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Rota-Baxter operators and decompositions of Lie algebras

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Abstract

In this thesis we study Rota-Baxter operators in the context of post-Lie algebra structures and decompositions of Lie algebras. These operators first appeared in the area of probability theory, but were later on dealt with in different settings. The aim of this thesis is to exhibit the relationship between Rota-Baxter operators and existence and classification problems of post-Lie algebra structures. For this purpose, we consider pairs of Lie algebras with different assumptions, such as semisimple, simple, nilpotent and solvable Lie algebras. Furthermore, we cover results that are connected to decompositions of Lie algebras. On the one hand we use decomposition results for statements on post-Lie algebra structures, on the other hand we examine nildecomposable Lie algebras and their derived length.

Key words: Lie theory, Rota-Baxter operators, post-Lie algebra structures, decompositions of Lie algebras

Zusammenfassung

In dieser Arbeit untersuchen wir Rota-Baxter Operatoren im Kontext von post-Lie Algebra Strukturen und Zerlegungen von Lie Algebren. Diese Operatoren wurden erstmals im Bereich der Wahrscheinlichkeitstheorie erwähnt, aber bald darauf in anderen Bereichen der Mathematik eingesetzt. Das Ziel dieser Arbeit ist es, die Beziehung zwischen Rota-Baxter Operatoren und Existenz- und Klassifikationsproblemen von post-Lie Algebra Strukturen aufzuweisen. Hierfür betrachten wir Paare von Lie Algebren unter verschiedenen Annahmen, wie halbeinfache, einfache, nilpotente und auflösbare Lie Algebren. Weiters behandeln wir Resultate, die verbunden sind mit Zerlegungen von Lie Algebren. Einerseits verwenden wir Zerlegungen von Lie Algebren um Aussagen über post-Lie Algebra Strukturen treffen zu können, andererseits um nilpotente zerlegbare Lie Algebren und deren Auflösbarkeitsklasse zu untersuchen.

Schlagwörter: Lie Theorie, Rota-Baxter Operatoren, post-Lie Algebra Strukturen, Zerlegungen von Lie Algebren

Introduction

The purpose of this thesis is to introduce Rota-Baxter operators and study their relationship to post-Lie algebra structures and decompositions of Lie algebras. Rota-Baxter operators are widely discussed in the literature, occuring in different areas in mathematics. We restrict ourselves to the algebraic perspective and present some results on classification and existence problems of post-Lie algebra structures.

In Chapter 1, we recall some general theory of Lie algebras and introduce Rota-Baxter operators. These preliminaries are essential for the foregoing discussion. The first section is a recollection of some important concepts of Lie theory, such as simple, semisimple and reductive Lie algebras. Furthermore, there is a brief discussion on nilpotent and solvable Lie algebras. The chapter on Rota-Baxter operators gives some examples and some elementary propositions. These are linear operators defined over a non-associative algebra of the form

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy).$$

where λ is a scalar. These operators are in strong connection to post-Lie algebra structures and decompositions of Lie algebras. The final part of this chapter covers Rota-Baxter operators and direct sums. Here a link between so-called split Rota-Baxter operators and decompositions is given.

Chapter 2 starts with some notions on PA-structures, i.e.,

Definition. Let $\mathfrak{g} = (V, [,])$ and $\mathfrak{n} = (V, \{,\})$ be two Lie brackets on a vector space V over a field K. A *post-Lie algebra structure*, or short PA-structure, on the pair $(\mathfrak{g}, \mathfrak{n})$ is a K-bilinear product $x \cdot y$ satisfying

(i) $x \cdot y - y \cdot x = [x, y] - \{x, y\}$,

(ii) $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$,

(iii) $x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}$.

Much emphasis lies here on classification and existence problems of post-Lie algebra structures. We start by introducing the left and right multiplication operator, to have an equivalent definition of a post-Lie algebra structure using the operator form. To see the relationship between post-Lie algebra structures and Rota-Baxter operators, inner post-Lie algebra structures are crucial. These are those post-Lie algebra structures that arise from an endomorphism, i.e. PA-structures of the form $x \cdot y = \{\phi(x), y\}$. For complete Lie algebras, these are exactly those arising from Rota-Baxter operators.

Theorem. Let $\mathfrak{n}$ be a Lie algebra with trivial center. Then any inner PA-structure on $(\mathfrak{g}, \mathfrak{n})$ comes from an RB-operator of weight 1. Moreover, if $\mathfrak{n}$ is complete, i.e. we have in addition $\mathrm{Der}(\mathfrak{n}) = \mathrm{ad}(\mathfrak{n})$, then every PA-structure on $(\mathfrak{g}, \mathfrak{n})$ is inner.

A counterexample of an inner PA-structure that does not arise from an RB-operator is given, using the three-dimensional Heisenberg Lie algebra.

Moreover, we investigate PA-structures on pairs of semisimple Lie algebras and nilpotent Lie algebras. The first uses results on decompositions and links the attributes of the Lie algebra by using the kernel and image of Rota-Baxter operators.

Theorem. Let $\mathfrak{n}$ be simple, $\mathfrak{g}$ semisimple and $x \cdot y$ a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then $\mathfrak{g}$ is also simple and $\mathfrak{g} \cong \mathfrak{n}$.

Whether or not this holds for $\mathfrak{g}$ and $\mathfrak{n}$ semisimple, but not simple, remains open. Based on the findings, this might be the case. Furthermore, classification results of PA-structures on two isomorphic Lie algebras, that are isomorphic to the direct sum of two simple isomorphic ideals, show that these PA-structures arise from Rota-Baxter operators.

The question of what happens if one of the Lie algebras is nilpotent and a PA-structure exists is also discussed in the subsequent chapter. Here we have an interesting result, namely

Theorem. Let $\mathfrak{g}$ be a nilpotent Lie algebra. Assume there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then $\mathfrak{n}$ is solvable.

This statement uses a result by Goto on nilpotent decompositions in the proof. Therefore it is only natural that we also study nildecomposable Lie algebras. As can be seen in the Theorem above, $\mathfrak{n}$ is solvable. A consequence is to look at what we need for $\mathfrak{n}$ to be nilpotent. Under certain assumptions this is the case, where new findings on arithmetically-free groups are applied.

The classification of PA-structures is a very hard task. As for existence questions, we provide a table at the end of Chapter 2 showing results on existence of PA-structures depending on the pair of Lie algebras,

$(\mathfrak{g}, \mathfrak{n})$	$\mathfrak{n}$ abe	$\mathfrak{n}$ nil	$\mathfrak{n}$ sol	$\mathfrak{n}$ sim	$\mathfrak{n}$ sem	$\mathfrak{n}$ red	$\mathfrak{n}$ com
$\mathfrak{g}$ abe	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ nil	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ sol	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{g}$ sim	-	-	-	✓	-	-	-
$\mathfrak{g}$ semi	-	-	-	-	✓	?	-
$\mathfrak{g}$ red	✓	?	?	-	?	✓	✓
$\mathfrak{g}$ com	✓	✓	✓	?	?	✓	✓

Note that the checkmark represents that there is some pair admitting a PA-structure.

The third chapter concerns decompositions of Lie algebras, in particular semisimple and reductive decompositions and nildecomposable Lie algebras. Based on results on

PA-structures for semisimple Lie algebras, we want to construct a generalization of the above mentioned Theorem, but this time for reductive Lie algebras. Regarding semisimple decompositions, we provide an explicit example of a semisimple decomposition of a Lie algebra that is itself not semisimple.

Example. Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes_\rho V(2)$, where $V(2)$ stands for the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, considered as an abelian Lie algebra. Then

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) + \mathfrak{sl}_2(\mathbb{C})$$

is a decomposition, where both summands are semisimple, but $\mathfrak{g}$ is not.

We study nildecomposable Lie algebras in a different context than RB-operators, that is the context of their derived length, and bounds that occur in analyzing the derived length. As mentioned before, Chapter 2 uses a result by Goto on nildecomposable Lie algebras, hence we give a survey of some open questions related to decompositions of nilpotent Lie algebras. Accordingly, we investigate bounds for the minimal dimension of a nilpotent Lie algebra where the derived length is given. This is done by using filiform nilpotent Lie algebras.

Another open question, that is discussed, is the case of whether or not it is possible to bound the derived length of a nildecomposable Lie algebra by the nilpotency classes of the subalgebras involved. The results for Lie algebras rely heavily on factorizations of groups. Therefore, we start by stating some results on groups, and then proceed with the case of Lie algebras.

Finally, the last chapter provides an outlook on open questions and ongoing research.

Contents

1 Preliminaries

In this chapter, we recall some definitions and lemmas from the theory of Lie algebras necessary for understanding the subsequent chapters. We also introduce the notion of Rota-Baxter operators and give some elementary examples and lemmas. The main literature used here for the theory of Lie algebras is [15], [44], [39] and [3]. Note that we have different notations for sums. We denote by $\oplus$ the direct sum as algebras, $\dot{+}$ the direct sum as vector spaces and $+$ the sum throughout this thesis.

1.1 Basic definitions and examples

1.1.1 General theory of Lie algebras

Definition 1.1. A Lie algebra $\mathfrak{g}$ over a field K is a K-vector space together with a K-bilinear map, the so called Lie bracket,

$$\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}, (x, y) \mapsto [x, y]$$

such that for all $x, y, z \in \mathfrak{g}$, we have

(i) anti-symmetry: $[x, x] = 0$,

(ii) Jacobi-identity: $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

Definition 1.2. A subspace $\mathfrak{a}$ of a Lie algebra $\mathfrak{g}$ is called a *subalgebra* if $[\mathfrak{a}, \mathfrak{a}] \subseteq \mathfrak{a}$. It is called an *ideal* if $[\mathfrak{g}, \mathfrak{a}] = [\mathfrak{a}, \mathfrak{g}] \subseteq \mathfrak{a}$.

Example 1.
- $\mathfrak{sl}_n(K)$ is a Lie subalgebra of the Lie algebra $\mathfrak{gl}_n(K)$, where $\mathfrak{gl}_n(K)$ denotes the general linear Lie algebra of dimension n^2 and $\mathfrak{sl}_n(K)$ the set of all matrices in $\mathfrak{gl}_n(K)$ with trace zero.
- The set of all upper triangular matrices of size n over a field K is a Lie algebra $\mathfrak{t}_n(K)$, where the strictly upper triangular matrices form a Lie-subalgebra $\mathfrak{n}_n(K)$.

Definition 1.3. Let A be an algebra. A linear map $D \in \mathrm{End}(A)$ is called a *derivation* of A if for all $x, y \in A$

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y).$$

We denote $\mathrm{Der}(A)$ as the set of all derivations of A. Note that this can also be viewed as a Lie algebra.

Definition 1.4. A *representation* of a Lie algebra $\mathfrak{g}$ is a K-vectorspace V together with a Lie-algebra homomorphism $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$, where $\mathfrak{gl}(V)$ denotes the Lie algebra whose elements are in $\mathrm{End}(V)$ and whose Lie bracket is given by the commutator of endomorphisms. Here a *Lie algebra homomorphism* is a linear map where the Lie brackets are preserved. If the homomorphism is injective, the representation is called *faithful.*
A *subrepresentation* of a representation (ρ, V) is a representation (π, W) such that W is a vector subspace of V and $\rho|_W = \pi$.

Note that by representation we mean, that an abstract Lie algebra can be viewed as a Lie algebra of matrices. Due to Ado [1] and Iwasawa [41], we know that every finite-dimensional Lie algebra over a field of characteristic zero or of prime characteristic possesses a faithful representation, thus making it possible to view every finite-dimensional Lie algebra concretely.

Example 2. The linear map $ad(x) : \mathfrak{g} \to \mathfrak{g}$ with $ad(x)(y) = [x, y]$ defines a representation

$$\mathrm{ad} : \mathfrak{g} \to \mathfrak{gl}(\mathfrak{g}), x \mapsto ad(x).$$

It is called the adjoint representation and $\mathrm{ad}(\mathfrak{g})$ is an ideal of $\mathrm{Der}(\mathfrak{g})$, often referred to as inner derivation, denoted as $\mathrm{ad}(\mathfrak{g}) = \mathrm{Inn}(\mathfrak{g})$. There is also a notion of outer derivations, that is $\mathrm{Out}(\mathfrak{g}) = {}^{\mathrm{Der}(\mathfrak{g})}\!\big/_{\mathrm{Inn}(\mathfrak{g})}$.

Definition 1.5. Let $\mathfrak{g}$ be a Lie algebra and $\mathfrak{a} \subseteq \mathfrak{g}$. We call

$$Z_{\mathfrak{g}}(\mathfrak{a}) = \{x \in \mathfrak{g} \mid [x, y] = 0 \,\forall y \in \mathfrak{a}\}$$

the *centralizer* of $\mathfrak{a}$ in $\mathfrak{g}$. $Z_{\mathfrak{g}}(\mathfrak{g}) = Z(\mathfrak{g})$ is then called the *center* of $\mathfrak{g}$.

Definition 1.6. A Lie algebra $\mathfrak{g}$ is called *complete*, if $Z(\mathfrak{g}) = 0$ and $\mathrm{Der}(\mathfrak{g}) = \mathrm{ad}(\mathfrak{g})$.

Definition 1.7. Let $\mathfrak{g}$ be a finite-dimensional Lie algebra over a field K. We call the form $\kappa : \mathfrak{g} \times \mathfrak{g} \to K$ with $\kappa(x, y) = tr(ad(x)ad(y))$ the *Killing-form*.

1.1.2 Simple, semisimple and reductive Lie algebras

Definition 1.8. A representation V of a Lie algebra $\mathfrak{g}$ is called *simple*, if $V \neq 0$, and the only subrepresentation different from V is the zero space.

Definition 1.9. A representation V of a Lie algebra $\mathfrak{g}$ is called *semisimple* if V is a sum of simple subrepresentations or equivalently every subrepresentation of V has a complement in V.

Definition 1.10. A Lie algebra $\mathfrak{g}$ is called *simple*, if its adjoint representation is simple. Equivalently, it is simple if and only if the only ideals of $\mathfrak{g}$ are 0 and $\mathfrak{g}$ itself and the commutator ideal is not zero, that is $[\mathfrak{g}, \mathfrak{g}] = \mathfrak{g}$. A Lie algebra that fulfills the last condition is called *perfect*.
A Lie algebra is called *reductive* if its adjoint representation is semisimple. This means that for every ideal $\mathfrak{a}$ in $\mathfrak{g}$, we have a complementary ideal $\mathfrak{b}$ in $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{a} \bigoplus \mathfrak{b}$.

Definition 1.11. A Lie algebra $\mathfrak{g}$ is called *semisimple*, if it is a direct sum of simple Lie algebras.

Definition 1.12. We call a real Lie algebra $\mathfrak{g}$, i.e. over the field of real numbers, *compact* if its Killing form is negative definite.

Remark 1. It follows that a compact Lie algebra is semisimple, see [44].

Note that it follows immediately that a simple Lie algebra is always semisimple.

Lemma 1.13. *[15] Let $\mathfrak{g}$ be a semisimple Lie algebra. Then $\mathfrak{g}$ is perfect, reductive and $Z(\mathfrak{g}) = 0$.*

Theorem 1.14. *[15] Let $\mathfrak{g}$ be reductive. Then the following holds:*

(i) If $\mathfrak{a}$ is an ideal in $\mathfrak{g}$, then $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are reductive.

(ii) $\mathfrak{g} = [\mathfrak{g}, \mathfrak{g}] \bigoplus Z(\mathfrak{g})$, where $[\mathfrak{g}, \mathfrak{g}]$ is semisimple.

(iii) $\mathfrak{g}$ is semisimple if and only if $Z(\mathfrak{g}) = 0$.

Remark 2. Using condition (ii), we can interpret that a reductive Lie algebra is a direct sum of a semisimple and an 'abelian' Lie algebra.

1.1.3 Nilpotent and solvable Lie algebras

We now state a few results on nilpotent and solvable Lie algebras. These are necessary in particular for Chapter 3 on nildecomposable Lie algebras.

Definition 1.15. We call a Lie algebra $\mathfrak{g}$ *abelian* if $[\mathfrak{g}, \mathfrak{g}] = 0$.

We now define the lower central series and derived series for a Lie algebra $\mathfrak{g}$.

Definition 1.16. Let $\mathfrak{g}$ be a Lie algebra. Then we define

(i) the *lower central series* $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$, ... , $\mathfrak{g}^{i+1} = [\mathfrak{g}, \mathfrak{g}^i]$;

(ii) the *derived series* $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$, ... , $\mathfrak{g}^{(i+1)} = [\mathfrak{g}^{(i)}, \mathfrak{g}^{(i)}]$.

A Lie algebra $\mathfrak{g} \neq 0$ is called *nilpotent of class k* or *k-step nilpotent*, if $\mathfrak{g}^k = 0$ and $\mathfrak{g}^{k-1} \neq 0$. It is called *$k$-step solvable* or *solvable of derived length k* if $\mathfrak{g}^{(k)} = 0$ and $\mathfrak{g}^{(k-1)} \neq 0$. Since $\mathfrak{g}^{(i)} \subset \mathfrak{g}^i$, we have that every nilpotent Lie algebra is solvable. Note that in Chapter 3, we use the notation $c(\mathfrak{g})$ for the nilpotency class and $d(\mathfrak{g})$ for the derived length of a solvable Lie algebra.

Example 3. The Heisenberg Lie algebra $\mathfrak{n}_3(K)$ is 2-step nilpotent. It has a basis (e_1, e_2, e_3) and Lie bracket $[e_1, e_2] = e_3$.

Theorem 1.17. *[39] Let $\mathfrak{g}$ be nilpotent. Then the following holds:*

(i) If $\mathfrak{a} \neq 0$ is an ideal in $\mathfrak{g}$, then $\mathfrak{a} \cap Z(\mathfrak{g}) \neq 0$. Additionally we have $Z(\mathfrak{g}) \neq 0$.

(ii) Every Lie subalgebra and homomorphic image of $\mathfrak{g}$ is nilpotent.

(iii) Let $\mathfrak{a}, \mathfrak{b}$ be nilpotent ideals in $\mathfrak{g}$, then $\mathfrak{a} + \mathfrak{b}$ are nilpotent.

Definition 1.18. The maximal nilpotent ideal in $\mathfrak{g}$ where $\mathfrak{g}$ is finite-dimensional is called the *nilradical* of $\mathfrak{g}$ and is denoted by $\mathrm{nil}(\mathfrak{g})$.

Definition 1.19. Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a representation of $\mathfrak{g}$. We call the representation *nilpotent*, if there exists an n such that $\rho(x_1) \cdots \rho(x_n) = 0$ for all $x_i \in \mathfrak{g}, i = 1, ...n$.

We are now ready to formulate the well-known result on nilpotent representations by Engel [33].

Theorem 1.20 (Engel)**.** *Let $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ be a finite-dimensional representation, where all $\rho(x)$ are nilpotent endomorphisms. Then ρ is a nilpotent representation.*

Corollary 1.21. *[39] Let $\mathfrak{g}$ be a finite-dimensional Lie algebra. Then $\mathfrak{g}$ is nilpotent if and only if $ad(x)$ is nilpotent.*

Analogously we can formulate a similar result for solvable Lie algebras:

Theorem 1.22. *[39] Let $\mathfrak{g}$ be a Lie algebra. Then the following holds:*

(i) If $\mathfrak{g}$ is solvable, then so is every subalgebra and homomorphic image.

(ii) Let $\mathfrak{a}$ be an ideal in $\mathfrak{g}$. If $\mathfrak{a}$ and $\mathfrak{g}/\mathfrak{a}$ are solvable, then so is $\mathfrak{g}$.

(iii) If $\mathfrak{a}$ and $\mathfrak{b}$ are solvable ideals in $\mathfrak{g}$, then so is $\mathfrak{a} + \mathfrak{b}$.

Definition 1.23. For a finite-dimensional Lie algebra $\mathfrak{g}$, we call the largest solvable ideal in $\mathfrak{g}$ the *solvable radical of* $\mathfrak{g}$ and we denote it by $\mathrm{rad}(\mathfrak{g})$.

Lemma 1.24. *[15] If $\mathfrak{g}$ is semisimple, then rad($\mathfrak{g}$) $= 0$.*

We conclude this chapter by recalling Lie's theorem [33] and a result on nilpotent derivations.

Theorem 1.25 (Lie)**.** *Let $\mathfrak{g}$ be a solvable Lie algebra over K, where K is algebraically closed and $char(K) = 0$. If $\rho : \mathfrak{g} \to \mathfrak{gl}(V)$ is a representation of $\mathfrak{g}$, then V has a basis for which all $\rho(x)$ for $x \in \mathfrak{g}$ have upper triangular matrix form.*

Lemma 1.26. *[15] Let $\mathfrak{g}$ be a Lie algebra and D a nilpotent derivation. Then e^D is an automorphism on $\mathfrak{g}$.*

1.2 Rota-Baxter operators

Rota-Baxter operators were first mentioned in a paper by G. Baxter [6] in 1960 to study some analytic problem stemming from probability theory. Later on, RB-operators were also used in terms of the classical and the modified Yang-Baxter-equation (CYBE and MYBE), see [62], [64]. Ever since, RB-operators have found their way in various branches of mathematics, reaching from combinatorics to mathematical physics. In this thesis, the aim is to introduce these operators and connect them to existence and classification problems of post-Lie Algebra structures. Furthermore, we study RB-operators in the context of decompositions of algebras.

For this section we restrict ourselves to non-associative algebras over fields with characteristic zero, unless stated otherwise. Let A be an arbitrary non-associative algebra over an arbitrary field K with characteristic zero see [60] for the definition. As a main reference for definitions and lemmas, we used [27], [8], [49], [35] and the references mentioned therein.

1.2.1 Some elementary results on Rota-Baxter operators

Definition 1.27. Let $\lambda \in K$ and $R : A \to A$ be a linear operator. The operator R is called a *Rota-Baxter operator of weight* λ if

$$R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$

for all $x, y \in A$. The algebra A is called the *Rota-Baxter algebra.* We write in short RB-operator for Rota-Baxter operator.

Remark 3. Rota-Baxter operators are also closely related to the operator form of CYBE and MYBE: Using the operator form of the classical Yang-Baxter equation makes sense when dealing with integrable systems. For further reading see [62] and [5].

Examples. (i) $R = 0$ and $R = -\lambda\,\mathrm{id}$ are RB-operators. We call them the trivial RB-operators. For $R = 0$ this is trivial. Now let $R = -\lambda id$, then

$$\begin{aligned}(-\lambda id(x))(-\lambda id(y)) &= \lambda^2 xy \\ &= -\lambda(-\lambda xy) \\ &= \lambda id(-\lambda xy - \lambda xy + \lambda xy) \\ &= -\lambda id(-\lambda id(x)y + x(-\lambda id(y)) + \lambda xy).\end{aligned}$$

(ii) The integration operator $R(f)(x) = \int_0^x f(t)dt$ is an RB-operator on A of weight zero, where A is the algebra of continous functions on $\mathbb{R}$.

$$R(f)(x)R(g)(x) = \int_0^x f(t)dt \int_0^x g(t)dt = \int_0^x R(f)(t)(R(g)(t))'dt + \int_0^x (R(f)(t))'R(g)(t)dt$$

and for the right hand side of the definition of Rota-Baxter operators we get

$$\begin{aligned}R(R(f)g + fR(g))(x) &= \int_0^x (R(f)g + fR(g))(t)dt = \int_0^x (R(f)g)(t)dt + \int_0^x (fR(g))(t)dt \\ &= \int_0^x R(f)(t)g(t)dt + \int_0^x f(t)R(g)(t)dt \\ &= \int_0^x R(f)(t)\left(\frac{d}{dt}\int_0^t g(s)ds\right)dt + \int_0^x \left(\frac{d}{dt}\int_0^t f(s)ds\right)R(g)(t)dt \\ &= \int_0^x R(f)(t)(R(g)(t))'dt + \int_0^x (R(f)(t))'R(g)(t)dt.\end{aligned}$$

(iii) Let d be an invertible derivation of an algebra A. Then d^{-1} is an RB-operator on A of weight zero.
Let $a, b \in A$. Then we have

$$d^{-1}(d(d^{-1}(a)d^{-1}(b))) = d^{-1}(d^{-1}(a)b + ad^{-1}(b)),$$

by definition of a derivation.

Now we prove some elementary lemmas before connecting RB-operators with decompositions of algebras, which will be used in Chapter 3.

Lemma 1.28. *Let R be an RB-operator of weight λ and $\phi \in Aut(A)$. Then $R^{(\phi)} = \phi^{-1}R\phi$ is an RB-operator on A of weight λ.*

Proof. We have to show that $R^{(\phi)}$ is linear and that $R^{(\phi)}$ satisfies the defining equation for RB-operators, namely

$$R^{(\phi)}(x)R^{(\phi)}(y) = R^{(\phi)}(R^{(\phi)}(x)y + xR^{(\phi)}(y) + \lambda xy).$$

Let $x, y \in A$ and $\phi \in Aut(A)$, then

$$\begin{aligned}
R^{(\phi)}(x+y) &= (\phi^{-1}R\phi)(x+y) = \phi^{-1}(R(\phi(x+y))) \\
&= \phi^{-1}(R(\phi(x)+\phi(y))) = \phi^{-1}(R(\phi(x)) + R(\phi(y))) \\
&= \phi^{-1}(R(\phi(x))) + \phi^{-1}(R(\phi(y))) = R^{(\phi)}(x) + R^{(\phi)}(y).
\end{aligned}$$

Let $c \in K$, then

$$R^{(\phi)}(cx) = (\phi^{-1}R\phi)(cx) = \phi^{-1}(R(c\phi(x))) = \phi^{-1}(c(R(\phi(x)))) = c\phi^{-1}(R(\phi(x))) = cR^{(\phi)}(x).$$

Hence, $R^{(\phi)}$ is linear. We still need to show that $R^{(\phi)}$ is indeed an RB-operator:

$$\begin{aligned}
R^{(\phi)}(x)R^{(\phi)}(y) &= (\phi^{-1}R\phi)(x)(\phi^{-1}R\phi)(y) \\
&= \phi^{-1}(R(\phi(x))R(\phi(y))) = \phi^{-1}(R(R(\phi(x))\phi(y) + \phi(x)R(\phi(y)) + \lambda\phi(x)\phi(y))) \\
&= \phi^{-1}(R(R(\phi(x))\phi(y) + \phi(x)R(\phi(y)) + \lambda\phi(xy))) \\
&= (\phi^{-1}R\phi)((\phi^{-1}R\phi)(x)y + x(\phi^{-1}R\phi)(y) + \lambda xy)) \\
&= R^{(\phi)}(R^{(\phi)}(x)y + xR^{(\phi)}(y) + \lambda xy).
\end{aligned}$$

□

Lemma 1.29. *Let $R : A \to A$ be an RB-operator of weight λ. Then $-R - \lambda\,\mathrm{id}$ is also an RB-operator of weight λ.*

Proof. Let $x, y \in A$. Then we have, by plugging in the definition,

$$\begin{aligned}
&(-R - \lambda\,\mathrm{id})((-R - \lambda\,\mathrm{id})(x)y + x(-R - \lambda\,\mathrm{id})(y) + \lambda xy) \\
&= (-R - \lambda\,\mathrm{id})(-R(x)y - \lambda xy - xR(y) - \lambda xy + \lambda xy) \\
&= (-R - \lambda\,\mathrm{id})(-R(x)y - xR(y) - \lambda xy) \\
&= (-R(-R(x)y - xR(y) - \lambda xy)) - \lambda\,\mathrm{id}(-R(x)y - xR(y) - \lambda xy) \\
&= R(R(x)y + xR(y) + \lambda xy) + \lambda R(x)y + \lambda xR(y) + \lambda^2 xy \\
&= R(x)R(y) + \lambda R(x)y + \lambda xR(y) + \lambda^2 xy \\
&= (-R(x) - \lambda x)(-R(y) - \lambda y) \\
&= (-R - \lambda id)(x)(-R - \lambda\,\mathrm{id})(y).
\end{aligned}$$

□

Proposition 1.30. *If R is an RB-operator of weight $\lambda \neq 0$ on A, then $\lambda^{-1}R$ is an RB-operator of weight 1 on A.*

Proof. Let $x, y \in A$. Then we have

$$\begin{aligned}(\lambda^{-1}R)(x)(\lambda^{-1}R)(y) &= (\lambda^{-1})^2 R(x)R(y)\\ &= (\lambda^{-1})^2 R(R(x)y + xR(y) + \lambda xy)\\ &= (\lambda^{-1}R)((\lambda^{-1}R)(x)y + x(\lambda^{-1}R)(y) + xy).\end{aligned}$$

□

1.2.2 Rota-Baxter operators and direct sums

Lemma 1.31. *Let A be an algebra and B be a countable direct sum of A. We define an operator R on B by*

$$R((a_1, a_2, \dots, a_n, \dots)) = (0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots).$$

Then R is an RB-operator on B of weight one for $a_i \in A$, $i = 1, 2, \dots$.

Proof. Let $(a_1, a_2, \dots, a_n, \dots), (b_1, b_2, \dots, b_n, \dots) \in B$, then

$$\begin{aligned}&R((a_1, a_2, \dots, a_n, \dots))R((b_1, b_2, \dots, b_n, \dots))\\ &= (0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots)(0, b_1, b_1 + b_2, b_1 + b_2 + b_3, \dots)\\ &= (0, a_1b_1, (a_1 + a_2)(b_1 + b_2), (a_1 + a_2 + a_3)(b_1 + b_2 + b_3), \dots)\\ &= (0, a_1b_1, a_1b_1 + (a_1 + a_2)(b_2 + a_2b_1, \dots)\\ &= R((a_1b_1, a_1b_2 + a_2b_1 + a_2b_2, (a_1 + a_2)b_3 + a_3(b_1 + b_2) + a_3b_3, \dots))\\ &= R((0, a_1b_2, (a_1 + a_2)b_3, (a_1 + a_2 + a_3)b_4, \dots) + (0, a_2b_1, a_3(b_1 + b_2), a_4(b_1\\ &\quad + b_2 + b_3), \dots) + (a_1b_1, a_2b_2, a_3b_3, \dots))\\ &= R((0, a_1, a_1 + a_2, a_1 + a_2 + a_3, \dots)(b_1, b_2, b_3, \dots) + (a_1, a_2, a_3, \dots)(0, b_1, b_1 + b_2, b_1 + b_2 + b_3, \dots)\\ &\quad + (a_1, a_2, a_3, \dots)(b_1, b_2, b_3, \dots))\\ &= R(R(a_1, a_2, a_3, ...)(b_1, b_2, b_3, \dots) + (a_1, a_2, a_3, \dots)R(b_1, b_2, b_3, \dots) + (a_1, a_2, a_3, \dots)(b_1, b_2, b_3, \dots)).\end{aligned}$$

□

Proposition 1.32. *Let A be an algebra, $B = A \oplus A$ and ϕ an automorphism on A. Let us define an operator R on B by*

$$R((a_1, a_2)) = (0, \phi(a_1)).$$

Then R is an RB-operator on B of weight 1.

Proof. Let $(a_1, a_2), (b_1, b_2) \in B$, then, since ϕ is a homomorphism, we get

$$\begin{aligned} R((a_1, a_2))R((b_1, b_2)) &= (0, \phi(a_1))(0, \phi(b_1)) = (0, \phi(a_1)\phi(b_1)) = (0, \phi(a_1 b_1)) \\ &= R((a_1 b_1, \phi(a_1)b_2 + a_2\phi(b_2) + a_2 b_2)) \\ &= R((0, \phi(a_1)b_2) + (0, a_2\phi(b_2)) + (a_1 b_1, a_2 b_2)) \\ &= R((0, \phi(a_1))(b_1, b_2) + (a_1, a_2)(0, \phi(b_2)) + (a_1, a_2)(b_1, b_2)) \\ &= R(R((a_1, a_2))(b_1, b_2) + (a_1, a_2)R((b_1, b_2)) + (a_1, a_2)(b_1, b_2)). \end{aligned}$$

□

Proposition 1.33. *Let $A = A_1 \oplus A_2$, R_1 an RB-operator of weight λ on A_1 and R_2 an RB-operator of weight λ on A_2. Then the operator R defined on A by*

$$R((a_1, a_2)) = (R_1(a_1), R_2(a_2))$$

is an RB-operator on A of weight λ.

Proof. Let $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, then

$$\begin{aligned} R((a_1, a_2))R((b_1, b_2)) &= (R_1(a_1), R_2(a_2))(R_1(b_1), R_2(b_2)) = (R_1(a_1)R_1(b_1), R_2(a_2)R_2(b_2)) \\ &= (R_1(R_1(a_1)b_1 + a_1R_1(b_1) + \lambda a_1 b_1), R_2(R_2(a_2)b_2 + a_2R_2(b_2) + \lambda a_2 b_2)) \\ &= R((R_1(a_1 b_1 + a_1R_1(b_1) + \lambda a_1 b_1, R_2(a_2)b_2 + a_2R_2(b_2) + \lambda a_2 b_2))) \\ &= R((R_1(a_1), R_2(a_2))(b_1, b_2) + (a_1, a_2)(R_1(b_1), R_2(b_2)) + \lambda(a_1 b_1, a_2 b_2)) \\ &= R(R((a_1, a_2))(b_1, b_2) + (a_1, a_2)R((b_1, b_2)) + \lambda(a_1 b_1, a_2 b_2)). \end{aligned}$$

□

Proposition 1.34. *Let A be an algebra and $A = A_1 \dot{+} A_2$. Then R defined on A by*

$$R(a_1 + a_2) = -\lambda a_2,$$

with $a_1 \in A_1$ and $a_2 \in A_2$, is an RB-operator on A of weight λ.

Proof. Let $a_1, b_1 \in A_1$ and $a_2, b_2 \in A_2$, then

$$\begin{aligned} &R(R(a_1 + a_2)(b_1 + b_2) + (a_1 + a_2)R(b_1 + b_2) + \lambda(a_1 + a_2)(b_1 + b_2)) \\ &= R(-\lambda a_2(b_1 + b_2) + (a_1 + a_2)(-\lambda b_2) + \lambda(a_1 + a_2)(b_1 + b_2)) \\ &= R(-\lambda a_2(b_1 + b_2) + (a_1 + a_2)(-\lambda b_2 + \lambda b_1 + \lambda b_2)) \\ &= R(-\lambda a_2(b_1 + b_2) + \lambda(a_1 + a_2)b_1) = R(-\lambda a_2 b_1 - \lambda a_2 b_2 + \lambda a_1 b_1 + \lambda a_2 b_1) \\ &= R(\lambda a_1 b_1 - \lambda a_2 b_2) = -\lambda(-\lambda a_2 b_2) = \lambda^2 a_2 b_2 = -\lambda a_2(-\lambda b_2) = R(a_1 + a_2)R(b_1 + b_2). \end{aligned}$$

□

Remark 4. Here is the first time we can link RB-operators to decompositions of algebras as a direct sum into two subalgebras. Because the operators mentioned above, which we call *split*, are in bijective correspondence to decompositions of algebras, cf. [8]

Lemma 1.35. *Let R be an RB-operator of weight $\lambda \neq 0$ on an algebra A. Then R is split if and only if $R(R + \lambda\,\mathrm{id}) = 0$.*

Proof. Let $A = A_1 \dot{+} A_2$ be the direct vector space sum of two subalgebras and $R(a_1 + a_2) = -\lambda a_2$ for $a_1 \in A_1$ and $a_2 \in A_2$. Then we have that

$$(R(R+\lambda id))(a_1+a_2) = R(R(a_1+a_2)+\lambda(a_1+a_2)) = R(-\lambda a_2+\lambda a_1+\lambda a_2) = R(\lambda a_1) = 0.$$

For the other direction we want to show that $A = \ker(R)\dot{+}R(A)$. Hence, we need to show that the intersection of both is empty. Let $x \in \ker(R) \cap R(A)$, then $x = R(y)$ for some $y \in A$ and $R(x) = R(R(y)) = 0$, since x lies in the kernel of R. Using the assumption we get that $x = R(y) = -\frac{1}{\lambda}R^2(y) = 0$. Thus the intersection is empty. Furthermore, we have $R(\ker(R)) = 0$ and $R(R(A)) = -\lambda\,\mathrm{id}(R(A))$, implying that R is split. $\square$

Proposition 1.36. *Let R be an RB-operator of weight λ. If $\lambda \neq 0$ and $R(1) \in K$, then R is splitting.*

Proof. By definition of an RB-operator, we have

$$R(1)R(1) = R(R(1)1+1R(1)+\lambda) = R(2R(1)+\lambda) = 2(R(1))^2+R(\lambda) = 2R(1)^2+\lambda R(1).$$

Reformulating this, we get $R(1)(R(1) + \lambda) = 0$, so $R(1) \in \{0, -\lambda\}$. If $R(1) = -\lambda$, then we can conclude that $(-R - \lambda\,\mathrm{id})(1) = 0$. Using Lemma 1.27, we obtain

$$\begin{aligned} 0 &= (-R - \lambda id)(1)(-R - \lambda\,\mathrm{id})(x) = (-R - \lambda\,\mathrm{id})((-R - \lambda\,\mathrm{id})(1)x + 1(-R - \lambda\,\mathrm{id})(x) + \lambda x) \\ &= (-R - \lambda\,\mathrm{id})((-R - \lambda\,\mathrm{id})(x) + \lambda x) = (-R - \lambda\,\mathrm{id})(-R(x)) = R(R(x)) + \lambda R(x). \end{aligned}$$

By Lemma 1.33, this implies that R is split.
Now if $R(1) = 0$, then

$$0 = R(1)R(x) = R(R(1)x + 1R(x) + \lambda x) = R(R(x) + \lambda x),$$

and it follows again from Lemma 1.33 that R is split.

$\square$

Lemma 1.37. *Let $A = A_-\dot{+}A_0\dot{+}A_+$. Assume that R is an RB-operator of weight λ on A_0, A_- is an $(R + \mathrm{id})(A_0)$-module and A_+ is an $R(A_0)$-module. Then the operator P defined on A by*

$$P\,|_{A_-} = 0, \quad P\,|_{A_0} = R, \quad P\,|_{A_+} = -\lambda\,\mathrm{id}$$

is an RB-operator on A of weight λ.

Proof. To prove this, we make a case distinction between three cases.

1. Let $x \in A_-$. If $y \in A_-$, then $P(x)P(y) = 0$ and

$$P(P(x)y + xP(y) + \lambda xy) = \lambda P(xy) = 0.$$

If $y \in A_0$, then $P(x)P(y) = 0$ and

$$\begin{aligned}P(P(x)y + xP(y) + \lambda xy) &= P(xR(y) + \lambda xy) \\ &= P(x(R(y) + \lambda \operatorname{id}(y))) = P(x(R(y) + id(\lambda y))) = 0.\end{aligned}$$

If $y \in A_+$, then $P(x)P(y) = 0$ and we have

$$P(P(x)y + xP(y) + \lambda xy) = P(x(-\lambda \operatorname{id}(y) + \lambda xy)) = P(0) = 0.$$

2. Let $x \in A_0$. If $y \in A_-$, then $P(x)P(y) = 0$ and

$$P(P(x)y + xP(y) + \lambda xy) = P(R(x)y + \lambda xy) = P((R(x) + \operatorname{id}(\lambda x))y) = 0.$$

If $y \in A_0$, then

$$P(x)P(y) = R(x)R(y) = R(R(x)y + xR(y) + \lambda xy)$$

and

$$P(P(x)y + xP(y) + \lambda xy) = P(R(x)y + xR(y) + \lambda xy) = R(R(x)y + xR(y) + \lambda xy).$$

If $y \in A_+$, then

$$P(x)P(y) = R(x)(-\lambda \operatorname{id})(y) = R(x)(-\lambda y) = -\lambda R(x)y$$

and

$$\begin{aligned}P(P(x)y + xP(y) + \lambda xy) &= P(R(x)y + x(-\lambda \operatorname{id}(y)) + \lambda xy) \\ &= P(R(x)y) = -\lambda \operatorname{id}(R(x)y) = -\lambda R(x)y.\end{aligned}$$

3. Let $x \in A_+$. If $y \in A_-$, then

$$P(x)P(y) = -\lambda \operatorname{id}(x)0 = 0$$

and

$$P(P(x)y + xP(y) + \lambda xy) = P(-\lambda \operatorname{id}(x)y + \lambda xy) = 0.$$

If $y \in A_0$, then

$$P(x)P(y) = -\lambda \operatorname{id}(x)R(y) = -\lambda xR(y)$$

and

$$\begin{aligned}P(P(x)y + xP(y) + \lambda xy) &= P(-\lambda \operatorname{id}(x)y + xR(y) + \lambda xy) = P(-\lambda xy + xR(y) + \lambda xy) \\ &= P(xR(y)) = -\lambda \operatorname{id}(xR(y)) = -\lambda xR(y).\end{aligned}$$

If $y \in A_+$, then

$$P(x)P(y) = -\lambda \operatorname{id}(x)(-\lambda \operatorname{id}(y)) = \lambda^2 xy$$

and

$$\begin{aligned}P(P(x)y + xP(y) + \lambda xy) &= P(-\lambda \operatorname{id}(x)y + x(-\lambda \operatorname{id}(y)) + \lambda xy) \\ &= P(-\lambda xy) = -\lambda P(xy) = -\lambda(-\lambda \operatorname{id}(xy)) = \lambda^2 xy.\end{aligned}$$

□

We end this chapter with an example on quasi-idempotent elements, that have been studied in [45]. The aim in [45] is to construct non-trivial Rota-Baxter algebras and connect them to Hopf algebras, by looking at quasi-idempotent elements .

Example 4. Now we assume that A is an associative algebra and $e \in A$ a quasi-idempotent element, i.e. $e^2 = -\lambda e$, where $\lambda \in K$. Then

$$R_e(x) = ex$$

is an RB-operator of weight λ on A. Let $x, y \in A$ and $c \in K$, then

$$R_e(x+y) = e(x+y) = ex + ey = R_e(x) + R_e(y) \, \text{and} \, R_e(cx) = cex = cR_e(x).$$

For the defining equation, we get

$$\begin{aligned} R_e(R_e(x)y + xR_e(y) + \lambda xy) &= R_e((ex)y + x(ey) + \lambda xy) = e(e(xy)) + e(x(ey)) + e(\lambda xy) \\ &= e^2(xy) + (ex)(ey) + \lambda exy = -\lambda exy + (ex)(ey) + \lambda exy \\ &= (ex)(ey) = R_e(x)R_e(y). \end{aligned}$$

Furthermore, we have

$$(R_e^2 + \lambda R_e)(x) = R_e(R_e(x)) + \lambda R_e(x) = R_e(ex) + \lambda ex = e^2 x + \lambda ex = -\lambda ex + \lambda ex = 0,$$

and thus R satisfies $R(R + \lambda \, \mathrm{id}) = 0$. By Lemma 1.33, this means if $\lambda \neq 0$, then R is split with subalgebras $A_1 = (1-e)A$ and $A_2 = eA$.

2 Post-Lie algebra structures

Post-Lie algebras were first mentioned in 2007 by Vallette [63] in connection to homology of generalized partition posets. They have also been studied in the context of differential geometry and geometric structures on Lie groups as a generalization of pre-Lie algebras and LR-algebras , see [37], [61], [47], [12], [17] and [20], [23].Also in terms of nil-affine actions of Lie groups they have been investigated, see [22]. Of interest for us are classification and existence problems of post-Lie algebras. Especially for commutative post-Lie algebra structures classification results have been found, see [19], [21], [28].

We will start this chapter with some general definitions, following a study on pairs of semisimple and pairs of nilpotent Lie algebras. For the sake of simplicity, we restrict ourselves to finite-dimensional Lie algebras over fields of characteristic zero, unless stated otherwise. Our main reference for this Chapter is [27]and [22], and the references therein.

2.1 Basic definitions

Let us start with the definition of a post-Lie algebra.

Definition 2.1. Let V be a vector space over a field K together with two bilinear operations $[,]$ and $\cdot$. V is called a *post-Lie algebra*, if the following holds for all $x, y, z \in V$:

$$\begin{aligned}
[x, y] &= -[y, x], \\
[x[x, z]] &= -[y, [z, x]] - [z, [x, y]], \\
[y, x] \cdot z &= (x \cdot y) \cdot z - x \cdot (y \cdot z) - (y \cdot x) \cdot z + y \cdot (x \cdot z), \\
x \cdot [y, z] &= [x \cdot y, z] + [y, x \cdot z].
\end{aligned}$$

Note that $[,]$ is a Lie bracket. Furthermore, we have another Lie bracket on V defined by

$$\{x, y\} = x \cdot y - y \cdot x + [x, y].$$

A simple reformulation of this definition brings us to the definition of a post-Lie algebra structure, which was defined in the context of geometric structures on Lie groups in 2012 in [22].

Definition 2.2. Let $\mathfrak{g} = (V, [,])$ and $\mathfrak{n} = (V, \{,\})$ be two Lie brackets on a vector space V over a field K. A *post-Lie algebra structure*, or short PA-structure, on the pair $(\mathfrak{g}, \mathfrak{n})$ is a K-bilinear product $x \cdot y$ satisfying

(i) $x \cdot y - y \cdot x = [x, y] - \{x, y\}$,

(ii) $[x, y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z)$,

(iii) $x \cdot \{y, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}$.

Remark 5. If we define by $L(x)(y) = x \cdot y$ the left multiplication operator of the algebra $A = (V, \cdot)$, then we have

$$L(x)(\{y,z\}) = x \cdot \{y,z\} = \{x \cdot y, z\} + \{y, x \cdot z\} = \{L(x)(y), z\} + \{y, L(x)(z)\},$$

and hence all $L(x)$ are derivations of $\mathfrak{n} = (V, \{,\})$. Moreover, we have

$$\begin{aligned} L([x,y])(z) = [x,y] \cdot z &= x \cdot (y \cdot z) - y \cdot (x \cdot z) \\ &= L(x)(L(y)(z)) - L(y)(L(x)(z)) \\ &= (L(x)L(y) - L(y)L(x))(z), \end{aligned}$$

from which we can conclude that

$$\begin{aligned} L : &\mathfrak{g} \to Der(\mathfrak{n}) \subseteq End(V) \\ &x \mapsto L(x) \end{aligned}$$

is a representation of $\mathfrak{g}$. This need not be the case for the right multiplication operator $R(x)(y) = y \cdot x$. Here the map $R : V \to V$, $x \mapsto R(x)$ is a linear map, but is not always a representation. For example, let $\mathfrak{g} = \mathfrak{n}_3(K)$ be the Heisenberg Lie algebra of dimension 3 over a field K with $[e_1, e_2] = e_3$, and let $\mathfrak{n} \cong \mathfrak{g}$ of type C, cf. [25]. Then by the classification results in [25], we have that

$$e_2 \cdot [e_1, e_2] = e_2 \cdot e_3 = -\frac{r_7}{2} e_1,$$

but we have

$$(e_2 \cdot e_2) \cdot e_1 - (e_2 \cdot e_1) \cdot e_2 = 0.$$

Therefore R is not a representation. For commutative post-Lie algebras both operators are representations, since $L(x) = R(x)$, see [25].

Lemma 2.3. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then we have:*

(i) $x \cdot \{y,z\} + y \cdot \{z,x\} + z \cdot \{x,y\} = \{[x,y],z\} + \{[y,z],x\} + \{[z,x],y\}$,

(ii) $$\begin{aligned} \{x,y\} \cdot z + \{y,z\} \cdot x + \{z,x\} \cdot y = &\{[x,y],z\} + \{[y,z],x\} + \{[z,x],y\} \\ &+ [\{x,y\},z] + [\{y,z\},x] + [\{z,x\},y], \end{aligned}$$

for all $x, y, z \in V$.

Proof. Let $x, y, z \in V$. Then using bilinearity and anti-symmetry of the Lie bracket $\{,\}$, the Jacoby-identity for $\{,\}$, and Condition (i) and (iii) of Definition 2.2, we see that

$$\begin{aligned}
&x\cdot\{y,z\}+y\cdot\{z,x\}+z\cdot\{x,y\}\\
&=\{x\cdot y,z\}+\{y,x\cdot z\}+\{y\cdot z,x\}+\{z,y\cdot x\}+\{z\cdot x,y\}+\{x,z\cdot y\}\\
&=\{[x,y]-\{x,y\}+y\cdot x,z\}+\{y,[x,z]-\{x,z\}+z\cdot x\}\\
&\quad+\{[y,z]-\{y,z\}+z\cdot y,x\}+\{z,[y,x]-\{y,x\}+x\cdot y\}\\
&\quad+\{[z,x]-\{z,x\}+x\cdot z,y\}+\{x,[z,y]-\{z,y\}+y\cdot z\}\\
&=\{[x,y],z\}+\{y\cdot x,z\}+\{y,[x,z]\}+\{y,z\cdot x\}\\
&\quad+\{[y,z],x\}+\{z\cdot y,x\}+\{z,[y,x]\}+\{z,x\cdot y\}\\
&\quad+\{[z,x],y\}+\{x\cdot z,y\}+\{x,[z,y]\}+\{x,y\cdot z\}\\
&=\{[x,y],z\}+\{y\cdot x,z\}+\{y,[x,z]+z\cdot x\}\\
&\quad+\{[y,z],x\}+\{z\cdot y,x\}+\{z,[y,x]+x\cdot y\}\\
&\quad+\{[z,x],y\}+\{x\cdot z,y\}+\{x,[z,y]+y\cdot z\}\\
&=\{[x,y],z\}+\{y\cdot x,z\}+\{y,x\cdot z\}+\{y,\{x,z\}\}\\
&\quad+\{[y,z],x\}+\{z\cdot y,x\}+\{z,y\cdot x\}+\{z,\{y,x\}\}\\
&\quad+\{[z,x],y\}+\{x\cdot z,y\}+\{x,z\cdot y\}+\{x,\{z,y\}\}\\
&=\{[x,y],z\}+\{[y,z],x\}+\{[z,x],y\}
\end{aligned}$$

The second assertion, follows from Lemma 2.3 (i), condition (i) from Definition 2.2 and the Jacobi-identity for $\mathfrak{n}=\{,\}$. □

Remark 6. If $\mathfrak{n}$ is abelian, that is $\{x,y\}=0$ for all $x,y\in V$, then a PA-structure on $(\mathfrak{g},\mathfrak{n})$ becomes a pre-Lie algebra strucutre on $\mathfrak{g}$. The conditions from Definition 2.2 are then,

$$\begin{aligned}
x\cdot y-y\cdot x&=[x,y]\\
[x,y]\cdot z&=x\cdot(y\cdot z)-y\cdot(x\cdot z),
\end{aligned}$$

which is the definition of a pre-Lie algebra, see [17].

2.1.1 Inner post-Lie algebra structures

We now connect Rota-Baxter operators to post-Lie algebra structures by introducing inner PA-structures. Here we have a strong statement on Lie algebras with trivial center, that will be stated in Proposition 2.6.

Definition 2.4. Let $x\cdot y$ be a PA-structure on $(\mathfrak{g},\mathfrak{n})$. Suppose there exists a $\phi\in End(V)$ such that

$$x\cdot y=\{\phi(x),y\},$$

for all $x,y\in V$, then $x\cdot y$ is called an *inner PA-structure* on $(\mathfrak{g},\mathfrak{n})$.

Proposition 2.5. *Let* $(\mathfrak{n}, \{,\})$ *be a Lie algebra and* R *a Rota-Baxter operator of weight 1 on* $\mathfrak{n}$*, that is for all* $x, y \in V$

$$\{R(x), R(y)\} = R(\{R(x), y\} + \{x, R(y)\} + \{x, y\}).$$

Then

$$x \cdot y = \{R(x), y\}$$

defines an inner PA-structure on $(\mathfrak{g}, \mathfrak{n})$*, where* $\mathfrak{g} = (V, [,])$ *with* $[x, y] = \{R(x), y\} - \{R(y), x\} + \{x, y\}$*.*

Proof. The above defined bracket $[,]$ is in fact a Lie bracket by Proposition 2.2 in [22] and $x \cdot y = \{R(x), y\}$ is a PA-structure, since

(i) $x \cdot y - y \cdot x = [x, y] - \{x, y\}$ by definition of $[,]$,

(ii) $$\begin{aligned}[x, y] \cdot z &= \{R([x, y]), z\} = \{R(\{R(x), y\} - \{R(y), x\} + \{x, y\}), z\} \\ &= \{\{R(x), R(y)\}, z\} = \{R(x), \{R(y), z\}\} - \{R(y), \{R(x), z\}\} \\ &= x \cdot (y \cdot z) - y \cdot (x \cdot z)\end{aligned}$$
since R is an RB-operator on $\mathfrak{n}$ and by the Jacobi-identity on $\mathfrak{n}$,

(iii) condition (iii) of Definition 2.2 follows immediately from the Jacobi-identity for $\mathfrak{n}$.

In particular, it is inner because R is an endomorphism. □

We now show that if an inner PA-structure on a pair of Lie algebras, where $\mathfrak{n}$ has trivial center, exists, then it automatically arises from an RB-operator of weight one. This will help us for results on PA-structures with semisimple Lie algebras, as we will see in Chapter 2.2.

Proposition 2.6. *Let* $\mathfrak{n}$ *be a Lie algebra with trivial center. Then any inner PA-structure on* $(\mathfrak{g}, \mathfrak{n})$ *comes from an RB-operator of weight 1. Moreover, if* $\mathfrak{n}$ *is complete, i.e. we have in addition* $\mathrm{Der}(\mathfrak{n}) = \mathrm{ad}(\mathfrak{n})$*, then every PA-structure on* $(\mathfrak{g}, \mathfrak{n})$ *is inner.*

In order to prove this statement, we introduce two lemmas, prove them and then the argument follows immediately.

Lemma 2.7. *Let* $(\mathfrak{g}, \mathfrak{n})$ *be a pair of Lie algebras such that* $Z(\mathfrak{n}) = 0$*. Then we have for* $\phi \in End(V)$ *that* $x \cdot y = \{\phi(x), y\}$ *is a PA-structure on* $(\mathfrak{g}, \mathfrak{n})$ *if and only if*

$$\begin{aligned}\{\phi(x), y\} + \{x, \phi(y)\} &= [x, y] - \{x, y\}, \\ \phi([x, y]) &= \{\phi(x), \phi(y)\},\end{aligned}$$

for all $x, y \in V$*.*

Proof. Let us assume $x \cdot y = \{\phi(x), y\}$ is a PA-structure, then using condition (i) from Definition 2.2, we get $\{\phi(x), y\} + \{x, \phi(y)\} = [x, y] - \{x, y\}$. To show that ϕ is a Lie algebra homomorphism from $\mathfrak{g}$ to $\mathfrak{n}$, we use that $Z(\mathfrak{n}) = 0$, condition (ii) from Definition 2.2 and the Jacobi-identity for $\mathfrak{n}$:

$$\begin{aligned}\{\phi([x,y]), z\} &= \{\phi(x), \{\phi(y), z\}\} - \{\phi(y), \{\phi(x), z\}\} \\ &= \{\phi(x), \{\phi(y), z\}\} + \{\phi(y), \{z, \phi(x)\}\} = \{\{\phi(x), \phi(y)\}, z\},\end{aligned}$$

from which we conclude that $\phi([x, y]) = \{\phi(x), \phi(y)\}$.
For the other direction we simply need to show the conditions for a PA-structure, using our assumptions:

(i) $x \cdot y - y \cdot x = \{\phi(x), y\} - \{\phi(y), x\} = \{\phi(x), y\} + \{x, \phi(y)\} = [x, y] - \{x, y\}$,

(ii) $[x, y] \cdot z = \{\phi([x, y]), z\} = \{\{\phi(x), \phi(y)\}, z\} = \{\phi(x), \{\phi(y), z\}\} + \{\phi(y), \{z, \phi(x)\}\} = \{\phi(x), \{\phi(y), z\}\} - \{\phi(y), \{\phi(x), z\}\} = x \cdot (y \cdot z) - y \cdot (x \cdot z)$,

(iii) $x \cdot \{y, z\} = \{\phi(x), \{y, z\}\} = -\{y, \{z, \phi(x)\}\} - \{z, \{\phi(x), y\}\} = \{y, \{\phi(x), z\}\} + \{\{\phi(x), y\}, z\} = \{x \cdot y, z\} + \{y, x \cdot z\}$.

□

Lemma 2.8. *Assume we have a PA-structure $x \cdot y$ on a pair of Lie algebras $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{n}$ is complete, i.e. $Der(\mathfrak{n}) = ad(\mathfrak{n})$ and $Z(\mathfrak{n}) = 0$. Then there exists a unique linear map ϕ on V, such that $L(x) = ad(\phi(x))$.*

Proof. By Remark 5, we know that all $L(x)$ are derivations of $\mathfrak{n}$, and thus $L(x) \in Der(\mathfrak{n}) = ad(\mathfrak{n})$. Let $\phi(x), m \in \mathfrak{n}$ such that $L(x) = ad(\phi(x)) = ad(m)$, then we have that $ad(m)(y) - ad(\phi(x))(y) = \{m - \phi(x), y\} = 0$. Since the center of $\mathfrak{n}$ is trivial, we obtain $\phi(x) = m$, which determines our unique map. We still want to show that ϕ is linear. Obviously,

$$\begin{aligned}\{\phi(x+y), z\} &= (x+y) \cdot z = x \cdot z + y \cdot z = \{\phi(x) + \phi(y), z\}, \\ \{\phi(\lambda x), z\} &= (\lambda x) \cdot z = \lambda(x \cdot z) = \lambda\{\phi(x), z\} = \{\lambda\phi(x), z\},\end{aligned}$$

holds for all $x, y, z \in V$ and $\lambda \in K$. Moreover, $Z(\mathfrak{n}) = 0$ implies $\phi(x + y) = \phi(x) + \phi(y)$ and $\phi(\lambda x) = \lambda\phi(x)$, which proves that ϕ is linear. □

Combining Lemma 2.7 and Lemma 2.8, Proposition 2.6 follows immediately. Note that the assumption, that the center of $\mathfrak{n}$ is trivial, is necessary for Proposition 2.6. We provide a counterexample using a nilpotent Lie algebra, which cannot have a trivial center by Theorem 1.17.

Example 5. Let (e_1, e_2, e_3) be a basis of V, $\mathfrak{g}$ be the 3-dimensional Heisenberg Lie algebra $\mathfrak{n}_3(K)$ and $\mathfrak{n}$ a Lie algebra with $\mathfrak{n} = \mathfrak{g}$, i.e. $\{e_1, e_2\} = e_3$. Let us find an inner

PA-structure $x \cdot y = \{\phi(x), y\}$, where ϕ need not be an RB-operator. Let $\alpha_i \in K$ with $i = 1, \dots, 9$ and

$$\begin{aligned}\phi(e_1) &= \alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3,\\ \phi(e_2) &= \alpha_4 e_1 + \alpha_5 e_2 + \alpha_6 e_3,\\ \phi(e_3) &= \alpha_7 e_1 + \alpha_8 e_2 + \alpha_9 e_3.\end{aligned}$$

- For condition (i) of Definition 2.2, we need

$$\begin{aligned}\{\phi(e_1), e_2\} - \{\phi(e_2), e_1\} &= 0,\\ \alpha_1 e_3 + \alpha_5 e_3 &= 0.\end{aligned}$$

  This means we have $\alpha_1 + \alpha_5 = 0$. Similiarly, we get $\alpha_7 = \alpha_8 = 0$.

- Condition (ii) and condition (iii) of Defintion 2.2 yield no new information on the coefficients, thus

$$\phi = \begin{pmatrix} \alpha_1 & \alpha_4 & 0 \\ \alpha_2 & -\alpha_1 & 0 \\ \alpha_3 & \alpha_6 & \alpha_9 \end{pmatrix}.$$

We want to examine when ϕ is an RB-operator. Let $x, y \in V$, then for ϕ to be an RB-operator, we need

$$\{\phi(x), \phi(y)\} = \phi(\{\phi(x), y\} + \{x, \phi(y)\} + \{x, y\}).$$

We obtain for e_1 and e_2

$$\begin{aligned}\{\phi(e_1), \phi(e_2)\} &= \{\alpha_1 e_1 + \alpha_2 e_2 + \alpha_3 e_3, \alpha_4 e_1 - \alpha_1 e_2 + \alpha_6\}\\ &= (-\alpha_1^2 - \alpha_2\alpha_4)e_3,\\ \phi(\{\phi(e_1), e_2\} + \{e_1, \phi(e_2)\} + e_3) &= \phi(e_3) = \alpha_9 e_3.\end{aligned}$$

This implies that ϕ is an RB-operator if and only if $\alpha_9 = -\alpha_1^2 - \alpha_2\alpha_4$.
Note that

$$L(e_1) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ -\alpha_2 & \alpha_1 & 0 \end{pmatrix}, \qquad L(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \\ \alpha_1 & \alpha_4 & 0 \end{pmatrix},$$

which coincides with the classification results in [25].

Hence, using Proposition 2.5 and 2.6, we can formulate the following statement.

Corollary 2.9. *Let $\mathfrak{n}$ be a complete Lie algebra. Then PA-structures on $(\mathfrak{g}, \mathfrak{n})$ are in bijective correspondence to RB-operators on $\mathfrak{n}$ of weight 1.*

Proposition 2.10. *Let $x \cdot y = \{R(x), y\}$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, where R is an RB-operator on $\mathfrak{n}$ of weight 1. Then R is also an RB-operator of weight 1 on $\mathfrak{g}$.*

Proof. By Proposition 2.5, we have that $\{R(x), R(y)\} = R([x, y])$ and $\{R(x), y\} + \{x, R(y)\} + \{x, y\} = [x, y]$. Therefore, we obtain

$$\begin{aligned}[R(x), R(y)] &= \{R(R(x)), R(y)\} + \{R(x), R(R(y))\} + \{R(x), R(y)\} \\ &= R([R(x), y]) + R([x, R(y)]) + R([x, y]) \\ &= R([R(x), y] + [x, R(y)] + [x, y]).\end{aligned}$$

□

Proposition 2.11. *Let $x \cdot y = \{R(x), y\}$ be an inner PA-structure on $(\mathfrak{g}, \mathfrak{n})$, where R is an RB-operator on $\mathfrak{n}$ of weight 1. Then we have*

(i) If $\mathfrak{g} \not\cong \mathfrak{n}$, then $\ker(R) \neq 0$ and $\ker(R + \mathrm{id}) \neq 0$.

(ii) If either $\mathfrak{g}$ or $\mathfrak{n}$ is not solvable, then either $\ker(R) \neq 0$ or $\ker(R + \mathrm{id}) \neq 0$.

Proof. (i) Suppose that $\ker(R) = 0$, then we have $dim(V) = dim(im(R))$. This implies that $im(R) = V$, which means R is an isomorphism, i.e. $\mathfrak{g} \cong \mathfrak{n}$, which is a contradiction to our assumption. For $R + \mathrm{id}$ the argument follows analogously.

(ii) Let us assume $\ker(R) = \ker(R + \mathrm{id}) = 0$, then R^{-1} exists and $\mathfrak{g} \cong \mathfrak{n}$. Since $\mathfrak{n}$ is not solvable, every automorphism of $\mathfrak{n}$ has a non-zero fixed point, see [42]. Thus, we get for $(R + \mathrm{id}) \circ R^{-1}$ a fixed point $x \in \mathfrak{n}$, $x \neq 0$, and hence

$$0 = ((R + \mathrm{id}) \circ R^{-1})(x) - x = x + R^{-1}(x) - x = R^{-1}(x).$$

But since $\ker(R) = 0$, we have that $\ker(R^{-1}) = 0$ implying $x = 0$, contradicting our assumption.

□

Corollary 2.12. *Let $\mathfrak{n}$ be simple and R an invertible RB-operator of weight $\lambda \neq 0$ on $\mathfrak{n}$. Then $R = -\lambda\,\mathrm{id}$.*

Proof. Let $P = \lambda^{-1}R$, then P is an RB-operator of weight 1 on $\mathfrak{n}$ by Proposition 1.30. By Proposition 2.5 we get an inner PA-structure on $(\mathfrak{g}, \mathfrak{n})$ arising from P. Since $\mathfrak{n}$ is simple, we conclude that $\mathfrak{n}$ is nonsolvable. Applying Proposition 2.11, we have that $\ker(P + \mathrm{id}) \neq 0$, since $\ker(P) = 0$ by assumption. We know that $\ker(P + \mathrm{id})$ is a non-zero ideal in $\mathfrak{n}$. Because $\mathfrak{n}$ is simple, this means $P + \mathrm{id} = 0$, which then again implies $R = -\lambda\,\mathrm{id}$. □

2.2 Post-Lie algebra structures on pairs of semisimple Lie algebras

In this section, we study PA-structures on $(\mathfrak{g}, \mathfrak{n})$, where we restrict ourselves to the cases of semisimplicity and simplicity. The algebras in this section are finite-dimensional and $K = \mathbb{C}$, unless stated otherwise.

Theorem 2.13. *Let $\mathfrak{n}$ be simple, $\mathfrak{g}$ semisimple and $x \cdot y$ a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then $\mathfrak{g}$ is also simple and $\mathfrak{g} \cong \mathfrak{n}$.*

Proof. Since $\mathfrak{n}$ is simple, we have that $\mathfrak{n}$ is complete. Using Proposition 2.6 we know that every PA-structure arises from an RB-operator R of weight 1 on $\mathfrak{n}$, so $x \cdot y = \{R(x), y\}$. Let us assume that $\mathfrak{g} \ncong \mathfrak{n}$. Then by Proposition 2.11 (i), we have that $ker(R) \neq 0$ and $ker(R + id) \neq 0$ and both are ideals in $\mathfrak{g}$ with $ker(R) \cap ker(R + id) = 0$. Since $\mathfrak{g}$ is semisimple, there exists a complementary ideal $\mathfrak{s}$ in $\mathfrak{g}$ such that

$$\mathfrak{g} = ker(R) \oplus ker(R + id) \oplus \mathfrak{s},$$

where $\mathfrak{s}$ is semisimple. Let $x \in \mathfrak{n}$, then we can write $x = R(-x) + (R + id)(x)$. Hence $\mathfrak{n} = im(R) + im(R + id)$ and using the isomorphism theorems, we get

$$im(R) \cong \mathfrak{g}/_{ker(R)} \cong ker(R + id) \oplus \mathfrak{s},$$

$$im(R + id) \cong \mathfrak{g}/_{ker(R + id)} \cong ker(R) \oplus \mathfrak{s}.$$

This means we can decompose $\mathfrak{n}$ into two semisimple parts. We have then the semisimple decomposition

$$\mathfrak{n} = (ker(R + id) \oplus \mathfrak{s}) + (ker(R) \oplus \mathfrak{s}).$$

Assume that $\mathfrak{s} \neq 0$, then both $ker(R + id) \oplus \mathfrak{s}$ and $ker(R) \oplus \mathfrak{s}$ are not simple. By Theorem 4.2 in [54] a semisimple decomposition of a simple Lie algebra has to have at least one simple summand, therefore we have a contradiction and $\mathfrak{s} = 0$. Thus we have a semisimple decomposition

$$\mathfrak{n} = im(R) \dot{+} im(R + id),$$

which is direct as a vector space sum since $ker(R) \cap ker(R + id) = 0$. In particular, it is reductive. Using a result by Koszul, see [48], on reductive decompositions, we obtain

$$\mathfrak{n} = im(R) \oplus im(R + id).$$

But this is a contradiction to $\mathfrak{n}$ being simple, hence $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic and $\mathfrak{g}$ is simple. □

For PA-structures on $(\mathfrak{g}, \mathfrak{n})$, where both Lie algebras are simple, this is much simpler, as the following statement shows. In order to prove the case for simple Lie algebras, we show correspondences between PA-structures and semidirect products, see also [22].

Proposition 2.14. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then*

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{n} \rtimes Der(\mathfrak{n}) \\ x &\mapsto (x, L(x)) \end{aligned}$$

is an injective homomorphism of Lie algebras. Conversely, if we have such an injective homomorphism with the identity map on the first factor, then we obtain a PA-structure on $(\mathfrak{g}, \mathfrak{n})$.

Note that the Lie bracket on $\mathfrak{n} \rtimes Der(\mathfrak{n})$ is given by

$$[(x,D),(y,D')] = (\{x,y\} + D(y) - D'(x), [D,D']).$$

Proof. Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, then we have

$$\begin{aligned}[\phi(x),\phi(y)] = [(x,L(x)),(y,L(y))] &= (\{x,y\} + L(x)(y) - L(y)(x), [L(x),L(y)]) \\ &= ([x,y],[L(x),L(y)]) = ([x,y],L([x,y])) = \phi([x,y]),\end{aligned}$$

by Remark 5.

For the other direction, let $\phi(x) = (x, L(x))$ be an injective homomorphism, where L(x) is a derivation. Now we define $x \cdot y = L(x)y$, then $x \cdot y$ is a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, because

(i) $([x,y],L([x,y])) = \phi([x,y]) = [\phi(x),\phi(y)] = [(x,L(x)),(y,(L(y)))] = (\{x,y\} + L(x)(y) - L(y)(x), [L(x),L(y)])$. This imples

$$x \cdot y - y \cdot x = [x,y] - \{x,y\}$$

and

$$[x,y] \cdot z = x \cdot (y \cdot z) - y \cdot (x \cdot z).$$

(ii) Since L(x) is a derivation, we have

$$x \cdot \{y,z\} = L(x)(\{y,z\}) = \{L(x)y,z\} + \{y,L(x)z\} = \{x \cdot y, z\} + \{y, x \cdot z\}.$$

□

Proposition 2.15. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then $x \cdot y$ is in one-to-one-correspondence with subalgebras $\mathfrak{h}$ of $\mathfrak{n} \rtimes Der(\mathfrak{n})$ for which the projection map*

$$\begin{aligned} p_1 : \mathfrak{n} \rtimes Der(\mathfrak{n}) &\to \mathfrak{n} \\ (x,D) &\mapsto x \end{aligned}$$

induces a Lie algebra isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$.

Proof. Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then by Proposition 2.14, the embedding

$$\begin{aligned} \phi : \mathfrak{g} &\to \mathfrak{n} \rtimes Der(\mathfrak{n}) \\ x &\mapsto (x, L(x)) \end{aligned}$$

is an injective homomorphism. Hence, the Lie subalgebra $\mathfrak{h} = im(\phi) = \{(x,L(x)) \mid x \in \mathfrak{g}\}$ of $\mathfrak{n} \rtimes Der(\mathfrak{n})$ is the subalgebra corresponding to $\mathfrak{g}$ and since $\mathfrak{h}$ is the image of ϕ, we get an isomorphism from $\mathfrak{g}$ onto $\mathfrak{h}$.
Now let $\mathfrak{h}$ be a subalgebra of $\mathfrak{n} \rtimes Der(\mathfrak{n})$ with $p_1|_{\mathfrak{h}} : \mathfrak{h} \to \mathfrak{g}$ being an isomorphism. Then the inverse of $p_1|_{\mathfrak{h}}$ is such an embedding as in Proposition 2.14. Therefore, this yields a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. □

Proposition 2.16. *Let $\mathfrak{n}$ be semisimple. Then PA-structures on $(\mathfrak{g}, \mathfrak{n})$ are in one-to-one correspondence with subalgebras $\mathfrak{h}$ of $\mathfrak{n} \oplus \mathfrak{n}$ for which the map*

$$\begin{aligned} p_1 - p_2 : \mathfrak{n} \oplus \mathfrak{n} &\to \mathfrak{n} \\ (x, y) &\mapsto x - y \end{aligned}$$

induces an isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$, where p_1 denotes the projection on the first component and p_2 denotes the projection on the second component.

Proof. Using Proposition 2.15, we know that PA-structures on $(\mathfrak{g}, \mathfrak{n})$ are in 1-1-correspondence with subalgebras $\mathfrak{h}$ of $\mathfrak{n} \rtimes \mathfrak{n}$ for which the projection onto the first factor induces an isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$. Since $\mathfrak{n}$ is semisimple, we have that all derivations are inner and hence $\mathfrak{n} \rtimes Der(\mathfrak{n}) = \mathfrak{n} \rtimes \mathfrak{n}$. This Lie algebra is isomorphic to the direct sum $\mathfrak{n} \oplus \mathfrak{n}$ via $\pi(x, y) = \pi(x + y, y)$. Therefore a subalgebra $\mathfrak{h}$ of $\mathfrak{n} \rtimes \mathfrak{n}$ amounts to a subalgebra $\mathfrak{h}'$ of $\mathfrak{n} \oplus \mathfrak{n}$ for which the map

$$\begin{aligned} p_1 - p_2 : \mathfrak{n} \oplus \mathfrak{n} &\to \mathfrak{n} \\ (x, y) &\mapsto x - y \end{aligned}$$

induces an isomorphism of $\mathfrak{h}'$ onto $\mathfrak{g}$. □

Now we can formulate structure results for PA-structures on pairs of simple Lie algebras.

Proposition 2.17. *Let $\mathfrak{g}$ and $\mathfrak{n}$ be simple Lie algebras and $x \cdot y$ a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then either $x \cdot y$=0 and $[x, y] = \{x, y\}$ for all $x, y \in \mathfrak{n}$ or $[x, y] = -\{x, y\}$.*

Proof. Since $\mathfrak{n}$ is simple, Proposition 2.16 implies that $x \cdot y$ corresponds to a subalgebra $\mathfrak{h}$ of $\mathfrak{n} \oplus \mathfrak{n}$ such that $p_1 - p_2$ induces an isomorphism of $\mathfrak{h}$ onto $\mathfrak{g}$. Now $\mathfrak{g}$ is simple, hence $\mathfrak{h}$ is simple too. The projection maps for $i \in \{1, 2\}$

$$p_i : \mathfrak{n} \oplus \mathfrak{n} \to \mathfrak{n}$$

are homomorphisms, so $\ker(p_1(\mathfrak{h}))$ and $\ker(p_2(\mathfrak{h}))$ are ideals in $\mathfrak{h}$. Since $\mathfrak{h}$ is simple, the kernels must be either 0 or $\mathfrak{h}$, resulting in three cases:

1. $p_2(\mathfrak{h}) = 0$. This implies that $\mathfrak{h} = \{(x, 0) \mid x \in \mathfrak{n}\}$. Since $\mathfrak{h}$ is simple, all derivations are inner, and we have $L(x) = \operatorname{ad}(0) = 0$ for all $x \in \mathfrak{n}$. Hence $x \cdot y = 0$ and $[x, y] = \{x, y\}$ for all $x, y \in \mathfrak{n}$. Therefore $\mathfrak{g} = \mathfrak{n}$.
2. $p_1(\mathfrak{h}) = 0$. Thus, we have $\mathfrak{h} = \{(0, x) \mid x \in \mathfrak{n}\}$, implying $L(x) = -\operatorname{ad}(x)$. Then we obtain $[x, y] = -\{x, y\}$ for all $x, y \in \mathfrak{n}$ and $\mathfrak{g} = -\mathfrak{n}$.
3. Both $p_1(\mathfrak{h}) \neq 0$ and $p_2(\mathfrak{h}) \neq 0$. Hence, $\ker(p_1|_{\mathfrak{h}}) = \ker(p_2|_{\mathfrak{h}}) = 0$. Therefore, under restriction to $\mathfrak{h}$, both p_1 and p_2 are isomorphisms. Hence, there exists a bijective linear map $\phi : \mathfrak{n} \to \mathfrak{n}$ such that $\mathfrak{h} = \{(x, \phi(x)) \mid x \in \mathfrak{n}\}$. Because $\mathfrak{h}$ is

a subalgebra of $\mathfrak{n} \oplus \mathfrak{n}$, we have $[(x, \phi(x)), (y, \phi(y))] \in \mathfrak{h}$ for all $x, y \in \mathfrak{n}$. Allowing us to write $(\{x, y\}, \{\phi(x), \phi(y)\}) = (z, \phi(z))$ for some $z \in \mathfrak{n}$. Since $z = \{x, y\}$, we have $\phi(\{x, y\}) = \phi(z) = \{\phi(x), \phi(y)\}$. This implies that ϕ is an automorphism on $\mathfrak{n}$. Using a result from Jacobson, see [42], $\lambda = 1$ is an eigenvalue of ϕ. But this is a contradiction to

$$\begin{aligned} p_1 - p_2 : \mathfrak{h} &\to \mathfrak{g} \\ (x, \phi(x)) &\mapsto x - \phi(x) \end{aligned}$$

being an isomorphism. So only Case 1 or Case 2 are possible.

□

Proposition 2.18. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{n}$ is semisimple, and $\mathfrak{g}$ is the direct sum of two simple ideals $\mathfrak{s}_1$ and $\mathfrak{s}_2$, i.e. $\mathfrak{g} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$. Then we have $\mathfrak{g} \cong \mathfrak{n}$.*

Proof. Since $\mathfrak{n}$ is semisimple, we can use Proposition 2.6 stating that all PA-structures on $(\mathfrak{g}, \mathfrak{n})$ are inner, i.e. $x \cdot y = \{R(x), y\}$ where R is an RB-operator of weight 1 on $\mathfrak{n}$. Let us assume that $\mathfrak{g} \ncong \mathfrak{n}$, then by Proposition 2.11 (i) $ker(R) \neq 0$ and $ker(R + id) \neq 0$ and the kernels are ideals in $\mathfrak{g}$ with the property, that the intersection is empty. Since $\mathfrak{g}$ is semisimple, we have that $\mathfrak{g} = ker(R) \oplus ker(R + id) \oplus \mathfrak{s} \cong \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\mathfrak{s}$ is semisimple. Analogously to the argument in the proof of Theorem 2.13, we infer

$$\mathfrak{n} = im(R) + im(R + id).$$

Hence, we have a semisimple decomposition

$$\mathfrak{n} = (ker(R + id) \oplus \mathfrak{s}) \dot{+} (ker(R + id) \oplus \mathfrak{s}).$$

If $\mathfrak{s} \neq 0$, then $ker(R + id) \oplus \mathfrak{s}$ and $ker(R) \oplus \mathfrak{s}$ are not simple, which is a contradiction to $\mathfrak{g} \cong \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are simple. Thus $\mathfrak{s} = 0$ and again using a result by Koszul, see [48], we have $\mathfrak{n} = im(R) \oplus im(R + id) \cong ker(R) \oplus ker(R + id) \cong \mathfrak{s}_1 \oplus \mathfrak{s}_2$, which concludes the proof. □

Proposition 2.19. *All PA-structures on $\mathfrak{g} \cong \mathfrak{n}$ with $\mathfrak{g} \cong \mathfrak{n} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, where $\mathfrak{s}_1$,$\mathfrak{s}_2$ are simple isomorphic ideals of $\mathfrak{n}$, arise from the trivial RB-operators or the RB-operators R on $\mathfrak{n}$ are of the following kind:*

$$\begin{aligned} R((s_1, s_2)) &= (-s_1, -\phi(s_1)), \\ R((s_1, s_2)) &= (0, \phi(s_1)), \\ R((s_1, s_2)) &= (-s_1, 0), \end{aligned}$$

where $\phi \in Aut(\mathfrak{n})$, up to permutation and applying $\phi(R) = -R - \mathrm{id}$.

Proof. The above defined operators are in fact RB-operators by Proposition 1.32, Proposition 1.34 and Propostion 2.5 in [27].
Since $\mathfrak{g}$ and $\mathfrak{n}$ are semisimple, it follows that $\mathfrak{g}$ and $\mathfrak{n}$ are not solvable. By Proposition 2.11 (i) at least one of the kernels of R or $R + \mathrm{id}$ is nonzero. Two prove the statement we will make a case distinction between the following cases

- Case 1: Let both kernels be nonzero, then $\mathfrak{g} = \ker(R) \oplus \ker(R + \mathrm{id})$ and $\mathfrak{n} = \ker(R) \dotplus \ker(R+\mathrm{id})$. Then by Koszul's result, see [48], we either have $ker(R) \cong s_1$ or $ker(R) \cong s_2$. Without loss of generality we can assume $ker(R) = s_2$. Since $im(R) = \mathfrak{g}/_{ker(R)}$, s_1 and s_2 are isomorphic ideals and again by Koszul [48], we have $R((s_1, s_2)) = (\phi_1(s_1), \phi_2(s_1))$ or $R((s_1, s_2)) = (\phi_1(s_1), 0)$ for some $\phi_1, \phi_2 \in Aut(\mathfrak{n})$. Since $im(R) = \ker(R + \mathrm{id})$, we have $R((s_1, s_2)) = (-s_1, -\phi(s_1))$ or $R((s_1, s_2)) = (-s_1, 0)$ for some $\phi \in Aut(\mathfrak{n})$.

- Case 2: The other case is that one of the kernels is zero. Suppose $\ker(R + \mathrm{id}) = 0$ and $ker(R) = \mathfrak{s}_1$. We know that $\mathfrak{g}/_{ker(R)} \cong \mathfrak{g}/_{s_1}$ is simple and $-R - \mathrm{id}$ is an invertible RB-operator of weight 1 on $\mathfrak{g}/_{ker(R)}$ by Lemma 1.29. Now using Corollary 2.12, we have $-R - id = -id$, hence $R = 0$ on $\mathfrak{g}/_{ker(R)} \cong im(R)$. We can conclude that $R^2 = 0$ on $\mathfrak{g}$. Since $ker(R) = \mathfrak{s}_1$, we either have $p_i = 0$ or p_i is an isomorphism for $i = 1, 2$ and the projection maps

$$p_i : im(R) \to \mathfrak{s}_1 \oplus \mathfrak{s}_2.$$

  Hence, $R((s, 0)) = (0, \phi(s))$ or $R((s, 0)) = (\phi_1(s), \phi_2(s))$, where $\phi_1, \phi_2, \phi \in Aut(\mathfrak{n})$. But the second one does not satisfy $R^2 = 0$.

□

In the following we mean by $Z^1(\mathfrak{g}, M)$ the set of all 1-cocycles, $B^1(\mathfrak{g}, M)$ the set of all 1-coboundaries and $H^1(\mathfrak{g}, M)$ the first cohomology group, see [16] for the definition.

Lemma 2.20. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ arising from an RB-operator R of weight 1 on $\mathfrak{n}$. Let M be an $\mathfrak{n}$-module. Then*

$$x \cdot_{\mathfrak{g}} m = R(x) \cdot_{\mathfrak{n}} m$$

for all $x \in V$ and $m \in M$, is a $\mathfrak{g}$-module structure on M. For $d \in Z^1(\mathfrak{n}, M)$ the linear map d_R defined by $d_R(x) = d(R(x))$ is a 1-cocycle in $Z^1(\mathfrak{g}, M)$.

Proof. Let $x, y \in V$, $m \in M$, then we have

$$\begin{aligned}
[x, y] \cdot_{\mathfrak{g}} m &= R([x, y]) \cdot_{\mathfrak{n}} m \\
&= R(\{R(x), y\} + \{x, R(y)\} + \{x, y\}) \cdot_{\mathfrak{n}} m \\
&= \{R(x), R(y)\} \cdot_{\mathfrak{n}} m \\
&= R(x) \cdot_{\mathfrak{n}} (R(y) \cdot_{\mathfrak{n}} m) - R(y) \cdot_{\mathfrak{n}} (R(x) \cdot_{\mathfrak{n}} m) \\
&= x \cdot_{\mathfrak{g}} (y \cdot_{\mathfrak{g}} m) - y \cdot_{\mathfrak{g}} (x \cdot_{\mathfrak{g}} m),
\end{aligned}$$

from which we can conclude that $\cdot_{\mathfrak{g}}$ defines a $\mathfrak{g}$-module structure on M and

$$\begin{aligned}
d_R([x, y]) = d(R([x, y])) &= d(\{R(x), R(y)\}) \\
&= d(R(x)) \cdot_{\mathfrak{n}} R(y) + R(x) \cdot_{\mathfrak{n}} R(y) \\
&= d_R(x) \cdot_{\mathfrak{g}} y + x \cdot_{\mathfrak{g}} d_R(y).
\end{aligned}$$

This implies that d_R is a 1-cocycle. □

Theorem 2.21. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is semisimple and $x \cdot y$ arises from an RB-operator of weight 1 on $\mathfrak{n}$. Then $\mathfrak{n}$ is semisimple.*

Proof. Let M be an $\mathfrak{n}$-module and $d \in Z^1(\mathfrak{n}, M)$. By Lemma 2.20, we know that $d_R(x) = d(R(x))$ is a 1-cocycle in $Z^1(\mathfrak{g}, M)$. By our assumption $\mathfrak{g}$ is semisimple, hence by Whitehead's Lemma [44] the first cohomology group is zero, i.e., $H^1(\mathfrak{g}, M) = 0$. This implies that d_R is a 1-coboundary, thus $d_R \in B^1(\mathfrak{g}, M)$. It follows that there exist $a, b \in M$ such that

$$\begin{aligned} d_R(x) &= d(R(x)) = ax, \\ d_{-R-id}(x) &= d((-R - id)(x)) = bx. \end{aligned}$$

From this we conclude that $d(x) = -(a+b)x$ and d is a 1-coboundary, i.e., $d \in B^1(\mathfrak{n}, M)$. Therefore, $\mathfrak{n}$ is reflexive over a field of charasteric zero, which then implies that $\mathfrak{n}$ is semisimple, see [38]. □

Using Proposition 2.6 and Theorem 2.21 we can immediately conclude

Theorem 2.22. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is complete. Then $\mathfrak{n}$ is semisimple too.*

One question that remains open is the following conjecture. It is yet unclear whether $\mathfrak{g}$ and $\mathfrak{n}$ are isomorphic, if both are semisimple, but not simple, and a PA-structure exists.

Conjecture 1. Suppose a PA-structure on a pair $(\mathfrak{g}, \mathfrak{n})$ exists, where $\mathfrak{g}$ and $\mathfrak{n}$ are semisimple, and both are not simple. Then $\mathfrak{g} \cong \mathfrak{n}$.

2.3 Post-Lie algebra structures where $\mathfrak{g}$ is nilpotent

In this section we study post-Lie algebra structures where $\mathfrak{g}$ is nilpotent. Again we restrict ourselves to finite-dimensional Lie algebras over fields of characteristic zero, unless stated otherwise. We introduce a notion of a left and right annihilator for PA-structures similiar to CPA-structures, see [28]. For this section we cite [25] and the literature therein, as our main reference point.

Definition 2.23. Let $A = (V, \cdot)$ be a post-Lie algebra on $(\mathfrak{g}, \mathfrak{n})$. We define the *left* and *right annihilators in A* as

$$\begin{aligned} Ann_L(A) &= \{x \in A \mid x \cdot A = 0\}, \\ Ann_R(A) &= \{x \in A \mid A \cdot x = 0\}. \end{aligned}$$

Remark 7. In the case of commutative PA-structures, we have that $Ann_L(A) = Ann_R(A)$ is an ideal of the given post-Lie algebra. For post-Lie algebras in general this is not the case.

Note that $\mathfrak{n}$ becomes a $\mathfrak{g}$-module with the bilinear operator $x \cdot y$ and using condition (ii) from Definition 2.2, where $x \in \mathfrak{g}$ and $y \in \mathfrak{n}$. Hence, we can formulate the zeroth Lie algebra cohomology

$$H^0(\mathfrak{g}, \mathfrak{n}) = \{y \in \mathfrak{n} \mid x \cdot y = 0 \ \ \forall x \in \mathfrak{g}\}.$$

Lemma 2.24. *The left resp. right annihilators in A equal the kernels of the left resp. right multiplicators, i.e.,*

$$\begin{aligned} Ann_L(A) &= ker(L) = \{x \in A \mid L(x) = 0\}, \\ Ann_R(A) &= ker(R) = \{x \in A \mid R(x) = 0\}. \end{aligned}$$

The left annihilator is a Lie algebra ideal of $\mathfrak{g}$ and the right annihilator coincides with the zeroth Lie algebra cohomology $H^0(\mathfrak{g}, \mathfrak{n})$.

Proof. The equalities follow immediately from the definition of the annihilator. L is a representation by Remark 5, and consequently, the kernel is an ideal of $\mathfrak{g}$. □

Example 6. Let V be 2-dimensional, (e_1, e_2) a basis of V such that $[e_1, e_2] = 0$ and $\{e_1, e_2\} = e_1$, and $(\mathfrak{g}, \mathfrak{n}) \cong (K^2, \mathfrak{r}_2(K))$. We want to classify all PA-structues on $\mathfrak{g}, \mathfrak{n}$. Let $\alpha_i, \beta_i \in K$, where $i = 1, 2, 3, 4$. We will expand $e_i \cdot e_j$ in our basis where $i, j = 1, 2$, i.e.,

$$\begin{aligned} e_1 \cdot e_1 &= \alpha_1 e_1 + \beta_1 e_2, \\ e_1 \cdot e_2 &= \alpha_2 e_1 + \beta_2 e_2, \\ e_2 \cdot e_1 &= \alpha_3 e_1 + \beta_3 e_2, \\ e_2 \cdot e_2 &= \alpha_4 e_1 + \beta_4 e_2. \end{aligned}$$

We want $e_i \cdot e_j$ to fulfill the defining properties of a PA-structure, so:

(i) For condition (i) of Definition 2.2, we get $e_1 \cdot e_2 - e_2 \cdot e_1 = -e_1$, which implies $\alpha_2 - \alpha_3 = 1$ and $\beta_2 - \beta_3 = 0$

(ii) For Definition 2.2 (iii), we have

$$\begin{aligned} e_1 \cdot \{e_2, e_3\} &= \{e_1 \cdot e_1, e_2\} + \{e_1, e_1 \cdot e_2\}, \\ e_1 \cdot e_1 &= \{\alpha_1 e_1 + \beta_1 e_2, e_2\} + \{e_1, \alpha_2 e_1 + \beta_2 e_2\}, \\ \alpha_1 e_1 + \beta_1 e_2 &= \alpha_1 e_1 + \beta_2 e_1. \end{aligned}$$

Since e_1, e_2 is our basis this implies $\beta_1 = 0$, $\beta_2 = 0$, and by the result above, we have $\beta_3 = 0$, therefore

$$e_1 \cdot e_1 = \alpha_1 e_1.$$

Furthermore, we have

$$\begin{aligned} e_2 \cdot \{e_1, e_2\} &= \{e_2 \cdot e_1, e_2\} + \{e_1, e_2 \cdot e_2\} \\ e_2 \cdot e_1 &= \{\alpha_3 e_1 + \beta_3 e_2, e_2\} + \{e_1, \alpha_4 e_1 + \beta_4 e_2\} \\ \alpha_3 e_1 + \beta_3 e_2 &= \alpha_3 e_1 + \beta_4 e_2 \end{aligned}$$

from which we can conclude that $\beta_4 = 0$

(iii) Condition (ii) of Definition 2.2 reduces to

$$\begin{aligned}[][e_1, e_2] \cdot e_2 &= e_1 \cdot (e_2 \cdot e_2) - e_2 \cdot (e_1 \cdot e_2) \\ 0 &= \alpha_4 \alpha_1 e_1 - \alpha_2(\alpha_2 - 1)e_1\end{aligned}$$

So we have $\alpha_4\alpha_1 = \alpha_2(\alpha_2 - 1)$.
This means every PA-structure on $(\mathfrak{g}, \mathfrak{n}) \cong (K^2, \mathfrak{r}_2(K))$ has the form

$$\begin{aligned} e_1 \cdot e_1 &= \alpha_1 e_1, & e_1 \cdot e_2 &= \alpha_2 e_1, \\ e_2 \cdot e_1 &= (\alpha_2 - 1)e_1, & e_2 \cdot e_2 &= \alpha_4, e_1 \end{aligned}$$

with $\alpha_4\alpha_1 = \alpha_2(\alpha_2 - 1)$.
To calculate the annihilator, we look at the kernel of the left multiplicator. Let $x, y \in K$ and $(xe_1 + ye_2) \in \ker(L)$. Then we get

$$\begin{aligned}(xe_1 + ye_2) \cdot e_1 &= 0, \\ (xe_1 + ye_2) \cdot e_2 &= 0.\end{aligned}$$

This immediately implies $x\alpha_1 + y(\alpha_2 - 1) = x\alpha_2 + y\alpha_4$. Using the result in Lemma 2.24, we have

$$Ann_L = \begin{cases} \langle \alpha_4 e_1 - \alpha_2 e_2 \rangle & \text{if } \alpha_4, \alpha_2 \neq 0 \\ \langle e_1 - \alpha_1 e_2 \rangle & \text{if } \alpha_4 = \alpha_2 = 0. \end{cases}$$

Similiarly, we get for the right annihilator

$$Ann_R = \begin{cases} \langle \alpha_2 e_1 - \alpha_1 e_2 \rangle & \text{if } \alpha_1, \alpha_2 \neq 0 \\ \langle \alpha_4 e_1 - e_2 \rangle & \text{if } \alpha_1 = \alpha_2 = 0. \end{cases}$$

Hence, for the dimension of these spaces, it holds that,

$$\dim Ann_L(A) = \dim Ann_R(A) = \dim H^0(\mathfrak{g}, \mathfrak{n}) = 1.$$

Proposition 2.25. *Let $\mathfrak{g}$ be a nilpotent Lie algebra. Assume there exists a post-Lie algebra structure on $(\mathfrak{g}, \mathfrak{n})$. Then $\mathfrak{n}$ is solvable.*

Proof. Let us look at the map $\phi : \mathfrak{g} \to \mathfrak{n} \rtimes Der(\mathfrak{n})$, $x \mapsto (x, L(x))$. By Proposition 2.14, this is an injective homomorphism. Since every homomorphic image and subalgebra of a nilpotent Lie algebra is nilpotent, we have that $\mathfrak{h} = L(\mathfrak{g})$ is nilpotent. We claim that $\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \oplus \mathfrak{h}$. Let $(x, y) \in \mathfrak{n} \rtimes \mathfrak{h}$, i.e., there exists a $g \in \mathfrak{g}$ such that $(x, y) = (x, L(g))$. We have

$$(x, L(g)) - \phi(x) = (x, L(g)) - (x, L(x)) = (0, L(g) - L(x)) = (0, L(g - x)),$$

which implies $(x, L(g)) = \phi(x) + (0, L(g - x)) \in \phi(\mathfrak{g}) \oplus \mathfrak{h}$.
For the other direction, let $(x, y) \in \phi(\mathfrak{g}) \oplus \mathfrak{h}$. Hence there exists $a, b \in \mathfrak{g}$ such that

$$(x, y) = \phi(a) + (0, L(b)) = (a, L(a)) + (0, L(b)) = (a, L(a) + L(b)) = (a, L(a + b),$$

implying $(x, y) \in \mathfrak{n} \rtimes \mathfrak{h}$. Combining these two, we get $\mathfrak{n} \rtimes \mathfrak{h} = \phi(\mathfrak{g}) \oplus \mathfrak{h}$. By a result from Goto, see [34], we have that the sum of two nilpotent Lie algebras is solvable. Therefore we can deduce that $\mathfrak{n} \rtimes \mathfrak{h}$ is solvable. Since every subalgebra of a solvable Lie algebra is solvable, we have that $\mathfrak{n}$ is solvable. □

We want to examine under which conditions $\mathfrak{n}$ is nilpotent, when a PA-structure is given. Example 6 shows that $\mathfrak{n}$ need not be nilpotent. To see which assumptions we have to impose on our PA-structure for $\mathfrak{n}$ to be nilpotent, we will use results on group-gradings of Lie algebras, that have been studied in [53], [52] .

Definition 2.26. We call a *grading of a Lie algebra* $\mathfrak{g}$ *by a group* $(G, \circ)$ a decomposition of $\mathfrak{g} = \bigoplus_{g \in G} \mathfrak{g}_g$ into homogenous subspaces, such that for all $g, h \in G$, $[\mathfrak{g}_g, \mathfrak{g}_h] \subseteq \mathfrak{g}_{g \circ h}$ holds. The *support* of a grading is defined as the set $X := \{g \in G \mid \mathfrak{g}_g \neq 0\}$.

Definition 2.27. For a subset X of an abelian group $(G, +)$, we define the *period set* as

$$P_G(X) := \{g \in G \mid \exists x \in X : x + \langle g \rangle \subseteq X\}.$$

If X is finite and $X \cap P_G(X) = \emptyset$, then X is called *arithmetically-free.* For arbitrary groups, we generalize this notion by taking abelian subgroups, see Definition 2.1 in [52].

Example 7. If $(G, +)$ is a torsion-free abelian group, then every finite subset $X \subseteq G \backslash \{e\}$ is arithmetically free.
Let $x \neq e \in X$ and $x \in P_G(X)$. Then there exists an $x' \in X$ such that $x' + < x > \subseteq X$. Since X is finite, there exists an element $n \in \mathbb{N}$ such that $nx = e$. But the only element of finite order in a torsion-free group is the identity element. Hence, $X \cap P_G(X) = \emptyset$, and thus X is arithmetically-free.

Since every free abelian group is torsion-free, we have that every finite subset $X \subseteq G \backslash \{e\}$ of a free abelian group $(G, \circ)$ is arithemtically-free.

Theorem 2.28. *([53], [52]) Let $\mathfrak{n}$ be a Lie algebra over a field K graded by a group G. If the support X is arithmetically-free, then $\mathfrak{n}$ is nilpotent of $|X|$-bounded class. If G is free abelian, then the nilpotency class is at most $|X|^{2^{|X|}}$.*

Theorem 2.29. *Assume we have a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g}$ nilpotent and the zeroth cohomology group is zero, i.e., $H^0(\mathfrak{g}, \mathfrak{n}) = 0$. Then $\mathfrak{n}$ is nilpotent of class at most $|X|^{2^{|X|}}$.*

Proof. Since $\mathfrak{g}$ is nilpotent, using a result by Jacobson, see [44], there exists a weight space decomposition of $\mathfrak{g}$. Since $\mathfrak{n}$ is a $\mathfrak{g}$-module, we also have a weight space decomposition of $\mathfrak{n}$, see [21], and hence

$$\mathfrak{n} = \bigoplus_{\alpha \in \mathfrak{g}^*} \mathfrak{n}_\alpha,$$

where $[\mathfrak{n}_\alpha, \mathfrak{n}_\beta] \subseteq \mathfrak{n}_{\alpha+\beta}$ with $\alpha, \beta \in \mathfrak{g}^*$ and $n_\alpha := \{y \in \mathfrak{n} \mid (L(x) - \alpha(x) id_{\mathfrak{g}})^{dim(\mathfrak{g})} \cdot y = 0 \; \forall x \in \mathfrak{g}\}$. We call α a weight, if $\mathfrak{n}_\alpha \neq 0$. There are only finitely many weights implying that the support X is finite. The grading group $\mathfrak{g}^*$ is free abelian, so it is also torsion-free. Since $H^0(\mathfrak{g},\mathfrak{n}) = 0$, we have for $v \in \mathfrak{n}$

$$L(x)^{dim(\mathfrak{g})} \cdot v = 0 \implies v = 0.$$

This means 0 is not a weight. Using Example 7, we know that the support $X \subset \mathfrak{g}^* \backslash 0$ is arithmetically-free. By Theorem 2.28, it follows that $\mathfrak{n}$ is nilpotent of class at most $|X|^{2^{|X|}}$. □

2.3.1 On the nilpotency of the left multiplication operator

We end this chapter by stating some results on PA- and CPA-structures. It seems that oftentimes, the operator $L(x)$ is nilpotent for nilpotent and indecomposable Lie algebras $\mathfrak{g}$ and $\mathfrak{n}$. For CPA-structures, this has been proven in [21], under the assumption that $Z(\mathfrak{g}) \subseteq [\mathfrak{g},\mathfrak{g}]$. Lie algebras with this property are called *stem Lie algebras*. Whether this holds for PA-structures as well, remains open. However, we study nilpotent Lie algebras for which we can associate a CPA-structure, and by that, having results on the nilpotency of $L(x)$, see also [25], which is our main reference for this section. Let K be a field of characteristic zero, unless stated otherwise.

Definition 2.30. For the operator form of the definition of a PA-structure, let $L(x)y = x \cdot y$, $R(x)y = y \cdot x$, and let ad denote the adjoint representation of $\mathfrak{g}$ and Ad the adjoint representation of $\mathfrak{n}$. Then we can formulate the properties of a PA-structure in operator form:

(i) $L(x) - R(x) = ad(x) - Ad(x)$,

(ii) $L([x,y]) = [L(x), L(y)]$,

(iii) $[L(x), Ad(y)] = Ad(L(x)y)$.

Lemma 2.31. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g},\mathfrak{n})$. Then the following holds:*

1. $[L(x), Ad(y)] + [Ad(x), L(y)] = Ad([x,y]) - Ad(\{x,y\})$.
2. $[R(x), ad(y)]+[ad(x), R(y)] = [L(x), ad(y)]+[ad(x), L(y)]+[Ad(x), ad(y)]+[ad(x), Ad(y)]-2[ad(x), ad(y)]$.

Proof. We use property (i) and (iii) from Definition 2.30, to prove the following,

$$\begin{aligned}
[L(x), Ad(y)] &\overset{3.}{=} Ad(L(x)y) \\
&\overset{1.}{=} Ad(ad(x)y - Ad(x)y + R(x)y) = Ad(ad(x)y) - Ad(Ad(x)y) + Ad(R(x)y) \\
&= Ad([x,y]) - Ad(\{x,y\}) + Ad(y \cdot x) = Ad([x,y]) - Ad(\{x,y\}) + Ad(L(y)x) \\
&= Ad([x,y]) - Ad(\{x,y\}) + [L(y), Ad(x)],
\end{aligned}$$

$$\begin{aligned}
[R(x), ad(y)] + [ad(x), R(y)] &\overset{1.}{=} [L(x) - ad(x) + Ad(x), ad(y)] + [ad(x), L(y) - ad(y) + Ad(y)] \\
&= [L(x), ad(y)] - [ad(x), ad(y)] + [Ad(x), ad(y)] \\
&+ [ad(x), L(y)] - [ad(x), ad(y)] + [ad(x), Ad(y)] \\
&= [L(x), ad(y)] + [ad(x), L(y)] + [Ad(x), ad(y)] \\
&+ [ad(x), Ad(y)] - 2[ad(x), ad(y)].
\end{aligned}$$

□

If $\mathfrak{g}$ and $\mathfrak{n}$ are 2-step nilpotent, then $ad([x, y]) = 0$ and $Ad(\{x, y\}) = 0$. Using this we can prove the following lemma:

Lemma 2.32. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ and $\mathfrak{n}$ are 2-step nilpotent and let the following identity hold*

$$[L(x) + R(x), ad(y)] = ad(x \cdot y + y \cdot x) \forall x, y \in V.$$

Then we have

(i) $[L(x) + R(x), ad(y)] = [L(y) + R(y), ad(x)]$,

(ii) $2[L(x), ad(y)] + 2[ad(x), L(y)] = [ad(y), Ad(x)] + [Ad(y), ad(x)]$.

Proof. Let $x, y \in V$, then

$$\begin{aligned}
[L(x) + R(x), ad(y)] &= ad(x \cdot y + y \cdot x) \\
&= ad(y \cdot x + x \cdot y) \\
&= [L(y) + R(y), ad(x)].
\end{aligned}$$

Now we use Lemma 2.31 (ii), nilpotency of $\mathfrak{g}$ and the above proven identity to show the second equality:

$$\begin{aligned}
2[L(x), ad(y)] + 2[ad(x), L(y)] &= [L(x), ad(y)] + [ad(x), L(y)] + [L(x), ad(y)] + [ad(x), L(y)] \\
&\overset{1.}{=} [L(x), ad(y)] + [ad(x), L(y)] + [R(y), ad(x)] + [ad(y), R(x)] \\
&= -[Ad(x), ad(y)] - [ad(x), Ad(y)] \\
&= [ad(y), Ad(x)] + [Ad(y), ad(x)].
\end{aligned}$$

□

Definition 2.33. We call a K-bilinear product $x \circ y$ a *commutative post-Lie algebra structure*, or short *CPA-structure*, on a Lie algebra $\mathfrak{g}$ over K, if the following identities for all $x, y, z \in V$ are satisfied:

$$\begin{aligned}
x \circ y &= y \circ x, \\
[x, y] \circ z &= x \circ (y \circ z) - y \circ (x \circ z), \\
x \circ [y, z] &= [x \circ y, z] + [y, x \circ z].
\end{aligned}$$

Let $l(x)y = x \circ y$ and $r(x)y = y \circ x$ denote the left resp. right multiplicator. Then we can formulate the definition of a CPA-structure in operator form as follows:

$$\begin{aligned} l(x) &= r(x), \\ l([x,y]) &= [l(x), l(y)], \\ [l(x), ad(y)] &= ad(l(x)y). \end{aligned}$$

Proposition 2.34. *Let $\mathfrak{g}$, $\mathfrak{n}$ be 2-step nilpotent and $x \cdot y$ a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$ defines a CPA-structure on $\mathfrak{g}$ if and only if $[L(x) + R(x), ad(y)] = ad(x \cdot y + y \cdot x)$.*

Proof. If $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$ defines a CPA-structure, then

$$\begin{aligned} [L(x) + R(x), ad(y)] &= [2l(x), ad(y)] \\ &= 2[l(x), ad(y)] = 2ad(l(x)y) \\ &= ad(2l(x)y) = ad(x \cdot y + y \cdot x). \end{aligned}$$

For the other direction, let us assume $[L(x) + R(x), ad(y)] = ad(x \cdot y + y \cdot x)$. We need to show the properties of a CPA-structure. We have the following identity

$$\begin{aligned} l(x) = \frac{1}{2}(L(x) + R(x)) &= \frac{1}{2}(L(x) + L(x) - ad(x) + Ad(x)) \\ &= L(x) - \frac{1}{2}ad(x) + \frac{1}{2}Ad(x). \end{aligned}$$

For the first property we simply use the symmetry of the left and right multiplicator:

$$\begin{aligned} l(x) &= L(x) - \frac{1}{2}ad(x) + \frac{1}{2}Ad(x) \\ &= R(x) + \frac{1}{2}ad(x) - \frac{1}{2}Ad(x) = \frac{1}{2}(R(x) + L(x)) = r(x). \end{aligned}$$

To examine $l([x,y])$, we use Lemma 2.31 (i) and the nilpotency of $\mathfrak{g}$:

$$\begin{aligned} l([x,y]) &= L([x,y]) - \frac{1}{2}ad([x,y]) + \frac{1}{2}Ad([x,y]) \\ &= L([x,y]) + \frac{1}{2}Ad([x,y]) \\ &= [L(x), L(y)] + \frac{1}{2}Ad([x,y]) \\ &= [L(x), L(y)] + \frac{1}{2}([L(x), Ad(y)] + [Ad(x), L(y)]). \end{aligned}$$

For $[l(x), l(y)]$ we use Lemma 2.32 (ii) and nilpotency of both $\mathfrak{g}$ and $\mathfrak{n}$:

$$\begin{aligned}
[l(x), l(y)] &= [L(x) - \frac{1}{2}ad(x) + \frac{1}{2}Ad(x), L(y) - \frac{1}{2}ad(y) + \frac{1}{2}Ad(y)] \\
&= [L(x), L(y)] - \frac{1}{2}[L(x), ad(y)] + \frac{1}{2}[L(x), Ad(y)] - \frac{1}{2}[ad(x), L(y)] \\
&\quad + \frac{1}{4}[ad(x), ad(y)] - \frac{1}{4}[ad(x), Ad(y)] + \frac{1}{2}[Ad(x), L(y)] \\
&\quad - \frac{1}{4}[Ad(x), ad(y)] + \frac{1}{4}[Ad(x), Ad(y)] \\
&= [L(x), L(y)] + \frac{1}{2}[L(x), Ad(y)] + \frac{1}{2}[Ad(x), L(y)] - \frac{1}{2}[L(x), ad(y)] \\
&\quad - \frac{1}{2}[ad(x), L(y)] - \frac{1}{4}[ad(x), Ad(y)] - \frac{1}{4}[Ad(x), ad(y)] \\
&= [L(x), L(y)] + \frac{1}{2}[L(x), Ad(y)] + \frac{1}{2}[Ad(x), L(y)] \\
&\quad - \frac{1}{4}(2[L(x), ad(y)] + 2[ad(x), L(y)] + [ad(x), Ad(y)] + [Ad(x), ad(y)]) \\
&= [L(x), L(y)] + \frac{1}{2}[L(x), Ad(y)] + \frac{1}{2}[Ad(x), L(y)].
\end{aligned}$$

We see that both coincide, and therefore property 2 of a CPA-structure is satisfied. It remains to show the third property:

$$\begin{aligned}
[l(x), ad(y)] &= [\frac{1}{2}(L(x) + R(x)), ad(y)] \\
&= \frac{1}{2}[L(x) + R(x), ad(y)] = \frac{1}{2}ad(x \cdot y + y \cdot x) \\
&= ad\left(\frac{1}{2}(x \cdot y + y \cdot x)\right) = ad(l(x)y).
\end{aligned}$$

□

Note that this holds for $\mathfrak{g}$ and $\mathfrak{n}$ isomorphic to the 3-dimensional Heisenberg Lie algebra, see Corollary 5.3 in [25]. In the same paper a counterexample has been given, which does not fulfill the identity stated in the above proposition.

Corollary 2.35. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ is 2-step nilpotent and $\mathfrak{n}$ is abelian. Then $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$ is a CPA-structure if and only if all $L(x)$ are derivations of $\mathfrak{g}$. If additionally we have $Z(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$, then all $L(x)$ are nilpotent.*

Proof. Note that the identity $[L(x) + R(x), ad(y)] = ad(x \cdot y + y \cdot x)$ holds also for $\mathfrak{n}$ abelian. Assume we have a CPA-structure, then by Proposition 2.34, we have

$$\begin{aligned}
[L(x) + R(x), ad(y)] &= [L(x) + L(x) - ad(x), ad(y)] \\
&= [2L(x), ad(y)] = 2[L(x), ad(y)]
\end{aligned}$$

and

$$ad(x \cdot y + y \cdot x) = ad(2L(x)y - ad(xy)) = ad(2L(x)y) = 2ad(L(x)y).$$

We can conclude that $[L(x), ad(y)] = ad(L(x)y)$. Therefore, we have

$$\begin{aligned}[L(x)y, z] = ad(L(x)y)(z) &= [L(x), ad(y)](z) \\ &= L(x)ad(y)(z) - ad(y)L(x)(z) \\ &= L(x)([y, z]) - [y, L(x)z].\end{aligned}$$

Hence, all $L(x)$ are derivations of $\mathfrak{g}$. Conversely, let $L(x)$ be a derivation of $\mathfrak{g}$, i.e., $[L(x), ad(y)] = ad(L(x)y)$. Then

$$\begin{aligned}[L(x) + R(x), ad(y)] &= [2L(x), ad(y)] = 2[L(x), ad(y)] \\ &= 2ad(L(x)y) = ad(2L(x)y) \\ &= ad(L(x)y + R(x)y + ad(x)(y)) = ad(L(x)y + R(x)y) = ad(x \cdot y + y \cdot x)\end{aligned}$$

We infer from Proposition 2.34 that $x \circ y$ is a CPA structure. Let us additionally assume that $Z(\mathfrak{g}) \subseteq [\mathfrak{g}, \mathfrak{g}]$. We have that

$$l(x) = L(x) - \frac{1}{2}ad(x).$$

Using our assumption and Theorem 3.6 in [21], this implies that all CPA structures are complete, i.e. all $l(x)$ are nilpotent. We want to show that $l(x)$ and $ad(x)$ are commutative operators. We have

$$\begin{aligned}[l(x), ad(y)](z) = ad(l(x)y)(z) = [l(x)y, z] &= [x \circ y, z] \\ &= \frac{1}{2}[x \cdot y + y \cdot x, z] \\ &= \frac{1}{2}(ad(x \cdot y)(z) + ad(y \cdot x)(z)) \\ &= 0,\end{aligned}$$

because of $ad(x \cdot y) = ad(y \cdot x + ad(x)y) = ad(y \cdot x)$. Hence, the operators are commuting. Because $L(x)$ is the sum of commutative nilpotent operators, it follows that $L(x)$ is nilpotent (This follows immediately from the binomial theorem). $\square$

Proposition 2.36. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ with $\mathfrak{g}$ and $\mathfrak{n}$ being 2-step nilpotent. Then $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$ defines a CPA-structure on $\mathfrak{n}$ if and only if*

(i) $[ad(x), Ad(y)] = Ad([x, y[)$ *and*

(ii) $L(\{x, y\}) - L([x, y]) = \frac{1}{2}(ad(\{x, y\}) + [ad(y), L(x)] + [L(y), ad(x)])$

for all $x, y \in V$.

Proof. Let us assume that $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$ is a CPA-structure on $\mathfrak{n}$. We want to show (i) and (ii), where we also use nilpotency of $\mathfrak{g}$ and $\mathfrak{n}$. We have

$$\begin{aligned} Ad([x,y]) &= Ad(ad(x)y) = Ad(2L(x)y - 2l(x)y + Ad(x)y) \\ &= 2Ad(L(x)y) - 2Ad(l(x)y) \\ &= 2[L(x), Ad(y)] - 2[l(x), Ad(y)] \\ &= [2L(x) - 2l(x), Ad(y)] \\ &= [L(x) - R(x), Ad(y)] = [ad(x) - Ad(x), Ad(y)] \\ &= [ad(x), Ad(y)]. \end{aligned}$$

Moreover,

$$\begin{aligned} L(\{x,y\}) - L([x,y]) &= \frac{1}{2}ad(\{x,y\}) + l(\{x,y\}) - L([x,y]) \\ &= \frac{1}{2}ad(\{x,y\}) + [l(x), l(y)] - L([x,y]) \\ &= \frac{1}{2}ad(\{x,y\}) + [L(x) - \frac{1}{2}ad(x) + \frac{1}{2}Ad(x), L(y) - \frac{1}{2} + \frac{1}{2}Ad(y)] - L([x,y]) \\ &= \frac{1}{2}ad(\{x,y\}) - \frac{1}{2}[L(x), ad(y)] + \frac{1}{2}[L(x), Ad(y)] - \frac{1}{2}[ad(x), L(y)] \\ &- \frac{1}{4}[ad(x), Ad(y)] + \frac{1}{2}[Ad(x), L(y)] - \frac{1}{4}[Ad(x), ad(y)] \\ &= \frac{1}{2}(ad(\{x,y\}) + [ad(y), L(x)] + [L(y), ad(x)]) + \frac{1}{2}[L(x) \\ &- \frac{1}{2}ad(x), Ad(y)] - \frac{1}{2}[L(y) - \frac{1}{2}ad(y), Ad(x)] \\ &= \frac{1}{2}(ad(\{x,y\}) + [ad(y), L(x)] + [L(y), ad(x)]) + \frac{1}{2}[l(x), Ad(y)] - \frac{1}{2}[l(y), Ad(x)] \\ &= \frac{1}{2}(ad(\{x,y\}) + [ad(y), L(x)] + [L(y), ad(x)]) + \frac{1}{2}Ad(l(x)y) - \frac{1}{2}Ad(l(y)x) \\ &= \frac{1}{2}(ad(\{x,y\}) + [ad(y), L(x)] + [L(y), ad(x)]). \end{aligned}$$

Now we want to prove the other direction. The first defining property of a CPA-structure is trivial. For the second one, we use Lemma 2.31, nilpotency and the two identities of our assumption:

$$\begin{aligned} l(\{x,y\}) &= \frac{1}{2}(L(\{x,y\}) + R(\{x,y\})) = \frac{1}{2}(L(\{x,y\}) + L(\{x,y\}) - ad(\{x,y\})) \\ &= L(\{x,y\}) - \frac{1}{2}ad(\{x,y\}) \\ &= L([x,y]) + \frac{1}{2}[ad(y), L(x)] + \frac{1}{2}[L(y), ad(x)] \\ &= [L(x), L(y)] + \frac{1}{2}[ad(y), L(x)] + \frac{1}{2}[L(y), ad(x)] \end{aligned}$$

On the other hand, we have

$$
\begin{aligned}
[l(x), l(y)] &= [L(x) - \frac{1}{2}ad(x) + \frac{1}{2}Ad(x), L(y) - \frac{1}{2}ad(y) + \frac{1}{2}Ad(y)] \\
&= [L(x), L(y)] - \frac{1}{4}[ad(x), Ad(y)] + \frac{1}{2}[Ad(x), L(y)] - \frac{1}{2}[L(x), ad(y)] \\
&- \frac{1}{4}[Ad(x), ad(y)] + \frac{1}{2}[L(x), Ad(y)] - \frac{1}{2}[ad(x), L(y)] \\
&= [L(x), L(y)] - \frac{1}{4}Ad([x, y]) + \frac{1}{2}Ad([x, y]) - \frac{1}{2}[L(x), ad(y)] - \frac{1}{2}[ad(x), L(y)] + \frac{1}{4}Ad([y, x]) \\
&= [L(x), L(y)] + \frac{1}{2}[ad(y), L(x)] + \frac{1}{2}[L(y), ad(x)].
\end{aligned}
$$

The only thing left is to show the third axiom of a CPA-structure:

$$
\begin{aligned}
[l(x), Ad(y)] &= [L(x) - \frac{1}{2}ad(x) + \frac{1}{2}Ad(x), Ad(y)] \\
&= [L(x), Ad(y)] - \frac{1}{2}[ad(x), Ad(y)] \\
&= Ad(L(x)y) - \frac{1}{2}Ad([x, y]) \\
&= Ad(L(x)y - \frac{1}{2}[x, y] + \frac{1}{2}\{x, y\}) \\
&= Ad(l(x)y).
\end{aligned}
$$

□

In general, the identities may not be satisfied. In [25], a PA-structure on a pair of Lie algebras isomorphic to the 3-dimensional Heisenberg Lie algebra is given, where the identities do not hold and thus a CPA-structure cannot be associated.

Corollary 2.37. *Let $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ where $\mathfrak{g}$ is abelian and $\mathfrak{n}$ is 2-step nilpotent. Then $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$ defines a CPA-structure on $\mathfrak{n}$ if and only if $\{\mathfrak{n}, \mathfrak{n}\} \cdot \mathfrak{n} = 0$.*

Proof. Let $x \circ y$ be a CPA-structure associated with a PA-structure. Then

$$\{x, y\} \cdot z = L(\{x, y\})z = \frac{1}{2}(ad(\{x, y\})z) = 0 \text{ for all } x, y, z \in \mathfrak{n}.$$

Conversely, the first identity of Proposition 2.36 is trivially satisfied, since $\mathfrak{g}$ is abelian. For the second one we have $L(\{x, y\}) - L([x, y]) = 0 = \frac{1}{2}(ad(\{x, y\} + [ad(y), L(x)] + [L(y), ad(x)]$. Hence, we can associate a CPA-structure to the PA-structure. □

Since $\mathfrak{g}$ is abelian, PA-structures on $(\mathfrak{g}, \mathfrak{n})$ amount to LR-structures on $\mathfrak{n}$ via the bilinear product $-x \cdot y$, see [23]. Hence, we can formulate the following corollary:

Corollary 2.38. *Let $\mathfrak{n}$ be a 2-step nilpotent Lie algebra with $Z(\mathfrak{n}) \subseteq \{\mathfrak{n}, \mathfrak{n}\}$ and $\{\mathfrak{n}, \mathfrak{n}\} \cdot \mathfrak{n} = 0$. Then every LR-structure on $\mathfrak{n}$ is complete.*

Proof. By Corollary 2.37 we have a CPA-structure on $\mathfrak{n}$ defined by $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$. Let $l(x)(y) = x \circ y$ be the left mulitplicator. Then we have $L(x) = l(x) - \frac{1}{2}Ad(x)$. Since $Z(\mathfrak{n}) \subseteq \{\mathfrak{n}, \mathfrak{n}\}$, we can use Theorem 3.6 in [21] to deduce that $l(x)$ is nilpotent. Analogously to the proof in Corollary 2.35, we have $[l(x), Ad(y)] = 0$, so $L(x)$ is the difference of commutative nilpotent operators, and hence is itself nilpotent. □

If we put some additional assumptions on $\mathfrak{n}$, we can see to which PA-structures a CPA-structure can be associated, by using Corollary 2.37.

Lemma 2.39. *Let $\mathfrak{g}$ be abelian, $\mathfrak{n}$ 2- step nilpotent and $x \cdot y$ be a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Then for each $p, q, x \in \mathfrak{n}$ with $\{x, p\} = \{x, q\} = 0.$, we have $x \cdot \{p, q\} = 0$.*

Proof. Using our assumption, nilpotency and condition 3 from the definition of a PA-structure we will show that $x \cdot \{p, q\} = 0$:

$$\begin{aligned}
x \cdot \{p, q\} &= \{x \cdot p, q\} + \{p, x \cdot q\} \\
&= \{p \cdot x, q\} + \{p, x \cdot q\} \\
&= p \cdot \{x, q\} - \{x, p \cdot q\} + \{p, x \cdot q\} \\
&= -\{x, p \cdot q\} + \{p, x \cdot q\} \\
&= -\{x, q \cdot p - Ad(p)(q)\} + \{p, x \cdot q\} \\
&= -\{x, q \cdot p\} + \{p, x \cdot q\} \\
&= \{q \cdot x, p\} - q \cdot \{x, p\} + \{p, x \cdot q\} \\
&= \{q \cdot x, p\} + \{p, x \cdot q\} \\
&= \{p, x \cdot q - q \cdot x\} = \{p, Ad(x)(q)\} \\
&= \{p, 0\} = 0.
\end{aligned}$$

□

In [25], we see a classification result of all possible PA-structures on pairs of Heisenberg Lie algebras. We recall this result from [25], where we have an explicit example of a 2-step nilpotent Lie algebra.

Corollary 2.40. *Let $\mathfrak{g} = \mathfrak{n}_3(K)$ be the 3-dimensional Heisenberg Lie algebra and $\mathfrak{g} \cong \mathfrak{n}$. Then all left multiplication operators $L(x)$ are nilpotent and for all $x, y, z \in V$ the following holds:*

$$\begin{aligned}
x \cdot \{y, z\} &= 0, \\
[x, y] \cdot z &= z \cdot [x, y], \\
[x, y \cdot z] + [x, z \cdot y] &= [y, x \cdot z] + [y, x \cdot z],
\end{aligned}$$

and $x \circ y = \frac{1}{2}(x \cdot y + y \cdot x)$ defines a CPA-structure.

We conclude this chapter by providing a table, in which we show what is known about the existence of PA-structures, see [13]. The checkmark represents whether there is some pair admitting a PA-structure, the questionmark means that it is still open.

$(\mathfrak{g}, \mathfrak{n})$	$\mathfrak{n}$ abe	$\mathfrak{n}$ nil	$\mathfrak{n}$ sol	$\mathfrak{n}$ sim	$\mathfrak{n}$ sem	$\mathfrak{n}$ red	$\mathfrak{n}$ com
$\mathfrak{g}$ abe	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ nil	✓	✓	✓	-	-	-	✓
$\mathfrak{g}$ sol	✓	✓	✓	✓	✓	✓	✓
$\mathfrak{g}$ sim	-	-	-	✓	-	-	-
$\mathfrak{g}$ semi	-	-	-	-	✓	?	-
$\mathfrak{g}$ red	✓	?	?	-	?	✓	✓
$\mathfrak{g}$ com	✓	✓	✓	?	?	✓	✓

The table shows whether post-Lie algebra structures exist or not exist for abelian, nilpotent, solvable, simple, semisimple, reductive and complete Lie algebras. As for the classification of PA-structures, this is a hard task. There have been some classification results in lower dimension for pairs of nilpotent Lie algebras where both are isomorphic to the 3-dimensional Heisenberg Lie algebra, see [25]. However, for commutative post-Lie algebra structures, better classification results have been obtained, see [13].

3 Decomposition of Lie algebras

The last chapter of this thesis will deal with decompositions of Lie algebras. We have seen in Chapter 2 that the study of PA-structures is strongly connected to decompositions of Lie algebras. In this chapter, we recall some results on decompositions to have a generalization of Theorem 2.13. The last part mentions a few results on the study of the derived length of a Lie algebra in the context of nildecomposable Lie algebras. We refer to [26] and [14] as our main reference point.

3.1 Basic concepts

Definition 3.1. Let (A, A_1, A_2) be a triple of algebras. The triple is called a *decomposition* of A if A_1 and A_2 are subalgebras of A and A decomposes into $A = A_1 + A_2$ as a vector space sum of A_1 and A_2. If we additionally have that A_1 and A_2 are proper subalgebras of A, then the decomposition is called *proper*. It is called *semisimple*, if A, A_1 and A_2 are semisimple. The decomposition is called *direct* if $A_1 \cap A_2 = 0$.

Let us recall two important theorems by Onishchik [54], [55], in his study on decompositions of reductive Lie groups.

Theorem 3.2. *Let L be a compact Lie algebra with two subalgebras L' and L''. Let $L = S \oplus Z$, $L' = S' \oplus Z'$ and $L'' = S'' \oplus Z''$ where Z, Z', Z'' denotes the centers and S, S', S'' are semisimple ideals. We denote by $\tilde{Z}'$ and $\tilde{Z}''$ the projections of Z' and Z'' on Z. Then we have*

$$L = L' + L'' \text{ if and only if } S = S' + S'' \text{ and } Z = \tilde{Z}' + \tilde{Z}''.$$

Theorem 3.3. *Let L be a compact Lie algebra and L', L'' two subalgebras. Then all proper decompositions are of the following kind:*

L	L'	L''	$L' \cap L''$
$A_{2n-1}, n > 1$	C_n	$A_{2n-2}, A_{2n-2} \oplus T$	$C_{n-1}, C_{n-1} \oplus T$
$D_{n+1}, n > 2$	B_n	$A_n, A_n \oplus T$	$A_{n-1}, A_{n-1} \oplus T$
$D_{2n}, n > 1$	B_{2n-1}	$C_n, C_n \oplus T, C_n \oplus A_1$	$C_{n-1}, C_{n-1} \oplus T, C_{n-1} \oplus A_1$
B_3	G_2	$B_2, B_2 \oplus T, D_3$	$A_1, A_1 \oplus T, A_2$
D_4	B_3	$B_2, B_2 \oplus T, B_2 \oplus A_1, D_3, D_3 \oplus T, B_3$	$A_1, A_1 \oplus T, A_1 \oplus A_1, A_2, A_2 \oplus T, G_2$
D_8	B_7	B_4	B_3

Definition 3.4. Let S be a subalgebra of a Lie algebra L. Then S is called *reductive in L*, if S is reductive and $ad(z)$ is semisimple in $End(L)$ for every $z \in Z(S)$.

Definition 3.5. A decomposition (L, L', L'') is called *reductive*, if L, L', L'' is reductive.

We recollect the following result by Koszul [48], that has also been used in the proof of Theorem 2.13.

Theorem 3.6. *Let (L, L', L'') be a direct reductive decomposition over a field of characteristic zero, and L', L'' are reductive in L. Then we have $L \cong L' \oplus L''$ and L', L'' are reductive ideals in L.*

Remark 8. Every subalgebra of a compact Lie algebra L is reductive in L, see also Remark 1.

3.2 Semisimple and reductive decompositions of Lie algebras

Our aim is to generalize Theorem 2.13, by using results on semisimple and reductive decompositions. We start by stating a result on semisimple decompositions of the matrix algebra, that has been studied in the context of post-associative structures, see [26].

Theorem 3.7. *The matrix algebra $M_n(\mathbb{C})$ has no proper semisimple decomposition.*

Proof. Let $A = M_n(\mathbb{C})$. Let us assume that there exists a proper semisimple decomposition $A = A_1 + A_2$ over $\mathbb{C}$. The Artin-Wedderburn theorem [7] tells us, that every semisimple algebra decomposes into a sum of quadratic matrix algebras, and hence we get

$$\begin{aligned} A_1 &= M_{i_1} \oplus ... \oplus M_{i_k}, \\ A_2 &= M_{j_1} \oplus ... \oplus M_{j_l}, \end{aligned}$$

for some natural numbers $i_1, ..., i_k, j_1, ..., j_l$. We obtain a decomposition of the Lie algebra $A^- = A_1^- + A_2^-$ where the Lie bracket is given by the commutator, that is,

$$\begin{aligned} A^- &= \mathfrak{gl}_n(\mathbb{C}) \cong \mathfrak{sl}_n(\mathbb{C}) \oplus \mathbb{C}, \\ A_1^- &= \mathfrak{sl}_{i_1}(\mathbb{C}) \oplus ... \oplus \mathfrak{sl}_{i_k}(\mathbb{C}) \oplus \mathbb{C}^k, \\ A_2^- &= \mathfrak{sl}_{j_1}(\mathbb{C}) \oplus ... \oplus \mathfrak{sl}_{j_l}(\mathbb{C}) \oplus \mathbb{C}^l. \end{aligned}$$

Since we are dealing with simple Lie algebras, we can look at the compact real form, see [44], and hence get a proper decomposition $B = B_1 + B_2$ with reductive Lie algebras over $\mathbb{R}$:

$$\begin{aligned} B &= \mathfrak{su}(n) \oplus \mathbb{R}, \\ B_1 &= \mathfrak{su}(i_1) \oplus ... \oplus \mathfrak{su}(i_k) \oplus \mathbb{R}^k, \\ B_2 &= \mathfrak{su}(j_1) \oplus ... \oplus \mathfrak{su}(j_l) \oplus \mathbb{R}^l. \end{aligned}$$

Since we have a decomposition over $\mathbb{R}$, we also obtain a decomposition for the complexification of the real Lie algebras involved by Lemma 1.3 in [54]. Therefore, we get a semisimple decomposition by Theorem 3.2

$$\mathfrak{su}(n) = \mathfrak{su}(i_1) \oplus ... \oplus \mathfrak{su}(i_k) + \mathfrak{su}(j_1) \oplus ... \oplus \mathfrak{su}(j_l),$$

which is a contradiction to results on proper reductive decompositions in Theorem 3.3.

□

Lemma 3.8. *Let $A = A_1 + A_2$ be a direct semisimple decompostion of associative algebras over $\mathbb{C}$. Then $A \cong A_1 \oplus A_2$.*

Proof. Similiarly to the proof of Theorem 3.7, we look at the compact real form of the decomposition $A^- = A_1^- + A_2^-$. By Theorem 3.6 we get for the reductive decompostion $B \cong B_1 \oplus B_2$, and therefore, we also have $A^- \cong A_1^- \oplus A_2^-$ over $\mathbb{R}$. From this we can immediately conclude $A \cong A_1 \oplus A_2$ over $\mathbb{C}$. $\square$

Note that by summing two semisimple Lie algebras, we need not get a semisimple Lie algebra. As an example we give a thorough computation of a Lie algebra below. Hence, a Lie algebra can decompose into two semisimple parts without being itself semisimple.

Example 8. Let $\mathfrak{g} = \mathfrak{s}_1 + \mathfrak{s}_2$ be a Lie algebra, where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ are two complex semisimple Lie algebras. Then $\mathfrak{g}$ need not be semisimple, as the following example shows:
Let $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes_\rho V(2)$, where $V(2)$ stands for the irreducible representation of $\mathfrak{sl}_2(\mathbb{C})$, considered as an abelian Lie algebra.
For ρ we will use the standard representation of $\mathfrak{sl}_2(\mathbb{C})$. Let (e_1, e_2, e_3) be a basis of $\mathfrak{sl}_2(\mathbb{C})$, that is,

$$\begin{aligned}
[e_1, e_2] &= e_3, \\
[e_3, e_1] &= 2e_1, \\
[e_3, e_2] &= -2e_2.
\end{aligned}$$

Then

$$\begin{aligned}
\rho : \mathfrak{sl}_2(\mathbb{C}) &\to \mathfrak{gl}(\mathbb{C}^2) \\
e_1 &\mapsto \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
e_2 &\mapsto \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
e_3 &\mapsto \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}
\end{aligned}$$

defines a 2-dimensional representation, the standard representation of $\mathfrak{sl}_2(\mathbb{C})$, because

$$\begin{aligned}
\rho([e_1, e_2]) &= \rho(e_3) = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \\
&= [\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}] \\
&= [\rho(e_1), \rho(e_2)], \\
\rho([e_3, e_1]) &= 2\rho(e_1) = 2 * \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \\
&= [\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}] \\
&= [\rho(e_3), \rho(e_1)], \\
\rho([e_3, e_2]) &= -2 * \rho(e_2) = -2 * \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \\
&= [\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}] \\
&= [\rho(e_3), \rho(e_2)].
\end{aligned}$$

We consider the 2-dimensional representation V(2) as an abelian Lie algebra, i.e. $\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) \ltimes_\rho \mathbb{C}^2$. Let (e_4, e_5) be a basis for $\mathbb{C}^2$. Since the Lie bracket for $\mathfrak{g}$ is defined as

$$[(a, b), (c, d)] := ([a, c]_{\mathfrak{sl}_2(\mathbb{C})}, \rho(a)(d) - \rho(c)(b)),$$

we have

$$
\begin{aligned}
[(e_1,0),(0,e_5)] &= ([e_1,0],\rho(e_1)(e_5)-\rho(0)0) \\
&= \left(0,\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right) = \left(0,\begin{pmatrix}1\\0\end{pmatrix}\right) = (0,e_4), \\
[(e_1,0),(0,e_4)] &= ([e_1,0],\rho(e_1)(e_4)-\rho(0)(0)) \\
&= \left(0,\begin{pmatrix}0&1\\0&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right) = 0, \\
[(e_2,0),(0,e_5)] &= (0,\rho(e_2)(e_5)) \\
&= \left(0,\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right) = 0, \\
[(e_2,0),(0,e_4)] &= (0,\rho(e_2)(e_4)) = \left(0,\begin{pmatrix}0&0\\1&0\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right) = \left(0,\begin{pmatrix}0\\1\end{pmatrix}\right) = (0,e_5), \\
[(e_3,0),(0,e_4)] &= (0,\rho(e_3)(e_4)) \\
&= \left(0,\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}1\\0\end{pmatrix}\right) = \left(0,\begin{pmatrix}1\\0\end{pmatrix}\right) = (0,e_4), \\
[(e_3,0),(0,e_5)] &= (0,\rho(e_3)(e_5)) \\
&= \left(0,\begin{pmatrix}1&0\\0&-1\end{pmatrix}\begin{pmatrix}0\\1\end{pmatrix}\right) = \left(0,\begin{pmatrix}0\\-1\end{pmatrix}\right) = (0,-e_5).
\end{aligned}
$$

This means the Lie bracket on the basis (e_1,e_2,e_3,e_4,e_5) for $\mathfrak{g}$ is given by

$$
\begin{aligned}
&[e_1,e_2]=e_3, && [e_1,e_5]=e_4, && [e_2,e_4]=e_5,\\
&[e_3,e_2]=-2e_2, && [e_1,e_4]=0, && [e_3,e_4]=e_4,\\
&[e_3,e_1]=2e_1, && [e_2,e_5]=0, && [e_3,e_5]=-e_5.
\end{aligned}
$$

Now let $x=e_4+e_5$, then we have

$$
\begin{aligned}
ad(x)(e_1) &= [e_4,e_1]+[e_5,e_1] = 0-e_4 = -e_4,\\
ad(x)(e_2) &= [e_4,e_2]+[e_5,e_2] = -e_5+0 = -e_5,\\
ad(x)(e_3) &= [e_4,e_3]+[e_5,e_3] = -e_4+e_5,\\
ad(x)(e_4) &= [e_4,e_4]+[e_5,e_4] = 0,\\
ad(x)(e_5) &= [e_4,e_5]+[e_5,e_5] = 0.
\end{aligned}
$$

Hence, the matrix has the form

$$
ad(x) = \begin{pmatrix}0&0&0&0&0\\0&0&0&0&0\\0&0&0&0&0\\-1&0&-1&0&0\\0&-1&1&0&0\end{pmatrix}.
$$

We have $ad(x)^2 = 0$, from which it follows that $ad(x)$ is a nilpotent derivation. By Lemma 1.26, we have that $e^{ad(x)}$ is an automorphism of $\mathfrak{g}$. Using nilpotency of ad(x), we obtain

$$e^{ad(x)} = \sum_{i=0}^{\infty} \frac{ad(x)^i}{i!} = id + ad(x).$$

Evaluating this automorphism on $\mathfrak{sl}_2(\mathbb{C})$, we obtain

$$\begin{aligned}
(id + ad(x))(e_1) &= e_1 + [x, e_1] = e_1 - e_4, \\
(id + ad(x))(e_2) &= e_2 + [x, e_2] = e_2 - e_5, \\
(id + ad(x))(e_3) &= e_3 + [x, e_3] = e_3 - e_4 + e_5.
\end{aligned}$$

The question now is, is $\mathfrak{sl}_2(\mathbb{C}) = \langle e_1 - e_4, e_2 - e_5, e_3 - e_4 + e_5 \rangle$? Let $e = e_1 - e_4$, $h = e_2 - e_5$ and $f = e_3 - e_4 + e_5$, then we have

$$\begin{aligned}
[e, h] &= [e_1 - e_4, e_2 - e_5] = [e_1, e_2] - [e_1, e_5] - [e_4, e_2] + [e_4, e_5] \\
&= e_3 - e_4 + e_5 = f, \\
[e, f] &= [e_1 - e_4, e_3 - e_4 + e_5] \\
&= [e_1, e_3] - [e_1, e_4] + [e_1, e_5] - [e_4, e_3] - [e_4, e_5] \\
&= -2e_1 - 0 + e_4 + e_4 = -2(e_1 - e_4) = -2e, \\
[h, f] &= [e_2 - e_5, e_3 - e_4 + e_5] \\
&= [e_2, e_3] - [e_2, e_4] + [e_2, e_5] - [e_5, e_3] + [e_5, e_4] \\
&= 2e_2 - e_5 - e_5 = 2(e_2 - e_5) = 2h.
\end{aligned}$$

We obtain a decomposition

$$\mathfrak{g} = \mathfrak{sl}_2(\mathbb{C}) + \mathfrak{sl}_2(\mathbb{C}).$$

Both summands are semisimple, but $\mathfrak{g}$ is not, since we have $rad(\mathfrak{g}) \neq 0$.

With these preliminaries, we can formulate the following generalization of Theorem 2.13, now using reductiveness, see also [26].

Theorem 3.9. *Assume there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$ over $\mathbb{R}$ or $\mathbb{C}$ with $\mathfrak{n}$ simple and $\mathfrak{g}$ reductive. Then $\mathfrak{g}$ is also simple and $\mathfrak{g} \cong \mathfrak{n}$.*

Proof. Since $\mathfrak{n}$ is simple, we have that $\mathfrak{n}$ is complete and by Corollary 2.9 we have a bijection between PA-structures on $(\mathfrak{g}, \mathfrak{n})$ and RB-operators on $\mathfrak{n}$ of weight 1. Now let R be that RB-operator of weight 1 on $(\mathfrak{g}, \mathfrak{n})$. We have $\mathfrak{n} = im(R) + im(R + id)$, which is a proper reductive decomposition (see also proof of Theorem 2.13). Assume $\mathfrak{g} \ncong \mathfrak{n}$. Then by Proposition 2.11 and reductiveness of $\mathfrak{g}$ we have $\mathfrak{g} = ker(R) \oplus ker(R + id) \oplus \mathfrak{c}$, where $\mathfrak{c}$ is an ideal in $\mathfrak{g}$. Let the field be $\mathbb{C}$ and consider the decomposition $\mathfrak{n} = im(R) + im(R + id)$ over $\mathbb{R}$, with semisimple and abelian parts, as in Theorem 3.2. Let us look at the compact real form of $\mathfrak{n}$, see [44]. We get a proper semisimple decomposition

$$\mathfrak{n}_{\mathbb{R}} = \mathfrak{s}_1 + \mathfrak{s}_2,$$

where $\mathfrak{s}_1$ and $\mathfrak{s}_2$ denote the semisimple parts of $im(R)$ and $im(R+id)$ over $\mathbb{R}$. Assume $\mathfrak{c}$ is an abelian ideal. Then, because $ker(R) \cap ker(R+id) = 0$ and $\mathfrak{n} = ker(R+id) \oplus \mathfrak{c} + ker(R) \oplus \mathfrak{c}$, we have that the decomposition is direct $\mathfrak{n}_{\mathbb{R}} = \mathfrak{s}_1 \oplus \mathfrak{s}_2$, which is a contradiction to Theorem 3.3. Therefore, $\mathfrak{c}$ is not abelian and non-zero. This means that $im(R)$ and $im(R+id)$ have a pair of isomorphic simple summands. Assume now that $ker(R)$ and $ker(R+id)$ are not abelian, then $im(R)$ and $im(R+id)$ contain at least two simple summands. This means that also s_1 and s_2 have at least two simple summands, which is a contradiction to Theorem 3.3. Suppose $ker(R)$ and $ker(R+id)$ are both abelian, from which we can conlude that s_1 and s_2 are isomoprhic. By Theorem 3.3 this holds only for $D_4 = B_3 + B_3$ over $\mathbb{R}$. Let $ker(R) = \mathbb{C}^k$, $ker(R+id) = \mathbb{C}^l$ and $\mathfrak{c} = B_3 \oplus \mathbb{C}^m$. We have the decomposition

$$D_4 = (B_3 \oplus \mathbb{C}^{l+m}) + (B_3 \oplus \mathbb{C}^{k+m}).$$

For the dimension of D_4 we have,

$$dim(\mathbb{C}^k \oplus \mathbb{C}^l \oplus B_3 \oplus \mathbb{C}m) = dim(D_4) = dim(\mathfrak{so}(8)) = \frac{8 * (8-1)}{2} = 28,$$

from which we can deduce that

$$k + l + m = dim(D_4) - dim(B_3) = 28 - 21 = 7.$$

So one of the summands contains the subalgebra $B_3 \oplus \mathbb{C}^4$. This implies that the centralizer $Z_{D_4}(B_3)$ s at least 4-dimensional, which is impossible by table 11 in [51]. Now suppose that either $ker(R)$ or $ker(R+id)$ is abelian. We can suppose that s_1 is simple, but this means that s_1 is a proper ideal in s_2, which is a contradiction to Theorem 3.3. Hence, by complexification, we get $\mathfrak{g} \cong \mathfrak{n}$. $\square$

3.3 Nilpotent decompositions

In Proposition 2.25 we used a result on the sum of two nilpotent Lie algebras by Goto, see [34], hence it only makes sense to study some related questions to the decomposition of such algebras. In this chapter we mention findings on the decomposition of nilpotent Lie algebras and provide some results on the derived length of nildecomposable Lie algebras, that have been studied in [14]. We start by investigating so-called filiform nilpotent Lie algebras. Let us again restrict ourselves to finite-dimensional Lie algebras over fields with characteristic zero, unless stated otherwise.

Definition 3.10. Let $\mathfrak{g}$ be a k-step nilpotent Lie algebra of dimension $n \geq 1$ for some $k \geq 1$, where $c(\mathfrak{g}) = k$ denotes the nilpotency class of $\mathfrak{g}$. If $c(\mathfrak{g}) = n-1$, then we call $\mathfrak{g}$ *filiform nilpotent.*

Let $\mathfrak{f}$ be the nilpotent Lie algebra with basis $\{e_1, ..., e_n\}$, where the Lie brackets are given by $[e_1, e_i] = e_{i+1}$ for $2 \leq i \leq n-1$ and all other are trivial. Let us write $\mathfrak{f}_k = span\{e_k, ..., e_n\}$. Using this notation we have $\mathfrak{f}^1 = \mathfrak{f}_3$, $\mathfrak{f}^2 = \mathfrak{f}_4$,...,$\mathfrak{f}^{n-2} = \mathfrak{f}_n$ and $\mathfrak{f}^{n-1} = 0$. Therefore, $c(\mathfrak{f}) = n-1$, which implies that $\mathfrak{f}$ is filiform nilpotent. We call this algebra the standard graded filiform nilpotent Lie algebra of dimension n.

Remark 9. We denote by $d(\mathfrak{g})$ the derived length of a Lie algebra $\mathfrak{g}$.

For the standard graded filiform nilpotent Lie algebra $\mathfrak{f}$ as mentioned above, we have

$$\begin{aligned} \mathfrak{f}^{(1)} &= [\mathfrak{f}, \mathfrak{f}] = span\{e_3, ..., e_n\} = \mathfrak{f}_3, \\ \mathfrak{f}^{(2)} &= [\mathfrak{f}^{(1)}, \mathfrak{f}^{(1)}] = [\mathfrak{f}_3, \mathfrak{f}_3] = 0. \end{aligned}$$

Hence, $\mathfrak{f}$ fulfills $d(\mathfrak{f}) = 2$.

Definition 3.11. Let $k \in \mathbb{N}_{>0}$. We denote by $\alpha(k)$ the *minimal dimension* of a nilpotent Lie algebra $\mathfrak{g}$ with derived length $d(\mathfrak{g}) = k$. Similiarly, we denote by $\beta(k)$ the *minimal dimension* of a solvable Lie algebra $\mathfrak{n}$ with derived length $d(\mathfrak{n}) = k$.

Remark 10. The definition of minimal dimension can be generalized to arbitrary fields K. In prime characteristic, Lie algebras with derived length k of smaller dimension exist, where the results vary to characteristic zero, as Bokut's example [9] will show, see Remark 12.

Definition 3.12. Let $\mathfrak{g}$ be a Lie algebra. We call a left-symmetric product $\mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies $[x, y] = x \cdot y - y \cdot x$ for all $x, y \in \mathfrak{g}$ an *affine structure on* $\mathfrak{g}$. Algebras with this product are also called *pre-Lie algebras*, see Remark 6.

Jacobson showed in [43] that, if we have a non-singuar derivation on a Lie algebra $\mathfrak{g}$ over a field K of characteristic zero, then $\mathfrak{g}$ is nilpotent. For further reading on affine structures and pre-Lie algebras, we refer to [17]. Using Corollary 3.4 in [17] and the result by Jacobson [43], we can formulate the following proposition.

Proposition 3.13. *Let* $\mathfrak{g}$ *be a Lie algebra. Suppose* $\mathfrak{g}$ *admits a non-singular derivation. Then* $\mathfrak{g}$ *is nilpotent and admits an affine structure.*

3.3.1 Lower and upper bounds for the minimal dimension of nilpotent Lie algebras with derived length k

Motivated by open questions on nildecomposable Lie algebras and their derived length, we want to analyze the minimal dimension $\alpha(k)$ of a nilpotent Lie algebra over a field K with characteristic zero, that has derived length $d(\mathfrak{g}) = k$. Explicit statements, in general, on the minimal dimension remain open. Nevertheless there are results on lower and upper bounds for $\alpha(k)$. Finding lower and upper bounds has been widely discussed. The following results rely mostly on the work of Bokut [9], who mentions the works of Hall [36], Dixmier [32] and Patterson [57]. Bradley and Sitzinger discussed the case of 2-generated Lie algebras, see [10]. We can estimate the minimal dimension $\alpha(k)$ with the upper bound $2^k - 1$ for all $k \geq 2$, as the following result shows:

Proposition 3.14. *For any $k \geq 2$ there is a filiform nilpotent Lie algebra $\mathfrak{f}$ graded by the natural numbers, defined over the field of rational numbers of dimension $2^k - 1$ with derived length $d(\mathfrak{f}) = k$. Moreover, $\mathfrak{f}$ admits a non-singular derivation, and therefore also an affine structure.*

Proof. Such a Lie algebra has been constructed in [24] with adapted basis $\{e_1, ..., e_n\}$, see [11] for the definition. Let $n \geq 3$. For the filiform nilpotent Lie algebra $\mathfrak{f}_{\frac{9}{10},n}$ of dimension n we have

$$\begin{aligned} [e_1, e_i] &= e_{i+1} && \text{for } 2 \leq i \leq n-1, \\ [e_i, e_j] &= \frac{6(j-i)}{j(j-1)\binom{j+i-2}{i-2}} e_{i+j} && \text{for } 2 \leq i \leq j, \\ & && i+j \leq n. \end{aligned}$$

Furthermore, $\mathfrak{f}_{\frac{9}{10},n}$ is in fact a Lie algebra, as seen in [24]. It has the basis elements e_1 and e_2 as generators. If $n \geq 7$, then we have

$$\begin{aligned} [e_1, e_2] &= e_3, & [e_2, e_3] &= e_5, \\ [e_2, e_4] &= e_6, & [e_2, e_5] &= \frac{9}{10} e_7, \end{aligned}$$

As before, we write again $\mathfrak{f}_k = span\{e_k, ..., e_n\}$. For $n = 2^k - 1$ the derived length of this Lie algebra is $d(\mathfrak{f}_{\frac{9}{10},n}) = k$, which we will show by induction. We claim that $\mathfrak{f}_{\frac{9}{10},n}^{(i)} = \mathfrak{f}_{2^{i+1}-1}$. For $i = 1$, we have

$$\mathfrak{f}_{\frac{9}{10},n}^{(1)} = [\mathfrak{f}_{\frac{9}{10},n}, \mathfrak{f}_{\frac{9}{10},n}] = \mathfrak{f}_3 = \mathfrak{f}_{2^2-1}.$$

Now let $i \geq 1$, by our induction hypothesis and the definition of the Lie bracket, we have

$$\begin{aligned} \mathfrak{f}_{\frac{9}{10},n}^{(i+1)} &= [\mathfrak{f}_{\frac{9}{10},n}^{(i)}, \mathfrak{f}_{\frac{9}{10},n}^{(i)}] \\ &= [\mathfrak{f}_{2^{i+1}-1}, \mathfrak{f}_{2^{i+1}-1}] = \mathfrak{f}_{2^{i+2}-1} \end{aligned}$$

Using this we obtain $\mathfrak{f}_{\frac{9}{10},n}^{(k-1)} = f_{2^k-1}$, and hence $\mathfrak{f}_{\frac{9}{10},n}^{(k)} = 0$. By definition, we get that $\mathfrak{f}_{\frac{9}{10},n}$ is positively graded. $D = diag(1, 2, ..., n)$ is an invertible derivation. To see this, consider for $2 \leq i \leq n-1$

$$D([e_1, e_i]) = D(e_{i+1}) = (i+1)e_{i+1} = ie_{i+1} + e_{i+1} = [D(e_1), e_i] + [e_1, D(e_i)],$$

and for $2 \leq i \leq j$ and $i + j \leq n$ we have

$$D([e_i, e_j]) = D(\frac{6(j-i)}{j(j-1)\binom{j+i-2}{i-2}} e_{i+j}) = (i+j)[e_i, e_j] = [D(e_i), e_j] + [e_i, D(e_j)].$$

By Proposition 3.13, this implies that $\mathfrak{f}_{\frac{9}{10},n}$ admits an affine structure. □

Example 9. For the explicit case $k = 3$, we obtain for $\mathfrak{f}_{\frac{9}{10},7}$:

$$\begin{aligned}
[e_1, e_i] &= e_{i+1} \text{ for } 2 \leq i \leq 6, \\
[e_2, e_3] &= e_5, \\
[e_2, e_4] &= e_6, \\
[e_2, e_5] &= \frac{9}{10} e_7, \\
[e_3, e_4] &= \frac{1}{10} e_7.
\end{aligned}$$

For $k = 4$, see [14].

Remark 11. In [56], Panferov also proved this upper bound for the minimal dimension. Nonetheless, we focus on the filiform nilpotent Lie algebras $\mathfrak{f}_{\frac{9}{10},n}$.

In [9], Bokut proved the following proposition, giving us a lower bound for the dimension.

Proposition 3.15. *Let $\mathfrak{g}$ be a nilpotent Lie algebra over a field K with $char(K) = 0$ and derived length $k \geq 4$. Then*

$$dim(\mathfrak{g}) \geq 2^{k-1} + 2k - 3.$$

Remark 12. This holds also for fields of prime characteristic $p \geq 5$, which can be seen in [9]. As a counterexample for $p = 2$, Bokut mentions the following nilpotent Lie algebra of dimension 12 with nilpotency class 10 and derived length 4.

$$\begin{aligned}
&[e_1, e_2] = e_4, && [e_3, e_4] = e_6, && [e_4, e_9] = e_{11}, \\
&[e_1, e_3] = e_5, && [e_3, e_7] = e_8, && [e_5, e_6] = e_8, \\
&[e_1, e_6] = e_7, && [e_3, e_9] = e_{10}, && [e_5, e_7] = e_9, \\
&[e_1, e_8] = e_9, && [e_3, e_{11}] = e_{12}, && [e_5, e_8] = e_{10}, \\
&[e_1, e_{10}] = e_{11}, && [e_4, e_5] = e_7, && [e_5, e_9] = e_{11}, \\
&[e_2, e_5] = e_6, && [e_4, e_6] = e_8, && [e_5, e_{10}] = e_{12}, \\
&[e_2, e_7] = e_8, && [e_4, e_7] = e_9, && [e_6, e_9] = e_{12}, \\
&[e_2, e_9] = e_{10}, && [e_4, e_8] = e_{10}, && [e_7, e_8] = e_{12}.
\end{aligned}$$

For $k \leq 3$, Proposition 3.15 cannot be applied. However, there are results for minimal dimensions of nilpotent Lie algebras of lower dimensions with derived length k due to classification results, see [31].

Proposition 3.16. *[14] Every nilpotent Lie algebra of dimension $n \leq 8$ has derived length $k \leq 3$. For the minimal dimension we get*

$$\alpha(1) = 1, \alpha(2) = 3, \alpha(3) = 6.$$

In [14], we see the that the derived length, of a filiform nilpotent Lie algebra over a field of charasteristic zero with dimension less than or equal to 14, is at most 3.

Proposition 3.17. *Let $\mathfrak{f}$ be a filiform nilpotent Lie algebra, whose dimension $n \leq 14$ over a field K with characteristic zero. Then we have for the derived lenght $d(\mathfrak{f}) \leq 3$.*

A question that arises now, is to study nilpotent Lie algebras of derived length $k = 4$, that are not filiform. In line with Bokut's result in [9], we consider

$$dim(\mathfrak{g}) = dim\left(\mathfrak{g}/\mathfrak{g}^{(1)}\right) + dim\left(\mathfrak{g}^{(1)}/\mathfrak{g}^{(2)}\right) + dim\left(\mathfrak{g}^{(2)}/\mathfrak{g}^{(3)}\right) + dim\left(\mathfrak{g}^{(3)}\right).$$

Now if we impose some assumptions on our Lie algebra, say that it is generated by two elements, then we can estimate these dimensions by results in [9], implying that

$$\begin{aligned} d_1 &= dim\left(\mathfrak{g}/\mathfrak{g}^{(1)}\right) \geq 2, \\ d_2 &= dim\left(\mathfrak{g}^{(1)}/\mathfrak{g}^{(2)}\right) \geq 4, \\ d_3 &= dim\left(\mathfrak{g}^{(2)}/\mathfrak{g}^{(3)}\right) \geq 6, \\ d_4 &= dim\left(\mathfrak{g}^{(3)}\right) \geq 1. \end{aligned}$$

The task is to find a nilpotent Lie algebra generated by two elements of minimal dimension, hence minimizing these dimensions. Until now no nilpotent Lie algebra with two generators is known that satisfies $(d_1, d_2, d_3, d_4) = (2, 4, 6, 1)$. However in [14] there is an example of a rational nilpotent Lie algebra of dimension 14 with $(d_1, d_2, d_3, d_4) = (2, 5, 6, 1)$.

Proposition 3.18. *There exists a rational, nilpotent Lie algebra $\mathfrak{g}$ of dimension 14 with derived length $k = 4$. For the nilpotency class we have $c(\mathfrak{g}) = 11$ and the Lie algebra is graded by positive integers.*

Proof. In [14], Burde has found such a Lie algebra, following the idea to find a rational nilpotent Lie algebra $\mathfrak{g}$ of dimension 14 generated by e_1 and e_2, and additional property that $(d_1, d_2, d_3, d_4) = (2, 5, 6, 1)$ and $c(\mathfrak{g}) \leq 12$. The following example illustrates the proof:

$$
\begin{aligned}
[e_1, e_2] &= e_3, & [e_2, e_7] &= 2e_8, & [e_3, e_{12}] &= -e_{14}, \\
[e_1, e_3] &= e_4, & [e_2, e_8] &= 2e_9, & [e_4, e_5] &= -3e_{10}, \\
[e_1, e_4] &= e_5, & [e_2, e_{10}] &= e_{11}, & [e_4, e_6] &= -3e_{11}, \\
[e_1, e_5] &= e_7, & [e_2, e_{13}] &= e_{14}, & [e_4, e_{10}] &= e_{13}, \\
[e_1, e_6] &= e_8, & [e_3, e_4] &= -e_6, & [e_4, e_{11}] &= e_{14}, \\
[e_1, e_8] &= e_{10}, & [e_3, e_5] &= -e_8, & [e_5, e_6] &= -3e_{12}, \\
[e_1, e_9] &= e_{11}, & [e_3, e_6] &= -2e_9, & [e_5, e_8] &= -e_{13}, \\
[e_1, e_{11}] &= e_{12}, & [e_3, e_7] &= 2e_{10}, & [e_5, e_9] &= -e_{14}, \\
[e_1, e_{12}] &= e_{13}, & [e_3, e_8] &= e_{11}, & [e_6, e_7] &= 2e_{13}, \\
[e_2, e_5] &= e_6, & [e_3, e_{10}] &= e_{12}, & [e_6, e_8] &= e_{14}.
\end{aligned}
$$

We have

$$
(dim(\mathfrak{g}^{(1)}), .., dim(\mathfrak{g}^{(3)})) = (12, 7, 1),
$$
$$
(dim(\mathfrak{g}^1), ..., dim(\mathfrak{g}^{10})) = (12, 11, 10, 9, 7, 6, 4, 3, 2, 1),
$$

implying $d(\mathfrak{g}) = 4$ and $c(\mathfrak{g}) = 11$. This Lie algebra also has invertible derivations. See [14] for an example. $\square$

Remark 13. The question of the minimal dimension for derived length $k = 4$ is still open. Bokut's result suggest that there is no rational nilpotent Lie algebra with $(d_1, d_2, d_3, d_4) = (2, 4, 6, 1)$. This would mean that the minimal dimension is 14 by Proposition 3.18, however this has to be varified yet.

This Lie algebra also implies a result concerning the degeneration theory of algebras, see Proposition 3.10 in [14]. For a thorough study on degenerations see [18].

Similar questions also arise in the context of the minimal dimension of a solvable Lie algebra with derived length k. It is clear that $\beta(k) \leq \alpha(k)$. Furthermore, over fields of characteristic zero, we know that if $\mathfrak{g}$ is solvable, then $\mathfrak{g}^{(1)}$ is nilpotent, see [15]. Connecting our results for nilpotent Lie algebras, we then can construct upper and lower bounds for the minimal dimension of a solvable Lie algebra with derived length k. We refer to [57] for more results. For nilpotent Lie algebras admitting non-singular derivations, results on solvability and the derived length can be found in [14] Lemma 5.1. For dimension $n \geq 7$, most cases do not admit a non-singular derivation. However, the Lie algebras in Proposition 3.14 and Proposition 3.18 do so, implying the existence of solvable rational Lie algebras of dimension 2^{k-1} and derived length k for any $k \geq 3$, and the existence of a rational, solvable Lie algebra $\mathfrak{g}$ of dimension 15 and derived length $k = 5$, see [14].

3.3.2 Nildecomposable groups and Lie algebras

We start by a brief discussion on nildecomposable groups. The study of those has resulted in similar observations in the study of nildecomposable Lie algebras. Let us recall the definition of a factorization of a group.

Definition 3.19. A *factorization of a group* G is a product $G = AB$, where A and B are two subgroups of G.

Note that the product need not be a subgroup. This is only the case if AB=BA, see [2].

The study of factorizations of a group is widely discussed in the literature, for further reading we refer to [50]. Putting assumptions on the subgroups yields different results for the group G. For G finite and A,B nilpotent, it follows from the Wielandt-Kegel theorem that G is solvable, see Theorem 2.4.4 in [2]. We call these groups *nildecomposable*.
The question of whether or not the derived length of G can be estimated by the nilpotency classes of the subgroups A and B is unclear. The first conjecture was to estimate it by the sum of the nilpotency classes, i.e.,

$$d(G) \leq c(A) + c(B).$$

In [2], there is a proof of the statement for A and B with coprime orders. The conjecture also holds for A and B abelian by Ito's result, see [40], also for infinite groups. Generally, this does not hold. The first counterexamples were given in [29]. There an example using p-groups is given. However one might be able to bound the derived length by a linear function dependent on the nilpotency classes. In [46] there is a result on bounding the derived length not by the nilpotency classes, but rather using the derived length of the subgroups.

Proposition 3.20. *Let $A, B \leq G$ be two subgroups of a solvable group G, that have coprime orders. Then the derived length $d(G)$ can be estimated through*

$$d(G) \leq 2d(A)d(B) + d(A) + d(B).$$

In [30], the following proposition is stated:

Proposition 3.21. *Let A be abelian, B nilpotent and 2-step solvable and G a factorization of A and B. Then G is solvable and the derived length is $d(G) \leq 4$.*

Similiarly to the group case, questions of the derived length of nildecomposable Lie algebras over characteristic zero were raised. Contrary to the group case, no counterexample in finite dimension for

$$d(\mathfrak{g}) \leq c(\mathfrak{a}) + c(\mathfrak{b}),$$

where $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ is a nildecomposable Lie algebra over a field K of characteristic zero, has yet been found. For the infinite case, following the counterexamples of the group case in [29], a counterexample is given in [4]. The conjecture is true if one of the summands is an ideal. If $\mathfrak{b}$ is an ideal, then we get for the derived length $d(\mathfrak{g}) \leq d(\mathfrak{g}/\mathfrak{b}) + d(\mathfrak{b})$. Using this and the fact that $\mathfrak{g}/\mathfrak{b} = \mathfrak{a}+\mathfrak{b}/\mathfrak{b} \cong \mathfrak{a}/\mathfrak{a}\cap\mathfrak{b}$, we get

$$d(\mathfrak{g}) \leq d(\mathfrak{g}/\mathfrak{b}) + d(\mathfrak{b}) \leq d(\mathfrak{a}) + d(\mathfrak{b}) \leq c(\mathfrak{a}) + c(\mathfrak{b}).$$

Analogously to the group case, Ito's result for $\mathfrak{a}$ and $\mathfrak{b}$ abelian also holds for Lie algebras, obtaining $d(\mathfrak{g}) \leq 2$, see [59]. For $\mathfrak{a}$ abelian and $\mathfrak{b}$ 2-step nilpotent, there is an estimate for the derived length, given in [58].

Proposition 3.22. *Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ be a Lie algebra over a field K with char$(K) \neq 2$, where $\mathfrak{a}$ is abelian and $\mathfrak{b}$ nilpotent of class 2. Then $d(\mathfrak{g}) \leq 10$.*

In [14], Burde suggests that this estimate might not be optimal, stating that no counterexample in this case has been found to $d(\mathfrak{g}) \leq 3$. We end this chapter with a simple case of a decomposition of a 2-step nilpotent and abelian Lie algebra.

Example 10. Let $\mathfrak{g} = \mathfrak{n}_3(\mathbb{C}) \ltimes_\rho \mathbb{C}^3$, where $\mathfrak{n}_3(\mathbb{C})$ is the Heisenberg Lie algebra of dimension 3 and $\mathbb{C}^3$ is viewed as an abelian ideal. By ρ we mean the representation

$$\rho(e_1) = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\rho(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{pmatrix},$$
$$\rho(e_3) = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix},$$

where (e_1, e_2, e_3) is a basis of $\mathfrak{n}_3(\mathbb{C})$ and $[e_1, e_2] = e_3$. Let (e_4, e_5, e_6) be a basis for $\mathbb{C}^3$, then we obtain for the Lie brackets of $\mathfrak{g}$

$$[e_1, e_2] = e_3, \qquad [e_2, e_6] = e_5$$
$$[e_1, e_5] = e_4, \qquad [e_3, e_6] = e_4.$$

The undefined Lie brackets are equal to zero. This Lie algebra is 2-step solvable, and hence $d(\mathfrak{g}) = 2 \leq c(\mathfrak{a}) + c(\mathfrak{b}) = 3$.

4 Outlook

We conclude this thesis by stating some open questions and ongoing research. We provide an overview of questions that have been raised in the course of the thesis. On the one hand, we have existence and classification problems of post-Lie algebra structures, on the other, we can look into nildecomposable Lie algebras.

Some questions remain open in the study of post-Lie algebra structures. It would be interesting to look at the case where both Lie algebras are semisimple. We know that, if $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is simple, and assuming a post-Lie algebra exists, then $\mathfrak{g}$ is isomorphic to $\mathfrak{n}$. We have also discussed the case where $\mathfrak{g}$ is semisimple and $\mathfrak{n}$ is complete. But what happens if $\mathfrak{n}$ is semisimple? No results on an isomorphism have yet been found.

Question 1. Assume there exists a PA-structure on $(\mathfrak{g}, \mathfrak{n})$, where $\mathfrak{g}$ and $\mathfrak{n}$ are semisimple, but not simple. Is $\mathfrak{g} \cong \mathfrak{n}$?

One approach could be to proceed as in the case of reductive and semisimple Lie algebras, to use Rota-Baxter operators in studying whether or not these Lie algebras are isomorphic, see Conjecture 1.

Furthermore, results on the nilpotency of the left multiplication operator for stem Lie algebras can be further investigated. We have seen that for the special case of CPA-structures this holds.

Question 2. Let $\mathfrak{g}$ and $\mathfrak{n}$ be nilpotent stem Lie algebras and $x \cdot y$ a PA-structure on $(\mathfrak{g}, \mathfrak{n})$. Are all left multiplication operators $L(x)$ nilpotent?

We have seen that by imposing certain properties on the post-Lie algebra structure, we can then associate a CPA-structure on the pair of Lie algebras. What conditions do $\mathfrak{g}$ and $\mathfrak{n}$ have to fulfill such that the anti-commutator defines a CPA-structure? This remains open.

As the table at the end of Chapter 2 shows, some existence questions of PA-structures are yet to be solved. It would be nice to see what happens when imposing different properties on the Lie algebras. The question marks can be either transformed in, there is a pair of Lie algebras that admit a post-Lie algebra structure, or, there is no pair of Lie algebras admitting a post-Lie algebra structure.

A general classification for post-Lie algebras is very hard. Even in low dimension, the classification results are complicated. However, for the special case of commutative post-Lie algebra structures, classification works a bit easier.

The discussion in Chapter 3 regarding the minimal dimension of a rational nilpotent Lie algebra, shows that one still needs to prove that there is no rational nilpotent Lie algebra with derived length 4 and $(d_1, d_2, d_3, d_4) = (2, 4, 6, 1)$, implying that the minimal dimension of a rational nilpotent Lie algebra with derived length 4 is 14.

Finally, though not directly connected to Rota-Baxter operators, bounding the derived length by a lower bound, see Proposition 3.22, could be explored. The question of bounding the derived length by the nilpotency classes in general in the case of Lie algebras is open.

Question 3. Let $\mathfrak{g} = \mathfrak{a} + \mathfrak{b}$ be a nildecomposable Lie algebra over a field of characteristic zero. Do we have

$$d(\mathfrak{g}) \leq c(\mathfrak{a}) + c(\mathfrak{b})?$$

Maybe one attempt could be to follow the group case, and inspired by that, get new results for Lie algebras.

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