

\[
\leq \|x_n\| \|y_n - y\| + \|x_n - x\| \|y\|
\]

\[
\downarrow \quad \downarrow \quad \downarrow
\]

\[
\|x\| \quad 0 \quad 0
\]
**Definition:** An inner product space $H$ is called a **Hilbert space**, if it is complete in the sense that every Cauchy sequence $(x_n)$ of elements of $H$ converges to some element $x \in H$, i.e.,

$$x_n, x_m \in H, \|x_m - x_n\| \to 0 \text{ as } m,n \to \infty \Rightarrow \exists x \in H: \|x_n - x\| \to 0.$$ 

**Example:** That the inner product space $\mathbb{R}^2$ is a Hilbert space can be seen as follows.

$$\left\| \begin{pmatrix} x_{m1} \\ x_{m2} \end{pmatrix} - \begin{pmatrix} x_{n1} \\ x_{n2} \end{pmatrix} \right\|^2 = (x_{m1} - x_{n1})^2 + (x_{m2} - x_{n2})^2 \to 0$$

$$\Rightarrow (x_{m1} - x_{n1})^2 \to 0, (x_{m2} - x_{n2})^2 \to 0$$

$$\Rightarrow \exists x_1, x_2 \in \mathbb{R}: x_{n1} \to x_1, x_{n2} \to x_2 \text{ (by the completeness of } \mathbb{R})$$

$$\Rightarrow \left\| \begin{pmatrix} x_{n1} \\ x_{n2} \end{pmatrix} - \begin{pmatrix} x_1 \\ x_2 \end{pmatrix} \right\|^2 = (x_{n1} - x_1)^2 + (x_{n2} - x_2)^2 \to 0$$

**Definition:** A linear subspace $S$ of a Hilbert space is said to be a **closed subspace**, if

$$x_n \in S, \|x_n - x\| \to 0 \Rightarrow x \in S.$$
Exercise: Show that the intersection $\bigcap_{i \in I} S_i$ of a family of closed subspaces of a Hilbert space is also a closed subspace.

Definition: The closed span of a subset $M$ of a Hilbert space is defined as the intersection of all closed subspaces which contain all elements of $M$. It is denoted by $\text{span}(M)$.

Definition: Two elements $x$ and $y$ of an inner product space are said to be orthogonal $(x \perp y)$, if $\langle x, y \rangle = 0$.

Proposition: The orthogonal complement

$$M^\perp = \{ x \in H : x \perp y \ \forall y \in M \}$$

of any subset $M$ of a Hilbert space $H$ is a closed subspace.

Proof: $M^\perp$ is a linear subspace, because

$$z \in M \Rightarrow \langle 0, z \rangle = 0 \Rightarrow 0 \perp z,$$

$$x, y \in M^\perp, \ z \in M \Rightarrow \langle x + \mu, z \rangle = \langle x, z \rangle + \langle y, z \rangle = 0 \Rightarrow x + y \perp z,$$

$$x \in M^\perp, \ \lambda \in \mathbb{R}, \ z \in M \Rightarrow \langle \lambda x, z \rangle = \lambda \langle x, z \rangle = 0 \Rightarrow \lambda x \perp z.$$ 

Moreover, $M^\perp$ is even a closed subspace, because

$$x_n \in M^\perp, \ |x_n - x| \to 0, \ z \in M \Rightarrow \langle x_n, z \rangle = 0 \ \text{for all } n \Rightarrow \langle x, z \rangle = \lim \langle x_n, z \rangle = 0.$$
**Projection Theorem:** If $S$ is a closed subspace of a Hilbert space $H$, then each $x \in H$ can be uniquely represented as

$$x = \hat{x} + u,$$

where $\hat{x} \in S$ and $u \in S^\perp$. Furthermore, $\hat{x}$ (which is called the projection of $x$ onto $S$) satisfies

$$\|x - \hat{x}\| < \|x - y\|$$

for any other element $y \in S$.

**Definition:** Let $S$ be a closed subspace of a Hilbert space $H$. The mapping

$$P_S(x) = \hat{x}, \ x \in H,$$

where $\hat{x}$ is the projection of $x$ onto $S$, is called the projection mapping of $H$ onto $S$.

**Properties of Projection Mappings:** If $S, S_1, S_2$ are closed subspaces of a Hilbert space $H$, $x, y, x_n \in H$, and $\lambda, \mu \in \mathbb{R}$, then:

(i) $P_S(\lambda x + \mu y) = \lambda P_S(x) + \mu P_S(y)$
(ii) $x \in S \iff P_S(x) = x$
(iii) $x \in S^\perp \iff P_S(x) = 0$
(iv) $x = P_S(x) + P_{S^\perp}(x)$
(v) $S_1 \subseteq S_2 \iff P_{S_1}(P_{S_2}(x)) = P_{S_1}(x)$
(vi) $\|x_n - x\| \rightarrow 0 \Rightarrow \|P_S(x_n) - P_S(x)\| \rightarrow 0$
Proposition: If \( M = \{u_1, \ldots, u_n\} \) is a set of mutually orthogonal elements of a Hilbert space \( H \) and \( 0 \notin M \), then

\[
P_{\text{span}(M)}(x) = \sum_{k=1}^{n} \frac{\langle x, u_k \rangle}{\|u_k\|^2} u_k \quad \forall x \in H.
\]

Proof: For each \( u_j \in M \) we have

\[
\langle x - \sum_{k=1}^{n} \frac{\langle x, u_k \rangle}{\|u_k\|^2} u_k, u_j \rangle = \langle x, u_j \rangle - \sum_{k=1}^{n} \frac{\langle x, u_k \rangle}{\|u_k\|^2} \langle u_k, u_j \rangle
\]

\[
= \langle x, u_j \rangle - \frac{\langle x, u_j \rangle}{\|u_j\|^2} \langle u_j, u_j \rangle
\]

\[
= 0.
\]

Thus

\[
\langle x - \sum_{k=1}^{n} \frac{\langle x, u_k \rangle}{\|u_k\|^2} u_k, \sum_{k=1}^{n} \lambda_k u_k \rangle = 0
\]

for each linear combination \( \sum_{k=1}^{n} \lambda_k u_k \in \text{span}(M) = \text{span}(M) \).