MASTER'S THESIS

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Engel Lie algebras

submitted by

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Summary

The purpose of this thesis is to present an investigation of Engel-n Lie algebras. In addition to the defining relations of Lie algebras these satisfy the so-called Engel-n identity $\mathrm{ad}(x)^n = 0$ for all x. Engel Lie algebras arise in the study of the Restricted Burnside Problem, which was solved by Efim Zelmanov in 1991. Beside a general introduction to the topic, special interest is taken in the exploration of the nilpotency classes of Engel-n Lie algebras for small values of n. At this, the primary objective is to elaborate the case of $n = 3$ explicitly.

Chapter 1 concerns the general theory of Lie algebras. In the course of this, the essential properties of solvability and nilpotency are explained as they will be central in the subsequent discussion. Further, the definition of free Lie algebras which contributes to the establishment of the concept of free-nilpotent Lie algebras. In the last section several notions of group theory are surveyed. These will be useful in Chapter 2.

The second chapter explains the origin and solution of the Burnside Problems. In that process, a historical survey on William Burnside and the first results on his fundamental questions are given. Next, the so-called Restricted Burnside Problem is considered and an overview of the most important steps to the solution is displayed. In particular, the connection to Lie theory is explained. As a last aspect, several results on Engel groups are discussed.

The main aim in the third chapter is to prove the nilpotency theorem of Engel-3 Lie algebras. That is, on the assumption that the underlying field is of characteristic different from two or five, an Engel-3 Lie algebra has nilpotency class at most 4. On top of that, the construction of an example of an Engel-3 Lie algebra that attains this upper bound is demonstrated in detail.

As a conclusion, Chapter 4 is devoted to an outlook on nilpotency classes of Engel-4 and Engel-5 Lie algebras. It is stated that for characteristics different from one of $2, 3$ or 5, the nilpotency class of Engel-4 Lie algebras is at most 7. For the case of Engel-5 Lie algebras an upper bound is given with regard to the number of generators of the Lie algebra.

In the appendix, a number of Mathematica codes are presented. The programs automise various tasks that have to be accomplished in connection with the construction of Engel Lie algebras.

Zusammenfassung

Der Zweck vorliegender Masterarbeit ist es Engel-n Lie-Algebren zu untersuchen. Dies sind Lie-Algebren, die, zusätzlich zu den definierenden Relationen von Lie-Algebren, der sogenannten Engel-Identität $\mathrm{ad}(x)^n = 0$ für alle x, genügen. Engel Lie-Algebren treten in Zusammenhang mit der Untersuchung des Eingeschränkten Burnside-Problems auf, welches von Efim Zelmanov 1994 gelöst wurde. Neben einer allgemeinen Einführung in das Thema ist insbesondere das Studium der Nilpotenzklassen von Engel-n Lie-Algebren für kleine n der Fokus dieser Arbeit. Dabei ist das primäre Ziel die explizite Ausarbeitung des Falles $n = 3$.

Kapitel 1 behandelt die allgemeine Theorie von Lie-Algebren. Im Zuge dessen werden die essentiellen Eigenschaften Auflösbarkeit und Nilpotenz beschrieben, welche eine zentrale Rolle in nachfolgenden Überlegungen einnehmen. Desweiteren wird die Definition freier Lie-Algebren erläutert, mit Hilfe welcher der Begriff frei-nilpotenter Lie-Algebren definiert wird. Abgesehen davon wird eine Übersicht wichtiger gruppentheoretischer Begriffe, welche in späteren Kapiteln Anwendung finden, gegeben.

Das zweite Kapitel befasst sich mit dem Ursprung und der Lösung der Burnside'schen Probleme. Es erfolgt ein historischer Überblick und erste Resultate bezüglich Burnsides fundamentaler Fragestellungen werden angegeben. Speziell wird das sogenannte Eingeschränkte Burnside'sche Problem betrachtet und die wichtigsten Schritte zu dessen Lösung präsentiert. Dabei wird insbesondere die Verbindung zur Lie-Theorie beschrieben. Darauffolgend werden einige Theoreme bezüglich Engel Gruppen diskutiert.

Ziel des dritten Kapitels ist der Beweis des Nilpotenz-Theorems für Engel-3 Lie-Algebren. Dieses besagt, dass eine Engel-3 Lie-Algebra höchstens Nilpotenzklasse 4 aufweist, sofern der zugrundeliegende Körper Charakteristik ungleich zwei oder fünf besitzt. Darüber hinaus erfolgt die detaillierte Konstruktion einer Engel-3 Lie-Algebra, welche diese obere Schranke tatsächlich annimmt.

Abschließend widmet sich Kapitel 4 einem Ausblick auf die Nilpotenzklassen von Engel-4 und Engel-5 Lie-Algebren. Für Charakteristik ungleich $2, 3$ oder 5 gilt die Behauptung, dass Engel-4 Lie-Algebren höchstens nilpotent der Stufe 7 sind. Im Fall von Engel-5 Lie-Algebren wird eine obere Schranke angegeben, die auf der Anzahl der Erzeuger der Lie-Algebra beruht.

Im Appendix werden mehrere hilfreiche Mathematica Codes präsentiert. Diese übernehmen verschiedene Aufgaben, die in Verbindung mit der Konstruktion von Engel Lie-Algebren zu bewältigen sind.

Contents

Chapter 1

Preliminaries on Lie algebras and groups

The Burnside Problems are group-theoretical questions but it turned out successful to work on the problems on the level of Lie algebras. Thus, for a discussion of the topic we need knowledge of both, group theory and Lie algebra theory. This section provides these preliminaries. Regarding Lie algebras, we give basic definitions and explain the properties of solvability and nilpotency. Next, we introduce free Lie algebras and study the concept of free-nilpotent Lie algebras. On the other hand, we recall necessary notions of group theory.

1.1 Basic theory of Lie algebras

We start with an introduction including basic definitions, examples and properties of Lie algebras. Moreover, we study representations and derivations of Lie algebras. The section is concluded by the definition of Engel Lie algebras which will be central in the further discussion. As references of the general theory of Lie algebras we mention [7], [12] or [39].

Definition 1.1.1. A *Lie algebra* $\mathfrak{g}$ over a field $\mathbb{F}$ is a $\mathbb{F}$-vector space together with a $\mathbb{F}$-bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \longrightarrow \mathfrak{g}$, the so-called Lie bracket, which satisfies the following two conditions for all $x, y, z \in \mathfrak{g}$:

(i) **Skew-symmetry:** $[x, x] = 0$,

(ii) **Jacobi identity:** $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$.

The name of property (i) is justified by seeing that, for $\mathrm{char}(\mathbb{F}) \neq 2$ and $x, y \in \mathfrak{g}$, we have

$$[x, x] = 0 \Longleftrightarrow [x, y] = -[y, x].$$

Indeed, for " $\Longrightarrow$ " one obtains

$$0 = [x + y, x + y] = [x, x] + [x, y] + [y, x] + [y, y] = [x, y] + [y, x],$$

using the bilinearity of the bracket. Regarding " $\Longleftarrow$ " we set $x = y$ and immediately arrive at $[x, x] = -[x, x]$ and thus, $2[x, x] = 0$. Since we required $\operatorname{char}(\mathbb{F}) \neq 2$, we conclude $[x, x] = 0$. Hence, if the field is of characteristic 2, only " $\Longrightarrow$ " holds.

The left side of the second property is often referred to as the Jacobian of x, y, z, written $J(x, y, z)$.

Notice that due to the non-associativity of the Lie product, the way of bracketing is important. To avoid confusion, we agree on using right normed notation, i.e. writing $[x_1, [x_2, \ldots, [x_{n-1}, x_n] \ldots]]$ for the Lie product of $x_1, \ldots, x_n$.

Example 1.1.2. Any vector space over a field can be turned into a Lie algebra if we endow it with the trivial Lie bracket, i.e. setting $[x, y] = 0$ for all its elements. Lie algebras with this property are called *abelian*.
Another example is the direct sum $\mathfrak{g} \oplus \mathfrak{h}$ of two Lie algebras $\mathfrak{g}, \mathfrak{h}$. Here, the vector space is simply $\mathfrak{g} \times \mathfrak{h}$ and the bracket operation is performed componentwise, that is $[(x_1, y_1), (x_2, y_2)] = ([x_1, x_2], [y_1, y_2])$.
We also mention the vector space $\mathbb{R}^3$ together with $[x, y] = x \times y$ being the cross product which is the Lie algebra $\mathfrak{so}(3, \mathbb{R})$, see Remark 1.1.10.

Definition 1.1.3. Let $\mathfrak{g}, \mathfrak{h}$ be two Lie algebras over a field $\mathbb{F}$. We define a *Lie algebra homomorphism* $\varphi : \mathfrak{g} \to \mathfrak{h}$ to be a linear map that preserves the bracket, that is, for $x, y \in \mathfrak{g}$ we require

$$\varphi([x, y]) = [\varphi(x), \varphi(y)].$$

The terms epi-, mono-, auto- and isomorphisms of Lie algebras are defined as usual.

Let $\mathfrak{k}, \mathfrak{l}$ be two subspaces of $\mathfrak{g}$. Define $[\mathfrak{k}, \mathfrak{l}]$ to be the subspace generated by the brackets $[x, y]$, $x \in \mathfrak{k}, y \in \mathfrak{l}$. Thus, each element of $[\mathfrak{k}, \mathfrak{l}]$ is a linear combination of brackets $[x_i, y_i]$ for $x_i \in \mathfrak{k}, y_i \in \mathfrak{l}$. This notation allows us to give the following definition.

Definition 1.1.4. A subspace $\mathfrak{k}$ of $\mathfrak{g}$ is called a *Lie subalgebra* of $\mathfrak{g}$ if

$$[\mathfrak{k}, \mathfrak{k}] \subseteq \mathfrak{k},$$

i.e. if it is closed under the bracket. Then $\mathfrak{k}$ itself becomes a Lie algebra in its own right with the inherited operations.
Moreover, $\mathfrak{i} \subseteq \mathfrak{g}$ is said to be an *ideal* or *Lie ideal* of $\mathfrak{g}$ if

$$[\mathfrak{g}, \mathfrak{i}] = [\mathfrak{i}, \mathfrak{g}] \subseteq \mathfrak{i},$$

written $\mathfrak{i} \trianglelefteq \mathfrak{g}$. In this definition we used that the bracket of subspaces is commutative. Indeed, $[\mathfrak{k}, \mathfrak{l}] \ni [x, y] = -[y, x] \in [\mathfrak{l}, \mathfrak{k}]$, hence $[\mathfrak{k}, \mathfrak{l}] \subseteq [\mathfrak{l}, \mathfrak{k}]$. Analogously one obtains $[\mathfrak{l}, \mathfrak{k}] \subseteq [\mathfrak{k}, \mathfrak{l}]$.

Similar to ring and group theory, one can define a *quotient Lie algebra*. To be more precise, if $\mathfrak{i} \trianglelefteq \mathfrak{g}$ we can construct the quotient algebra $\mathfrak{g}/\mathfrak{i}$ by setting

$$[x + \mathfrak{i}, y + \mathfrak{i}] = [x, y] + \mathfrak{i}$$

for two equivalence classes $x + \mathfrak{i}, y + \mathfrak{i}$. Let $x' = x + i, y' = y + j$. Then,

$$[x', y'] = [x + i, y + j] = [x, y] + [x, j] + [i, y] + [i, j]$$

applying bilinearity of the bracket. This prompts the independence of the choice of the representative since $[x, j] + [i, y] + [i, j] \in \mathfrak{i}$ by the ideal property.

The following auxiliary result enables us to state the isomorphism theorems.

Lemma 1.1.5. *Let $\mathfrak{i}, \mathfrak{j}$ be ideals of a Lie algebra $\mathfrak{g}$. Then:*

(i) The bracket of $\mathfrak{i}$ and $\mathfrak{j}$ is again an ideal of $\mathfrak{g}$, i.e. $[\mathfrak{i}, \mathfrak{j}] \trianglelefteq \mathfrak{g}$.

(ii) The intersection of $\mathfrak{i}$ and $\mathfrak{j}$ is again an ideal of $\mathfrak{g}$, i.e. $\mathfrak{i} \cap \mathfrak{j} \trianglelefteq \mathfrak{g}$.

Proof. To begin with, we prove $[[\mathfrak{i}, \mathfrak{j}], \mathfrak{g}] \subseteq [\mathfrak{i}, \mathfrak{j}]$. By the Jacobi identity, $[[i, j], x] + [[j, x], i] + [[x, i], j] = 0$ and therefore $[[i, j], x] = [i, [j, x]] + [[i, x], j]$ which lies in $[\mathfrak{i}, \mathfrak{j}]$. Concerning (ii), note that if we are given $i \in \mathfrak{i} \cap \mathfrak{j}$ then $[i, x] \in \mathfrak{i}$ and $[i, x] \in \mathfrak{j}$ and hence $[i, x]$ is contained in the intersection. □

Theorem 1.1.6 (Isomorphism theorems)**.** *Let $\varphi : \mathfrak{g} \to \mathfrak{h}$ be a Lie algebra homomorphism and let $\mathfrak{i}, \mathfrak{j}$ be two ideals of $\mathfrak{g}$. Then:*

1. $\mathfrak{g}/\mathrm{ker}\varphi \cong \mathrm{im}\varphi$.
2. $(\mathfrak{g}/\mathfrak{i})/(\mathfrak{j}/\mathfrak{i}) \cong \mathfrak{g}/\mathfrak{j}$ *whenever* $\mathfrak{i} \subseteq \mathfrak{j}$.
3. $(\mathfrak{i} + \mathfrak{j})/\mathfrak{j} \cong \mathfrak{i}/(\mathfrak{i} \cap \mathfrak{j})$.

Proof. The proof is omitted since the isomorphisms are given in a canonical way and the procedure is analogous to the case of groups. □

Further on, we want to obtain more explicit examples of Lie algebras. For this purpose, we make use of the following Lemma which claims that by defining the Lie bracket as the commutator any associative $\mathbb{F}$-algebra is turned into a Lie algebra.

Lemma 1.1.7. *Let A be an associative algebra over a field $\mathbb{F}$, i.e. a $\mathbb{F}$-vector space with an associative, bilinear map $(x, y) \mapsto x \cdot y$ for elements $x, y \in A$. Then $[x, y] := x \cdot y - y \cdot x$ defines a Lie algebra structure on A.*

Proof. By definition, $[x, y] = -[y, x]$ hence the bracket is skew-symmetric. The Jacobi identity is verfied by expanding the product. □

Example 1.1.8. The pair $(M(n, \mathbb{F}), .)$ of $n \times n$ matrices together with the matrix multiplication is an associative algebra over $\mathbb{F}$. Then Lemma 1.1.7 yields a Lie algebra denoted by $\mathfrak{gl}(n, \mathbb{F})$, called the *general linear Lie algebra*. Moreover, the Lie subalgebra $[\mathfrak{gl}(n, \mathbb{F}), \mathfrak{gl}(n, \mathbb{F}))]$ gives another important example of a Lie algebra.

Lemma 1.1.9. *The commutator algebra of* $\mathfrak{gl}(n, \mathbb{F})$ *takes the following form:*

$$[\mathfrak{gl}(n, \mathbb{F}), \mathfrak{gl}(n, \mathbb{F}))] = \{A \in \mathfrak{gl}(n, \mathbb{F}) : \mathrm{tr}(A) = 0\} =: \mathfrak{sl}(n, \mathbb{F}),$$

called the special linear Lie algebra*. Additionally,*

$$\dim(\mathfrak{sl}(n, \mathbb{F})) = n^2 - 1.$$

Proof. Note that $\dim(\mathfrak{gl}(n, \mathbb{F})) = n^2$ since the matrices E_{ij}, that have a 1 in the i-th row and j-th column and 0 in all other positions, form a basis. For this matrices we have the relation

$$[E_{ij}, E_{kl}] = \delta_{jk} E_{il} - \delta_{li} E_{kj}.$$

In particular, for $i \neq j$, we see $[E_{ij}, E_{ji}] = E_{jj} - E_{ii}$ and $[E_{ik}, E_{kj}] = E_{ij}$. It can be verified that the matrices E_{ij} for $i \neq j$ and the $E_{ii} - E_{i+1,i+1}$ for $1 \leq i \leq n-1$ form a basis of $\mathfrak{sl}(n, \mathbb{F})$. However, according to the calculation above, all of them can be expressed in terms of the Lie bracket, whence $\mathfrak{sl}(n, \mathbb{F}) \subseteq [\mathfrak{gl}(n, \mathbb{F}), \mathfrak{gl}(n, \mathbb{F})]$. For the dimension, we count

$$\dim(\mathfrak{sl}(n, \mathbb{F})) = (n^2 - n) + (n - 1) = n^2 - 1.$$

Regarding the opposite inclusion, we take $A, B \in \mathfrak{gl}(n, \mathbb{F})$ and compute

$$\mathrm{tr}([A, B]) = \mathrm{tr}(AB - BA) = \mathrm{tr}(AB) - \mathrm{tr}(BA) = 0,$$

using basic properties of the trace functional. □

Remark 1.1.10. There are several other well-studied matrix Lie algebras we mention at this point:

- The Lie algebra of skew-symmetric matrices $\mathfrak{so}(n, \mathbb{F})$.
- The Lie algebra of upper triangular matrices $\mathfrak{t}(n, \mathbb{F})$.
- The Lie algebra of strictly upper triangular matrices $\mathfrak{n}(n, \mathbb{F})$.
- The Lie algebra of diagonal matrices $\mathfrak{d}(n, \mathbb{F})$.

Apart from that, we denote by $\mathfrak{gl}(A)$ the Lie algebra that arises from $\mathrm{End}(A)$ using Lemma 1.1.7. Here,

$$\mathrm{End}(A) = \{\varphi : A \to A \text{ such that } \varphi \text{ is a homomorphism}\}.$$

Choosing $A = \mathbb{F}^n$ results in $\mathfrak{gl}(A) = \mathfrak{gl}(\mathbb{F}^n) = \mathfrak{gl}(n, \mathbb{F})$.

Definition 1.1.11. A *representation* of a Lie algebra $\mathfrak{g}$ is defined as a $\mathbb{F}$-vector space V together with a Lie algebra homomorphism $\varphi : \mathfrak{g} \to \mathfrak{gl}(V)$. The representation is called *faithful* if φ is injective. In this context we state Ado's theorem which gives an important insight into finite-dimensional Lie algebras.

Theorem 1.1.12 (Ado, [39],[3])**.** *Every finite-dimensional Lie algebra over a field* $\mathbb{F}$ *of characteristic* 0 *possesses a finite-dimensional faithful representation.*

Remark 1.1.13. In particular, Ado's theorem implies that any finite-dimensional Lie algebra is isomorphic to a subalgebra of endomorphisms, allowing us to view any such Lie algebra as a Lie algebra of square matrices.
The mentioned theorem also holds for fields of prime characteristic as K. Iwasawa proved in [29].

Definition 1.1.14. Let $\mathfrak{g}$ be a Lie algebra over $\mathbb{F}$. For $x \in \mathfrak{g}$ we define the *adjoint endomorphism*

$$\begin{aligned} &\mathrm{ad}(x) : \mathfrak{g} \to \mathfrak{g}, \\ &\mathrm{ad}(x)(y) := [x, y]. \end{aligned}$$

Then the linear mapping ad: $\mathfrak{g} \to \mathfrak{gl}(\mathfrak{g})$ with $x \mapsto \mathrm{ad}(x)$ defines a representation, called the *adjoint representation of* $\mathfrak{g}$. Observe that

$$\begin{aligned} \ker(\mathrm{ad}(x)) &= \{x \in \mathfrak{g} : \mathrm{ad}(x)(y) = 0 \ \ \forall y \in \mathfrak{g}\} \\ &= \{x \in \mathfrak{g} : [x, y] = 0 \ \ \forall y \in \mathfrak{g}\} \\ &=: Z(\mathfrak{g}), \end{aligned}$$

the *center* of the Lie algebra $\mathfrak{g}$.

It is important to add that the adjoint endomorphisms are *derivations* of $\mathfrak{g}$ in the sense of the following definition.

Definition 1.1.15. A linear map $D : \mathfrak{g} \to \mathfrak{g}$ satisfying

$$D([x, y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in \mathfrak{g}$ is called a *derivation* of the Lie algebra $\mathfrak{g}$. The set of all derivations of $\mathfrak{g}$ is denoted by $\mathfrak{der}(\mathfrak{g})$ and is easily seen to be a vector space. One actually has more structure:

Proposition 1.1.16. *Let* $\mathfrak{g}$ *be a Lie algebra over* $\mathbb{F}$*. Then the following assertions hold:*

1. *The derivations* $\mathfrak{der}(\mathfrak{g})$ *form a Lie subalgebra of* $\mathfrak{gl}(\mathfrak{g})$*.*
2. *For all* $x \in \mathfrak{g}$ *the adjoint endomorphism* $\mathrm{ad}(x) \in \mathfrak{der}(\mathfrak{g})$*.*

3. $\mathrm{ad}(\mathfrak{g})$ *is a Lie ideal in* $\mathfrak{der}(\mathfrak{g})$*.*

Proof. Regarding the first statement, we have to show that the commutator of two derivations is again a derivation. Let $D_1, D_2 \in \mathfrak{der}(\mathfrak{g})$ and $x, y \in \mathfrak{g}$. Then one calculates

$$\begin{aligned} D_1D_2([x,y]) - D_2D_1([x,y]) = \\ &= D_1([D_2(x),y] + [x,D_2(y)]) - D_2([D_1(x),y] + [x,D_1(y)]) \\ &= [D_1D_2(x),y] + [D_2(x),D_1(y)] + [D_1(x),D_2(y)] + [x,D_1D_2(y)] \\ &- [D_2D_1(x),y] - [D_1(x),D_2(y)] - [D_2(x),D_1(y)] - [x,D_2D_1(y)] \\ &= [D_1D_2(x) - D_2D_1(x),y] + [x,D_1D_2(y) - D_2D_1(y)]. \end{aligned}$$

In order to see the second assertion, let $x, y, z \in \mathfrak{g}$ and observe

$$\begin{aligned} \mathrm{ad}(x)([y,z]) &= [x,[y,z]] = -[y,[z,x]] - [z,[x,y]] \\ &= [[x,y],z] + [y,[x,z]] \\ &= [\mathrm{ad}(x)(y),z] + [y,\mathrm{ad}(x)(z)]. \end{aligned}$$

To prove the third item we show that for every $D \in \mathfrak{der}(\mathfrak{g})$ we have $[D,\mathrm{ad}(x)] = \mathrm{ad}(D(x))$. To this end, compute

$$\begin{aligned} [D,\mathrm{ad}(x)](y) &= (D \cdot \mathrm{ad}(x))(y) - (\mathrm{ad}(x)D)(y) \\ &= D([x,y]) - [x,D(y)] \\ &= [D(x),y] + [x,D(y)] - [x,D(y)] \\ &= [D(x),y] = \mathrm{ad}(D(x))(y). \end{aligned}$$

□

Remark 1.1.17. Notice that we can define derivations for any $\mathbb{F}$-algebra A. In fact, one simply defines a linear map $D \in \mathrm{End}(A)$ to be a *derivation* of A if it satisfies

$$D(x \cdot y) = D(x) \cdot y + x \cdot D(y)$$

for all $x, y \in A$. For instance, let $A = C^\infty(\mathbb{R})$. Then A is an algebra over $\mathbb{R}$ and the map $D : C^\infty(\mathbb{R}) \to C^\infty(\mathbb{R})$ with $D(f) = f'$ is a derivation of $C^\infty(\mathbb{R})$ by the product rule.

In the following example we display the procedure to compute the adjoint representation of a given Lie algebra.

Example 1.1.18. Consider the special linear Lie algebra $\mathfrak{sl}(2, \mathbb{F})$ of 2×2 matrices over a field $\mathbb{F}$ together with the commutator as Lie product. From 1.1.9 we know that $\{e_1, e_2, e_3\}$ with

$$e_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \ e_2 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \ e_3 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

is a basis of $\mathfrak{sl}(2, \mathbb{F})$. That means we can view them as three independent vectors that span this 3-dimensional space. Calculating the Lie brackets of the basis elements gives:

$[\ ,\ ]$	e_1	e_2	e_3
e_1	0	e_3	$-2e_1$
e_2	$-e_3$	0	$2e_2$
e_3	$2e_1$	$-2e_2$	0

By linear extension, this table defines the Lie bracket on all of $\mathfrak{sl}(2, \mathbb{F})$. As a next step, we compute the adjoint endomorphisms $\mathrm{ad}(e_1)$, $\mathrm{ad}(e_2)$, $\mathrm{ad}(e_3)$. These are linear transformations of $\mathfrak{sl}(2, \mathbb{F})$ into itself. Hence, one actually obtains a matrix representation of them in terms of the above basis. For example, $\mathrm{ad}(e_1)$ results from our table by looking at the effect of bracketing with e_1, i.e.

$$\begin{aligned} [e_1, e_1] &= 0 = 0 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3, \\ [e_1, e_2] &= e_3 = 0 \cdot e_1 + 0 \cdot e_2 + 1 \cdot e_3, \\ [e_1, e_3] &= -2e_1 = -2 \cdot e_1 + 0 \cdot e_2 + 0 \cdot e_3. \end{aligned}$$

Thus, we get columns $(0, 0, 0), (0, 0, 1)$ and $(-2, 0, 0)$. Continuing this process for the actions of e_2 and e_3 we get

$$\mathrm{ad}(e_1) = \begin{pmatrix} 0 & 0 & -2 \\ 0 & 0 & 0 \\ 0 & 1 & 0 \end{pmatrix}, \ \mathrm{ad}(e_2) = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 2 \\ -1 & 0 & 0 \end{pmatrix}, \ \mathrm{ad}(e_3) = \begin{pmatrix} 2 & 0 & 0 \\ 0 & -2 & 0 \\ 0 & 0 & 0 \end{pmatrix}.$$

So we have a 3-dimensional adjoint representation of $\mathfrak{sl}(2, \mathbb{F})$.

Many of the results in Chapter 2 hold for Lie rings which constitute a generalisation of Lie algebras.

Definition 1.1.19. A *Lie ring* is an abelian group $(\mathfrak{g}, +)$ together with a Lie bracket $[,] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$ that satisfies

(i) $\mathbb{Z}$-bilinearity,
(ii) skew-symmetry,
(iii) the Jacobi identity.

Thus, any Lie algebra is a Lie ring if we consider it over an abelian group instead of a field.

We conclude the chapter with the definition of the eponymous notion of Engel Lie algebras which are of major interest in the following work.

Definition 1.1.20. A Lie algebra $\mathfrak{g}$ in which

$$\mathrm{ad}(x)^n = 0$$

holds for all $x \in \mathfrak{g}$ is called an *Engel-n Lie algebra*. The equation $\mathrm{ad}(x)^n = 0$ is called the *Engel-n identity*. This type of Lie algebra is central to the solution of the Restricted Burnside Problem, see Section 2.2. Besides, Engel Lie algebras are of interest in their own right and will be further investigated in Chapter 3.

1.2 Solvable and nilpotent Lie algebras

In this section we introduce the notions of solvable and nilpotent Lie algebras as they will be important in the subsequent discussion. As in group theory, this properties are defined in terms of the derived series and the lower central series.

Definition 1.2.1. Let $\mathfrak{g}$ be a Lie algebra. Set $\mathfrak{g}^{(0)} := \mathfrak{g}$ and define $\mathfrak{g}^{(m+1)}$ inductively as $\mathfrak{g}^{(m+1)} := [\mathfrak{g}^{(m)}, \mathfrak{g}^{(m)}]$. Certainly, it is $\mathfrak{g}^{(m)} \subseteq \mathfrak{g}^{(m-1)}$ and by Lemma 1.1.5 we know that $\mathfrak{g}^{(m)} \trianglelefteq \mathfrak{g}$ for all m. The decreasing sequence

$$\mathfrak{g} = \mathfrak{g}^{(0)} \supseteq \mathfrak{g}^{(1)} \supseteq \cdots \supseteq \mathfrak{g}^{(m)} \supseteq \cdots$$

is called the *derived series* of $\mathfrak{g}$. If there exists an integer $c \in \mathbb{N}$ such that $\mathfrak{g}^{(c)} = 0$ and $\mathfrak{g}^{(c-1)} \neq 0$ then we say $\mathfrak{g}$ is *of derived length c* or *solvable of class c*.

Further, define $\mathfrak{g}^0 := \mathfrak{g}$ and $\mathfrak{g}^{m+1} := [\mathfrak{g}, \mathfrak{g}^m] \subseteq \mathfrak{g}^m$ in an inductive manner. Again, by means of Lemma 1.1.5, all $\mathfrak{g}^m$ are ideals in $\mathfrak{g}$. The decreasing sequence

$$\mathfrak{g} = \mathfrak{g}^0 \supseteq \mathfrak{g}^1 \supseteq \cdots \supseteq \mathfrak{g}^m \supseteq \cdots$$

is called *lower central series* of $\mathfrak{g}$. If we have $\mathfrak{g}^c = 0$ and $\mathfrak{g}^{c-1} \neq 0$ for some $c \in \mathbb{N}$, we say $\mathfrak{g}$ is *nilpotent of class c* or *c-nilpotent*.

Lemma 1.2.2. *Let $\mathfrak{g}$ be a Lie algebra. Then $\mathfrak{g}^{(m)} \subseteq \mathfrak{g}^m$ for all $m \in \mathbb{N}$. Consequently, every nilpotent Lie algebra is solvable.*

Proof. We use induction on m. For $m = 0$ we have $\mathfrak{g}^{(0)} = \mathfrak{g} = \mathfrak{g}^0$. Now suppose $\mathfrak{g}^{(m-1)} \subseteq \mathfrak{g}^{m-1}$. By definition, $\mathfrak{g}^{(m-1)}$ and $\mathfrak{g}^{m-1}$ are both subsets of $\mathfrak{g}$, thus

$$\mathfrak{g}^{(m)} = [\mathfrak{g}^{(m-1)}, \mathfrak{g}^{(m-1)}] \subseteq [\mathfrak{g}, \mathfrak{g}^{m-1}] = \mathfrak{g}^m.$$

□

Lemma 1.2.3. *Let* $\mathfrak{g}$ *be a Lie algebra. Then*

$$\mathfrak{g} \text{ is } c\text{-nilpotent} \Longleftrightarrow \mathrm{ad}(x_1)\cdots \mathrm{ad}(x_{c-1}) = 0$$

for all $x_1, \ldots, x_c \in \mathfrak{g}$. *Note that here multiplication denotes the composition of morphisms.*

Proof. By definition,

$$\mathfrak{g}^c = [\mathfrak{g}, \mathfrak{g}^{c-1}] = [\mathfrak{g}^{c-1}, \mathfrak{g}] = \{\text{linear combinations of } [x_1, [x_2, [\ldots, [x_{c-1}, y]] \ldots]]\}.$$

So $\mathfrak{g}^c = 0 \Longleftrightarrow [x_1, [x_2, [\ldots, [x_{c-1}, y]] \ldots]] = 0 \Longleftrightarrow \mathrm{ad}(x_1)\cdots \mathrm{ad}(x_{c-1}) = 0$. Equivalently, $\mathfrak{g}$ is of nilpotency class c if and only if every Lie product of c elements vanishes. □

Example 1.2.4. The *Heisenberg Lie algebra* $\mathfrak{n}(3, \mathbb{F})$ of strictly upper triangular 3×3 matrices is nilpotent of class 2. Indeed, for arbitrary matrices $A, B, C \in \mathfrak{n}(3, \mathbb{F})$ one directly computes

$$\begin{aligned}[A, [B, C]] &= [\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, [\begin{pmatrix} 0 & r & s \\ 0 & 0 & t \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & x & y \\ 0 & 0 & z \\ 0 & 0 & 0 \end{pmatrix}]] \\ &= [\begin{pmatrix} 0 & a & b \\ 0 & 0 & c \\ 0 & 0 & 0 \end{pmatrix}, \begin{pmatrix} 0 & 0 & -tx + rz \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}] = 0.\end{aligned}$$

By Lemma 1.2.3 this is equivalent to the 2-nilpotency of the Heisenberg Lie algebra. Notice that $\mathfrak{n}(3, \mathbb{F})$ is also an example of a solvable Lie algebra.

Proposition 1.2.5. *Let* $\mathfrak{g}$ *be a nilpotent Lie algebra. Then:*

(i) Lie subalgebras and homomorphic images of $\mathfrak{g}$ *are nilpotent.*

(ii) Let $0 \neq \mathfrak{i} \trianglelefteq \mathfrak{g}$. *Then* $\mathfrak{i} \cap Z(\mathfrak{g}) \neq 0$. *In particular, the center of* $\mathfrak{g}$ *is nontrivial.*

(iii) Let $0 \longrightarrow \mathfrak{i} \longrightarrow \mathfrak{k} \longrightarrow \mathfrak{g} \longrightarrow 0$ *be a short exact sequence of Lie algebras such that* $\mathfrak{i} \subset Z(\mathfrak{g})$ *and* $\mathfrak{g} \cong \mathfrak{k}/\mathfrak{i}$. *Then* $\mathfrak{k}$ *is nilpotent.*

Proof. (i): Let $\mathfrak{k}$ be a Lie subalgebra of $\mathfrak{g}$. Then the fact that $\mathfrak{k}^m \subseteq \mathfrak{g}^m$ for all m yields the nilpotency of $\mathfrak{k}$. If $\varphi : \mathfrak{g} \longrightarrow \mathfrak{k}$ is a surjective homomorphism of Lie algebras then $\varphi(\mathfrak{g}^m) = \mathfrak{k}^m$ and hence, $\mathfrak{h}$ is nilpotent.
(ii): For this part we refer the reader to [12] where this is shown in detail.
(iii): Recall that, by exactness, the morphism ι: $\mathfrak{i} \longrightarrow \mathfrak{k}$ is acutally a monomorphism and $\pi : \mathfrak{k} \longrightarrow \mathfrak{g}$ is an epimorphism. Furthermore, imι =kerπ and $\mathfrak{g} \cong \mathfrak{k}/\mathrm{im}\iota$. By assumption, we have $\mathfrak{g}^m = (\mathfrak{k}/\mathfrak{i})^m = 0$ for some $m \in \mathbb{N}$ and due to the surjectivity of π, one obtains $\pi(\mathfrak{k}^m) = (\mathfrak{k}/\mathfrak{i})^m = 0$, hence $\mathfrak{k}^m \subseteq \mathfrak{i} \subseteq Z(\mathfrak{k})$. Therefore, $\mathfrak{k}^{m+1} = [\mathfrak{k}, \mathfrak{k}^m] \subseteq [\mathfrak{k}, Z(\mathfrak{k})] = 0$, by definition of the center. □

Remark 1.2.6. There is a similar proposition for solvable Lie algebras stating that subalgebras and homomorphic images of solvable Lie algebras are solvable (which is proven analogously to (i)). Moreover, the proposition asserts that solvability is an extension property of Lie algebras which means the following: Let $\mathfrak{g}$ be a Lie algebra and let $\mathfrak{i} \trianglelefteq \mathfrak{g}$. If $\mathfrak{i}$ and $\mathfrak{g}/\mathfrak{i}$ are solvable then $\mathfrak{g}$ itself is solvable.
This is not true in the case of nilpotency. The condition $\mathfrak{i} \subseteq Z(\mathfrak{k})$ in 1.2.5 (iii) is indispensable in general, making nilpotency not an extension property of Lie algebras.

As a conclusion, we state Engel's and Lie's theorem which are of major importance in the structure theory of Lie algebras. For this purpose, we need the following notion:

Definition 1.2.7. A representation (V, φ) is said to be *nilpotent* if there is $c \in \mathbb{N}$ such that $\varphi(x_1) \cdots \varphi(x_c) = 0$ for all $x_1, \ldots, x_c \in \mathfrak{g}$.

Theorem 1.2.8 (Engel's theorem). *Let $\varphi : \mathfrak{g} \longrightarrow \mathfrak{gl}(V)$ be a finite-dimensional representation of $\mathfrak{g}$ such that for all $x \in \mathfrak{g}$ the homomorphism $\varphi(x)$ is a nilpotent endomorphism. Then φ is a nilpotent representation. In particular, for a finite-dimensional Lie algebra $\mathfrak{g}$, we have*

$$\mathfrak{g} \text{ is nilpotent} \Longleftrightarrow \mathrm{ad}(x) \text{ is nilpotent for all } x \in \mathfrak{g}.$$

Proof. See for example [7] or [39]. □

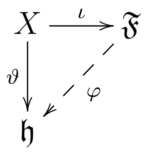
Theorem 1.2.9 (Lie's theorem). *Let $\mathfrak{g}$ be a solvable Lie algebra over an algebraically closed field of characteristic 0 and let (φ, V) be a finite dimensional representation of $\mathfrak{g}$. Then there exists a basis B of V such that for any $x \in \mathfrak{g}$ the endomorphism $\varphi(x)$ can be expressed as an upper triangular matrix in terms of the basis B.*

Proof. We refer to [12]. □

1.3 Free-nilpotent Lie algebras

In order to conceive the work in Section 3.2 we have to understand the notion of *free* Lie algebras. They are thought of the generic Lie algebras whose elements are in no relation apart from the defining equations of Lie algebras. Moreover, we study *free-nilpotent* Lie algebras which additionally provide information on the nilpotency class.

Definition 1.3.1. Let X be an arbitrary set. A Lie algebra $\mathfrak{F} = \mathfrak{F}(X)$ is called *free on* X if it satisfies the following universal property:



That is, given morphisms of sets $\iota : X \longrightarrow \mathfrak{F}$ and $\vartheta : X \longrightarrow \mathfrak{h}$ for any Lie algebra $\mathfrak{h}$, there exists a morphism of Lie algebras $\varphi : \mathfrak{F} \longrightarrow \mathfrak{h}$ such that $\vartheta = \varphi \circ \iota$.

An alternative definition involves the *free algebra* with the set X of *free generators* over a fixed field. Factoring with respect to the ideal generated by the elements of the form x^2 and $x(yz)+y(zx)+z(xy)$ yields the free Lie algebra on X. This makes sense since in the quotient algebra all elements of the above forms vanish, hence the skew-symmetry and the Jacobi identity are satisfied. More details to this construction can be found in Bourbaki's book [11] or in [7].

Remark 1.3.2. Analogously to group presentations we can construct Lie algebras with specific relations. More precisely, let R be a set of *Lie-words*, that is, a set of finite linear combinations of elements of X or Lie brackets of elements of X. For instance, x, $[x,[y,z]]$ or $[x,y]+[u,v]$ are Lie words. Then the generated subspace of R is an ideal in the free Lie algebra on X, i.e. $\langle R\rangle \trianglelefteq \mathfrak{F}(X)$. The quotient Lie algebra

$$\mathfrak{F}(X|R) := \mathfrak{F}(X)/\langle R\rangle$$

is called the *free Lie algebra generated by X with relations R.*

Example 1.3.3.

- Let $R = \varnothing$. Then

$$\mathfrak{F}(X|R) = \mathfrak{F}(X|\varnothing) = \mathfrak{F}(X).$$

- If $R = \{[x_1,x_2] : x_1, x_2 \in X\}$ then $\mathfrak{F}(X|R)$ is the abelian Lie algebra generated by X because all Lie brackets equal 0 in the quotient algebra.

- Let $X = \{x_1, x_2\}$. Then $\mathfrak{F}(X)$ is the set of all Lie words

$$x_1, x_2, [x_1,x_2], [x_1,[x_1,x_2]], [x_2,[x_1,x_2]], [x_1,[x_1,[x_1,x_2]]], \ldots$$

  and so on, such that skew-symmetry and the Jacobi identity are satisfied.

Next, we want to study Lie algebras that have g-many generators and are of nilpotency class c for given values $c, g \in \mathbb{N}$. By results of the article [21], any such Lie algebra can be viewed as a quotient of a universal nilpotent Lie algebra.

Definition 1.3.4. Let $c, g \in \mathbb{N}$ and let $\mathfrak{F}_g = \mathfrak{F}(\{x_1, \dots, x_g\})$ be the free g-generator Lie algebra. Then

$$\mathfrak{F}_{g,c} := \mathfrak{F}_g / \mathfrak{F}_g^{c+1}$$

is called the *free-nilpotent g-generator Lie algebra of class c.*

As in Definition 1.2.1, $\mathfrak{F}_g^{c+1}$ denotes the $(c+1)$-st term of the lower central series of $\mathfrak{F}_g$, whereas we know that it is indeed an ideal in $\mathfrak{F}_g$. Observe that $\mathfrak{F}_g^{c+1}$ consists of all Lie words of length $\geq c+1$. Thus, only Lie words of length c appear in the quotient. One can elaborate free-nilpotent Lie algebras with small values for g, c by hand. For instance,

$$\mathfrak{F}_{3,2} = \operatorname{span}(\{x_1, x_2, x_3, [x_1, x_2], [x_1, x_3], [x_2, x_3]\})$$

since all words of length > 2 vanish.

Remark 1.3.5. Note that $\mathfrak{F}_{g,c}$ is finite-dimensional and therefore by no means *free* in the category-theoretical sense, but rather in the meaning of the comment right before Definition 1.3.4. To emphasize this fact, we use a hyphen in the phrase *free-nilpotent.*

The next result gives a formula for calculating the dimension of free-nilpotent Lie algebras without having a basis a priori.

Theorem 1.3.6 (Witt,[42])**.** *Let $\mathbb{F}$ be a field of characteristic* 0*. Then*

$$\dim(\mathfrak{F}_{g,c}) = \sum_{k=1}^{c} \frac{1}{k} \sum_{d|k} \mu(d) g^{k/d}.$$

Remark 1.3.7. We make a few comments on Witt's theorem. First, μ denotes the number-theoretical Möbius function defined as follows:

$$\mu : \mathbb{N} \longrightarrow \{0, \pm 1\}, \; n \longmapsto \begin{cases} (-1)^r & \text{if } n \text{ is square-free} \\ 0 & \text{otherwise} \end{cases},$$

where r is the number of prime factors of n.
Second, Theorem 1.3.6 is actually a corollary of the original theorem. Denote by $\mathfrak{F}(n)$ the subspace of the free Lie algebra that is generated by all elements of form $[x_{i_1}, [x_{i_2}, [\dots [x_{i_{n-1}}, x_{i_n}] \dots]]]$. One can think of $\mathfrak{F}(n)$ as the length-n part of the free Lie algebra. Then Witt originally proved

$$\dim(\mathfrak{F}(n)) = \frac{1}{n} \sum_{d|n} \mu(d) g^{n/d}.$$

This fact is of interest because this formula also appears as the number of monic irreducible polynomials of degree n over a finite field with g elements and also as the number of aperiodic necklaces of length n in g colors, just to name a couple. For the method of the transition from the original Witt-formula to the case of free-nilpotent Lie algebras, the reader is recommended to look at [21]. Furthermore, in Appendix A.6 the formula is programmed in Mathematica and a table of the dimensions is presented for $c, g \in \{1, 2, \ldots, 10\}$.

Example 1.3.8. We construct $\mathfrak{F}_{3,3}$ in detail. By Witt's theorem one immediately obtains that $\mathfrak{F}_{3,3}$ is 14-dimensional and by the above considerations we know that all Lie brackets of length greater than three vanish. Listing the Lie brackets in the generators x_1, x_2, x_3 gives

$$\begin{aligned} x_4 &= [x_1, x_2] & x_5 &= [x_1, x_3] & x_6 &= [x_2, x_3] \\ x_7 &= [x_1, [x_1, x_2]] & x_8 &= [x_2, [x_1, x_2]] & x_9 &= [x_3, [x_1, x_2]] \\ x_{10} &= [x_1, [x_1, x_3]] & x_{11} &= [x_2, [x_1, x_3]] & x_{12} &= [x_3, [x_1, x_3]] \\ x_{13} &= [x_1, [x_2, x_3]] & x_{14} &= [x_2, [x_2, x_3]] & x_{15} &= [x_3, [x_2, x_3]], \end{aligned}$$

However, by Witt, we should have only 14 elements and indeed, we get a restriction caused by the Jacobi identity:

$$J(x_1, x_2, x_3) = 0 \Longleftrightarrow [x_2, [x_1, x_3]] = [x_3, [x_1, x_2]] + [x_1, [x_2, x_3]],$$

telling us that $x_{11} = x_9 + x_{13}$ which is why we delete it from the list of basis elements and end up with the correct 14 elements.

1.4 Some group theory

We provide a survey of group-theoretical definitions and results that are used in the later chapters. For more information on general group theory we refer the reader to Lang's book [31] or [14] among other works on group theory.

Definition 1.4.1. Let G be a group. G is called *solvable* if it admits a subnormal series such that all factors are commutative. To be more precise, G is solvable if there is a chain

$$1 = G_n \trianglelefteq G_{n-1} \trianglelefteq \cdots \trianglelefteq G_1 \trianglelefteq G_0 = G,$$

such that each quotient group G_i/G_{i+1} is abelian.

Equivalently, G is solvable if its *derived series* terminates in the trivial group. The derived series of a group G is the descending sequence

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \cdots$$

where for all n, $G^{(n)} = [G^{(n-1)}, G^{(n-1)}]$ is the commutator subgroup. So G is solvable if and only if

$$G = G^{(0)} \trianglerighteq G^{(1)} \trianglerighteq G^{(2)} \trianglerighteq \cdots \trianglerighteq G^{(d)} = 1$$

for some $d \in \mathbb{N}$. If d is the least value such that $G^{(d)} = 1$ then we say G is of *derived length* d. For a proof of the equivalence of the definitions see [14].

Example 1.4.2. All abelian groups are solvable. Moreover, it can be shown that every subgroup and quotient group of a solvable group is again solvable. To have a few explicit examples of solvable groups, consider the following:

- The symmetric group $\mathcal{S}_4$ has derived length 3 since it admits the sequence

  $$\mathcal{S}_4 \trianglerighteq \mathcal{A}_4 \trianglerighteq \mathcal{V}_4 \trianglerighteq 1,$$

  where $\mathcal{A}_4$ is the alternating group and $\mathcal{V}_4$ denotes the Klein four-group.

- The dihedral group D_n is 2-step solvable because we have

  $$D_n \trianglerighteq \langle r^2 \rangle \trianglerighteq 1,$$

  where $D_n = \langle r, s : r^n = s^2 = srs^{-1}r = 1 \rangle$.

Definition 1.4.3. Let G be a group. The *lower central series* of G is the descending series

$$G^0 = G \supseteq G^1 \supseteq G^2 \supseteq \cdots$$

where for all n, the n-th member is defined as $G^n = [G^{n-1}, G] = [G, G^{n-1}] = \langle [g,h] : g \in G, h \in G^{n-1} \rangle$. It is common to use the notation $\gamma_n = [G^{n-1}, G]$. By induction, one can check that $\gamma_i \trianglelefteq G$ for all i.

We call G *nilpotent* if $\gamma_c = 1$ for some $c \geq 0$. The minimal such $c \geq 0$ is called *nilpotency class* of G.

Example 1.4.4. Certainly, all abelian groups are nilpotent. As another instance, consider the quaternion group

$$Q_8 = \langle i, j, k, -1 : (-1)^2 = 1, i^2 = j^2 = k^2 = ijk = -1 \rangle.$$

Then $Q_8 \supseteq \{1, -1\} \supseteq 1$ and hence Q_8 is of nilpotency class 2.

Proposition 1.4.5.

1. *Every subgroup and every homomorphic image of a nilpotent group is nilpotent. In particular, all quotients of nilpotent groups are nilpotent.*

2. *Every finite p-group is nilpotent.*

3. *Every nilpotent group is solvable.*

Proof. For a proof of the properties we refer to [14]. □

Definition 1.4.6. A group G is said to be *residually nilpotent* if the nilpotent residual of the group is trivial, i.e. if the intersection of all members of the lower central series is trivial.

Definition 1.4.7. A group G is called *locally nilpotent* if every finitely generated subgroup of G is nilpotent. For example, all abelian groups and all nilpotent groups are locally nilpotent. Further, the Fitting subgroup (cf.[5]) of a finite group is locally nilpotent.

Next, we explain *free groups*. These are groups in which elements are in no relation other than the defining relations of a group. In this sense, free groups constitute the generic examples of groups.

Definition 1.4.8. A group F is said to be *free* if there is a *free basis* of F, that is, if there is $S \subseteq F$ such that every function $\varphi : S \to G$ to some group G can be extended uniquely to a group homomorphism $\widetilde{\varphi} : F \to G$ so that $\widetilde{\varphi}(s) = \varphi(s)$ for all $s \in S$. In other words, S is a free basis of F if the following universal property holds:

$$\begin{array}{ccc} S & \xrightarrow{\iota} & F \\ {\scriptstyle\varphi}\downarrow & \swarrow {\scriptstyle\widetilde{\varphi}} & \\ G & & \end{array}$$

where $\varphi = \widetilde{\varphi} \circ \iota$ so that the diagram commutes.

Example 1.4.9.

- The trivial group is a free group with free basis $S = \varnothing$.

- The infinite cyclic group $C_\infty = \langle x^{-1} \rangle = \langle x \rangle = \{x^n : n \in \mathbb{Z}\}$ is free with free basis $S = \{x\}$. To see this, let G be a group and let $\varphi : S \to G$ be an arbitrary function with $\varphi(x) = g$ for a $g \in G$. If we define $\widetilde{\varphi}(x^n) = g^n$ for all $n \in \mathbb{Z}$ then φ extends uniquely to $\widetilde{\varphi} : C_\infty \to G$.

- The group $(\mathbb{Z}/m\mathbb{Z}, +)$ is not free for $m \geq 2$. We prove this indirectly and suppose that there is a free basis S of $\mathbb{Z}/m\mathbb{Z}$. Observe that $S \neq \varnothing$ since $m \geq 2$. Next, let $x \in S$ and consider the map $\varphi : S \to \mathbb{Z}$ with $\varphi(x) = 1$. This extends uniquely to a morphism $\widetilde{\varphi} : \mathbb{Z}/m\mathbb{Z} \to \mathbb{Z}$, but $\mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z}) = 0$ since for $\vartheta \in \mathrm{Hom}(\mathbb{Z}/m\mathbb{Z}, \mathbb{Z})$ we have

$$\vartheta(0) = \vartheta(x^m) = \vartheta(x)^m = 0$$

and so $\vartheta(x) = 0$ for all x.

Proposition 1.4.10.

1. *Let S be any set. Then there is a free group F_S that has S as a free basis.*
2. *Every group is a quotient of a free group, i.e. $G \cong F_G/N$ for a group G and a normal subgroup N.*

Proof. Again, it is referred to [14] for the proof of the statements and more details to free groups. □

Definition 1.4.11. A group G is said to be *linear* if it is embeddable into the group $\mathrm{GL}_n(\mathbb{F})$ for some $n > 1$ and some field $\mathbb{F}$.

Example 1.4.12. All matrix groups are linear, e.g. $\mathrm{SL}_n(\mathbb{R})$ or $\mathrm{O}_n(\mathbb{R})$. One can proof that for $m \geq 1$ any m-generator free group is linear. In fact, we have

$$F_m \hookrightarrow F_2 \hookrightarrow \mathrm{SL}_2(\mathbb{C}).$$

A proof can be found in [14].

In the discussion of Burnside's problems the notion of periodic groups takes a central role.

Definition 1.4.13. A group G is said to be *periodic* or *torsion* if for all elements $g \in G$ there is an integer $n \in \mathbb{N}$ such that $g^n = 1$. In other words, G is periodic if every element in G has finite order. We remark that, in contrast to the above, a group G is of *bounded exponent* if there exists $n \in \mathbb{N}$ such that $g^n = 1$ for all group elements g. The least such n is called the *exponent of G*. Notice that the property of a group being of bounded exponent implies the periodicity of the group.

Example 1.4.14. Certainly, a finite group is of bounded exponent and hence periodic. Instances for infinite periodic groups are the direct sum of cyclic groups $\bigoplus_{i=1}^{\infty} C_2$ or the quotient group $\mathbb{Q}/\mathbb{Z}$. Indeed, in the first group all elements have order 2 while for the second example observe that for $q \in \mathbb{Q}$, $q = \frac{a}{b}$ we have $q \cdot b = a$ and hence finite order in the quotient.

As a conclusion to this introductory chapter, we define the notions of profinite groups and ordered groups as they appear in a later section. To this end, we recall that a group G is said to be a *topological group* if it is equipped with a topology and the maps $x \mapsto x^{-1}$ as well as $(x, y) \mapsto xy$ are both continuous. Note that the second map has domain $G \times G$ which is equipped with the product topology. As examples we mention $(\mathbb{Q}, +)$, $(\mathbb{R}, +)$ or $(\mathbb{C}, +)$ with respect to the topology induced by the Euclidean metric. Further, the matrix group $\mathrm{GL}(n, \mathbb{R})$ is a topological group regarding it as a subset of $\mathbb{R}^{n^2}$ and, evidently, every finite group together with the discrete topology is a topological group.

Definition 1.4.15. A topological group G is *profinite* if G is isomorphic to a projective limit of finite groups. Without going into much detail, forming the projective limit of objects in a category can be viewed as an operation that "glues" these objects together. In this sense, one can think of a profinite group as a group that arises from "gluing" finite groups together. It can be proved that a group G is profinite if and only if it is a topological group that is compact, totally disconnected and possesses the Hausdorff property. Therefore, a finite group together with the discrete topology is profinite. Apart from that, the group $\mathbb{Z}$ is not profinite because it is not compact as it is unbounded. However, it is possible to form a profinite completion of $\mathbb{Z}$. For more details to this topic the reader is referred to [13].

Definition 1.4.16. A group G is defined to be an *ordered group* if there is a total order $\leq$ on G such that $a \leq b$ implies $\alpha a \beta \leq \alpha b \beta$ for all $a, b, \alpha, \beta \in G$. For example, it can be proved that all torsion-free abelian groups and all free groups are orderable.

Chapter 2

Burnside problems

In this chapter a short introduction on the history of the person William Burnside and the origin of his famous Burnside Problems is given. We display the key steps and methods to the solution of the Restricted Burnside Problem in a summarizing manner. In particular, we demonstrate the idea that connects this group-theoretical problem to Lie theory, especially to Engel Lie algebras which turn out to be of major importance and thus, are further investigated in Chapter 3. The last section regarding Burnside's Problems is dedicated to Engel-n groups. These are groups that satisfy the additional condition

$$\underbrace{[x,[x,\dots,[x}_{n\text{-times}},y]]]=1,$$

where $[x,y]=x^{-1}y^{-1}xy$ is the commutator, *not* the Lie bracket. We state numerous results in order to present a purely group-theoretical perspective on the Burnside Problems.

2.1 Historical survey

The English mathematician William Burnside was born in London in 1852. He became an orphan at the age of six and attended Christ's Hospital, a school especially designed for boys in a situation like Burnside's. Aged twenty-three, Burnside won a scholarship by means of which he entered St. John's College, a department of the University of Cambridge. Most prominently, he was educated by George Stokes, J.C. Maxwell and Arthur Cayley. Notably, the affinity for applied mathematics, passed on by the above, should influence Burnside's future research fundamentally. However, in 1875 Burnside joined Pembroke College, another constituent institute of Cambridge University, to push ahead his athletic career as an oarsman. For the next ten years he lectured at Pembroke on hydrodynamics before he was granted a mathematics professorship at Greenwich's Royal Naval College. His work on hydrodynamics was often confronted with group theoretical aspects and it became more and more apparent that this was his eventual research area. A whole sequence of group theoretical articles ensued, for instance, the nowadays standard result of the solvability of groups of order $p^a \cdot q^b$ for integers $a, b \in \mathbb{N}$ and primes p, q. On top of that, in 1902, the English mathematician stated the famous *Burnside Problem* which occupied the minds of mathematicians for the next

seven decades and, in variations, is the central topic of this chapter. Eventually, William Burnside died in August 1927 at the age of seventy-five leaving behind an outstanding mathematical legacy that influences group theory until today.
More details to the person William Burnside can be found in [34].

As indicated, in 1902, William Burnside posed the following question in his article [16]:

> *"A still undecided point in the theory of discontinuous groups is whether the group order of a group may be not finite, while the order of every operation it contains is finite."*

In modern mathematical language this translates to the question:

Problem 2.1.1 (General Burnside Problem)**.** *Are all finitely generated periodic groups finite?*

In 1964, E.S. Golod answered this question in his paper [22] by constructing a family of finitely generated infinite p-groups. Today, numerous counter examples are known, see [4] or [23] among others.

In order to obtain more structural information about the group, Burnside required the group to be periodic of bounded exponent. He then asked the question:

Problem 2.1.2 (Bounded Burnside Problem)**.** *Are all finitely generated periodic groups of bounded exponent finite?*

To provide a brief insight, let $\mathcal{F}_r$ be the free group of rank r and let $\mathcal{F}_r^n$ be the normal subgroup of $\mathcal{F}_r$ that is generated by all g^n for $g \in \mathcal{F}_r$ and fixed $n \in \mathbb{N}$. The quotient $\mathcal{F}_r/\mathcal{F}_r^n$ is called the *r-generator Burnside group of exponent* n and is denoted by $B(r,n)$. Any given r-generator group G of exponent n is a homomorphic image of the Burnside group. So the Bounded Burnside Problem can be rephrased as: *For which values of r and n is $B(r,n)$ finite?* One immediately observes that $B(1,n)$ is the cyclic group of order n. As a next step, let G be of bounded exponent 2 and let $x, y \in G$. Then

$$ab = (ab)^{-1} = b^{-1}a^{-1} = ba.$$

Thus G is abelian. Further, let G be generated by r-many elements. Then $|G| \leq 2^r$. Taking $G = \bigoplus_{i=1}^r C_2$ yields an r-generator group of exponent 2 satisfying $|G| = |\bigoplus_{i=1}^r C_2| = 2^r$ and hence, G is a homomorphic image of $B(r,2)$, but on the other hand, $|B(r,2)| \leq 2^r$. Consequently, for $r \geq 1$ we found $B(r,2) = \bigoplus_{i=1}^r C_2$.

Many other cases are elaborated, in fact, Burnside himself showed in [16] that the cardinality of $B(r,3)$ is bounded by 3^{2^r-1} or that $|B(2,4)| \leq 2^{12}$. It can be looked up in [40] that if $n = 2,3,4,6$ then $B(r,n)$ is finite for all values of r.

However, Adjan proves in [2] that if $r > 1, n \geq 665$ and n odd then $B(r, n)$ turns out to be infinite and so answering the Bounded Burnside Problem in the negative.

The connection to Lie algebras lies within the following question:

Problem 2.1.3 (Restricted Burnside Problem)**.** *Are there only finitely many finite r-generator groups of exponent n?*

Assuming that the above has an affirmative answer, let M be the intersection of all normal subgroups of $B(r, n)$ that are of finite index. Certainly, M is again a normal subgroup of finite index. The quotient $B_0(r, n) := B(r, n)/M$ is called the *universal finite r-generator group of exponent n.* All finite r-generator groups of exponent n are homomorphic images of $B_0(r, n)$. Note that $B_0(r, n) = B(r, n)$ whenever $B(r, n)$ is finite. So Problem 2.1.3 turns into the question of the existence of $B_0(r, n)$. Much of the work on this problem was done in the 1950s by A.I. Kostrikin who established the existence of $B_0(2, 5)$ and later proved that $B_0(r, p)$ exists for every prime p. Additionally, in 1956, Hall and Higman published a reduction theorem to the case of prime powers and it was in 1991 that Efim Zelmanov settled the Restricted Burnside Problem for arbitrary values of r, n affirmatively.

2.2 The Restricted Burnside Problem

We present a short outline of the solution of the Restricted Burnside Problem. In the course of this, we describe the early development, give necessary definitions and state the most crucial results up to Zelmanov's celebrated theorem that proves the Restricted Burnside Problem. Moreover, we define Engel groups and give a survey of numerous results on the topic. We begin with the central theorem of Hall-Higman:

Theorem 2.2.1 (Hall, Higman, [25])**.** *Let $p_1, \ldots, p_k$ be distinct primes, $m_1, \ldots, m_k \geq 1$ and let $n = p_1^{m_1} \cdots p_k^{m_k}$.*

(i) The Restricted Burnside Problem has an affirmative solution for groups of exponents $p_i^{m_i}$.

(ii) There are only finitely many finite simple groups of exponent n.

(iii) For any finite simple group G of exponent n the outer automorphism group $\mathrm{Out}(G) = \mathrm{Aut}(G)/\mathrm{Inn}(G)$ is sovable.

Assuming hypotheses (i),(ii) and (iii), the Restricted Burnside Problem holds true for all groups of exponent n.

Assumptions (ii) and (iii) are dealt with by the classification of finite simple groups. In particular, if n is odd, Feit-Thompson's *odd-order-theorem* (1962) implies that there are no finite simple groups in (ii),(iii).

However, the above result from 1956 gives a reduction to the case of prime powers and so caused euphoria to more extensive work on the Restricted Burnside Problem. Another driving force that contributed to the later success was the link to the theory of Lie algebras as they provide the mindset and methods from linear algebra, see Chapter 1.

At this point, we highlight that for groups the term $[x, y] = x^{-1}y^{-1}xy$ denotes the commutator of two elements x, y. To avoid confusion, we use the notation xy for the Lie product of x and y until the end of this chapter.

Lemma 2.2.2. *Consider a group G and its lower central series. Then*

(i) $\gamma_i/\gamma_{i+1} \subseteq Z(G/\gamma_{i+1})$.

(ii) $[\gamma_i, \gamma_j]$ *is a subgroup of* γ_{i+j} *for all* $i, j \geq 1$.

Proof. For (i) let $x \in \gamma_i/\gamma_{i+1}$ and $y \in G/\gamma_{i+1}$. So they have form $x = h \cdot \gamma_{i+1}$ for some $h \in \gamma_i$ and $y = g \cdot \gamma_{i+1}$ for some $g \in G$. Then

$$[x, y] = [h \cdot \gamma_{i+1}, g \cdot \gamma_{i+1}] = [h, g] \cdot \gamma_{i+1} = 1,$$

by definition of γ_{i+1}. So $[x, y] = x^{-1}y^{-1}xy = 1$ or equivalently, $xy = yx$ and thus, $x \in Z(G/\gamma_{i+1})$. For (ii) we refer to [40]. □

Lemma 2.2.3 ([40])**.** *Let G be the group generated by the set $X = \{x_1, x_2, \ldots\}$. Then γ_i equals the normal closure of $\{[\ldots, [[x_1, x_2], x_3] \ldots, x_i] : x_j \in X\}$ for all i. Furthermore, these commutators generate γ_i modulo γ_{i+1}.*

By means of the above auxiliary results we are ready to construct the associated Lie ring of a group.

Definition 2.2.4. Let G be a group and consider its lower central series. For all i we set $\mathfrak{g}_i = \gamma_i/\gamma_{i+1}$. The groups $\mathfrak{g}_i$ are abelian whence we write them additively and regard them as modules over $\mathbb{Z}$. The *associated Lie ring of G* is defined as

$$\mathfrak{g} = \mathfrak{g}(G) := \bigoplus_{i \geq 1} \mathfrak{g}_i.$$

In order to turn $\mathfrak{g}$ into a Lie ring we need to define a multiplication. The two elements $x \in \mathfrak{g}_i$ and $y \in \mathfrak{g}_j$ have form $x = g \cdot \gamma_{i+1}$ and $y = h \cdot \gamma_{j+1}$, respectively, for some $g \in \gamma_i$ and $h \in \gamma_j$. Lemma 2.2.2 (ii) gives $[g, h] \in \gamma_{i+j}$ and so enables us to define the Lie product

$$xy := [g, h] \cdot \gamma_{i+j+1} \in \mathfrak{g}_{i+j}.$$

It is easily checked that this is a well-defined expression. We extend this product linearly to all of $\mathfrak{g}$. The verification that this product satisfies bilinearity, skew-symmetry and the Jacobi identity can be looked up in Vaughan-Lee's book [40].

Observe that $\mathfrak{g}_{i+1} = \mathfrak{g}_i\mathfrak{g}_1$ for all possible i. This holds since γ_{i+1} is generated by the commutators $[g, h]$ for $g \in \gamma_i$ and $h \in \gamma_1$ by definition. Now assume that G is generated by a set X. Then $\mathfrak{g}_1$ is generated by all elements of form $x \cdot \gamma_2$ for $x \in X$. Similarly, $\mathfrak{g}_2$ is generated by all elements of form $x_1x_2 \cdot \gamma_2$. Continuing this process we obtain that $\mathfrak{g}_i$ is generated by the elements $x_1 \cdots x_i \cdot \gamma_2$. This fact corresponds to Lemma 2.2.3.

Returning to the context of the Restricted Burnside Problem, let $G = B(r, q)$ be the r-generator Burnside group of exponent q. Recall that due to the reduction theorem of Hall-Higman, we may restrict the exponent q to prime powers, i.e. $q = p^m$ for a prime p and some $m \in \mathbb{N}$. Consider the lower central series of $B(r, q)$

$$\gamma_1 = B(r, q) \geq \gamma_2 \geq \cdots \geq \gamma_i \geq \cdots$$

and denote by $\mathfrak{g}(r, q)$ the associated Lie ring. By Lemma 2.2.3, all quotients $\mathfrak{g}_i = \gamma_i/\gamma_{i+1}$ are finitely generated abelian groups that have exponent dividing q and thus are periodic. By the classification of finitely generated abelian groups, all $\mathfrak{g}_i$ are finite. Consequently, also $B(r, q)/\gamma_i$ is finite for $i \geq 1$. As proceeded in [40], two cases are distinguished:

1. γ_{i+1} is a proper subgroup of γ_i for all $i \geq 1$.
2. There is an $i \geq 1$ such that $\gamma_{i+1} = \gamma_i$.

In case 1 the quotients $B(r, q)/\gamma_i$ are of unbounded order for all $i \geq 1$. That means there is no bound for the order of r-generator groups of exponent q and the associated Lie ring $\mathfrak{g}(r, q)$ has infinitely many elements. In the second case one obtains a lower central series of form

$$B(r, q) \geq \gamma_2 \geq \cdots \geq \gamma_i \geq \gamma_i \geq \cdots$$

for some i. So for all $j \geq i$ this gives $\mathfrak{g}_j = 0$ and therefore, $\mathfrak{g}(r, q)$ is finite. If we are given any finite r-generator group G of exponent $q = p^m$ then, by the arguments in Section 2.1, $G \cong B(r, q)/N$ for some normal subgroup N of G. Additionally, G is a p-group and thus nilpotent. Hence, there exists an integer $k \geq 1$ such that $\gamma_k \leq N$ but on the other hand, $\gamma_i \leq N$ because $\gamma_j = \gamma_i$ for all $j \geq i$ by assumption. As a result, G is isomorphic to a homomorphic image of $B(r, q)/\gamma_i$. This considerations prompt that $B(r, q)/\gamma_i$ is the largest such group containing every other finite r-generator group of exponent q as homomorphic image. Apart from that, the associated Lie ring $\mathfrak{g}(r,q)$ has the same nilpotency class as $B(r,q)/\gamma_i$ by definition and, also, they are of equal order as is shown by:

$$|\mathfrak{g}(r,q)| = \prod_{n=1}^{i} |\mathfrak{g}_n| = \prod_{n=1}^{i} |\gamma_n/\gamma_{n+1}| = |B(r,q)/\gamma_i|.$$

where the third identity is due to a standard result in group theory [31].

A.I. Kostrikin utilizes this connection in his proof for the Restricted Burnside Problem for prime exponents. To be more precise, Kostrikin established the following results:

Theorem 2.2.5. *Let G be a group of prime exponent p. Then its associated Lie ring is an Engel-$(p-1)$ Lie algebra over the field $\mathbb{Z}/p\mathbb{Z}$, cf. Definition 1.1.20.*

Theorem 2.2.6 (Kostrikin,[30])**.** *Let $\mathfrak{g}$ be a finitely generated Engel-$(p-1)$ Lie algebra over a field of characteristic p. Then $\mathfrak{g}$ is nilpotent.*

Exploring the latter for $p = 2$ and $p = 3$, one observes that on the one hand, for $p = 2$, $\mathfrak{g}$ satisfies the Engel-1 condition $xy = 0$ which is equivalent to $\mathfrak{g}$ being abelian. Therefore, $\mathfrak{g}$ has nilpotency class at most 1. On the other hand, if $\mathfrak{g}$ is an Engel-2 Lie algebra then it is nilpotent of class at most 3. We prove this fact explicitly in Chapter 3.

Kostrikin's theorems imply that $\mathfrak{g}(r,p)$ is nilpotent. On top of that, due to our above considerations, we deduce that $\mathfrak{g}(r,p)$ is finite which proves the existence of a largest finite r-generator group of exponent p. For the proofs of 2.2.5 and 2.2.6 the reader is referred to Kostrikin's book [30] which contains all details and more.

Efim Zelmanov extended Kostrikin's methods in his solution to the Restricted Burnside Problem for prime-power exponents. We now state his famous theorem and emphasize that this result is explicitly on Lie algebras while solving a group-theoretical problem.

Theorem 2.2.7 (Zelmanov)**.** *Let $\mathfrak{g}$ be a Lie algebra over a field $\mathbb{F}$ generated by $a_1, a_2, \ldots, a_k$ for some $k \in \mathbb{N}$. Suppose there exist integers $n, m \geq 1$ such that:*

1. *For all $x, x_1, \ldots, x_m \in \mathfrak{g}$ we have*

$$\sum_{\sigma \in \mathrm{Sym}(m)} [x_{\sigma(1)}, [x_{\sigma(2)}, \ldots [x_{\sigma(m)}, x] \ldots]] = 0.$$

2. *For all $x \in \mathfrak{g}$ and all $y \in \mathfrak{g}$ that have form $y = [a_1, [a_2, \ldots [a_{i-1}, a_i], \ldots]]$, $i \in \{1, \ldots, n\}$, the Engel-n identity holds:*

$$[x^n, y] = 0.$$

Then $\mathfrak{g}$ *is nilpotent.*

Efim Zelmanov proved this result in 1989 in his series of articles [44], [45] for which he was awarded a fields medal in 1994. To see that this implies the existence of a largest r-generator group of prime-power exponent q, we reference [40] where this is presented in detail. Apart from that, Theorem 2.2.7 proves that any finitely generated Engel-n Lie algebra is nilpotent since the requirements are met with $n = m$.

2.3 Engel groups

As for Lie algebras, it certainly makes sense to consider groups that satisfy the Engel identity. Define $e_0(x, y) = x$ and $e_{n+1}(x, y) = [e_n(x, y), y]$, where here, the bracket $[,]$ denotes the commutator of group elements. A group G is called an *Engel group* if for all $x, y \in G$ one has $e_n(x, y) = 1$ for an integer $n = n(x, y) \in \mathbb{N}$. If it is possible to choose the positive integer n independently of the elements x, y then we say G is an *Engel-n group.*
This part of the text concerns this type of groups and their relation to the Burnside Problems.

As mentioned in Section 2.1, in his 1901 paper [16] W. Burnside proved that a finitely generated group of exponent 3 is necessarily finite. In the course of this, Burnside remarks that in this type of group any two conjugates x, x^y commute, where $x^y = y^{-1}xy$. Indeed, for $x, y \in G$ we have

$$1 = (yx)^3 = y^3 x^{y^2} x^y x.$$

Consequently,

$$x^{y^2} x^y x = 1 \tag{2.1}$$

and replacing y by y^2 in (2.1) we additionally obtain

$$x^y x^{y^2} x = 1, \tag{2.2}$$

since $x^{y^4} = x^y$ in G. Using (2.1) and (2.2) one calculates

$$x^y x = (x^{y^2})^{-1} = y^2 x^{-1} y^{-2}$$

and on the other hand,

$$x x^y = x(x^{y^2} x)^{-1} = x x^{-1} y^2 x^{-1} y^{-2} = y^2 x^{-1} y^{-2}.$$

Thus, x and x^y commute. Furthermore, x and $[y, x]$ commute. To see this, observe that $[y, x] = y^{-1}x^{-1}yx = (x^{-1})^y x$ and consider the following computations:

$$(x^{-1})^y x = (x^{-1})^y x^{-1} x^2 = x^{-1}(x^{-1})^y x^2 = x^{-1}(x^{-1})^y x^{-1},$$
$$x(x^{-1})^y = x^2 x^{-1}(x^{-1})^y = x^2(x^{-1})^y x^{-1} = x^{-1}(x^{-1})^y x^{-1}.$$

Notice that we used commutativity of x and x^y in each of the equations. As a result,

$$[[y, x], x] = [x, [x, y]] = 1$$

for all $x, y \in G$. So every group of exponent 3 is an Engel-2 group. Burnside picked up on this fact in his article [15] where he showed that in an Engel-2 group the following relations hold:

$$[x, y, z] = [x, [y, z]] = [y, z, x],$$
$$[x, y, z]^3 = 1.$$

He then concluded that if G is an Engel-2 group that contains no elements of order 3 then G has nilpotency class at most 2. Note that, in general, such groups have nilpotency class at most 3, although Burnside did not prove this fact.

Established a connection between Burnside's questions and Engel groups, we state a first result:

Theorem 2.3.1 (Zorn,[46]). *A finite Engel group is nilpotent.*

Due to this theorem, we can view the Engel property as a generalization of nilpotency. In order to observe that the Engel identity poses a weaker property, in [37] Traustason gives an example of a group that is an Engel-$(p + 1)$ p-group which is not nilpotent. In fact, the example is given by

$$G(p) = C_p \mathrm{wr}\ C_p^{\infty},$$

where wr denotes the wreath product. However, there are the following generalizations of Zorn's theorem.

Theorem 2.3.2 (Gruenberg,[24]). *Any finitely generated solvable Engel group is nilpotent.*

Theorem 2.3.3 (Baer,[6]). *Any Engel group that satisfies the maximum condition is nilpotent.*

Theorem 2.3.4 (Garaščuk, Suprunenko, [20]). *Any linear Engel group is nilpotent.*

Before continuing our survey, we recall the Burnside Problems and state their equivalents for Engel groups as done in [37]. Here, the values r and n are positive integers.

Problem 2.3.5 (The Burnside Problems)**.**

(B1) **The General Burnside Problem.** *Are all finitely generated periodic groups finite?*

(B2) **The Bounded Burnside Problem.** *Are all finitely generated periodic groups of bounded exponent finite?*

(B3) **The Restricted Burnside Problem.** *Are there only finitely many finite r-generator groups of exponent n?*

Problem 2.3.6 (Analogues for Engel groups)**.**

(E1) **The general local nilpotence problem.** *Are all finitely generated Engel groups nilpotent?*

(E2) **The local nilpotence problem.** *Are all finitely generated Engel-n groups nilpotent?*

(E3) **The restricted local nilpotence problem.** *Are there only finitely many nilpotent r-generator Engel-n groups?*

Golod's counter-examples in [22] answer the questions (B1) and (E1) in the negative. For more information on Golod's construction the survey article [19] is recommended. However, Zelmanov's theorem affirms the Restricted Burnside Problem (B3) and its equivalent (E3), where we refer to Section 2.1 for the first question and to [37] for the restricted local nilpotence problem (E3). We point out that there are counter-examples for the Bounded Burnside Problem given by Adian [2], but none are known for its equivalent for Engel groups (E2).

While the above results from Gruenberg, Baer and Garaščuk-Suprunenko are from the 1950s and 60s, more recent work on the topic is mostly based on Zelmanov's Theorem 2.2.7. For instance, the following are known:

Theorem 2.3.7 ([37])**.** *Any finitely generated residually nilpotent Engel-n group is nilpotent.*

Theorem 2.3.8 (Wilson,[41])**.** *Any profinite Engel group is locally nilpotent.*

Theorem 2.3.9 (Medvedev, [32])**.** *Any compact Engel group is locally nilpotent.*

Theorem 2.3.10. *Any orderable Engel-n group is nilpotent.*

We refer to Section 1.4 to recall the above notions.

As a conclusion to our survey on Engel groups, we briefly discuss the following theorem.

Theorem 2.3.11 (Heineken,[27]). *Any Engel-*3 *group* G *that is* $\{2,5\}$*-free has nilpotency class at most* 4.

Remark 2.3.12. As explained in [37], we can view G as a finite Engel-3 p-group for $p \neq 2, 5$. Now consider its associated Lie ring $\mathfrak{g}(G)$ as in Definition 2.2.4 and solve the problem with Lie theory. In fact, we prove this result in the next chapter, see 3.1.6.

Chapter 3

Nilpotency index of Engel Lie algebras

As mentioned in Chapter 2, Engel-n Lie algebras arise naturally in terms of the Burnside Problems. Beside Theorem 2.2.7, regarding the Restricted Burnside Problem, Zelmanov provides two more results:

Theorem 3.0.1 (Zelmanov,[43]). *Any Engel-n Lie algebra over a field of characteristic* 0 *is nilpotent.*

Theorem 3.0.2 (Zelmanov,[44]). *Any Engel-n Lie algebra over an arbitrary field is locally nilpotent.*

Naturally, the following question arises:

> *What can be said about the nilpotency classes and how do they depend on n or the number of generators?*

We will explore this issue more intensively for small values of n. For now, we primarily focus on the case of Engel-3 Lie algebras which turn out to have nilpotency class at most 4. At this, the construction of an Engel-3 Lie algebra that actually attains this bound is demonstrated explicitly. Moreover, we briefly discuss the impact of the characteristic of the underlying field.

Before that, we present an easy example of an Engel Lie algebra so that we have a concrete instance of this type of algebra.

Example 3.0.3. Consider the associative algebra $\mathfrak{n}(3,\mathbb{F})$ of strictly upper triangular 3×3 matrices, say over a field of characteristic 0, that is

$$\mathfrak{n}(3,\mathbb{F}) = \{ \begin{pmatrix} 0 & \alpha & \beta \\ 0 & 0 & \gamma \\ 0 & 0 & 0 \end{pmatrix} : \alpha,\beta,\gamma \in \mathbb{F}\}.$$

By Lemma 1.1.7 this is turned into a Lie algebra by using the commutator as

Lie bracket, i.e. $[A, B] = AB - BA$ for $A, B \in \mathfrak{n}(3, \mathbb{F})$. It is easy to see that

$$\begin{aligned}\mathfrak{n}(3,\mathbb{F}) &= \langle \underbrace{\begin{pmatrix}0&1&0\\0&0&0\\0&0&0\end{pmatrix}}_{=:e_1}, \underbrace{\begin{pmatrix}0&0&0\\0&0&1\\0&0&0\end{pmatrix}}_{=:e_2}, \underbrace{\begin{pmatrix}0&0&1\\0&0&0\\0&0&0\end{pmatrix}}_{=:e_3} \rangle \\ &= \langle e_1, e_2, e_3 : [e_1, e_2] = e_3 \rangle.\end{aligned}$$

Thus, we have the simple multiplication table:

[,]	e_1	e_2	e_3
e_1	0	e_3	0
e_2	$-e_3$	0	0
e_3	0	0	0

Extracting the adjoint endomorphisms as in 1.1.18 gives

$$\mathrm{ad}(e_1) = \begin{pmatrix}0&0&0\\0&0&0\\0&1&0\end{pmatrix}, \mathrm{ad}(e_2) = \begin{pmatrix}0&0&0\\0&0&0\\0&0&0\end{pmatrix}, \mathrm{ad}(e_3) = \begin{pmatrix}0&0&0\\0&0&0\\-1&0&0\end{pmatrix}.$$

Since every $A \in \mathfrak{n}(3, \mathbb{F})$ is a linear combination of the matrices e_1, e_2, e_3, say $A = \lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3$ for $\lambda_i \in \mathbb{F}$, one calculates

$$\mathrm{ad}(A) = \mathrm{ad}(\lambda_1 e_1 + \lambda_2 e_2 + \lambda_3 e_3) = \lambda_1 \mathrm{ad}(e_1) + \lambda_2 \mathrm{ad}(e_2) + \lambda_3 \mathrm{ad}(e_3)$$

using linearity. It is easily checked that $\mathrm{ad}(A)^3 = 0$, implying that

$$[A, [A, [A, B]]] = 0$$

for all A, B. Thus, the matrix Lie algebra $\mathfrak{n}(3, \mathbb{F})$ is indeed an Engel-3 Lie algebra.

Notice that the above example is not only a commonly known instance of an Engel Lie algebra, but also demonstrates how simple methods of linear algebra are used to obtain information about Lie algebras.

3.1 Engel-3 Lie algebras

In this section our main aim is to prove that Engel-3 Lie algebras are nilpotent of class at most 4. To this end, we use the methods from the paper [38] by G. Traustason of the University of Bath who is an active contributor to recent developments in the theory of Engel Lie algebras and Engel groups. However, already in 1954 P.J. Higgins showed in [28] that Engel-3 Lie algebras have nilpotency class at most six. Traustason utilizes Higgins' argument in his proof of the revised upper bound as is presented below.

Recall that, until mentioned otherwise, the underlying field is of characteristic 0. We use the common notation $X = \mathrm{ad}(x)$ for the adjoint endomorphism for an element x in the Lie algebra. Similarly, Y denotes $\mathrm{ad}(y)$ and so on. We will make use of the following well-known result:

Theorem 3.1.1. *Let $\mathfrak{g}$ be an Engel-2 Lie algebra over $\mathbb{F}$. Then $\mathfrak{g}^3 = 0$, i.e. $\mathfrak{g}$ is of nilpotency class at most 2.*

Proof. The Engel-2 identity says that for any element $x \in \mathfrak{g}$ the adjoint endomorphism satisfies $X^2 = 0$. Thus, also $(X+Y)^2 = 0$ which prompts

$$(X+Y)^2 = X^2 + XY + YX + Y^2 = XY + YX = 0.$$

For arbitrary $x, y, z \in \mathfrak{g}$ one then calculates

$$\begin{aligned} 0 = (XY + YX)(z) &= [x,[y,z]] + [y,[x,z]] \\ &= -[y,[z,x]] - [z,[x,y]] + [y,[x,z]] \\ &= -2 \cdot [y,[z,x]] + [z,[y,x]] \\ &= (ZY - 2 \cdot YZ)(x), \end{aligned}$$

where the Jacobi identity was used in the third equality. By interchanging the variables we obtain the two equations

$$XY + YX = 0, \tag{3.1}$$

$$XY - 2 \cdot YX = 0 \tag{3.2}$$

which imply that $XY = 0$ and thus, $\mathfrak{g}^3 = 0$ as claimed. □

Remark 3.1.2. Note that in the proof of 3.1.1 it is important that the field is of characteristic 0. Certainly, if $\mathrm{char}(\mathbb{F}) = 2$ instead, equation (3.2) yields the result and also, for $\mathrm{char}(\mathbb{F}) > 3$ the above argument works properly. On the other hand, if the characteristic is 3 then the two relations imply $3 \cdot XY = 0$ which is why we have to alter our proof as follows: Since $XY + YX = 0$ also $X[Y,Z] + [Y,Z]X = 0$. Observe that, as in 1.1.16, for the adjoint endomorphisms we have $[Y,Z] = YZ - ZY$. Therefore,

$$XYZ - XZY + YZX - ZYX = 0.$$

Let σ be a permutation in the symmteric group S_3. Then

$$X_{\sigma(1)}X_{\sigma(2)}X_{\sigma(3)} = \mathrm{sgn}(\sigma)X_1X_2X_3$$

using the relation $XY + YX = 0$ repeatedly. This fact together with the above equation gives $4 \cdot X_1X_2X_3 = 0$. Thus, in the case of characteristic 3 we have that an Engel-2 Lie algebra is 4-nilpotent.

In order to show our main theorem we prove the following Lemma.

Lemma 3.1.3. *Let $\mathfrak{g}$ be a Lie algebra and $x, y, z \in \mathfrak{g}, n \in \mathbb{N}$. Then*

$$\sum_{i=0}^{n} [x^{n-i}, [z, [x^i, y]]] = \sum_{j=0}^{n} (-1)^{j+1} \binom{n+1}{j+1} [x^{n-j}, [y, [x^j, z]]].$$

Proof. At first, one considers the inner part of the Lie bracket:

$$\begin{aligned} [z, [x^i, y]] &= -[[x^i, y], z] = -[\underbrace{X, [X, \ldots, [X}_{i \text{ times}}, Y], \ldots,]](z) \\ &= -\sum_{j=0}^{i} (-1)^j \binom{i}{j} X^i Y X^{i-j}(z). \end{aligned}$$

To verfiy the last identity, take e.g. $i = 3$ and observe

$$\begin{aligned} [x^3, y] &= [X, [X, [X, Y]]] = [X, [X, XY - YX]] \\ &= [X, X(XY - YX) - (XY - YX)X] \\ &= [X, X^2Y - 2XYX + YX^2] \\ &= X(X^2Y - 2XYX + YX^2) - (X^2Y - 2XYX + YX^2)X \\ &= X^3Y - 2X^2YX + XYX^2 - X^2YX + 2XYX^2 - YX^3 \\ &= X^3Y - 3X^2YX + 3XYX^2 - YX^3. \end{aligned}$$

If we compose with X^{n-i} and sum over i we get

$$\begin{aligned} \sum_{i=0}^{n} [x^{n-i}, [z, [x^i, y]]] &= \sum_{i=0}^{n} \sum_{j=0}^{i} (-1)^{j+1} \binom{i}{j} X^j Y X^{i-j} X^{n-i}(z) \\ &= \sum_{j=0}^{n} (-1)^{j+1} (\sum_{i=j}^{n} \binom{i}{j}) X^j Y X^{n-j}(z) \\ &= \sum_{j=0}^{n} (-1)^{j+1} \binom{n+1}{j+1} [x^{n-j}, [y, [x, z]]]. \end{aligned}$$

□

Corollary 3.1.4. *Let $\mathfrak{g}$ be a Lie algebra over a field of characteristic prime to $n!$ for $n \in \mathbb{N}$. If $X^{n+1} = 0$ then the following hold:*

1. $\sum_{i=0}^{n} X^i Y X^{n-i} = 0,$
2. $\sum_{j=0}^{n} (-1)^{j+1} \binom{n+1}{j+1} X^j Y X^{n-j} = 0.$

Proof. For the first part, consider $(X + \lambda Y)^{n+1} = 0$ and collect the terms with coefficient λ. This is displayed below in the case of $n = 2$.

$$\begin{aligned} 0 &= (X + \lambda Y)^3 \\ &= X^3 + \lambda X^2 Y + \lambda XYX + \lambda^2 XY^2 + \lambda YX^2 + \lambda^2 YXY + \lambda^2 Y^2 X + \lambda^3 Y^3 \\ &= \lambda \cdot (X^2 Y + XYX + YX^2) + \lambda^2 \cdot (XY^2 + YXY + Y^2 X). \end{aligned}$$

To obtain 2. one uses Lemma 3.1.3:

$$\begin{aligned} \sum_{j=0}^{n} (-1)^{j+1} \binom{n+1}{j+1} X^j Y X^{n-j}(z) &= \sum_{i=0}^{n} [x^{n-1}, [z, [x^i, y]]] \\ &= \sum_{i=0}^{n} X^i Z X^{n-i}(y) = 0. \end{aligned}$$

□

Remark 3.1.5. One should observe that Corollary 3.1.4 deals with the general case of the calculation in the proof of Theorem 3.1.1. Indeed, by setting $n = 1$ the above result yields the two equations

$$XY + YX = 0, \quad XY - 2YX = 0$$

as we computed in the mentioned proof.

Now we acquired all prerequisites to prove the nilpotency theorem for Engel-3 Lie algebras.

Theorem 3.1.6 (Higgins, Traustason)**.** *Every Engel-3 Lie algebra over a field $\mathbb{F}$ of characteristic different from 2 and 5 is of nilpotency class at most 4.*

Proof. As indicated in the label of the theorem, the proof of it is very similar to Higgin's from 1954, although in his paper [28] he only showed $\mathfrak{g}^7 = 0$ whereas Traustason proves the correct upper bound in [38] using Higgin's argument as we will see below.
Certainly, the Engel-3 identity can be expressed as $X^3 = 0$ which implies the two relations

$$X^2 Y + XYX + YX^2 = 0, \tag{3.3}$$

$$X^2 Y - 3XYX + 3YX^2 = 0 \tag{3.4}$$

by Corollary 3.1.4.
For the moment, let $\text{char}(\mathbb{F}) = 3$ whence $X^2 Y = 0$ by equation (3.4). As shown in Lemma 4 of Higgin's article, the subspace

$$\mathfrak{i} := \langle T^2 \rangle = \text{span}\{[t, [t, s]] : t, s \in \mathfrak{g}\}$$

forms an ideal of $\mathfrak{g}$. Within the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$ every element of type $[t,[t,s]]$ is 0, hence $\mathfrak{g}/\mathfrak{i}$ is an Engel-2 Lie algebra by definition. By means of Remark 3.1.2, which considers the characteristic 3 case of the nilpotency theorem for Engel-2 Lie algebras, we have

$$(\mathfrak{g}/\mathfrak{i})^4 = 0.$$

Equivalently, every four-element Lie product $[x_1,[x_2,[x_3,x_4]]]$ lies in the ideal $\mathfrak{i}$ and so has form $[t,[t,s]]$ by definition. Thus,

$$[x_1,[x_2,[x_3,[x_4,x_5]]]] = [x_1,[t,[t,s]]] = 0$$

because we already calculated $X^2Y = 0$. So we elaborated that any given product of five elements equals zero, i.e. $\mathfrak{g}^5 = 0$ as claimed.
Now consider the case of $\mathrm{char}(\mathbb{F}) \neq 3$ and also $\mathrm{char}(\mathbb{F})$ is not 2 or 5 by assumption. At first, we eliminate X^2Y in equation (3.3) and YX^2 in (3.4).

$$\begin{aligned} (3.3)-(3.4):&\ 4XYX - 2YX^2 = 0 \Longleftrightarrow YX^2 = 2XYX && (3.5)\\ 3\cdot(3.3)-(3.4):&\ 6XYX + 3X^2Y = 0 \Longleftrightarrow X^2Y = -3XYX. && (3.6)\end{aligned}$$

Therefore, one immediately has

$$3YX^2 = -2X^2Y \tag{3.7}$$

and certainly, we can interchange X and Y in (3.5) and (3.6), respectively, to obtain

$$XY^2 = 2YXY, \tag{3.8}$$
$$Y^2X = -3YXY. \tag{3.9}$$

Operating with Y from the left on (3.5) and with X from the right on relation (3.9) yields

$$\begin{aligned} Y^2X^2 &= 2YXYX,\\ Y^2X^2 &= -3YXYX.\end{aligned}$$

Eliminating the right side gives $5Y^2X^2 = 0$ which implies $Y^2X^2 = 0$ as we assumed $\mathrm{char}(\mathbb{F}) \neq 5$. Again by Theorem 3.1.1, the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$ satisfies $(\mathfrak{g}/\mathfrak{i})^3 = 0$ where $\mathfrak{i}$ denotes the same Lie ideal as above. So one calculates

$$\begin{aligned} Y^2X_1X_2(z) &= [y,[y,[x_1,[x_2,z]]]]\\ &= [y,[y,[t,[t,s]]]]\\ &= Y^2T^2(s) = 0\end{aligned}$$

since every product of three elements lies in the ideal $\mathfrak{i}$ and hence has form $[t,[t,s]]$ for elements $t,s \in \mathfrak{g}$. With the help of relation (3.7) we further compute

$$9X_1X_2Y^2 = -4Y^2X_1X_2 = 0.$$

We use 3.1.1 and the quotient Lie algebra once more, i.e. the fact that every product of three elements is of form $[t,[t,s]]$ and so one eventually obtains

$$\begin{aligned} X_1X_2Y_1Y_2(z) &= [x_1,[x_2,[y_1,[y_2,z]]]] \\ &= [x_1,[x_2,[t,[t,s]]]] \\ &= 0, \end{aligned}$$

implying that every product of five elements equals zero, hence

$$\mathfrak{g}^5 = 0.$$

□

3.2 A construction of an Engel-3 Lie algebra of nilpotency class 4

We demonstrate the construction of an Engel-3 Lie algebra of class 4 in detail. Thus, Theorem 3.1.6 gives the best possible upper bound. With the help of the theory of free-nilpotent Lie algebras we are able to explicitly work out a Lie algebra of nilpotency class 4. To obtain the Engel-3 property we will factor this Lie algebra by a certain ideal such that the resulting quotient satisfies both, Engel-3 and nilpotency 4. Accomplished this, we try to reduce its dimension in order to present a convenient Lie algebra.

We start off by elaborating a 4-nilpotent Lie algebra. To be more precise, we consider the free-nilpotent Lie algebra $\mathfrak{F}_{3,4}$ on three generators and of nilpotency class four. By Witt's Theorem 1.3.6, we already know that its dimension is 32 and to list its elements and Lie brackets, we follow the procedure from Example 1.3.8. That is,

$$
\begin{aligned}
x_1&,\\
x_2&,\\
x_3&,\\
x_4&=[x_1,x_2],\\
x_5&=[x_1,x_3],\\
x_6&=[x_2,x_3],\\
x_7&=[x_1,x_4]=[x_1,[x_1,x_2]],\\
x_8&=[x_1,x_5]=[x_1,[x_1,x_3]],\\
x_9&=[x_1,x_6]=[x_1,[x_2,x_3]],\\
x_{10}&=[x_2,x_4]=[x_2,[x_1,x_2]],\\
x_{11}&=[x_2,x_5]=[x_2,[x_1,x_3]],\\
x_{12}&=[x_2,x_6]=[x_2,[x_2,x_3]],\\
&[x_3,x_4]=[x_3,[x_1,x_2]]=-x_9+x_{11} \text{ by } (J_1),\\
x_{13}&=[x_3,x_5]=[x_3,[x_1,x_3]],\\
x_{14}&=[x_3,x_6]=[x_3,[x_2,x_3]],\\
x_{15}&=[x_1,x_7]=[x_1,[x_1,[x_1,x_2]]],\\
x_{16}&=[x_1,x_8]=[x_1,[x_1,[x_1,x_3]]],\\
x_{17}&=[x_1,x_9]=[x_1,[x_1,[x_2,x_3]]],\\
x_{18}&=[x_1,x_{10}]=[x_1,[x_2,[x_1,x_2]]],\\
x_{19}&=[x_1,x_{11}]=[x_1,[x_2,[x_1,x_3]]],\\
x_{20}&=[x_1,x_{12}]=[x_1,[x_2,[x_2,x_3]]],\\
x_{21}&=[x_1,x_{13}]=[x_1,[x_3,[x_1,x_3]]],\\
x_{22}&=[x_1,x_{14}]=[x_1,[x_3,[x_2,x_3]]],\\
&[x_2,x_7]=[x_2,[x_1,[x_1,x_2]]]=x_{18} \text{ by } (J_2),\\
x_{23}&=[x_2,x_8]=[x_2,[x_1,[x_1,x_3]]],\\
x_{24}&=[x_2,x_9]=[x_2,[x_1,[x_2,x_3]]],\\
x_{25}&=[x_2,x_{10}]=[x_2,[x_2,[x_1,x_2]]],\\
x_{26}&=[x_2,x_{11}]=[x_2,[x_2,[x_1,x_3]]],\\
x_{27}&=[x_2,x_{12}]=[x_2,[x_2,[x_2,x_3]]],\\
x_{28}&=[x_2,x_{13}]=[x_2,[x_3,[x_1,x_3]]],\\
x_{29}&=[x_2,x_{14}]=[x_2,[x_3,[x_2,x_3]]],\\
&[x_3,x_7]=-x_{17}+2\cdot x_{19}-x_{23} \text{ by } (J_3),(J_5),\\
&[x_3,x_8]=[x_3,[x_1,[x_1,x_3]]]=x_{21}\text{by } (J_6),
\end{aligned}
$$

$$\begin{aligned}
x_{30} = [x_3, x_9] &= [x_3, [x_1, [x_2, x_3]]], \\
[x_3, x_{10}] &= x_{20} - 2 \cdot x_{24} + x_{26} \text{ by } (J_4), (J_8), \\
[x_3, x_{12}] &= [x_3, [x_2, [x_2, x_3]]] = x_{29} \text{ by } (J_{10}), \\
x_{31} = [x_3, x_{13}] &= [x_3, [x_3, [x_1, x_3]]], \\
x_{32} = [x_3, x_{14}] &= [x_3, [x_3, [x_2, x_3]]].
\end{aligned}$$

Observe that $\{x_1, \ldots, x_6\}$ is the basis of $\mathfrak{F}_{3,2}$ and $\{x_1, \ldots, x_{14}\}$ spans $\mathfrak{F}_{3,3}$, so:

$$\mathfrak{F}_{3,1} \subseteq \mathfrak{F}_{3,2} \subseteq \mathfrak{F}_{3,3} \subseteq \mathfrak{F}_{3,4} \subseteq \cdots$$

Moreover, as indicated in the list, several valid Lie brackets can be expressed as linear combination of other elements which is why they do not get added to the list of basis elements. These restrictions are caused by the Jacobi identity. The corresponding calculations are performed below.

$$\begin{aligned}
J(x_1, x_2, x_3) = 0 &\Longleftrightarrow [x_1, [x_2, x_3]] + [x_2, [x_3, x_1]] + [x_3, [x_1, x_2]] = 0 \\
&\Longrightarrow [x_3, x_4] = -[x_1, [x_2, x_3]] + [x_2, [x_1, x_3]] \\
&\Longrightarrow [x_3, x_4] = -x_9 + x_{11} \qquad (J_1)
\end{aligned}$$

$$\begin{aligned}
J(x_1, x_2, x_4) = 0 &\Longleftrightarrow [x_1, [x_2, x_4]] + [x_2, [x_4, x_1]] + [x_4, [x_1, x_2]] = 0 \\
&\Longrightarrow x_{18} = [x_2, [x_1, x_4]] - [x_4, [x_1, x_2]] \\
&\Longrightarrow x_{18} = [x_2, [x_1, [x_1, x_2]]] = [x_2, x_7] \qquad (J_2)
\end{aligned}$$

$$\begin{aligned}
J(x_1, x_2, x_5) = 0 &\Longleftrightarrow [x_1, [x_2, x_5]] + [x_2, [x_5, x_1]] + [x_5, [x_1, x_2]] = 0 \\
&\Longrightarrow [x_1, [x_2, x_5]] = [x_2, [x_1, x_5]] - [x_5, [x_1, x_2]] \\
&\Longrightarrow [x_4, x_5] = [x_1, x_{11}] - [x_2, x_8] = x_{19} - x_{23} \qquad (J_3)
\end{aligned}$$

$$\begin{aligned}
J(x_1, x_2, x_6) = 0 &\Longleftrightarrow [x_1, [x_2, x_6]] + [x_2, [x_6, x_1]] + [x_6, [x_1, x_2]] = 0 \\
&\Longrightarrow [x_1, [x_2, x_6]] = [x_2, x_9] - [x_6, x_4] \\
&\Longrightarrow [x_4, x_6] = [x_1, x_{12}] - [x_2, x_9] = x_{20} - x_{24} \qquad (J_4)
\end{aligned}$$

$$\begin{aligned}
J(x_1, x_3, x_4) = 0 &\Longleftrightarrow [x_1, [x_3, x_4]] + [x_3, [x_4, x_1]] + [x_4, [x_1, x_3]] = 0 \\
&\Longrightarrow [x_1, [x_3, x_4]] = [x_3, x_7] - [x_4, x_5] \\
&\Longrightarrow [x_1, x_{11} - x_9] = [x_3, x_7] - [x_4, x_5] \qquad (J_5)
\end{aligned}$$

$$\begin{aligned}
J(x_1, x_3, x_5) = 0 &\Longleftrightarrow [x_1, [x_3, x_5]] + [x_3, [x_5, x_1]] + [x_5, [x_1, x_3]] = 0 \\
&\Longrightarrow [x_1, x_{13}] = [x_3, x_8] - [x_5, x_5] \\
&\Longrightarrow [x_3, x_8] = x_{21} \qquad (J_6)
\end{aligned}$$

$$\begin{aligned} J(x_1,x_3,x_6)=0 \Longleftrightarrow & [x_1,[x_3,x_6]]+[x_3,[x_6,x_1]]+[x_6,[x_1,x_3]]=0 \\ \Longrightarrow & [x_1,x_{14}]=[x_3,x_9]-[x_6,x_5] \\ \Longrightarrow & [x_5,x_6]=x_{22}-x_{30} \end{aligned} \tag{J_7}$$

$$\begin{aligned} J(x_2,x_3,x_4)=0 \Longleftrightarrow & [x_2,[x_3,x_4]]+[x_3,[x_4,x_2]]+[x_4,[x_2,x_3]]=0 \\ \Longrightarrow & [x_2,-x_9+x_{11}]=[x_3,x_{10}]-[x_4,x_6] \\ \Longrightarrow & [x_3,x_{10}]=[x_2,-x_9+x_{11}]+[x_4,x_6] \end{aligned} \tag{J_8}$$

$$\begin{aligned} J(x_2,x_3,x_5)=0 \Longleftrightarrow & [x_2,[x_3,x_5]]+[x_3,[x_5,x_2]]+[x_5,[x_2,x_3]]=0 \\ \Longrightarrow & [x_2,x_{13}]=[x_3,x_{11}]-[x_5,x_6] \\ \Longrightarrow & [x_3,x_{11}]=x_{28}+[x_5,x_6] \end{aligned} \tag{J_9}$$

$$\begin{aligned} J(x_2,x_3,x_6)=0 \Longleftrightarrow & [x_2,[x_3,x_6]]+[x_3,[x_6,x_2]]+[x_6,[x_2,x_3]]=0 \\ \Longrightarrow & [x_2,x_{14}]=[x_3,x_{12}]-[x_3,x_3] \\ \Longrightarrow & x_{29}=[x_3,x_{12}] \end{aligned} \tag{J_{10}}$$

Additionally, if we combine equations (J_3) and (J_5) we obtain

$$\begin{aligned} [x_3,x_7] &= [x_4,x_5]+[x_1,x_{11}-x_9] \\ &= x_{19}-x_{23}+[x_1,x_{11}]-[x_1,x_9] \\ &= -x_{17}+2\cdot x_{19}-x_{23}. \end{aligned}$$

Similarly, by considering $(J_4),(J_8)$ and $(J_7),(J_9)$ respectively, we get

$$\begin{aligned} [x_3,x_{10}] &= [x_4,x_6]-[x_2,x_9]+[x_2,x_{11}] \\ &= x_{20}-2\cdot x_{24}+x_{26}, \\ [x_3,x_{11}] &= x_{22}+x_{28}-x_{30}. \end{aligned}$$

Now we are able to display the multiplication table for the Lie algebra $\mathfrak{F}_{3,4}$.

$[\,,\,]$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}	x_{12}	x_{13}	x_{14}	$x_{15}\cdots$
x_1	0	x_4	x_5	x_7	x_8	x_9	x_{15}	x_{16}	x_{17}	x_{18}	x_{19}	x_{20}	x_{21}	x_{22}	$0\cdots$
x_2	$-x_4$	0	x_6	x_{10}	x_{11}	x_{12}	x_{18}	x_{23}	x_{24}	x_{25}	x_{26}	x_{27}	x_{28}	x_{29}	$0\cdots$
x_3	$-x_5$	$-x_6$	0	$-x_9+x_{11}$	x_{13}	x_{14}	$-x_{17}$ $+2x_{19}$ $-x_{23}$	x_{21}	x_{30}	x_{20} $-2x_{24}$ $+x_{26}$	x_{22} $+x_{28}$ $-x_{30}$	x_{29}	x_{31}	x_{32}	$0\cdots$
x_4	$-x_7$	$-x_{10}$	x_9-x_{11}	0	$x_{19}-x_{23}$	$x_{20}-x_{24}$	0	0	0	0	0	0	0	0	$0\cdots$
x_5	$-x_8$	$-x_{11}$	$-x_{13}$	$-x_{19}+x_{23}$	0	$x_{22}-x_{30}$	0	0	0	0	0	0	0	0	$0\cdots$
x_6	$-x_9$	$-x_{12}$	$-x_{14}$	$-x_{20}+x_{24}$	$-x_{22}+x_{30}$	0	0	0	0	0	0	0	0	0	$0\cdots$
x_7	$-x_{15}$	$-x_{18}$	x_{17} $-2x_{19}$ $+x_{23}$	0	0	0	0	0	0	0	0	0	0	0	$0\cdots$
x_8	$-x_{16}$	$-x_{23}$	x_{21}	0	0	0	0	0	0	0	0	0	0	0	$0\cdots$
x_9	$-x_{17}$	$-x_{24}$	$-x_{30}$	0	0	0	0	0	0	0	0	0	0	0	$0\cdots$
x_{10}	$-x_{18}$	$-x_{25}$	$-x_{20}$ $+2x_{24}$ $-x_{26}$	0	0	0	0	0	0	0	0	0	0	0	$0\cdots$
x_{11}	$-x_{19}$	$-x_{26}$	$-x_{22}-x_{28}$ $+x_{30}$	0	0	0	0	0	0	0	0	0	0	0	$0\cdots$
x_{12}	$-x_{20}$	$-x_{27}$	$-x_{29}$	0	0	0	0	0	0	0	0	0	0	0	$0\cdots$
x_{13}	$-x_{21}$	$-x_{28}$	$-x_{31}$	0	0	0	0	0	0	0	0	0	0	0	$0\cdots$
x_{14}	$-x_{22}$	$-x_{29}$	$-x_{32}$	0	0	0	0	0	0	0	0	0	0	0	$0\cdots$
x_{15}	0	0	0	0	0	0	0	0	0	0	0	0	0	0	$0\cdots$
$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\vdots$	$\ddots$

Now we have full information about our 4-nilpotent Lie algebra. The next step is to establish the Engel-3 property. The idea is to factor $\mathfrak{F}_{3,4}$ by an ideal in a way such that the quotient Lie algebra satisfies Engel-3 and does not lose nilpotency class 4. The first problem we encounter is the non-multilinearity of the Engel identity. That means that it is not enough to have $[x,[x,[x,y]]] = 0$ for all basis elements x, y in order for it to hold on all elements of the Lie algebra. To see that, consider the example

$$\mathfrak{sl}(2,\mathbb{F}) = \langle \underbrace{\begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}}_{=e_1}, \underbrace{\begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}}_{=e_2}, \underbrace{\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}}_{=e_3} \rangle$$

together with $[,]$ being the commutator of matrices. One easily checks that the Engel-3 identity is satisfied for all combinations of basis elements. However, for $A = e_2 + e_3$ we have

$$[A,[A,[A,e_3]]] = \begin{pmatrix} 0 & 0 \\ 8 & 0 \end{pmatrix}.$$

Thus, it is not enough to verify Engel-3 on the basis elements of the Lie algebra. To resolve this issue, we linearize the Engel identity:

Proposition 3.2.1 (Full linearization of the Engel identity)**.** *Let $\mathfrak{g}$ be an Engel-n Lie algebra over a field $\mathbb{F}$ such that* $\mathrm{char}(\mathbb{F}) = 0$ *or* $\mathrm{char}(\mathbb{F})$ *prime to n. Let u, v be arbitrary elements of $\mathfrak{g}$ and let $x_1, \ldots, x_n, y$ be basis elements of $\mathfrak{g}$. Then*

$$[u,[u,\ldots[u,[u,v]]\ldots]] = 0 \iff \sum_{\sigma\in\mathrm{Sym}(n)} [x_{\sigma(1)},[x_{\sigma(2)},\ldots,[x_{\sigma(n)},y]\ldots]] = 0.$$

Proof. The following proof is due to Traustason as it is presented in his article [38]. However, the result was known for a long time, notice the similarity to Zelmanov's Theorem 2.2.7. We begin with "$\Longrightarrow$": For this purpose, take n indeterminates $\lambda_1, \ldots, \lambda_n$ and expand the Lie bracket $P := [(\lambda_1 x_1 + \cdots + \lambda_n x_n)^n, y]$. Then we get

$$P = a_0\lambda_1 a_1 + \lambda_1\lambda_2 a_2 + \cdots + \lambda_1\lambda_2\cdots\lambda_n \sum_{\sigma\in\mathrm{Sym}(n)} [x_{\sigma(1)},[x_{\sigma(2)},\ldots,[x_{\sigma(n)},y]\ldots],$$

where a_0 denotes the sum of all monomials that are not divisible by λ_1, $\lambda_1 a_1$ is the sum of all monomials of $P - a_0$ that are not divisible by λ_2 and proceed in this manner for all other a_i. By definition, $P = 0$ for all $\lambda_1, \ldots, \lambda_n$. So if we set $\lambda_1 = 0$, we obtain $a_0 = 0$. For the values $\lambda_1 = 1, \lambda_2 = 0$ we get $a_1 = 0$ and continuing this way gives $\sum_{\sigma\in\mathrm{Sym}(n)}[x_{\sigma(1)},[x_{\sigma(2)},\ldots,[x_{\sigma(n)},y] = 0$.
For "$\Longleftarrow$" we set $y = x_1 = x_2 = \cdots = x_n$. Then

$$\sum_{\sigma\in\mathrm{Sym}(n)} [x_{\sigma(1)},[x_{\sigma(2)},\ldots,[x_{\sigma(n)},y] = n!\cdot[x,[x,\ldots,[x,y]\ldots]].$$

Since $\mathrm{char}(\mathbb{F})$ does not divide $n!$ we conclude $[x,[x,\ldots,[x,y]\ldots]] = 0$. □

Summarizing, by the full linearization 3.2.1 and by Corollary 3.1.4, an Engel-3 Lie algebra $\mathfrak{g}$ satisfies the two equations

$$\sum_{\sigma\in \mathrm{Sym}(3)} [x_{\sigma(1)}, [x_{\sigma(2)}, [x_{\sigma(3)}, y]]] = 0, \tag{3.10}$$

$$[x, [x, [y, z]]] + [x, [y, [x, z]]] + [y, [x, [x, z]]] = 0. \tag{3.11}$$

Conversely, if $\mathfrak{g}$ admits the Engel-3 identity, (3.10) and (3.11) for basis elements x_1, x_2, x_3, x, y, z then $\mathfrak{g}$ is an Engel-3 Lie algebra, cf. [38].

These considerations motivate us to look at the ideal $\mathfrak{i}$ of $\mathfrak{F}_{3,4}$ that is generated by all basis elements of $\mathfrak{F}_{3,4}$ that satisfy the Engel-3 identity, (3.10) and (3.11). To this end, we calculate this identities for all elements of $\mathfrak{F}_{3,4}$. Looking at the list of elements, one immediately observes that $x_{15}, x_{16}, x_{25}, x_{27}, x_{31}, x_{32}$ are contained in $\mathfrak{i}$ by the Engel-3 identity and hence vanish in the quotient. For the linearized Engel identity we begin by setting $y = x_1$ and compute

$$\begin{aligned} 0 = {} & [x_3, [x_2, [x_1, x_1]]] + [x_2, [x_3, [x_1, x_1]]] + [x_3, [x_1, [x_2, x_1]]]+ \\ & [x_1, [x_3, [x_2, x_1]]] + [x_2, [x_1, [x_3, x_1]]] + [x_1, [x_2, [x_3, x_1]]] \\ = {} & 2 \cdot x_{17} - 4 \cdot x_{19}. \end{aligned}$$

Similarly, by putting $y = x_2$, we obtain the relation

$$-4 \cdot x_{24} + 2 \cdot x_{26} = 0$$

and $y = x_3$ implies

$$2 \cdot x_{22} + 2 \cdot x_{28} = 0.$$

All other choices of y are trivially satisfied.

The computation for (3.11) is more comprehensive, but we show two cases: Set for example $x = x_2, y = x_3, z = x_1$. Then equation (3.11) gives

$$\begin{aligned} 0 &= [x_2, [x_2, [x_3, x_1]]] + [x_2, [x_3, [x_2, x_1]]] + [x_3, [x_2, [x_2, x_1]]] \\ &= -x_{20} + 3 \cdot x_{24} - 3 \cdot x_{26}. \end{aligned}$$

If we take $x = x_2, y = x_1, z = x_1$ we get

$$\begin{aligned} 0 &= [x_2, [x_2, [x_1, x_1]]] + [x_2, [x_1, [x_2, x_1]]] + [x_1, [x_2, [x_2, x_1]]] \\ &= -2 \cdot x_{18}. \end{aligned}$$

Since the computations are performed analogously, we give the list of relations obtained by all other choices of x, y, z below.

$$\begin{aligned}
x_{15} &= 0, & x_{20} + x_{24} + x_{26} &= 0,\\
x_{16} &= 0, & -2x_{22} - 3x_{28} + 3x_{30} &= 0,\\
x_{18} &= 0, & x_{28} - 3x_{30} &= 0,\\
x_{21} &= 0, & -3x_{17} + 3x_{19} - x_{23} &= 0,\\
x_{25} &= 0, & x_{17} + x_{19} + x_{23} &= 0,\\
x_{27} &= 0, & x_{29} &= 0,\\
x_{31} &= 0, & x_{32} &= 0.
\end{aligned}$$

So in the quotient Lie algebra $\mathfrak{F}_{3,4}/\mathfrak{i}$ all the above equations hold true. Therefore, beside all the elements that get mapped to 0, in $\mathfrak{F}_{3,4}/\mathfrak{i}$ we have the relations

$$\begin{aligned}
x_{17} &= 2 \cdot x_{19},\\
x_{23} &= -3 \cdot x_{19},\\
x_{26} &= 2 \cdot x_{24},\\
x_{20} &= -3 \cdot x_{24},\\
x_{22} &= -x_{28},\\
x_{28} &= 3 \cdot x_{30},
\end{aligned}$$

induced by the full linearization (3.10) and (3.11). As a consequence, we delete the respective elements from the list of basis elements. More explicitly, we do not have to add x_{17} to our list because it can be expressed by x_{19}. Continuing this way gives the following basis of $\mathfrak{F}_{3,4}/\mathfrak{i}$:

$$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}, x_{12}, x_{13}, x_{14}, x_{19}, x_{24}, x_{30}\}$$

Since the Engel-3 identity, its linearization and equation (3.11) are satisfied for all these basis elements, the Lie algebra $\mathfrak{F}_{3,4}/\mathfrak{i}$ is an Engel-3 Lie algebra. Apart from that, it is of nilpotency class 4 because we have x_{19}, x_{24} and x_{30} as witnesses for the fact that not all Lie words of length four vanish.

However, in order to have a more convenient Lie algebra, we try to reduce its dimension. To this end, we set $x_{24} = x_{30} = 0$ leaving us with x_{19} as only witness for 4-nilpotency. Additionally, we can set $x_{10} = x_{12} = x_{13} = x_{14} = 0$ as it has no impact on the remaining elements. Thus,

$$\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{11}, x_{19}\}$$

is a basis of an 11-dimensional Engel-3 Lie algebra of nilpotency class 4, henceforth denoted by $\mathfrak{g}_{3,4}$.

Remark 3.2.2. Note that if, instead of x_{19}, we would choose x_{24} as remaining witness for 4-nilpotency, the Lie algebra loses the Engel-3 property. The program in Appendix A.2 verifies this. Similarly, the code can be used to show that taking x_{30} as only witness, also yields no Engel-3 Lie algebra.

To verify correctness of the above work, we prove the following assertion:

Proposition 3.2.3. *The obtained Lie algebra $\mathfrak{g}_{3,4}$ is Engel-3 and 4-nilpotent. In particular, Theorem 3.1.6 gives the best possible upper bound.*

Proof. We rename the elements in sequence to obtain $\{x_1, x_2, x_3, x_4, x_5, x_6, x_7, x_8, x_9, x_{10}, x_{11}\}$ as basis. Then we have the following table of Lie brackets:

$[,]$	x_1	x_2	x_3	x_4	x_5	x_6	x_7	x_8	x_9	x_{10}	x_{11}
x_1	0	x_4	x_5	x_7	x_8	x_9	0	0	$2x_{11}$	x_{11}	0
x_2	$-x_4$	0	x_6	0	x_{10}	0	0	$-3x_{11}$	0	0	0
x_3	$-x_5$	$-x_6$	0	$x_{10} - x_9$	0	0	$3x_{11}$	0	0	0	0
x_4	$-x_7$	0	$x_9 - x_{10}$	0	$4x_{11}$	0	0	0	0	0	0
x_5	$-x_8$	$-x_{10}$	0	$-4x_{11}$	0	0	0	0	0	0	0
x_6	$-x_9$	0	0	0	0	0	0	0	0	0	0
x_7	0	0	$-3x_{11}$	0	0	0	0	0	0	0	0
x_8	0	$3x_{11}$	0	0	0	0	0	0	0	0	0
x_9	$-2x_{11}$	0	0	0	0	0	0	0	0	0	0
x_{10}	$-x_{11}$	0	0	0	0	0	0	0	0	0	0
x_{11}	0	0	0	0	0	0	0	0	0	0	0

It is easy to see that the skew-symmetry and the Jacobi identity are satisfied, hence $\mathfrak{g}_{3,4}$ is a Lie algebra. Additionally, one immediately observes that $\mathfrak{g}_{3,4}^2 = \langle x_4, x_5, \ldots, x_{11} \rangle$. Thus,

$$\begin{aligned}
\mathfrak{g}_{3,4}^3 &= [\mathfrak{g}_{3,4}, \mathfrak{g}_{3,4}^2] = \langle x_7, x_8, \ldots, x_{11} \rangle, \\
\mathfrak{g}_{3,4}^4 &= [\mathfrak{g}_{3,4}, \mathfrak{g}_{3,4}^3] = \langle x_{11} \rangle, \\
\mathfrak{g}_{3,4}^5 &= [\mathfrak{g}_{3,4}, \mathfrak{g}_{3,4}^4] = 0.
\end{aligned}$$

Consequently, $\mathfrak{g}_{3,4}$ is nilpotent of class 4. To prove the Engel-3 property, we calculate the adjoint endomorphisms as demonstrated in Example 1.1.18.

$$
\underbrace{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&1&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&1&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&2&1&0
\end{pmatrix}}_{\mathrm{ad}(x_1)},
\underbrace{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&1&0&0&0&0&0&0\\
0&0&0&0&0&0&0&-3&0&0&0
\end{pmatrix}}_{\mathrm{ad}(x_2)},
$$

$$
\underbrace{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&-1&0&0&0&0&0&0&0\\
0&0&0&1&0&0&0&0&0&0&0\\
0&0&0&0&0&0&3&0&0&0&0
\end{pmatrix}}_{\mathrm{ad}(x_3)},
\underbrace{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&1&0&0&0&0&0&0&0&0\\
0&0&-1&0&0&0&0&0&0&0&0\\
0&0&0&0&4&0&0&0&0&0&0
\end{pmatrix}}_{\mathrm{ad}(x_4)},
$$

$$
\underbrace{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&-1&0&0&0&0&0&0&0&0&0\\
0&0&0&-4&0&0&0&0&0&0&0
\end{pmatrix}}_{\mathrm{ad}(x_5)},
\underbrace{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
-1&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0
\end{pmatrix}}_{\mathrm{ad}(x_6)},
$$

$$
\underbrace{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&-3&0&0&0&0&0&0&0&0
\end{pmatrix}}_{\mathrm{ad}(x_7)},
\underbrace{\begin{pmatrix}
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&0&0&0&0&0&0&0&0&0&0\\
0&3&0&0&0&0&0&0&0&0&0
\end{pmatrix}}_{\mathrm{ad}(x_8)},
$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\mathrm{ad}(x_9)}, \underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\mathrm{ad}(x_{10})},$$

$$\underbrace{\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}}_{\mathrm{ad}(x_{11})}.$$

Let y be an arbitrary element of $\mathfrak{g}_{3,4}$. Then y is a linear combination of the eleven basis elements, i.e. $y = \alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{11} x_{11}$ and therefore

$$\mathrm{ad}(y) = \mathrm{ad}(\alpha_1 x_1 + \alpha_2 x_2 + \cdots + \alpha_{11} x_{11}) = \alpha_1 \mathrm{ad}(x_1) + \cdots + \alpha_{11} \mathrm{ad}(x_{11}).$$

Now it is easy to compute $\mathrm{ad}(y)^3 = 0$, whence we established the Engel-3 identity for all elements of $\mathfrak{g}_{3,4}$. So by constructing an explicit example we proved that Theorem 3.1.6 indeed gives the best possible bound as stated. □

Remark 3.2.4. The Mathematica code in Appendix A.2 automatises the procedure in the proof of 3.2.3 for arbitrary Lie algebras. However, it remains an open question whether the so obtained Lie algebra is of minimal dimension.

As a last step, we compute a basis of the derivation algebra $\mathfrak{Der}(\mathfrak{g}_{3,4})$. Recall that a derivation D of $\mathfrak{g}_{3,4}$ is a linear map $\mathfrak{g}_{3,4} \longrightarrow \mathfrak{g}_{3,4}$ satisfying

$$D([x,y]) = [D(x), y] + [x, D(y)]$$

for all $x, y \in \mathfrak{g}_{3,4}$. Equivalently, we have the condition

$$\mathrm{ad}(D(x)) + \mathrm{ad}(x).D - D.\mathrm{ad}(x) = 0. \tag{3.12}$$

Let D be an arbitrary derivation, i.e. an 11×11 matrix in 121 indeterminates $d_{i,j}$. Then for each basis element x_i, relation (3.12) gives an equation, hence, we end up with a linear system that can be solved. The code in Appendix A.5 performs this task and returns the following general derivation:

$$D = \begin{pmatrix}
-d_{9,9}+d_{11,11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_{9,8} & -d_{8,8}+d_{11,11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-d_{6,4} & 0 & T_1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-d_{6,3}+d_{9,5} & d_{9,6} & 0 & T_2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_{5,1} & 0 & d_{9,6}+d_{10,6} & 0 & T_3 & 0 & 0 & 0 & 0 & 0 & 0 \\
d_{6,1} & d_{9,4} & d_{6,3} & d_{6,4} & d_{9,8} & 2d_{9,9}-d_{11,11} & 0 & 0 & 0 & 0 & 0 \\
d_{7,1} & d_{7,2} & d_{7,3} & d_{9,6} & 0 & 0 & T_4 & 0 & 0 & 0 & 0 \\
d_{8,1} & d_{8,2} & d_{8,3} & 0 & d_{10,6}+d_{9,6} & 0 & 0 & d_{8,8} & 0 & 0 & 0 \\
d_{9,1} & d_{9,2} & d_{9,3} & d_{9,4} & d_{9,5} & d_{9,6} & 2d_{6,4} & d_{9,8} & d_{9,9} & 0 & 0 \\
d_{10,1} & T_5 & T_6 & -d_{5,1} & d_{6,3}-d_{9,5} & d_{10,6} & -d_{6,4} & d_{9,8} & 0 & d_{9,9} & 0 \\
d_{11,1} & d_{11,2} & d_{11,3} & d_{11,4} & d_{11,5} & T_7 & T_8 & T_9 & T_{10} & T_{11} & d_{11,11}
\end{pmatrix}$$

where we abbreviated

$$\begin{aligned}
T_1 &= d_{8,8} + 2d_{9,9} - 2d_{11,11},\\
T_2 &= -d_{8,8} - d_{9,9} + 2d_{11,11},\\
T_3 &= d_{8,8} + d_{9,9} - d_{11,11},\\
T_4 &= -d_{8,8} - 2d_{9,9} + 3d_{11,11},\\
T_5 &= -3d_{8,1} - 2d_{9,2} + d_{11,4},\\
T_6 &= 3d_{7,1} - 2d_{9,3} + d_{11,5},\\
T_7 &= -3d_{7,2} - 3d_{8,3},\\
T_8 &= -5d_{5,1} + 2d_{9,4},\\
T_9 &= -3d_{6,3} + 5d_{9,5},\\
T_{10} &= d_{10,6} + 2d_{9,6},\\
T_{11} &= -3d_{10,6} + d_{9,6}.
\end{aligned}$$

There are twenty-seven distinct indeterminate entries in our derivation. Pick one of them at a time and set it equal to 1 and let the remaining entries vanish in D. For instance, if we set $d_{9,9} = 1$ and $d_{i,j} = 0$ for all other possible i, j, we get

$$D_1 = \begin{pmatrix}
-1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 2 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & -2 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

or for only $d_{8,1} = 1$ we end up with

$$D_2 = \begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & -3 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0\\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}.$$

Notice that D_1 and D_2 are linearly independent. Indeed, for scalars c_1, c_2 we have

$$c_1 \cdot D_1 + c_2 \cdot D_2 = 0 \Longleftrightarrow c_1 = c_2 = 0.$$

If we continue this process, we obtain twenty-seven matrices $D_1, \dots, D_{27}$. These are all linearly independent as is verified in the code in A.5. Thus, $\{D_1, D_2, \dots, D_{27}\}$ forms a basis of $\mathfrak{Der}(\mathfrak{g}_{3,4})$.

3.3 A counter-example for characteristic 2

The nilpotency theorem for Engel-3 Lie algebras 3.1.6 requires $\mathrm{char}(\mathbb{F}) \neq 2, 5$. In this paragraph we provide an example of an Engel-3 Lie algebra over a field of characteristic 2 that is not nilpotent. To this end, we follow Traustason's construction in [38]. On the other hand, for an Engel-3 Lie algebra over a field of characteristic 5, Traustason shows that for every x in the Lie algebra we have

$$\langle x \rangle^3 = 0,$$

where $\langle x \rangle$ denotes the ideal generated by x. In fact, Traustason proves that the above assertion holds for every Engel-3 Lie algebra $\mathfrak{g}$ with $\mathrm{char}(\mathbb{F}) \neq 2, 3$. Under the assumption that $\mathfrak{g}$ is generated by r elements he further concludes that $\mathfrak{g}$ is of nilpotency class $\leq 2r$. For more details to this case see [38].

Regarding the construction of the counter-example in characteristic 2, we recall from Section 3.2 that it suffices to verify

$$[x, [x, [x, y]]] = 0, \tag{3.13}$$

$$[x, [x, [y, z]]] + [x, [y, [x, z]]] + [y, [x, [x, z]]] = 0, \tag{3.14}$$

$$\sum_{\sigma \in \mathrm{Sym}(3)} [x_{\sigma(1)}, [x_{\sigma(2)}, [x_{\sigma(3)}, y]]] = 0. \tag{3.15}$$

for all basis elements x, y, z, x_1, x_2, x_3 for the purpose of showing that a Lie algebra is Engel-3. First, we define basis elements $\Sigma_A, \Lambda_A, \Omega_A$ for every non-empty subset $A \subseteq \mathbb{N}$ and add another basis element denoted by x. Then we let $\mathfrak{g}$ be the $\mathbb{F}$-vector field with the above basis. Second, we define a Lie product as follows: For all basis elements a, b we set $[a, b] = [b, a]$ and $[a, a] = 0$. In addition to that, we define

$$
\begin{aligned}
&[\Sigma_A, x] = [\Lambda_A, x] = 0, \\
&[\Omega_A, x] = \Sigma_A, \\
&[\Sigma_A, \Sigma_B] = [\Lambda_A, \Lambda_B] = [\Omega_A, \Omega_B] = 0, \\
&[\Sigma_A, \Lambda_B] = \begin{cases} \Sigma_{A\cup B} & \text{if } A \cap B = \varnothing \\ 0 & \text{otherwise,} \end{cases} \\
&[\Sigma_A, \Omega_B] = \begin{cases} \Lambda_{A\cup B} & \text{if } A \cap B = \varnothing \\ 0 & \text{otherwise,} \end{cases} \\
&[\Lambda_A, \Omega_B] = \begin{cases} \Omega_{A\cup B} & \text{if } A \cap B = \varnothing \\ 0 & \text{otherwise} \end{cases}
\end{aligned}
$$

and extend this Lie product linearly to all of $\mathfrak{g}$.

Proposition 3.3.1. *The algebra $\mathfrak{g}$ as defined above is an Engel-3 Lie algebra. In particular, the ideal $\langle x \rangle$ is not nilpotent.*

Proof. One immediatley observes that for all $y, z \in \mathfrak{g}$ we have

$$yz = zy = -zy$$

due to $\mathrm{char}(\mathbb{F}) = 2$. It is sufficient to check the Jacobi identity and the Engel identities (3.13), (3.14), (3.15) only for products of basis elements with pairwise disjoint integer subsets since, by definition, if $A \cap B \neq \varnothing$ for two basis elements then their product is zero. Moreover, it is enough to check the relations for every possible combination of the four different types of basis elements. For instance,

$$
\begin{aligned}
&[\Sigma_A, [\Lambda_B, \Omega_C]] + [\Lambda_B, [\Omega_C, \Sigma_A]] + [\Omega_C, [\Sigma_A, \Lambda_B]] = \\
=&[\Sigma_A, \Omega_{B\cup C}] + [\Lambda_B, \Lambda_{A\cup C}] + [\Omega_C, \Sigma_{A\cup B}] = 0
\end{aligned}
$$

verifies the Jacobi identity for one combination of basis elements. We demonstrate another case for the linearized Engel identity:

$$
\begin{aligned}
&[x, [\Sigma_A, [\Omega_B, \Omega_C]]] + [x, [\Omega_B, [\Sigma_A, \Omega_C]]] + \\
&[\Sigma_A, [x, [\Omega_B, \Omega_C]]] + [\Sigma_A, [\Omega_B, [x, \Omega_C]]] + \\
&[\Omega_B, [\Sigma_A, [x, \Omega_C]]] + [\Omega_B, [x, [\Sigma_B, \Omega_C]]] = 2 \cdot \Sigma_{A\cup B\cup C} = 0.
\end{aligned}
$$

All other cases are shown analogously. As a consequence, $\mathfrak{g}$ is an Engel-3 Lie algebra. However, consider

$$[\Omega_{\{1\}}, [\Omega_{\{2\}}, [x, [\Omega_{\{3\}}, [\Omega_{\{4\}}, [x, [\ldots [\Omega_{\{2n\}}, [x, \Omega_{\{2n+1\}}]] \ldots] = \Omega_{\{1,2,\ldots,2n+1\}}.$$

The basis element x appears n times in the above nonzero Lie product. As a result, $\langle x \rangle^n \neq 0$ for all $n \in \mathbb{N}$ which implies that $\langle x \rangle$ is not nilpotent. Furthermore, $\mathfrak{g}$ is not nilpotent. □

Chapter 4

Perspectives

As a conclusion to the thesis, we provide some information on further topics and ongoing research. On the one hand, we present some results regarding Engel-4 and Engel-5 Lie algebras and on the other hand, we mention a number of articles concerning related questions such as the problem of classifying the class of finite solvable groups in terms of so-called *Engel-like identities*.

4.1 Engel-4 and Engel-5 Lie algebras

We give some results on nilpotency classes of Engel-4 and Engel-5 Lie algebras. To this end, we make use of the following auxiliary fact:

Theorem 4.1.1 (Higgins, [28]). *Let $\mathfrak{g}$ be an Engel-n Lie ring with characteristic prime to $n!$. If $\mathfrak{g}^{(s)} = 0$ then $\mathfrak{g}^t = 0$, where $t = \frac{n^s-1}{n-1} + 1$.*

So if we can elaborate the solvable class of an Engel Lie algebra, Theorem 4.1.1 allows us to deduce an upper bound for the nilpotency class. Higgins uses this fact in his proof of the following theorem regarding Engel-4 Lie algebras.

Theorem 4.1.2 (Higgins). *Let $\mathfrak{g}$ be an Engel-4 Lie algebra over a field $\mathbb{F}$ such that $\mathrm{char}(\mathbb{F}) \neq 2, 3, 5$. Then $\mathfrak{g}$ is nilpotent.*

Proof. We provide a sketch of the proof as in [38] and show how 4.1.1 is utilized. For details it is referred to [28]. As above, we let capital letters denote adjoint endomorphisms, i.e. $X = \mathrm{ad}(x)$ for $x \in \mathfrak{g}$. Further, we denote $[X, Y] = XY - YX$ the Lie product in $\mathrm{ad}(\mathfrak{g})$. Using Lemma 3.1.4, Higgins derives the following relations:

$$
\begin{aligned}
X^3Y^3 &= -Y^3X^3 = 13V, & 2X^2Y^2XY &= -2Y^2X^2YX = -25V,\\
X^2Y^3X &= -Y^2X^3Y = -V, & XYXYXY &= -YXYXYX = -3V,\\
XY^3X^2 &= -YX^3Y^2 = 7V, & 2XY^2XYX &= -2YX^2YXY = -5V,\\
2X^2YXY^2 &= -2Y^2XYX^2 = V, & 2XY^2X^2Y &= -2YX^2Y^2X = 17V,\\
2XYX^2Y^2 &= -2YXY^2X^2 = -13V, & 2XYXY^2X &= -2YXYX^2Y = -7V,
\end{aligned}
$$

where $V = -X^2Y^3X$. Consider the subspace generated by all cubes of adjoint endomorphisms

$$\mathfrak{i} = \langle T^3 \rangle.$$

With the help of the above equations it is easy to calculate

$$[X, Y]^3 = (XY - YX)^3 = -21V$$

and since 21 is coprime to char($\mathbb{F}$) it follows that $V \in \mathfrak{i}$. Moreover, $13V = X^3Y^3 = -Y^3X^3$ and therefore, $X^3Y^3 \in \mathfrak{i}$ implying that X^3Y^3 is a cube. For cubes we have anti-commutativity and thus,

$$X^3Y^3Z^3 = (X^3Y^3)Z^3 = -Z^3(X^3Y^3) = X^3Z^3Y^3 = -X^3Y^3Z^3.$$

Consequently, $X^3Y^3Z^3 = 0$. As in the proof of the nilpotency theorem of Engel-3 Lie algebras 3.1.6, the quotient Lie algebra $\mathfrak{g}/\mathfrak{i}$ is an Engel-3 Lie algebra of nilpotency class at most 4 and hence solvable of class at most 3. Therefore,

$$X^3Y^3Z^3 = 0 \Longrightarrow \mathfrak{g}^{(3)}Y^3Z^3 = 0 \Longrightarrow \mathfrak{g}^{(6)}Z^3 = 0 \Longrightarrow \mathfrak{g}^{(9)} = 0.$$

By 4.1.1, $\mathfrak{g}$ is nilpotent of class at most 87382. □

Remark 4.1.3. In [38] Traustason shows that *an* upper bound for the nilpotency class is 7. So one encounters the analogue problem for Engel-4 Lie algebras:

Question 4.1.4. *Is there an Engel-*4 *Lie algebra that attains nilpotency class* 7*?*

For the purpose of proving this upper bound one could try to construct an example in the same manner as we demonstrated in the case of Engel-3 Lie algebras. As in Section 3.2, the linearization of the Engel identity 3.2.1 and Corollary 3.1.4 can be used to generate an ideal such that the quotient Lie algebra satisfies the Engel-4 identity. However, our procedure requires to explicitly elaborate the free-nilpotent Lie algebra. It is easy to see that this is impossible with only two generators. So, assuming that this is possible using three generators, that means one has to calculate $\mathfrak{F}_{3,7}$. By Witt's formula 1.3.6, we know that $\dim(\mathfrak{F}_{3,7}) = 508$. If four generators are needed, the code in Appendix A.6 gives $\dim(\mathfrak{F}_{4,7}) = 3304$. Thus, one still could try to follow the construction in Section 3.2, but it is of course very complex and probably only feasible using computers.

Remark 4.1.5. Recently, Dietrich Burde from the University of Vienna and supervisor of this thesis together with Willem A. de Graaf from the University of Trento found that the largest 4-generator Engel-4 Lie algebra has dimension 484 and is of nilpotency class 7, thus answering Question 4.1.4 affirmatively.

Remark 4.1.6. Traustason's proof of the upper bound is very complex as one has to show that every product of 8 elements vanishes and hence, one has to work in 8-generator Engel-4 Lie algebras. Therefore, Traustason introduces the notions of *superalgebras* and *colour algebras* that can be used to reduce the number of generators to 4 and so ease computations. Below, we give the idea of Traustason's reduction and refer to [38] for more information.

Let $\{x_1, x_2, \ldots, x_m\}$ be a basis of a Lie algebra $\mathfrak{g}$. A Lie product is assigned a multiweight $\Omega = (\omega_1, \omega_2, \ldots, \omega_m)$ if there are ω_i occurences of x_i in the product. Now let $\mathfrak{g}$ be the free Engel-4 Lie algebra generated by $x_1, x_2, \ldots, x_8$ over a field $\mathbb{F}$. Denote by A the subspace generated by all Lie products of multiweight $(1, 1, \ldots, 1)$ in the generators. We define an action on A by the symmetric group $\mathrm{Sym}(8)$. Take a Lie product $p(x_1, x_2, \ldots, x_8)$ and a permutation $\pi \in \mathrm{Sym}(8)$. Then set

$$\pi p(x_1, x_2, \ldots, x_8) := p(x_{\pi(1)}, x_{\pi(2)}, \ldots, x_{\pi(8)}).$$

For instance,

$$(1346)[x_1, [x_2, [x_3, [x_4, [x_5, [x_6, [x_7, x_8]]]]]]] = [x_3, [x_2, [x_4, [x_6, [x_5, [x_1, [x_7, x_8]]]]]]].$$

So A is a $\mathrm{Sym}(8)$-module over the field $\mathbb{F}$.

Definition 4.1.7. An element $\alpha \in \mathbb{F}\mathrm{Sym}(8)$ is called *symmetrized* if there exist $\sigma, \tau \in \mathrm{Sym}(8)$ such that

$$\alpha = \sigma(\mathrm{id} + (12) + (13) + (23) + (123) + (132))\sigma^{-1}\tau.$$

α is called *skew-symmetrized* if there are σ, τ such that

$$\alpha = \sigma(\mathrm{id} - (12) - (13) - (23) + (123) + (132))\sigma^{-1}\tau.$$

Further, a in A is called *(skew-)symmetrized* if $a = \alpha p$ for a (skew-)symmetrized element $\alpha \in \mathbb{F}\mathrm{Sym}(8)$ and a Lie product p of $x_1, x_2, \ldots, x_8$.

Example 4.1.8. Let $\sigma = (14)(25), \tau = (135)$ and let $\alpha = \sigma(\mathrm{id} + (12) + (13) + (23) + (123) + (132))\sigma^{-1}\tau$. Then we have

$$\tau[x_1, [x_2, [x_3, [x_4, [x_5, [x_6, [x_7, x_8]]]]]]] = [x_3, [x_2, [x_5, [x_4, [x_1, [x_6, [x_7, x_8]]]]]]].$$

Moreover, $\sigma(\mathrm{id} + (12) + (13) + (23) + (123) + (132))\sigma^{-1} = (\mathrm{id} + (45) + (43) + (53) + (453) + (435))$. This is easy to see as for example

$$\sigma(12)\sigma^{-1} = (14)(25)(12)(41)(52) = (14)(25)(4251) = (54).$$

So we obtain

$$\alpha[x_1, [x_2, [x_3, [x_4, [x_5, [x_6, [x_7, x_8]]]]]]] = \sum_{\pi \in \mathrm{Sym}(\{3,4,5\})} [x_{\pi(3)}, [x_2, [x_{\pi(5)}, [x_{\pi(4)}, [x_1, [x_6, [x_7, x_8]]]]]]]$$

Traustason suggests to think of symmetrized and skew-symmetrized elements of A as follows: Given an arbitrary Lie product of $x_1, x_2, \ldots, x_8$ in some order, take any three elements in it and form all possible permutations of them. So we end up with six products. Adding this products up, we obtain a symmetrized element and if we form the alternating sum, we get a skew-symmetrized element. Recall that we have to show that every product of 8 elements vanishes in order to prove that an Engel-4 Lie algebra is of nilpotency class at most 7. The key to Traustason's reduction is the fact that the term $720 \cdot [x_1, [x_2, [x_3, [x_4, [x_5, [x_6, [x_7, x_8]]]]]]]$ can be expressed as a sum of symmetrized and skew-symmetrized products. Since $720 = 2^4 \cdot 3^2 \cdot 5$ it is enough to prove that for a field of characteristic different from $2, 3$ and 5 all symmetrized and skew-symmetrized products vanish. In [38] Traustason proves this facts in detail and utilizes them to prove the nilpotency theorem for Engel-4 Lie algebras.

As in the case of Engel-3 Lie algebras, we can find counter examples if we change the characteristic of the field. Clearly, every Engel-3 Lie algebra is also an Engel-4 Lie algebra. Thus, the counter example given in Section 3.3 for characteristic 2 also illustrates an instance of a non-nilpotent Engel-4 Lie algebra. Further, Traustason shows the following results:

Theorem 4.1.9 (Traustason, [38]). *Let $\mathfrak{g}$ be an Engel-4 Lie algebra over a field of characteristic different from 2 or 5. Then for all $x \in \mathfrak{g}$ we have*

$$\langle x \rangle^4 = 0.$$

Remark 4.1.10. So by Theorem 4.1.9 every ideal generated by one element is of nilpotency class at most 3 provided that the field is of characteristic different from 2 or 5. By a result in the article [26], for all x in an Engel-4 Lie algebra the ideal $\langle x \rangle$ is nilpotent of class at most 7 if the characteristic is equal to 5.

In the case of Engel-5 Lie algebras calculations get very complex and various reduction steps need to be performed in order to ease computations. However, Trausaston proves the following results in the article [35].

Theorem 4.1.11 (Traustason). *Let $\mathfrak{g}$ be an r-generator Engel-5 Lie algebra over a field $\mathbb{F}$ such that $\operatorname{char}(\mathbb{F}) \neq 2, 3, 5, 7$. Then $\mathfrak{g}$ is nilpotent of class at most $59r$. If the field is of characteristic 7 then $\mathfrak{g}$ is of class at most $80r$.*

Corollary 4.1.12 (Traustason). *Let $\mathfrak{g}$ be an Engel-5 Lie algebra over a field $\mathbb{F}$ such that $\operatorname{char}(\mathbb{F}) = 0$ or $\operatorname{char}(\mathbb{F}) > 195113$. Then $\mathfrak{g}$ has nilpotency class at most 975563.*

The essence of the statement in Corollary 4.1.12 is that the nilpotency class can be elaborated independently of the number of generators.

4.2 Further literature

We provide a number of articles that address related topics.

First, we mention Moravec's and Traustason's articles [33], [36] regarding *powerful Engel-2 groups*. Recall that a finite p-group G is said to be *powerful* if the subgroup $G^p = \langle g^p : g \in G\rangle$ contains the commutator subgroup $[G, G]$ for odd p. The starting point of this paper is Zorn's theorem:

Theorem 4.2.1 (Zorn, [46])**.** *A finite group is nilpotent if and only if there is an integer n such that $e_n(x, y) = 1$.*

Recall from Section 2.3 that the equation $e_n(x, y) = 1$ is the Engel-n identity defined inductively as $e_0(x, y) = x$ and $e_{n+1}(x, y) = [e_n(x, y), y]$ for group elements x, y. Note that here, the bracket $[,]$ denotes the commutator of group elements. This result gives a characterization of finite nilpotent groups. However, for $n \geq 3$, the nilpotency class is not bounded in terms of the integer n. In [1] it is shown that for a powerful group we indeed have that the nilpotency class is n-bounded. Moravec and Traustason investigate this topic and show that every powerful Engel-2 group generated by three elements is of nilpotency class at most 2.

Next, we cite Crosby's and Traustason's papers [17] and [18] regarding *right Engel-n subgroups*. An element $g \in G$ is called *right Engel-n* if

$$\underbrace{[x, [x, \ldots, [x}_{n-\text{times}}, g] \ldots]] = 1$$

for all $x \in G$. The group G is called *right Engel-n* if every element $g \in G$ is right Engel-n. A subgroup $H \leq G$ is a *right Engel-n subgroup* of G if all $h \in H$ are right Engel-n elements of G. The authors provide a boundedness condition for a number of specific types of right Engel-n groups.

Another related problem of recent success is the characterization of finite solvable groups by Engel-like identities. Again, Zorn's theorem constitutes the starting point of the topic. The goal was to replace nilpotency by solvability in Theorem 4.2.1 and to find a similar, "*Engel-like*", identity that characterizes this property. After the series of papers [8], [10], Bandman, Greuel, Grunewald, Kunyavskiĭ, Pfister and Plotkin succeeded in doing so and proved the following result in their 2006 article [9]:

Theorem 4.2.2 (Bandman, Greuel, Grunewald, et al)**.** *A finite group G is solvable if and only if for some n the identity $u_n(x, y) = 1$ holds in G. Here, $u_1(x, y) := x^{-2}y^{-1}x$ and inductively, $u_{n+1} := [xu_n(x, y)x^{-1}, yu_n(x, y)y^{-1}]$.*

Appendix A

Codes

We present several Mathematica codes that automatise calculations and procedures that were used in the construction of an Engel-3 Lie algebra of nilpotency class 4. In code A.1 we define a Lie bracket that is applied in A.2, A.3 and A.4. The input is given in form of Lie brackets, only program A.5 processes the adjoint endomorphisms. However, as shown in Chapter 1, it is easy to convert one into the other. The exemplary input displayed represents the 11-dimensional Lie algebra $\mathfrak{g}_{3,4}$ constructed in Section 3.2. In A.6 Witt's formula 1.3.6 is programmed.

A.1 Definition of the Lie product

The below code defines a Lie bracket, that is, a function which is bilinear and skew-symmetric. We do not include the Jacobi identity in the definition.

```
In[1]:=
    LB[0,U_] := 0;

    LB[U_Plus,V_] := Block[{LB},Distribute[LB[U_,V_]]];
    LB[U_,V_Plus] := Block[{LB},Distribute[LB[U_,V_]]];

    LB[r_?NumericQ*U_,V_] := r*LB[U,V];
    LB[V_,r_?NumericQ*U_] := r*LB[V,U];

    LB[U_,U_] = 0;

    LB[U_,V_] /; Sort[{U,V}] = ! = {U, V} :=
    -LB[Apply[Sequence, Sort[{U,V}]]];
```

A.2 Verification of an Engel-3 Lie algebra

This program verifies whether a given input defines an Engel-3 Lie algebra. To this end, it is checked if the Jacobian of any three elements yields 0. To validate the Engel-3 property, the procedure from the proof of Claim 3.2.3 is implemented.

Notice that the code allows to vary the number of input Lie brackets and can easily be extended to the case of Engel-n Lie algebras for other small values of n. However, for large n the code has a long computational time.

```
In[1]:=
    (* Input: in this example, the Lie brackets of the
     basis of g3,4 are entered *)
    n = 11;
    LB[x1, x2] = x4;
    LB[x1, x3] = x5;
    LB[x1, x4] = x7;
    LB[x1, x5] = x8;
    LB[x1, x6] = x9;
    LB[x1, x7] = 0;
    LB[x1, x8] = 0;
    LB[x1, x9] = 2*x11;
    LB[x1, x10] = x11;
    LB[x1, x11] = 0;

    LB[x2, x3] = x6;
    LB[x2, x4] = 0;
    LB[x2, x5] = x10;
    LB[x2, x6] = 0;
    LB[x2, x7] = 0;
    LB[x2, x8] = -3*x11;
    LB[x2, x9] = 0;
    LB[x2, x10] = 0;
    LB[x2, x11] = 0;

    LB[x3, x4] = -x9 + x10;
    LB[x3, x5] = 0;
    LB[x3, x6] = 0;
    LB[x3, x7] = 3*x11;
    LB[x3, x8] = 0;
    LB[x3, x9] = 0;
    LB[x3, x10] = 0;
    LB[x3, x11] = 0;
```

In[1]:=
```
LB[x4, x5] = 4*x11;
LB[x4, x6] = 0;
LB[x4, x7] = 0;
LB[x4, x8] = 0;
LB[x4, x9] = 0;
LB[x4, x10] = 0;
LB[x4, x11] = 0;
```

In[2]:=
```
(* Set remaining brackets equal to 0 *)
For[j=5, j≤n, j++,
   For[k=j+1, k≤n, k++,
      LB[xj, xk] = 0;
   ]
]
For[j=1, j≤n, j++,
   For[k=12, k≤n, k++,
      LB[xj, xk] = 0;
   ]
]

(* Check Jacobi *)
For[i=1, i≤n-2, i++,
  For[j=i+1, j≤n-1, j++,
    For[k=j+1, k≤n, k++,
      If[PossibleZeroQ[
        LB[xi,LB[xj,xk]]+LB[xj,LB[xk,xi]]+LB[xk,LB[xi,xj]]],,
      Print[
        LB[xi,LB[xj,xk]]+LB[xj, LB[xk,xi]]+LB[xk,LB[xi,xj]]];
      Print["Input gives no Lie algebra."];
      ]
    ]
  ]
]
```

In[3]:=
```
(* Table of Lie brackets *)
Table[LB[xk,xi], {k,1,n}, {i,1,n}] // MatrixForm
```

```
In[4]:=
    (* Check if the given input satisfies the Engel-3
    identity. Here, subs[] is an auxiliary function
    that returns the index of a variable. *)
    subs[x_]:=Flatten[Cases[#, (Subscript)[_, y_] ↦
    y, {0, Infinity}] & /@ {x}];

    For[o=1, o≤n, o++, vec_o=Table[0, n];]
    adz=0;

    For[i=1, i≤n, i++,
     adx_i= {};
     For[j=1, j≤n, j++,
      If[PossibleZeroQ[LB[x_i,x_j]], ,
        If[Part[subs[LB[x_i,x_j]],1]≠0,
          vec_j=ReplacePart[vec_j,Part[subs[LB[x_i,x_j]],1]
          →Coefficient[LB[x_i, x_j],x_Part[subs[LB[x_i,x_j]], 1]]];
        ]
        If[Length[subs[LB[x_i, x_j]]]≥2 &&
         Part[subs[LB[x_i,x_j]],2]≠0,
          vec_j=ReplacePart[vec_j,Part[subs[LB[x_i,x_j]], 2]
            →Coefficient[LB[x_i,x_j],x_Part[subs[LB[x_i,x_j]],2]]];
        ]
        If[Length[subs[LB[x_i, x_j]]] ≥ 3 &&
          Part[subs[LB[x_i,x_j]], 3]≠0,
          vec_j=ReplacePart[vec_j,Part[subs[LB[x_i,x_j]], 3]
            →Coefficient[LB[x_i,x_j],x_Part[subs[LB[x_i,x_j]],3]]];
        ]
       ]
       AppendTo[adx_i,vec_j];
      ];
    adz = adz + l_i*Transpose[adx_i];
    For[o=1, o≤n, o++, vec_o=Table[0,{n}];]
    ]
    (* Output ad(z)^3 for an arbitrary element in the
     given Lie algebra. *)
    adz.adz.adz // MatrixForm
```

The output of the above code gives the multiplication table of the input Lie algebra, that is:

Out[1]=

$$\begin{pmatrix}
0 & x_4 & x_5 & x_7 & x_8 & x_9 & 0 & 0 & 2x_{11} & x_{11} & 0 \\
-x_4 & 0 & x_6 & 0 & x_{10} & 0 & 0 & -3x_{11} & 0 & 0 & 0 \\
-x_5 & -x_6 & 0 & -x_9+x_{10} & 0 & 0 & 3x_{11} & 0 & 0 & 0 & 0 \\
-x_7 & 0 & x_9-x_{10} & 0 & 4x_{11} & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_8 & -x_{10} & 0 & -4x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_9 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & -3x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 3x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-2x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
-x_{11} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

and on the other hand, it computes the result of $\mathrm{ad}(z)^3$:

Out[2]=

$$\begin{pmatrix}
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\
0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0
\end{pmatrix}$$

Therefore, the exemplary input Lie brackets of $\mathfrak{g}_{3,4}$ indeed yield an Engel-3 Lie algebra.

A.3 Computation of the linearized Engel identity

We briefly recall the linearized Engel-3 identity:

$$\sum_{\sigma \in \mathrm{Sym}(3)} [x_{\sigma(1)}, [x_{\sigma(2)}, [x_{\sigma(3)}, y]]] = 0.$$

The following program takes the task to calculate this formula. Again, it is of no issue to extend the method for other values of n, however, for a large input the computation time gets very long.

```
In[1]:=
        (* Compute the linearized Engel-3 identity *)
        LinEngel[y_] := (
        res=0;
        For[i=1, i≤Length[Permutations[{1, 2, 3}]], i++,
          res = res + LB[x_Permutations[{1,2,3}][[i]][[1]],
                        LB[x_Permutations[{1, 2, 3}][[i]][[2]],
                        LB[x_Permutations[{1, 2, 3}][[i]][[3]],y]]];

          Print[LB[x_Permutations[{1, 2, 3}][[i]][[1]],
                LB[x_Permutations[{1,2,3}][[i]][[2]],
                LB[x_Permutations[{1, 2, 3}][[i]][[3]],y]]]]
        ];
        Print["Linear Engel for ",y," is ",res];
        )
```

A.4 Computation of the mixed Engel identity

By the mixed Engel identity we denote the relation obtained by Corollary 3.1.4. Recall that for the case of $n = 3$ this means:

$$[x, [x, [y, z]]] + [x, [y, [x, z]]] + [y, [x, [x, z]]] = 0.$$

The following short code computes this equation.

```
In[1]:=
        (* Compute the "Mixed" Engel-3 identity *)
        MixEngel[y_]:=(
        For[i=1, i≤n, i++,
          For[j=1, j≤n, j++,
            If[PossibleZeroQ[LB[x_j,LB[x_j,LB[x_i, y]]]+
             LB[x_j,LB[x_i,LB[x_j,y]]]+LB[x_i,LB[x_j,LB[x_j, y]]]], ,
            Print["Not satisfied for",y,x_i,x_j];
            ]
          ];
        ];)
```

A.5 Calculation of a basis of the derivation algebra

In contrast to the other codes, here the input is given in form of the adjoint endomorphisms. Therefore, another way of defining a Lie bracket is presented. Next, a general derivation of the input Lie algebra is generated which can be used to obtain a basis of the derivation algebra as explained in Section 3.2. A code that verfies linear independency of matrices is added.

```
In[1]:=
    (* We define unit vectors, a derivation in 121
    indeterminates and set up the input *)
    e1 = {1, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0};
    e2 = {0, 1, 0, 0, 0, 0, 0, 0, 0, 0, 0};
    e3 = {0, 0, 1, 0, 0, 0, 0, 0, 0, 0, 0};
    e4 = {0, 0, 0, 1, 0, 0, 0, 0, 0, 0, 0};
    e5 = {0, 0, 0, 0, 1, 0, 0, 0, 0, 0, 0};
    e6 = {0, 0, 0, 0, 0, 1, 0, 0, 0, 0, 0};
    e7 = {0, 0, 0, 0, 0, 0, 1, 0, 0, 0, 0};
    e8 = {0, 0, 0, 0, 0, 0, 0, 1, 0, 0, 0};
    e9 = {0, 0, 0, 0, 0, 0, 0, 0, 1, 0, 0};
    e10 = {0, 0, 0, 0, 0, 0, 0, 0, 0, 1, 0};
    e11 = {0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 1};
```

In[2]:= $D = \begin{pmatrix} d_{1,1} & d_{1,2} & d_{1,3} & d_{1,4} & d_{1,5} & d_{1,6} & d_{1,7} & d_{1,8} & d_{1,9} & d_{1,10} & d_{1,11} \\ d_{2,1} & d_{2,2} & d_{2,3} & d_{2,4} & d_{2,5} & d_{2,6} & d_{2,7} & d_{2,8} & d_{2,9} & d_{2,10} & d_{2,11} \\ d_{3,1} & d_{3,2} & d_{3,3} & d_{3,4} & d_{3,5} & d_{3,6} & d_{3,7} & d_{3,8} & d_{3,9} & d_{3,10} & d_{1,11} \\ d_{4,1} & d_{4,2} & d_{4,3} & d_{4,4} & d_{4,5} & d_{4,6} & d_{4,7} & d_{4,8} & d_{4,9} & d_{4,10} & d_{4,11} \\ d_{5,1} & d_{5,2} & d_{5,3} & d_{5,4} & d_{5,5} & d_{5,6} & d_{5,7} & d_{5,8} & d_{5,9} & d_{5,10} & d_{5,11} \\ d_{6,1} & d_{6,2} & d_{6,3} & d_{6,4} & d_{6,5} & d_{6,6} & d_{6,7} & d_{6,8} & d_{6,9} & d_{6,10} & d_{6,11} \\ d_{7,1} & d_{7,2} & d_{7,3} & d_{7,4} & d_{7,5} & d_{7,6} & d_{7,7} & d_{7,8} & d_{7,9} & d_{7,10} & d_{7,11} \\ d_{8,1} & d_{8,2} & d_{8,3} & d_{8,4} & d_{8,5} & d_{8,6} & d_{8,7} & d_{8,8} & d_{8,9} & d_{8,10} & d_{8,11} \\ d_{9,1} & d_{9,2} & d_{9,3} & d_{9,4} & d_{9,5} & d_{9,6} & d_{9,7} & d_{9,8} & d_{9,9} & d_{9,10} & d_{9,11} \\ d_{10,1} & d_{10,2} & d_{10,3} & d_{10,4} & d_{10,5} & d_{10,6} & d_{10,7} & d_{10,8} & d_{1,9} & d_{10,10} & d_{10,11} \\ d_{11,1} & d_{11,2} & d_{11,3} & d_{11,4} & d_{11,5} & d_{11,6} & d_{11,7} & d_{11,8} & d_{11,9} & d_{11,10} & d_{11,11} \end{pmatrix};$

$$\texttt{adx1} = \begin{pmatrix} 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&1&0&0&0&0&0&0&0&0&0\\ 0&0&1&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&1&0&0&0&0&0&0&0\\ 0&0&0&0&1&0&0&0&0&0&0\\ 0&0&0&0&0&1&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&0&0&0\\ 0&0&0&0&0&0&0&0&2&1&0 \end{pmatrix};$$

$$\texttt{adx2} = \begin{pmatrix} \texttt{-adx1[[1,2]]}&0&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[2,2]]}&0&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[3,2]]}&0&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[4,2]]}&0&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[5,2]]}&0&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[6,2]]}&0&1&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[7,2]]}&0&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[8,2]]}&0&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[9,2]]}&0&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[10,2]]}&0&0&0&1&0&0&0&0&0&0\\ \texttt{-adx1[[11,2]]}&0&0&0&0&0&0&-3&0&0&0 \end{pmatrix};$$

In[3]:=

$$\texttt{adx3} = \begin{pmatrix} \texttt{-adx1[[1,3]]}&\texttt{-adx2[[1,3]]}&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[2,3]]}&\texttt{-adx2[[2,3]]}&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[3,3]]}&\texttt{-adx2[[3,3]]}&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[4,3]]}&\texttt{-adx2[[4,3]]}&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[5,3]]}&\texttt{-adx2[[5,3]]}&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[6,3]]}&\texttt{-adx2[[6,3]]}&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[7,3]]}&\texttt{-adx2[[7,3]]}&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[8,3]]}&\texttt{-adx2[[8,3]]}&0&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[9,3]]}&\texttt{-adx2[[9,3]]}&0&-1&0&0&0&0&0&0&0\\ \texttt{-adx1[[10,3]]}&\texttt{-adx2[[10,3]]}&0&1&0&0&0&0&0&0&0\\ \texttt{-adx1[[11,3]]}&\texttt{-adx2[[11,3]]}&0&0&0&0&3&0&0&0&0 \end{pmatrix};$$

$$\texttt{adx4} = \begin{pmatrix} \texttt{-adx1[[1,4]]}&\texttt{-adx2[[1,4]]}&\texttt{-adx3[[1,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[2,4]]}&\texttt{-adx2[[2,4]]}&\texttt{-adx3[[2,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[3,4]]}&\texttt{-adx2[[3,4]]}&\texttt{-adx3[[3,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[4,4]]}&\texttt{-adx2[[4,4]]}&\texttt{-adx3[[4,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[5,4]]}&\texttt{-adx2[[5,4]]}&\texttt{-adx3[[5,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[6,4]]}&\texttt{-adx2[[6,4]]}&\texttt{-adx3[[6,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[7,4]]}&\texttt{-adx2[[7,4]]}&\texttt{-adx3[[7,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[8,4]]}&\texttt{-adx2[[8,4]]}&\texttt{-adx3[[8,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[9,4]]}&\texttt{-adx2[[9,4]]}&\texttt{-adx3[[9,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[10,4]]}&\texttt{-adx2[[10,4]]}&\texttt{-adx3[[10,4]]}&0&0&0&0&0&0&0&0\\ \texttt{-adx1[[11,4]]}&\texttt{-adx2[[11,4]]}&\texttt{-adx3[[11,4]]}&0&4&0&0&0&0&0&0 \end{pmatrix};$$

[...]

```
In[4]:=
    (* Define a Lie bracket with regard to the
    ad-matrices *)
    LB[{a1_,a2_,a3_,a4_,a5_,a6_,a7_,a8_,a9_,a10_,a11_},
    {b1_,b2_,b3_,b4_,b5_,b6_,b7_,b8_,b9_,b10_,b11_}]:=
    a1 adx1.{b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11}+
    aa2 adx2.{b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11}+
    a3 adx3.{b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11}+
    a4 adx4.{b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11}+
    a5 adx5.{b1,b2,b3,b4,bb5,b6,b7,b8, b9, b10, b11}+
    a6 adx6.{b1,b2,bb3,b4,b5,b6,b7,b8,b9,b10,b11}+
    a7 adx7.{b1,bb2,b3,b4,b5,b6,b7,b8,b9,b10,b11}+
    a8 adx8.{b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11}+
    a9 adx9.{b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11} +
    a10 adx10.{b1,b2,b3,b4,b5,b6,b7,b8,b9,b10, b11}+
    a11 adx11.{b1,b2,b3,b4,b5,b6,b7,b8,b9,b10,b11}

    Do[
     Print[Simplify[Derivation.LB[y,z]]-
     LB[Derivation.y,z]-LB[y,Derivation.z]];,
     {y, {e1,e2,e3,e4,e5,e6,e7,e8,e9,e10,e11}},
     {z, {e1,e2,e3,e4,e5,e6,e7,e8,e9,e10,e11}}
    ]
```

The Do-loop in the programm returns 121 vectors of length 11. Each of these vectors has to be equal to 0. For instance, we get the equation

$$\begin{pmatrix} d_{1,4} \\ d_{2,4} \\ d_{3,4} \\ -d_{1,1} - d_{2,2} - d_{4,4} \\ -d_{3,1} + d_{6,4} \\ -d_{4,2} + d_{7,4} \\ -d_{5,2} + d_{8,4} \\ -d_{6,2} + d_{9,4} \\ d_{5,1} + d_{10,4} \\ -3 \cdot d_{8,1} - 2 \cdot d_{9,2} + d_{10,4} + d_{11,4} \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \\ 0 \end{pmatrix}$$

from which we can derive $d_{1,4} = d_{2,4} = d_{3,4} = 0$ or $d_{3,1} = d_{6,4}$ and so on. By regarding all the relations resulting in this way, a general derivation in the input Lie algebra is returned, as displayed in 3.2.

To prove linear independency of a set of matrices one can use the below code.

```
In[1]:=
    (* Check linear independence of a given set
     of matrices *)

    sc = Table[c_k, {k, 1, Length[matSet]}];

            Length[matSet]
    Solve[     ∑         c_j*matSet[[j]]==Table[0,{11},{11}],sc]
              j=1
```

In the case of $\mathfrak{g}_{3,4}$ and the set of matrices that span $\mathfrak{Der}(\mathfrak{g}_{3,4})$ the code returns:

```
Out[1]=
    {c1 → 0, c2 → 0, c3 → 0, c4 → 0, c5 → 0,
    c6 → 0, c7 → 0, c8 → 0, c9 → 0, c10 → 0,
    c11 → 0, c12 → 0, c13 → 0, c14 → 0, c15 → 0,
    c16 → 0, c17 → 0, c18 → 0, c19 → 0, c20 → 0,
    c21 → 0, c22 → 0, c23 → 0, c24 → 0, c25 → 0,
    c26 → 0, c27 → 0}
```

Thus, the 27 derivations we derived earlier are indeed linearly independent and thus form a basis of the derivation algebra.

A.6 Dimension of free-nilpotent Lie algebras

We present a code that realizes Witt's formula 1.3.6.

```
In[1]:=
    (* Create a 10x10 table of the dimensions of the
    free-nilpotent Lie algebras *)
    Table[Sum[(1/m)*DivisorSum[m,MoebiusMu[#] n^(m/#) &],{m,1,c}],
     {c,1,10},{n,1,10}] // MatrixForm
```

As output we obtain the table:

$$
\begin{pmatrix}
1 & 2 & 3 & 4 & 5 & 6 & 7 & 8 & 9 & 10 \\
1 & 3 & 6 & 10 & 15 & 21 & 28 & 36 & 45 & 55 \\
1 & 5 & 14 & 30 & 55 & 91 & 140 & 204 & 285 & 385 \\
1 & 8 & 32 & 90 & 205 & 406 & 728 & 1212 & 1905 & 2860 \\
1 & 14 & 80 & 294 & 829 & 1960 & 4088 & 7764 & 13713 & 22858 \\
1 & 23 & 196 & 964 & 3409 & 9695 & 23632 & 51360 & 102153 & 189343 \\
1 & 41 & 508 & 3304 & 14569 & 49685 & 141280 & 350952 & 785433 & 1617913 \\
1 & 71 & 1318 & 11464 & 63319 & 259475 & 861580 & 2447592 & 6165453 & 14116663 \\
1 & 127 & 3502 & 40584 & 280319 & 1379195 & 5345276 & 17360616 & 49212093 & 125227663 \\
1 & 226 & 9382 & 145338 & 1256567 & 7425032 & 33591116 & 124731516 & 397884621 & 1125217654
\end{pmatrix}
$$

So, for example, $\mathfrak{F}_{3,7}$ has already 508 basis elements and the dimensions grow very fast in both respects, the number of generators and the nilpotency class.

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