$\Sigma_3^1$-Absoluteness in Forcing Extensions

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Abstract

We investigate the consistency strength of the forcing axiom for $\Sigma^1_3$ formulas, for various classes of forcings. We review that the consistency strength of $\Sigma^1_3$-absoluteness for all set forcing or even just for $\omega_1$-preserving forcing is that of a reflecting cardinal. To get the same strength from the forcing axiom restricted to proper forcing, one can add the hypotheses that $\omega_1$ is inaccessible to reals. Then we investigate the strength of the forcing axiom restricted to ccc forcing notions under this additional hypothesis; to gauge it we introduce a weak version of a weak compact cardinal, namely, a lightface $\Sigma^1_2$-indescribable cardinal.
1 Preliminary facts

This section contains some definitions, well known facts and basic theorems which we will use throughout the present paper.

First, we take a little time to fix our notation (we hope to include all possible sources of confusion). For a structure $M = \langle M, \in; P_0, \ldots, P_k \rangle$, $L^M$ denotes the language $L = \{ \in, P_0, \ldots, P_k \}$. $L^M_X$ denotes the same language with constants for all the elements of $X$. We write $M \preceq N$ to denote elementarity (i.e. $M$ and $N$ have the same theory in $L^M_M$), and we write $\pi : M \to \Sigma^\omega N$ to express $\pi$ is an elementary embedding (i.e. $\langle \text{ran}(\pi), \pi'' \in ; \pi''P_0, \ldots, \pi''P_k \rangle \preceq N$). For transitive models, we call the least ordinal $\alpha \in M$ such that $\pi(\alpha) > \alpha$ the critical point of $\pi$, and denote it by $\text{crit}(\pi)$.

For a structure $N = \langle N, E, \ldots \rangle$ such that $E$ is extensional and well-founded (and, in the case of a class sized structure, set-like), we denote its transitive collapse (that is, the unique structure $\langle M, \in, \ldots \rangle$ isomorphic to $N$ and such that $M$ is transitive) by $tcoll(N)$. By $(\phi)^M$ we mean the formula $\phi$ holds relativized to a (possibly class-sized) structure $M$ and by $t^M$ we mean the interpretation of a term $t$ as from the viewpoint of $M$. When we talk about structures $M = \langle M, \in; P_0, \ldots, P_k \rangle$, we sometimes omit listing the $\in$ as a predicate. When we use notation that is defined for structures in a context when we talk about sets $M$, it is implied we mean the structure $\langle M, \in \rangle$.

We denote the usual Zermelo-Fraenkel axioms by $ZF$, and that theory with $AC$ (the axiom of choice) added by $ZFC$. $ZF^-$ and $ZFC^-$ denote the corresponding theories with the power set axiom deleted.

We use the familiar $\Sigma_k, \Pi_k, \Delta_k$ notation for the classes formulas that can be written with $k$ blocks of equal quantifiers starting with $\exists$ or $\forall$ and for the intersection of these first two classes, respectively. We shall give a more thorough review of the analytical hierarchy in section 1.3; in section 5.1 we quickly review the general $\Sigma_n^k$ hierarchies of higher order formulas over a structure.

We write $\text{Lim}$, $\text{Card}$, $\text{Reg}$ for the class of limit ordinals, cardinals and regular cardinals respectively. For a set $x$, $|x|$ is its cardinality, and we sometimes write $x \preceq y$ to mean that $|x| < |y|$. $TC(x)$ denotes the transitive closure of $x$. For any cardinal $\kappa$, let $H_\kappa$ denote the set of all $x$ such that
$TC(x)$ has size less than $\kappa$ (i.e., the $H_\kappa$ hierarchy is continuously defined also at singular cardinals). $HC$ denotes $H_{\omega_1}$, the set of hereditarily countable sets.

As far as forcing is concerned, we sometimes write $V^P$ for a generic extension $V[G]$, where $G$ is $P$-generic over $V$, if the particular $G$ concerned is of no importance. We write dots over names (e.g. $\dot{q}$) and $\dot{q}[G]$ for the interpretation of the name $\dot{q}$ by a generic $G$. For sets $a$ in the ground model we write $\check{a}$ for the “standard name” for $a$.

### 1.1 General set theory

#### 1.1 Definition

1. $C \subseteq [A]^\omega$ is called cub if, and only if, there are predicates $P_1, \ldots, P_k \subseteq A$ such that
   $$ C = \{ X \in [A]^\omega \mid \langle X, P_1, \ldots, P_k \rangle \prec \langle A, P_1, \ldots, P_k \rangle \} $$

2. $S \subseteq [A]^\omega$ is called stationary if, and only if,
   $$ \text{for all cub sets } C \subseteq [A]^\omega, S \cap C \neq \emptyset $$

#### 1.2 Fact

Let $\mathcal{N} = \langle N, R_0, \ldots, R_k \rangle$, $X \in H_\kappa$ and $X \subseteq N$. Then there exists a transitive $M \in H_\kappa$ s.t. $X \subseteq M$ together with $R'_0, \ldots, R'_k \subseteq M$, and an elementary embedding $i : \langle M, R'_0, \ldots, R'_k \rangle \rightarrow \Sigma_\omega \langle N, R_0, \ldots, R_k \rangle$ s.t. $i$ is the identity on $X$.

**Proof.** First observe that since $X \in H_\kappa$, also $TC(X) \in H_\kappa$, whence we can assume $X$ to be transitive. Now start by taking the Skolem-hull (with respect to the language where the additional predicates have been added): let $\bar{M} := h_{\Sigma_\omega}^\mathcal{N}(X)$. $\bar{M}$ has size less than $\kappa$. Set $\bar{M} := tcoll(\bar{M})$, the transitive collapse of $\bar{M}$. Being transitive and of size less than $\kappa$, $M \in H_\kappa$. Let $i$ be the inverse of the collapsing map (the new predicates are just the pullbacks of the original predicates). As $X \subseteq \bar{M}$ was transitive, $i$ is the identity on $X$. \hfill $\square$

#### 1.3 Fact

If $N$ is transitive and $\lambda + 1 \subseteq M \prec N$, then $(H_{\lambda^+})^M$ is transitive.

**Proof.** Let $x \in (H_{\lambda^+})^M$ be arbitrary, we show $x \subseteq M$. $M \models \exists f : \lambda \rightarrow TC(x)$", and so by elementarity, for some $f \in M$, $N \models \exists f : \lambda \rightarrow TC(x)"$. Now for arbitrary $y \in x$, we have $y = f(\xi)$ for some $\xi \in \lambda$, while $\lambda \subseteq M$. It is easily checked that $f, \xi \in M$ by elementarity implies $f(\xi) \in M$, so $y \in M$. \hfill $\square$

#### 1.4 Corollary

Let $A$ be any set of ordinals, $\kappa$ a successor cardinal. Then $C := \{ \alpha < \kappa \mid L_\alpha[A] \prec L_\kappa[A] \}$ is cub in $\kappa$. 

3
Proof. For $\xi < \kappa$, let $\chi(\xi)$ denote the least ordinal such that Skolem functions for $L_\kappa[A]$ on $L_\xi[A]$ have values in $L_{\chi(\xi)}[A]$. Then $\chi : \kappa \to \kappa$ and $C$ is precisely the set of closure points of $\chi$, hence cub.

1.5 Fact. If $M \models ZF^-$ is transitive, $\pi : M \to \Sigma_\omega N$ an elementary embedding with critical point $\alpha$, then $\alpha \in Reg^M$ and $\pi \upharpoonright H_\alpha^M = id$.

Proof. Assume $M$ thinks $\alpha$ is singular; then there is a function $f$ in $M$, with domain an ordinal below $\alpha$ and values below $\alpha$ whose range is cofinal in $\alpha$. But as all these ordinals are not moved by $\pi$, $\pi(f) = f$ and thus $\alpha = sup(ran(f)) = sup(ran(\pi(f))) = \pi(sup(ran(f))) = \pi(\alpha)$, contradiction. As $\pi \upharpoonright \alpha = id$, $\alpha \subseteq ran(\pi)$. By hypothesis $ran(\pi) < N$; we can apply 1.3: for each $\lambda < \alpha$, $(H_{\lambda^+})^{ran(\pi)}$ is transitive. So $(H_{\pi(\alpha)})^{ran(\pi)} = \pi''(H_\alpha)^M$ (as the union of these transitive sets) is seen to be transitive. Note that as the transitive collapse of any set $A$ is uniquely defined as the transitive set isomorphic to $A$, we have that $M$ must be the transitive collapse of $ran(\pi)$ and $\pi$ must be the inverse of the collapsing map. As the collapsing map is just the identity on any transitive set, $\pi$ is the identity on $(H_\alpha)^M$.

1.6 Fact. Let $M = \langle M, R_1, \ldots, R_k \rangle$, $N = \langle N, S_1, \ldots, S_r \rangle$, be models such that $k \leq r$, $R_i$ has the same arity as $S_i$ for $i \leq k$, and $M$ is countable. Then the statement

$$\exists \pi, \exists \langle R_{k+1}, \ldots, R_r \rangle \rightarrow \langle \pi(M, R_1, \ldots, R_r) \rangle \rightarrow_{\Sigma_\omega} \langle N, S_1, \ldots, S_r \rangle$$

(1)

is absolute for transitive models $U$ of $ZF^-$ such that $U$ contains both $N$ and $M$ and $U \models M \cong \omega$.

Proof. Let $m : \omega \to M$ be an enumeration of $M$, $(\phi_i)_{i \in \omega}$ an enumeration of formulas of the language of $N$. Let $\gamma : \omega \to \omega$, $\delta : \omega \to \omega$ be such that together, they enumerate all formulas of the language $\mathcal{L}_M^N$ (the language of $N$ with constants for all the elements of $M$) in the following sense: for all $n$, $ran(\delta(n)) \subseteq n$, the number of free variables of $\phi_{\gamma(n)}$ is no larger than $dom(\delta(n))$, and $\gamma$ and $\delta$ are onto in the sense that $\phi_{\gamma(n)}(m(s(0)), \ldots, m(s(l)))$, for $s = \delta(n)$ and $l = lh(s) - 1$, runs through all formulas of $\mathcal{L}_M^N$ as $n$ runs through $\omega$. Think of the elementary embedding as an interpretation of the constant symbols of $\mathcal{L}_M^N$.

We inductively define a tree $T$ searching for the embedding and the new predicates. Let $M_k$ denote the set $\{m(0), \ldots, m(k-1)\}$. We let $f \in T_{n+1}$, the $n + 1$-th level of $T$, if and only if $f : n + 1 \to N$ such that

- $f \upharpoonright n \in T_n$
- $f \circ m^{-1} : \langle M_{n+1}; R_1, \ldots, R_k \rangle \to \langle N; S_1, \ldots, S_k \rangle$ is a homomorphism
• setting \( s = \delta(n) \) and \( l = lh(s) - 1 \), if \( N \models \exists x \phi_{\gamma(n)}(x, f(s(0)), \ldots, f(s(l))) \)
then \( f(n) \) is such that \( \phi_{\gamma(n)}(f(n), f(s(0)), \ldots, f(s(l))) \). Otherwise, \( f(n) \) is allowed to be an arbitrary element of \( N \).

If \((f_i)_{i \in \omega}\) is an infinite branch through \( T \), \( \pi := (\bigcup_{i \in \omega} f_i) \circ m^{-1} \) is an elementary embedding of \( \langle M, R_1, \ldots, R_r \rangle \) into \( N \), where the additional predicates are obtained by taking the (componentwise) pre-image of the predicates of \( N \) under \( \pi \). On the other hand, any such relations together with an embedding define an infinite branch through \( T \). So (1) is equivalent to the statement

\[ \supseteq \text{ is not well-founded on } T \] (2)

Any \( U \) as in the hypothesis contains \( T \), whose definition is \( \Delta_1 \) and thus absolute. Then by juggling infinite branches and ranking functions for \( T \), (2) is seen to be absolute.

\[ \square \]

1.2 Forcing

This section reviews the basic properties of some well-known forcing notions that are used in this paper.

General forcing facts

We use the largely standardized forcing notation as developed for partial orders; we shall take for granted basic facts about forcing and finite iterations as found in texts such as [Kun80]. We shall only venture under the hood of the forcing machinery on one occasion, namely to describe what is known as universality of the Lévy-Collapse (corollary 1.30). To this end, we repeat some basic facts. A proof of the next fact can be found in [Kun80, p. 220f.]

1.7 Definition. Let \( P, Q \) be p.o.’s. A map \( i : P \to Q \) is called a complete embedding if \( i \) is order-preserving and

1. \( \forall p, q \in P \quad i(p) \parallel i(q) \Rightarrow p \parallel q \)

2. \( \forall q \in Q \quad \exists p_0 \in P \text{ s.t. } \forall p \in P \quad p \leq p_0 \Rightarrow i(p) \parallel q \)

A map \( i : P \to Q \) is called a dense embedding if it is order preserving and \( i''P \) is dense in \( Q \). Let \( \sim \) be the least equivalence relation of partially ordered sets extending the relation “there exists a dense embedding from \( P \) into \( Q \)”. Equivalently, \( P \sim Q \) if \( r.o.(P) \) and \( r.o.(Q) \) are the same (up to isomorphism) i.e. they are canonically associated with the same Boolean algebra (see [II,3.3, p.62][Kun80] or [Jec78, lemma 17.2, p.152]). If \( P \sim Q \), they are called equivalent.
1.8 Fact. 1. Let $i : P \rightarrow Q$ be a complete embedding. Then if $H$ is $Q$-generic, $G := (i^{-1})''H$ is $P$-generic and $V[G] \subseteq V[H]$.

2. Let $d : P \rightarrow Q$ be a dense embedding. Then not only is $d$ complete and the above holds, but for any $P$-generic $G$, $H := \{ q \in Q \mid \exists p \in P \ d(p) \leq q \}$ is $Q$-generic and $V[G] = V[H]$. If $P \sim Q$, $P$ and $Q$ yield the same extensions and a generic for either can be used to construct a generic for the other.

1.9 Fact. Suppose $P$ is a p.o., $\hat{Q}$ a $P$-name, $\models_P " \hat{Q} \text{ is a p.o."}$ and $M$ a transitive model of $ZF$ (and let $p_i$ denote the canonical projection to the $i$-th coordinate). Then:

1. If $I$ is generic for $P \ast \hat{Q}$ over $M$, $G := \{ p \mid \exists \hat{q}(p, \hat{q}) \in I \}$ is $P$-generic over $M$, $H := I[G]$ is $\hat{Q}[G]$-generic over $M[G]$ and $M[I] = M[G][H]$.

2. If $G$ is $P$-generic over $M$ and $H$ is $\hat{Q}[G]$-generic over $M[G]$,

$$G \ast H := \{ (p, \hat{q}) \in P \ast \hat{Q} \mid p \in G \wedge \hat{q}[G] \in H \}$$

is $P \ast \hat{Q}$-generic over $M$ and $M[G \ast H] = M[G][H]$.

Moreover, $i_P : P \rightarrow P \ast \hat{Q}$, defined by $p \mapsto (p, \hat{1}_Q)$ is a complete embedding of $P$ into $P \ast \hat{Q}$. For $Q \in M$, all of the above holds for $P \times Q$, $G \times H$ and $H := \{ q \mid \exists p(p, q) \in I \}$, and $i_Q$ (defined in complete analogy to $i_P$) is a complete embedding.

**Almost disjoint coding**

Let $\alpha$ a regular cardinal, $\beta > \alpha$ some ordinal. Let $A = (a_\xi)_{\xi < \beta}$ be a family of unbounded subsets of $\alpha$ such that for $\xi \neq \xi' < \beta$, $|a_\xi \cap a_{\xi'}| < \alpha$. This is called an almost disjoint (or a.d.) family on $\alpha$. Further, let $B$ be any subset of $\beta$. Using $A$ we can force to add a subset $A$ of $\alpha$, such that $A$ codes $B$ in the following sense:

$$B = \{ \xi < \beta \mid |A \cap a_\xi| < \alpha \}$$

(3)

We shall call this forcing $P_{A,B}$, the almost disjoint coding of $B$ using $A$. Of course this method of adding a set coding $B$ depends on the existence of an appropriate a.d. family. For basic results about constructing a.d. families, see [Kun80, p. 47]. We will use the easy facts that there always exists an a.d. family on $\alpha$ of size $\alpha^+$ (by a maximality argument using Zorn’s lemma), and that there is an a.d. family on $\omega$ of size $2^\omega$ (just take sets of code numbers of finite strings approximating a real - the same works with $\omega$ replaced by some
strong limit cardinal). We shall use the notation of the preceding paragraph throughout this section without repeating the conditions imposed on, e.g., \( \alpha \) and \( \beta \). For the moment, let \( A \upharpoonright q \) denote \( \{ a_\xi \mid \xi \in q \} \).

1.10 Definition. Almost Disjoint Coding. Define

\[ P_{A,B} := [\alpha]^{<\alpha} \times [B]^{<\alpha} \]

ordered by

\[ (p,q) \leq (\bar{p},\bar{q}) \iff p \text{ end-extends } \bar{p}, \ q \supseteq \bar{q}, \text{ and } (p \setminus \bar{p}) \cap \bigcup A \upharpoonright \bar{q} = \emptyset \]

The forcing consists of pairs, the first part of which is an approximation of the set \( A \) to be added, the second part indexes the \( a_\xi \) we wish to avoid by further approximations. We must now show that the generic has the property promised in (3).

1.11 Lemma. For each \( \sigma \in B \), the set \( D_\sigma := \{(p,q) \in P_{A,B} \mid \sigma \in q\} \) is dense in \( P_{A,B} \).

Proof. Given \((p_0,q_0)\), just extend it to \((p_0,q_0 \cup \{\sigma\})\) to hit \( D_\sigma \). \(\square\)

1.12 Lemma. For each \( \rho < \alpha, \sigma \in (\beta - B) \), the set

\[ D_{\rho,\sigma} := \{(p,q) \in P_{A,B} \mid p \cap a_\sigma \text{ has order type at least } \rho\} \]

is dense in \( P_{A,B} \).

Proof. Let \((p_0,q_0)\) be a condition, \( \rho \) and \( \sigma \) given as above. Look at \( S := a_\sigma - \bigcup A \upharpoonright q_0 = a_\sigma - \bigcup_{\xi \in q_0} (a_\sigma \cap a_\xi) \). As \( \sigma \notin B \), for \( \xi \in q_0 \), \( a_\sigma \cap a_\xi \preceq \alpha \). Thus \( S \) is obtained by taking away the union of less than \( \alpha \) sets of size less than \( \alpha \) from a set of size \( \alpha \), and thus has size \( \alpha \), by regularity of \( \alpha \). Add a subset of \( S \) of order type \( \rho \) to \( p_0 \) and call the set so obtained \( p_1 \). As we have extended inside \( a_\sigma \), \( a_\sigma \cap p_1 \) will have the desired order type, and as we have avoided all the \( a_\xi \) indexed by \( q_0 \), \((p_1,q_0)\) is a condition stronger than \((p_0,q_0)\). \(\square\)

Now let \( G \) be generic for \( P_{A,B} \), and let \( A := \bigcup \{p \mid (p,q) \in G, \text{ some } q\} \). As \( G \) meets all of the above dense sets, \( A \) codes \( B \) in the sense of (3).

1.13 Fact. \( P_{A,B} \) is \( \alpha \)-closed.

Proof. Let \( \rho < \alpha, \ (p_\xi,q_\xi)_{\xi<\rho} \) a decreasing sequence of conditions. Consider \( p' \) and \( q' \), the union of the first and second parts of the conditions in this sequence, respectively. \((p',q')\) is a condition extending the conditions of the sequence: let \( \xi \in q_\lambda, \lambda < \rho \). Then \( p' \cap a_\xi = p_\lambda \cap a_\xi \). So \((p',q') \leq (p_\lambda,q_\lambda)\). \(\square\)
1.14 Fact. $P_{A,B}$ is ([α]<α)-linked. If [α]<α = α, $P_{A,B}$ is α-centered.

Proof. Recall a forcing $P$ is called λ-centered if there is $f : P \rightarrow \lambda$ s.t. \(\forall \xi < \lambda, \forall W \in [f^{-1}(\xi)]^\lambda\), there exists $\bigwedge W$, a weakest condition stronger than all the conditions in $W$. A forcing is λ-linked if there is such a function s.t. if $f(p) = f(p')$, $p$ and $p'$ are compatible. If $[\alpha]^\alpha = \alpha$, let $f : P_{A,B} \rightarrow [\alpha]^\alpha \cong \alpha$ be the projection to the first part. For a set of conditions $W \lesssim \alpha$ with the same first part $p_0$, $(p_0, \bigcup_{(p,q) \in W} q)$ is $\bigwedge W$. If $[\alpha]^\alpha > \alpha$, we can't find $\bigwedge W$ for $W \cong [\alpha]^\alpha$, but $P_{A,B}$ will at least be linked, as conditions with the same first part are compatible.

Note that λ-centeredness implies the λ-cc, as any two conditions with the same image under $f$ must be compatible (centered implies linked). So if $\alpha$ is a strong limit (or $\omega$), $P_{A,B}$ has the $\alpha^+\text{-cc}$

Another kind of almost disjoint coding

We have seen that by forcing with $P_{A,B}$, we can code some given object of size $2^\omega$ in the ground model by a single, generic real. Basically, we could use this forcing to code a function $f : 2^\omega \rightarrow 2^\omega$. For example, we could use a canonical bijection $g : 2^\omega \times 2^\omega \rightarrow 2^\omega$ and code the graph of $f$ into a set of reals $B$ before forcing with $P_{A,B}$.

What if we want to code a function $f : 2^\omega \rightarrow 2^\omega$ into a real $A$ such that for any model $M$ with sufficient closure properties and containing $A$ and $\mathcal{A}$, for any real $r \in M$, we also have $f(r) \in M$? That is, we want to make sure that $f(r)$ can be decoded inside $M$ from $A$. Using method described above, this will in general not be the case.

We will use a special almost disjoint family. Fix some arithmetical enumeration $(s_n)_{n \in \omega}$ of $\preceq^\omega \omega$, and some arithmetical partition $(H_i)_{i \in \omega}$ of $\omega$, where each $H_i$ is infinite. For $r \in 2^\omega$, $i \in \omega$, let

\[ a_r^i := \{ n \in \omega \mid r \upharpoonright lh(s_n) = s_n \land lh(s_n) \in H_i \}, \quad a_r := \bigcup_{i \in \omega} a_r^i = \{ n \in \omega \mid r \upharpoonright lh(s_n) = s_n \} \]

Clearly, \(\{a_r^i \mid r \in 2^\omega, i \in \omega\}\) is an a.d.-family: let $(r, i) \neq (s, j) \in 2^\omega \times \omega$. If $i \neq j$, $a_r^i \cap a_s^j = \emptyset$. Otherwise $s \neq r$. Note that $a_r^i \subseteq a_r$, $a_s^j \subseteq a_s$. Assume $k_0 \in r \Delta s$; then for all $k \geq k_0$, $s \upharpoonright k \neq r \upharpoonright k$, whence $a_r^i \cap a_s^j \subseteq a_r \cap a_s$ can only contain indices of sequences with length shorter than $k_0$ and hence must be finite.

1.15 Definition. Let $f : A \rightarrow 2^\omega$ be a function, $A \subseteq 2^\omega$. Define $P_f$

\[ P_f := \left< \omega \times \bigcup_{r \in A} \left( \{r\} \times f(r) \right) \right< \omega, \]

8
ordered by

\[(s, g) \leq (t, h) \iff s \supseteq t \text{ and for all } (r, i) \in h, \ s \setminus t \cap a^i_r = \emptyset \]

The first component of a condition should be thought of as a finite approximation to the generic real; the second component contains pairs \((r, i)\) indexing the sets \(a^i_r\) that must be avoided by any stronger approximation.

For any two reals \(r, s\) define

\[r \odot s := \{ i \in \omega \mid s \cap a^i_r \text{ is finite} \}\]

It remains to check the forcing does what was promised and behaves nicely.

1.16 Fact. If \(B = \bigcup \{ p \mid \exists q (p, q) \in G \}\) for \(G\) that is \(P_f\)-generic, then for all \(r \in A\), \(f(r) = r \odot B\).

This fact is an immediate consequence of the following two lemmas.

1.17 Lemma. For each \((r, i) \in 2^\omega \times \omega\) such that \(i \notin f(r)\) and for each \(n \in \omega\), the set \(D := \{(p, h) \in P_f \mid (p \cap a^i_r) \setminus n \neq \emptyset\}\) is dense in \(P_f\).

Proof. Like lemma 1.12. Let \((p, h)\) be arbitrary. As \(i \notin f(r)\), certainly \((r, i) \notin h\); so \(a^i_r \setminus \bigcup_{(s, j) \in h} a^j_s\) must be infinite, as these are almost disjoint subsets of \(\omega\). So, picking \(k > n\) in that latter set, \((p \cup \{k\}, h)\) is a condition in \(D\) extending \((p, h)\).

1.18 Lemma. For each \(r \in 2^\omega\) and each \(i \in f(r)\), the set \(D := \{(p, h) \in P_f \mid (r, i) \in h\}\) is dense in \(P_f\).

Proof. For any condition \((p, h)\) and any pair \((r, i)\) as above, \((p, h \cup \{(r, i)\})\) is a condition extending \((p, h)\).

1.19 Fact. \(P_f\) is \(\omega\)-centered (and thus has the ccc).

Proof. Again, let \(f : P_f \to \langle \omega \omega, \omega \rangle\) be the projection to the first coordinate. If \((p_i, h_i)\), for \(i \in k\), are conditions with the same first component, indeed \((p_0, \bigcup_{i \in k} h_i)\) is also a condition.

Shooting a club through a stationary set

Any superset of a cub set is stationary, but not necessarily all stationary sets are obtained in this way: if \(cf(\kappa) > \omega_1\), the set of ordinals below \(\kappa\) with uncountable cofinality is stationary; it cannot contain a cub subset as any such set must have points of cofinality \(\omega\). For \(\omega_1\), the situation is different: we shall show that for any stationary subset \(S\) of \(\omega_1\), you can force to add a cub subset of \(S\). If \(S\) does not contain a cub subset, \(\omega_1 - S\) is also stationary, but ceases to be in the extension - so the forcing won’t preserve stationary subsets of \(\omega_1\). As we shall see, it does preserve \(\omega_1\).
1.20 **Definition. Adding a club.** Let $S$ be a stationary subset of $\omega_1$. Define

$$P := \{ s \subseteq S \mid s \text{ is closed and bounded in } \kappa \},$$

ordered by

$$s \leq t \iff t = s \cap (\bigcup t).$$

In short, conditions should be thought of as closed initial segments of a cub subset of $S$ to be added, ordered by end-extension.

1.21 **Fact.** If $G$ is $P$-generic, $\bigcup G$ is a cub subset of $S$.

*Proof.* As, for any $\xi < \kappa$, the set $D_\xi := \{ s \in P \mid \xi < \text{sup}(s) \}$ is dense, $\bigcup G$ is unbounded in $\omega_1$. If $\gamma$ is a limit point of $\bigcup G$, take some $s \in G$ such that $\text{sup}(s) > \gamma$. As $(\bigcup G) \cap \text{sup}(s) = s$, and $s$ is closed, it must be the case that $\gamma \in \bigcup G$, so $\bigcup G$ is closed. \qed

1.22 **Fact.** $P$ is $\omega_1$-distributive.

*Proof.* Why is $P$ not $\sigma$-closed? Because if we let $\gamma$ be a limit point of $S$ that is not contained in $S$, we can choose a sequence of conditions such that any candidate for a condition containing all of them as subsets must either fail to be closed or must contain $\gamma \notin S$.

With this in mind, given a sequence $(D_n)_{n<\omega}$ of dense subsets of $P$ and a condition $q_0$, we shall construct a descending chain of conditions $(p_n)_{n<\omega}$ below $q_0$ with sufficient care so that at the limit, we can find a condition extending what has previously been chosen.

To this end, we build a chain of models $(M_\xi)_{\xi<\omega_1}$: for successor stages $\eta = \xi + 1$ for $\xi < \omega_1$, choose $M_\eta$ s.t.

$$\langle M_\eta, p_0, P, (D_n)_{n<\omega} \rangle \prec \langle H_{\omega_1}, p_0, P, (D_n)_{n<\omega} \rangle,$$

and $\xi \cup M_\xi \subseteq M_\eta$.

At limit stages $\eta \in \omega_1 \cap \text{Lim}$, take unions:

$$M_\lambda = \bigcup_{\xi<\lambda} M_\xi$$

Thus, (4) will hold for all $\eta < \omega_1$. Note that by construction the set

$$C := \{ \bigcup (M_\xi \cap \text{On}) \mid \xi < \omega_1 \}$$

is cub in $\omega_1$. So we can choose $\gamma \in S \cap C$. Let $\delta$ be such that $\bigcup (M_\delta \cap \text{On}) = \gamma$. Then, for each $n \in \omega$ we can choose $\gamma_n \in M_\delta$ such that $\bigcup_{n \in \omega} \gamma_n = \gamma$. Now by (4), we have that

$$M_\delta \models "\forall n \in \omega \ D_n \text{ is dense in } P", \quad 10$$
and for each \( n \in \omega \),
\[
M_\delta \models \{ p \in P \mid \sup(p) > \gamma_n \} \text{ is dense in } P.
\]
So we can choose a sequence of conditions, starting with \( p_0 \), such that for each \( n \in \omega - \{0\} \),
\[
p_{n+1} \leq p_n,
\]
\[
p_{n+1} \in D_n \cap M_\delta, \text{ and }
\]
\[
\sup(p_{n+1}) > \gamma_n.
\]
Finally, we set \( p_\omega := (\bigcup_{n\in\omega} p_n) \cup \{\gamma\} \). This \( p_\omega \) is a closed and thus a condition, and clearly \( p_\omega \in \bigcap_{n\in\omega} D_n \). \( p_\omega \leq p_0. \]

**1.23 Corollary.** \( P \) does not collapse \( \omega_1 \).

**Collapsing orders**

**1.24 Definition.** For \( S \subseteq \text{On}, \gamma \text{ regular, define} \)
\[
Coll(\gamma, S) := \{ f \mid f \text{ is a function, } \text{dom}(f) \in [S \times \gamma]^{<\gamma} \text{ and } \forall (\xi, \eta) \in S \times \gamma \ f(\xi, \eta) < \xi \},
\]
ordered by inclusion. \( Coll(\gamma, \{\kappa\}) \) is usually called the *Lévy Collapse of \( \kappa \) onto \( \gamma \)*, while \( Coll(\gamma, \kappa) \) is called the *gentle Lévy Collapse of \( \kappa \) (onto \( \gamma^+ \)).

**1.25 Fact.**

1. \( \models_{Coll(\gamma, S)} \forall \xi \in \tilde{S} \ \xi \cong \gamma \)

2. \( Coll(\gamma, S) \) is \( \gamma \)-closed.

3. If \( \kappa^{<\gamma} = \kappa \), \( Coll(\gamma, \{\kappa\}) \) has the \( \kappa^+ \)-cc.

4. Let \( \kappa \) be regular and greater than \( \gamma \), and assume \( \forall \xi < \kappa, \xi^{<\gamma} < \kappa \). Then \( Coll(\gamma, \kappa) \) has the \( \kappa \)-cc.

**Proof.**

1. Obviously, for any \( \xi \in S \) and any \( \chi \in \xi \), the set of conditions \( q \) where for some \( \eta \in \gamma \), \( p(\xi, \eta) = \chi \) is dense. Thus, if \( G \) is the generic, \( F := \bigcup G \) is a function \( F : S \times \gamma \to S \) such that for each \( \xi \in S \), the function \( \eta \mapsto F(\xi, \eta) \) is a surjection from \( \gamma \) to \( \xi \).

2. Clear, as the union of less than \( \gamma \) many functions of size less than \( \gamma \) has itself size less than \( \gamma \) by regularity.

3. Clear, as \( Coll(\gamma, \{\kappa\}) \subseteq [\{\kappa\} \times \gamma \times \kappa]^{<\gamma} \cong \kappa^{<\gamma} \).
4. By way of contradiction, let $A$ be an antichain of $\text{Coll}(\gamma, \kappa)$ of size $\kappa$. The unwieldy hypothesis that $\forall \xi < \kappa \quad \xi^{<\gamma} < \kappa$ (and regularity of $\kappa$) allows us to use the $\Delta$-System-Lemma for $\{\text{dom}(p) | p \in A\}$: we can find a root $r \in [\kappa]^{<\gamma}$ and $A' \subseteq A$ of size $\kappa$ such that for any two distinct $p, q \in A'$, $\text{dom}(p) \cap \text{dom}(q) = r$. But as $r$ must be bounded in $\kappa$, say, by $\kappa_0$, there are only $\kappa^{<\gamma}_0 < \kappa$ possibilities for $p \upharpoonright r$, $p \in A'$. Thus (once more by regularity of $\kappa$) $\kappa$ many $p \in A'$ agree on $r$ and thus, as their domains are disjoint outside $r$, they must be be compatible, in contradiction to the fact that $A$ is an antichain.

A simple proof of the following simple fact is strangely absent from the usual texts on forcing, but one is included in [Kan97, p. 129]. More general theorems can be proved along similar lines, both about the Lévy-Collapse (see [Jec78, p. 280]) and about weak homogeneity (see [Jec78, p. 270]). The class of standard names for $P$ is defined by induction: a $P$-name $q$ is a standard name exactly if for all $x, y$ s.t. $(x, y) \in q$, $x$ is a standard name and $y = 1_P$, the maximal element of $P$. Every set $a$ in the ground model has a standard name $\check{a}$, and for any $P$-generic $G$, $\check{a}[G] = a$.

1.26 Fact. If $P$ is weakly homogeneous, that is, if for any $p, q \in P$ there is an automorphism $e : P \to P$ such that $i(p) \| q$, then for any formula $\phi$ of the forcing language mentioning only standard names, and for any $p \in P$,

$$p \Vdash_P \phi \iff \Vdash_P \phi.$$  

1.27 Fact. The Lévy-Collapse is weakly homogeneous.

Proof. Let $p, q \in \text{Coll}(\lambda, S)$. We need to find a map $e$ which is an automorphism of this collapsing order such that $e(p)$ and $q$ are compatible. Pick any bijection $f : \lambda \to \lambda$ such that for all $(\chi, \xi) \in \text{dom}(p)$ and all $\chi^,' \in S$, $(\chi^,' f(\xi)) \notin \text{dom}(q)$ (this is possible as $\text{dom}(p), \text{dom}(q) \subseteq \lambda$). This induces an automorphism $e$: we define $e(p) : \lambda \times S \to \bigcup S$ to be such that

$$e(p)(\chi, \xi) = p(\chi, f(\xi))$$

As $e$ can be viewed as taking images under the bijection $id \times f \times id$ on $S \times \lambda \times \bigcup S$, $e$ clearly preserves $\subseteq$ on $P(S \times \lambda \times \bigcup S)$. Clearly, $e : \text{Coll}(\lambda, S) \to \text{Coll}(\lambda, S)$ is an automorphism of $\text{Coll}(\lambda, S)$ and $\text{dom}(e(p)) \cap \text{dom}(q) = \emptyset$. The Lévy-Collapse is very easily decomposable as a product, and the projections and complete embeddings take a very simple form:
1.28 Fact. Let \( S = R \cup T \) and \( R \cap T = \emptyset \). Then the map \( i : \langle p, q \rangle \mapsto p \cup q \) is an isomorphism from \( Coll(\lambda, R) \times Coll(\lambda, T) \) onto \( Coll(\lambda, S) \) and \( Coll(\lambda, T)^{V} = Coll(\lambda, T)^{V[G]} \), for any \( G \) that is \( Coll(\lambda, S) \)-generic over \( V \).

Proof. That the mapping \( i \) is a bijection and order-preserving is obvious. The second fact holds since by \( \lambda \)-closedness of the first Lévy-Collapse, no new subsets of \( S \times \lambda \times \bigcup S \) of size less than \( \lambda \) are added by \( G \).

One of properties of the Lévy-Collapse, its so-called universality, can be loosely described thus: The Lévy-Collapse forces every upwards absolute formula that can be forced by a comparatively small forcing, or, in other words, the Lévy-Collapse adds a generic for every small forcing.

1.29 Fact. Let \( P \) be a separative p.o. and \( \alpha \) uncountable such that \( |P| \leq \alpha \) but \( \forces_P \omega \cong \alpha \). Then there is a dense subset of \( Coll(\omega, \{\alpha\}) \) which can be densely embedded into \( P \).

Proof. First, observe that below every condition, we can find an antichain of size \( \alpha \). Otherwise, if there were some \( q \in P \) such that the \( \alpha \)-cc holds below \( q \), then for some regular \( \beta \leq \alpha \), even the \( \beta \)-cc holds below \( q \). (We tacitly use a well-known fact about the chain condition see [Jec78, p. 157]. It would, on the other hand, be of no harm to prove the present fact only for regular \( \alpha \).) Let \( f \) be a \( P \)-name s.t. \( \forces_P f : \vec{\omega} \to \vec{\beta} \). The set of conditions below \( p \) which decide \( \dot{f}(\vec{n}) \) with different value is clearly an antichain; if for each \( n \), this chain is small, i.e. the set \( A_n := \{\xi < \alpha | \exists q \leq p \text{ s.t. } q \forces f = \xi\} \) is bounded in \( \beta \), by regularity of \( \beta \), \( \dot{f} \) will be forced by \( q \) to be bounded in \( \beta \), a contradiction.

The embedding - call it \( i \) - will be defined on \( \preceq^\omega \alpha \), which is of course dense in \( Coll(\omega, \{\alpha\}) \). Fix a name \( \dot{g} \) such that \( \forces_P \dot{g} : \vec{\omega} \to \vec{G} \). Now we define \( i \) inductively on the length of conditions. Of course, the empty sequence \( 1_{Coll(\omega, \{\alpha\})} \) can be mapped simply to \( 1_P \). Now let’s assume we have defined \( i \) on \( ^n \alpha \). For each \( q \in ^n \alpha \), let \( A^{i(q)} := \{a^{i(q)}_\xi | \xi < \alpha\} \) be a maximal pairwise incompatible subset of \( \{p \in P | p \leq i(q)\} \), of size \( \alpha \), and such that each element of \( A^{i(q)} \) decide the value of \( \dot{g}(\vec{n}) \) (enlarge an antichain below \( i(q) \) until it is maximal and refine it to decide the values of \( \dot{g} \)). Then for \( p \in ^{n+1} \alpha \), let \( i(p) := a^{i(p(n))}_{p(n)} \).

It is obvious from the construction that \( i \) is injective and \( i \) and \( i^{-1} \) are order-preserving. It thus remains to show that \( i''(\preceq^\omega \alpha) \) is dense in \( P \). So take any \( p \in P \). As \( p \forces \dot{p} \in \vec{G} \), for some \( n \) and some \( p' \leq p \), \( \dot{p}' \forces \dot{g}(\vec{n}) = \vec{p} \). Of course, \( p' \) must be compatible to some element \( a \) of \( A^{\vec{p}} \); then again, to some element of \( A^{\vec{a}} \), as this is a maximal antichain below \( a \); going on like this, \( p' \) has to be compatible with one of the elements, say \( a' \), of \( A^{\vec{a}} \), for some \( \vec{a} \). As
\(a'\) itself also decides the value of \(\dot{g}(\dot{n})\), it must agree with \(p'\) on this value, whence \(a' \models \dot{p} \in \dot{G}\). But then, by separativity of \(P\), even \(a' \leq p\).

1.30 Corollary. Universality. Let \(P\) be a p.o. and \(\alpha\) regular such that \(|P| < \alpha\). Then there exists a p.o. \(Q\) and a complete embedding \(i : P \to Q\) where \(Q \sim \text{Coll}(\omega, \{\alpha\})\), i.e. \(Q\) is equivalent to the Lévy-Collapse of \(\alpha\) onto \(\omega\).

Proof. Consider \(Q := P \times \text{Coll}(\omega, \{\alpha\})\). By the previous fact, there exists a dense embedding \(d : \omega \to Q\), and \(\omega\) is dense in \(\text{Coll}(\omega, \{\alpha\})\). So obviously \(Q \sim \text{Coll}(\omega, \{\alpha\})\), and \(p_0 : P \to Q\), the canonical projection to the first coordinate, is a complete embedding.

1.3 Reals

To make notation easier, for this section, let variables \(m, n\) always denote natural numbers, and variables \(s, w\) elements of \(\text{Seq} := \omega^\omega\). By reals we usually mean members of \(2^\omega\), and we will issue a warning before we switch to \(\omega^\omega\). Fix a bijection \(\Gamma : \omega^2 \to \omega\), which is arithmetical, and an arithmetical enumeration of \(\omega^\omega\), \((s_n)_{n \in \omega}\). Thus we are allowed to talk about finite sequences of natural numbers as if they were natural numbers. We denote the length of an \(\omega\)-sequence \(s\) by \(lh(s)\).

We shall need a normal form for \(\Sigma^1_n\) relations, which we can easily state as a Definition 1.31. \(\Pi^1_n, \Delta^1_n\) and boldface and lightface versions of the hierarchy are then defined as usual (see, e.g., [Jec78, p. 500] for a survey of descriptive set theory and [Jec78, p. 509ff.] for definitions compatible with the one given here). By \(\Delta^1_n(a_1, \ldots, a_l)\) or \(\text{arithmetical in } a_1, \ldots, a_l\), we mean definable in the model \(\langle \omega, a_1, \ldots, a_l, \epsilon \rangle\). When we talk about trees, for simplicity of notation, consider them to be ordered by reverse inclusion. Thus our trees will grow downward, as they do in Israel. An infinite branch through a tree is an infinite descending chain of nodes. By a \(\text{ranking function}\), we mean an order-preserving function into the ordinals always taking the least possible value.

1.31 Definition. A \(k\)-ary relation on \(\omega^\omega\) is \(\Sigma^1_n(a_1, \ldots, a_l)\) exactly if it can be written in the following form:

\[
(x_1, \ldots, x_k) \in A \iff \exists z_1 \in \omega^\omega \ldots Q \exists z_n \in \omega^\omega \ B(x_1, \ldots, x_k, z_1, \ldots, z_n),
\]

where each block consists exclusively of quantifiers of one type, either \(\forall\) or \(\exists\) (we have written \(Q\) for the last quantifier, \(\forall\) for even \(n\), \(\exists\) for odd \(n\)) and
where $B(x, z_1, \ldots, z_n)$ is a relation arithmetical in $a_1, \ldots, a_l$. Moreover for $n > 0$, we can demand that $B$ be equivalent to the following statement if $n$ is even and equivalent to the negation if $n$ is odd:

$$\exists m \in \omega \ (x_1 \upharpoonright m, \ldots, x_k \upharpoonright m, z_1 \upharpoonright m, \ldots, z_n \upharpoonright m) \not\in T,$$

for a tree $T$ on $\text{Seq}^{k+n}$, arithmetical in $a_1, \ldots, a_l$.

**Hereditarily countable sets and reals**

**1.32 Definition.** We say that $r \in 2^\omega$ codes $x \in HC$ if, and only if, the relation $E_r := \{(m, n) \in \omega \times \omega \mid \Gamma(m, n) \in r\}$ is well-founded, extensional and the Mostowski-collapse of $E_r$ is just $TC(\{x\})$. We say that $r$ codes $x$ via $g : TC(\{x\}) \longrightarrow \omega$ if all of the above holds and $g$ is the inverse of the collapsing map. We write $r_0 \cong r_1$, if $E_{r_0}$ and $E_{r_1}$ are extensional and $E_r \subseteq \omega \times \omega$ is the graph of a partial function $f_r$, s.t. $\forall m, n \in \text{dom}(f_r) [mE_{r_0}n \iff f_r(m)E_{r_1}f_r(n)]$. Denote the $m$ with no successors in $E_r$ by $\text{top}(r)$ (if $r$ codes $x$ via $g$, $g(x) = \text{top}(r)$). Let $\text{field}(r) := \text{dom}(E_r) \cup \text{range}(E_r)$, and $r \cup s := \{\Gamma(2^n, 2^m)|nE_r m\} \cup \{\Gamma(3^n, 3^m)|nE_s m\}$. For $m \in \text{field}(r)$, let

$$\text{tc}_r(m) := \{n \mid \exists k \exists s \in \omega^{k+1} \land s(0) = n \land s(k) = m \land \forall l < k \ s(l) \in E_r s(l+1)\}$$

Observe all the notions presented are arithmetical (with real parameters).

Of course, for any set $x \in HC$, we can find a real $r_x$ coding $x$. Just choose some $g : TC(\{x\}) \xrightarrow{\sim} \omega$ and let $r_x := \Gamma^x E_x$, where $\langle TC(\{x\}), \in \rangle \cong \langle \omega, E_x \rangle$.

Assume $r_0, r_1$ code $p_0, p_1$. “$p_0 \in p_1$” is $\Delta_1^1(r_0, r_1)$:

$$\exists r \ r_0 \cong r_1 \land \text{field}(r_0) \subseteq \text{dom}(f_r) \land f_r(\text{top}(r_0))E_{r_1} \text{top}(r_1)$$

$$\forall s \forall t \ [r_0 \cup r_1 \cong t \land \text{field}(r_0 \cup r_1) = \text{dom}(f_s) \land \text{ran}(f_s) = \text{field}(t) \implies \text{true}]$$

$$\text{true} \iff f_s(3^{\text{top}(r_0)})E_r f_s(3^{\text{top}(r_1)})$$

“$p_0 = p_1$” is also $\Delta_1^1(r_0, r_1)$:

$$\exists r \ r_0 \cong r_1 \land \text{field}(r_0) \subseteq \text{dom}(f_r) \land f_r(\text{top}(r_0)) = \text{top}(r_1)$$

$$\forall s \forall t \ [r_0 \cup r_1 \cong t \land \text{field}(r_0 \cup r_1) = \text{dom}(f_s) \land \text{ran}(f_s) = \text{field}(t) \implies \text{true}]$$

$$\text{true} \iff f_s(3^{\text{top}(r_0)}) = f_s(3^{\text{top}(r_1)})$$

Moreover, for variables $x_i$ ranging over $TC(\{p_i\})$, respectively $(i \in 2)$, we can replace them by variables ranging over $\omega$ and express “$x_0 \in x_1$” and “$x_0 = x_1$” using the formulas above, with $x_i$ instead of $\text{top}(r_i)$ and $\text{tc}_{r_0}(x_0)$ instead of $\text{field}(r_0)$. 
1.33 Fact. Let $\Psi(x_0,\ldots,x_k)$ be a statement in the language of set theory, $(p_0,\ldots,p_k) \in HC$. Then there exists a formula $\Phi(x_0,\ldots,x_k)$ s.t. for any reals $r_0,\ldots,r_k$ coding $p_0,\ldots,p_k$,

1. $(HC,\in) \models \Psi(p_1,\ldots,p_k) \iff \Phi(r_1,\ldots,r_k)$

2. If $\Psi(p_1,\ldots,p_k)$ is $\Sigma_n$, $\Phi(r_1,\ldots,r_k)$ is $\Sigma_{n+1}^1$

   If $\Psi(p_1,\ldots,p_k)$ is $\Pi_n$, $\Phi(r_1,\ldots,r_k)$ is $\Pi_{n+1}^1$.

Proof. Let $\Psi = \Psi(x_0,\ldots,x_k)$, $p_0,\ldots,p_k$, $r_0,\ldots,r_k$ be given as above. The proof goes by induction on formula complexity. For the $\Delta_0$ case, replace atomic formulas in the way indicated in the preceding discussion. From what has been said there, 1. is immediate. By shifting quantifiers to the front, we see we have a $\Delta_1^1$ statement. Now assume $\Psi$ is $\Sigma_{n+1}$ and we can already convert $\Pi_n$ statements. For simplicity, assume $\Psi(p) = \exists x \Theta(x,p)$, where $\Theta(x,p)$ is $\Pi_n$. We want to express

$$\exists x \in HC \quad HC \models \Theta(x,p).$$

(5)

By induction, $HC \models \Theta(x,p)$ converts to a $\Pi_{n+1}^1$ formula. We see that (5) is equivalent to saying “$\exists R$ s.t. $R$ codes an extensional, well-founded relation and $\bar{\Theta}(R,r_p)$ holds”, where $\bar{\Theta}$ is $\Pi_{n+1}^1$.

“$R \in 2^\omega$ codes a well founded relation” $\iff$ “there is no $R' \in 2^\omega$ such that $f(m,n) \in R'$ just if $n$ is the $m$-th element of an infinite branch through $R'$, i.e., this is a $\Pi_{n+1}^1$-property of $R$. So we get that (5) is equivalent to a statement of the form

$$\exists R \in 2^\omega \quad \forall R' \in 2^\omega \quad \neg \Upsilon(R',R) \land \bar{\Theta}(R,r_p)$$

(6)

where $\Upsilon(R',R)$ is a $\Delta_1^1$ statement asserting that $R'$ codes an infinite branch through $R$, and thus the whole of (6) is $\Sigma_{n+2}^1$. This completes the inductive step for the $\Sigma_{n+1}^1$-case, and the $\Pi_{n+1}^1$-case works just the same, mutatis mutandis. \qed

Levy-Shoenfield absoluteness theorem

This paper deals with $\Sigma_3^1$-absoluteness between a ground model and the forcing extension. What about $\Sigma_2^1$-absoluteness? In this case, Theorem 1.34, called Levy-Shoenfield Absoluteess Theorem, shows absoluteness holds not only between a ground model and its extension, but for a large class of transitive models.

For this section, reals will be elements of $\omega^\omega$, rather than elements of $2^\omega$. 

16
1.34 Theorem. Let $A$ be a $\Sigma^1_2(a)$-predicate, for a real $a$, and let $M$ be a transitive model of $ZF^-$, $a \in M$, $\omega_1 \subseteq M$. Then

$$A^M = A \cap M$$

Proof. Let $U$ a model of $ZF^-$ (possibly $U = V$) s.t. $a \in U$, $\omega_1 \subseteq U$ and work in $U$. For a $\Sigma^1_2(a)$-predicate $A$ the following equivalences clearly hold, with $\kappa$ denoting the $\omega_1$ of $V$,

$$x \in A \iff \exists y \in \omega^\omega \forall z \in \omega^\omega \exists n \in \omega \ (x \upharpoonright n, y \upharpoonright n, z \upharpoonright n) \not\in T$$

$$\iff \exists y \text{ s.t. } T(x, y) \text{ is well founded.}$$

$$\iff \exists y \exists f : \text{Seq} \to \kappa \text{ s.t. } f \text{ is order-preserving on } T(x, y).$$

Here $T$ is arithmetical in a parameter $a$ and

$$T(x, y) := \{ w \in \text{Seq} \mid (x \upharpoonright \text{lh}(w), y \upharpoonright \text{lh}(w), w) \in T \}.$$ 

The last equivalence holds as the $ZF^-$-model $U$ can build a rank function for the countable tree $T(x, y)$ just if it is well-founded.

Now we define another tree $\bar{T}$ in $U$ (the nodes correspond to initial segments of a ranking function):

$$\bar{T} := \{(u, v, r) \in \text{Seq} \times \text{Seq} \times < \omega \kappa \mid \text{lh}(u) = \text{lh}(v) = \text{lh}(r) \text{ and } \bar{r} : \text{Seq} \to \kappa \text{ is order-preserving on } T(u, v)\}.$$ 

where $\bar{r}$ denotes the partial function $\bar{r}(s_n) = r(n)$ ($\ (s_k)_{k \in \omega}$ is your favorite canonical enumeration of sequences of natural numbers), and $\bar{T}(u, v) := \{ w \in \text{lh}(u) \omega \mid (u \upharpoonright \text{lh}(w), v \upharpoonright \text{lh}(w), w) \in T \}$. This allows us to write

$$x \in A \iff \bar{T}(x) \text{ not well-founded},$$

where of course $\bar{T}(x) := \{ (v, r) \mid (x \upharpoonright \text{lh}(t), v, r) \in \bar{T} \}$. Now we leave $U$. Observe that, as $\bar{T}$ is $\Delta_1(a)$, $\bar{T}^U = \bar{T}^V$. The usual way of showing absoluteness of well-foundedness (via a rank function on $\bar{T}(x)$) easily yields that for $M$ as in the hypotheses of the theorem, and $x \in M$,

$$x \in A \iff M \models x \in A$$

$\square$
1.4 Large cardinals

Zero sharp and the Covering Lemma

$0^\#$ is a set of natural numbers satisfying a certain, absolute syntactical property. $0^\#$ is not in $L$; in fact, we may regard it as a truth predicate for $L$. Its existence is equivalent to the existence of a cub class of ordinals, containing all uncountable $V$-cardinals, which are indiscernible in $L$ and whose Skolem hull in $L$ is all of $L$. An introduction to $0^\#$ and indiscernibles can be found in [Jec78, sec. 30] or [Dev84, V]. The next two facts can serve as a black box for our purposes. For a proof, see [DJ75].

1.35 Fact. If $0^\#$ exists, $V \neq L$ and for any two uncountable $V$-cardinals $\alpha < \beta$, $L_\alpha \prec L_\beta$.

1.36 Fact. Covering Lemma. If $0^\#$ does not exist, $\forall X \subseteq L \exists Y \in L$ s.t. $|Y| = |X| + \omega_1 \wedge X \subseteq Y$

We shall use the following consequences of the covering lemma. For sake of completeness, we include the proof.

1.37 Fact. Assume $0^\#$ does not exist. Then

1. If $2^{c_f \kappa} \leq \kappa^+$, then $\kappa^{c_f \kappa} = \kappa^+$ (i.e., the Singular Cardinal Hypothesis holds, implying GCH at strong limit singular cardinals).

2. If $\omega < c_f \lambda < |\lambda|$, $(c_f \lambda < \lambda)^L$.

3. If $\kappa$ is a singular cardinal, then $(\kappa^+)^L = \kappa^+$.

Proof. 1. For each $s \in [\kappa]^{c_f \kappa}$, let $s \subset Y_s \cong c_f \kappa \cdot \omega_1$, $Y_s \in L \cap P(\kappa)$. We have

$[\kappa]^{c_f \kappa} = \bigcup \{[Y_s]^{c_f \kappa} \mid Y_s \in L_{\kappa^+}, s \in [\kappa]^{c_f \kappa}\}$, whence

$[\kappa]^{c_f \kappa} \leq \kappa^+ \cdot (c_f \kappa \cdot \omega_1)^{c_f \kappa} = \kappa^+ \cdot 2^{c_f \kappa} = \kappa^+$

2. Let $C \subset \lambda$, $otp C = c_f \lambda$. Take $C \subseteq Y \in L$, $Y \cong c_f \lambda$. Then $otp Y < (c_f \lambda)^+ \leq |\lambda| \leq \lambda$, but the order type is absolute between $L$ and $V$.

3. By 2, any ordinal between $\kappa$ and $\kappa^+$ must have cofinality less than itself in $L$. In particular, it cannot be a regular cardinal in $L$. □
Some small large cardinals

The consistency proofs in this paper will deal with reflecting, remarkable and with lightface $\Sigma^1_2$-indescribable cardinals (which will be introduced and treated in section 5.1). We would like to give convenient upper and lower bounds for the strength of these notions; any definitions we omit here can be found in [Kan97].

1.38 Definition. We say $\kappa \in \text{Reg}$ is reflecting if, and only if, for all formula $\phi(x)$ and $\forall p \in H_\kappa$, whenever $\exists \Theta H_\Theta \models \phi(p)$, $\exists \delta < \kappa H_\delta \models \phi(p)$.

1.39 Fact. Let $\kappa$ be a regular cardinal. Then $\kappa$ is reflecting $\iff V_\kappa \prec \Sigma_2 V$.

Proof. First assume $\kappa$ is reflecting. Observe that $H_\kappa = V_\kappa$: $\kappa$ is inaccessible, as $\forall x \in H_\kappa \exists \delta < \kappa$ s.t. $H_\delta \models \exists \mathcal{P}(x)$, and $\mathcal{P}(x)^{H_\delta} = \mathcal{P}(x) \cap H_\delta = \mathcal{P}(x)$. $\kappa$ inaccessible implies $H_\kappa = V_\kappa$, as in that case, $H_\kappa$ is closed under power set and small unions. As $\kappa$ is an uncountable regular cardinal, $H_\kappa \prec \Sigma_1 V$, and so upwards absoluteness of $\Sigma_2$-statements is immediate. Now if $V \models \exists x \phi(x)$ (where $\phi(x)$ is $\Pi_1$ with parameters in $H_\kappa$), there is $x_0 \in H_\delta$, $\delta < \kappa$ s.t. $H_\delta \models \phi(x_0)$, and $\phi(x_0)$ is absolute between $H_\delta$ and $H_\kappa$.

Now, conversely, suppose $V_\kappa \prec \Sigma_2 V$. First of all, again $\kappa$ must be inaccessible: asserting existence of a surjection from an ordinal to $\mathcal{P}(x)$ is $\Sigma_2$, while $\mathcal{P}(x) \subseteq V_\kappa$ for $x \in V_\kappa$ (as $\kappa$ is limit). So again $H_\kappa = V_\kappa$. Secondly, suppose $H_\delta \models \phi$, allowing parameters in $H_\kappa$. “$y = H_\delta$” is $\Pi_1$ (and hence absolute between $H_\kappa$ and $V$) so $\exists \delta H_\delta \models \phi$ is $\Sigma_2$, and thus is reflected by $H_\kappa \prec \Sigma_2 V$.

1.40 Fact. If both a reflecting and a Mahlo cardinal exist, then the least Mahlo is strictly below the least reflecting, which itself is not Mahlo. Existence of a Mahlo implies consistency of a stationary class of reflecting cardinals.

Proof. Let $\theta$ be the least Mahlo and $\kappa$ the least reflecting. If $\theta > \kappa$, as $H_{\theta^+} \models \exists$ a Mahlo”, some $H_{\eta}$, with regular $\eta < \kappa$ reflects this; as being Mahlo is absolute between $H_\eta$ and $V$, there is a Mahlo below $\kappa$, contradiction. Now assume $\kappa$ itself is Mahlo. For any ordinal $\xi < \kappa$, let $f(\xi)$ be least such that for all $p \in H_{\xi^+}$ and all formulas $\phi(x)$, if there is a regular $\delta$ s.t. $H_\delta \models \phi(p)$, then there is such $\delta$ below $f(\xi)$. By the reflection property and inaccessibility of $\kappa$, $f$ maps $\kappa$ into $\kappa$; as $\kappa$ is Mahlo, there must be a regular fixed point of $f$ below $\kappa$, but such a point must be a reflecting cardinal itself. If $\theta$ is any Mahlo (not reflecting), look at a similar $f$ such that its regular fixed points are reflecting in $H_\theta$; $H_\theta$ is a model of $\text{ZFC}$ with stationarily many reflecting cardinals.

19
Let us call a cardinal Σ$_k$-Mahlo (resp. Π$_k$-Mahlo) if every cub subset of κ with a Σ$_k$ (resp. Π$_k$) definition (with a parameter from $V_κ$) contains an inaccessible cardinal.

1.41 Fact. If κ is reflecting, κ is Σ$_2$-Mahlo.

Proof. Assume $ξ ∈ C ⇐⇒ φ(ξ)$, where φ(x) is Σ$_2$. As $V_κ ⊆ V$, we have $V_κ ⊆ Σ_2 V$, so for some inaccessible $η < κ$, $V_η$ believes the same thing, and by upwards absoluteness of Σ$_2$ statements for $V_η$, $C$ is unbounded in $η$, whence $η ∈ C$.

1.42 Definition. [Sch00b] κ is called θ-remarkable for a regular cardinal θ, if, and only if, there exist countable transitive models $M$, $N$, together with elementary embeddings $π : M → Σ_ω H_θ$, $σ : M → Σ_ω N$ such that for $κ := π^{-1}(κ)$, $θ := M ∩ On$,

- $crit(σ) = κ$,
- $σ(κ) > θ$
- $θ ∈ Reg^N$, $M ∈ N$ and $N ⊨ “M = H_θ”$.

κ is remarkable if, and only if κ is θ-remarkable for all regular $θ > κ$.

1.43 Fact. If κ is remarkable, κ is reflecting.

Proof. Assume κ is remarkable and $H_θ ⊨ φ(p)$, where $p ∈ H_κ$ and $θ > κ$ regular. Take $N, M$ countable transitive, together with elementary embeddings $π : M → Σ_ω H_θ$, $σ : M → Σ_ω N$, as in the definition of remarkability. Let $κ := π^{-1}(κ)$. By elementarity,

$$M ⊨ φ(π^{-1}(p)).$$

But $N ⊨ M = H_{M ∩ On}$, and $M ∩ On < σ(κ)$ is a regular cardinal in $N$. Thus

$$N ⊨ “∃λ < σ(κ) H_λ ⊨ φ(π^{-1}(p))”$$

As $π^{-1}(p) ∈ H_κ$, we have $σ(π^{-1}(p)) = π^{-1}(p)$. So we can pull back via $σ$ to obtain

$$M ⊨ “∃λ < κ H_λ ⊨ φ(π^{-1}(p))”.$$

Once more applying $π$, we can see that $H_λ ⊨ φ(p)$ for some regular cardinal $λ$ below $κ$.

To show relative consistency of a remarkable cardinal, we mention the ω-Erdős cardinal $κ(ω)$, i.e. the least κ such that $κ → (ω)^{≤ω}_2$ (meaning every $f : [κ]^{<ω} → 2$ has an infinite homogeneous set). Such κ are inaccessible (by a combinatorial argument, see [Kan97, 7.15, p. 82]).
1.44 Fact. $\exists \kappa$ s.t. $\kappa \rightarrow (\omega)^{\omega}_2$ implies the consistency of “$ZFC \land$ there is a remarkable cardinal”.

Proof. Observe that if $\kappa \rightarrow (\omega)^{\omega}_2$, then the same holds relativized to $L$: let $f$ be a constructible coloring of finite sequences of $\kappa$. A homogeneous set of order-type $\omega$ exists if and only if $\{ p \in [\kappa]^{<\omega} | p \text{ is } f\text{-homogeneous} \}$, ordered by $\supset$, has an infinite branch. As has been shown before (see the proof of 1.34), such statements are sufficiently absolute, so a homogeneous set exists in $L$. We work in $L$ for the rest of the proof. Let $\kappa$ be least such that there is $\langle L_\kappa, \iota_i \rangle_{i \in \omega}$, a structure with $\omega$ indiscernibles. Let $\pi : L_\gamma \rightarrow_{\Sigma_\omega} L_\kappa$ be an elementary embedding obtained by collapsing the Skolem hull of the indiscernibles with respect to $L_\kappa$ (observe $\gamma < \omega_1$). Let $\alpha, \beta$ denote the pre-image under $\pi$ of $\iota_0, \iota_1$. As $L_\gamma$ is the Skolem hull of the pre-image of the indiscernibles, the map $\iota_k \mapsto \iota_{k+1}$ induces an elementary embedding $\sigma : L_\gamma \rightarrow_{\Sigma_\omega} L_\gamma$ such that $\sigma(\alpha) = \beta$ and (by minimality of $\kappa$) $\alpha = \text{crit}(\sigma)$ (we omit some details here; see [Kan97, sec. 9]). Note that by 1.5, $\iota_0$ is regular in $L_\kappa$, and playing with $\sigma$ and $\pi$ shows $\iota_0$ and hence every indiscernibles is inaccessible in $L_\kappa$ (and thus in $L$). Now we show that $L_\beta \models \alpha \text{ is remarkable}$ (it also is a model of $ZFC$ by inaccessibility). Let $\theta$ be regular in $L_\beta$. Set $\bar{\pi} := \pi \upharpoonright L_\theta$, $\bar{\sigma} := \sigma \upharpoonright L_\theta$. It remains to observe that as $\beta$ is inaccessible in $L_\gamma$, $\theta$ is also regular in $L_\gamma$ and thus in $L_{\sigma(\theta)}$. We have:

$$\exists \hat{\theta}, \hat{\theta} < \omega_1 \ \exists \bar{\pi} : L_{\bar{\theta}} \rightarrow L_{\pi(\theta)}, \bar{\sigma} : L_{\bar{\theta}} \rightarrow L_{\hat{\theta}} \text{ s.t. } \text{crit}(\bar{\sigma}) = \bar{\pi}^{-1}(\pi(\alpha)), \bar{\sigma}(\text{crit}(\bar{\sigma})) > \hat{\theta} \text{ and } \hat{\theta} \text{ is regular in } L_{\hat{\theta}}$$

(7)

By inaccessibility of $\pi(\beta)$, $L_{\pi(\beta)} \prec_{\Sigma_1} L$, so (7) holds in $L_{\pi(\beta)}$; pulling back via $\pi$ shows that $\alpha$ is $\theta$-remarkable in $L_\beta$; but $\theta$ was arbitrary. \hfill $\square$

1.45 Fact. Every Silver indiscernible is remarkable in $L$.

Proof. Let $(\iota_k)_{k \in \omega+1}$ be the first $\omega+1$ indiscernibles; it suffices to show $\iota_0$ is remarkable in $L_{\iota_1}$. Let the real $r$ consist of the G"odel-numbers $\{ \sharp \phi(\iota_0, \iota_1) \}$. Then $r \in L$. The following sentence is $\Sigma_1(r)$: there exists a well-founded model $M$ with $\omega$ many indiscernibles with height a limit ordinal such that the theory of $M$ includes $r$. Therefore, by Levy-Shoenfield absoluteness 1.34, this is true in $L$. The previous proof shows that the G"odel-number of the formula saying “$\iota_0$ is remarkable in $\iota_1$” is in $r$ and we are done. \hfill $\square$
2 Equiconsistency for set forcing

2.1 Theorem. [FMW92] The following are equiconsistent, modulo ZFC:

1. $\Sigma^1_3$-absoluteness for all set forcing holds

2. there exists a reflecting cardinal

Proof. First, assume $\Sigma^1_3$-absoluteness for all set forcing holds. We will show that $L \models \“\kappa \text{ is reflecting}\”$, for $\kappa = \omega_1^L$, using the characterization of reflect-
ingness given in Fact 1.39 (on p. 19).

Firstly, $\kappa$ is a limit cardinal in $L$: Else, assume $L \models \“\kappa = \lambda^+ \land \lambda \in \text{Card}\”$. The following formula $\psi(\lambda)$ expresses that there exists another countable $L$-cardinal above $\lambda$:

$$\exists x < \omega_1 \ x > \lambda \land \forall \alpha < \omega_1 \ L_\alpha \models x \in \text{Card}$$

As both “$y = L_x^y$” and $y \models \phi(x)$ (for any formula $\phi$) are $\Delta_1$, $\psi(\lambda)$ is $\Sigma_2$ over $HC$, in the parameter $\lambda \in HC$. So by fact 1.33 we can find a real $r_\lambda$ coding $\lambda$ and a formula $\bar{\psi}(r_\lambda)$ which is equivalent to $\psi(\lambda)$ and $\Sigma^1_3$ in $r_\lambda$.

Now force with $Col(\omega, \{\kappa\})$: In the extension, $\kappa < \omega_1$ holds (by fact 1.25) so $\psi(\lambda)$ holds. As this has been shown to be equivalent to a $\Sigma^1_3(r_\lambda)$ formula, it must also hold in the ground model, by $\Sigma^1_3$-absoluteness. If $\alpha < \omega_1$ and is not an $L$-cardinal, this must be reflected by some $L_\gamma$, for $\gamma < \omega_1$. So any $\alpha$ witnessing $\psi(\lambda)$ really has to be an $L$-cardinal, $\lambda < \alpha < \omega_1$, contradicting $(\kappa = \lambda^+)^L$.

Secondly, we show $L_\kappa \prec_{\Sigma_2} L$. So let $p \in H^L_\kappa = L_\kappa$, $L \models \exists x \phi(x, p)$, where $\phi(x)$ is $\Pi_1$ with parameter $x$. Using reflection, choose some regular $\delta$ such that $L_\delta \models \exists x \phi(x, p)$; Now force with $Col(\omega, \{\delta\})$. In the extension, $\delta \cong \omega$, and so the formula following $\psi(p)$ holds:

$$\exists \chi < \omega_1 \ \forall \alpha < \omega_1 \ \ L_\alpha \models \chi \in \text{Card} \land p \in L_\chi \land L_\chi \models \phi\exists(x, p)$$

Just like above, this formula can be seen to be $\Sigma_2$ over $HC$ in its parameter $p$, and thus equivalent to a formula $\bar{\psi}(r_\kappa)$ which is $\Sigma^1_3$ in a parameter $r_\kappa$ coding $p$. Interestingly we have not yet used the fact that $\phi$ itself has bounded complexity. As before, we can now argue that $\psi(p)$ must also hold in the ground model and that there must exist an $L$-cardinal $\bar{\delta} < \kappa = \omega_1^L$ such that $L_{\bar{\delta}} \models \exists x \phi(x, p)$. But as $\delta$ is an $L$-cardinal, $L_{\bar{\delta}} = H^L_{\bar{\delta}} \prec_{\Sigma_1} L_\kappa$, so $\Sigma_2$ formulas with parameters from the smaller model are upwards absolute between these two models, whence $L_\kappa \models \exists x \phi(x, p)$. This completes the proof that $L \models \“\kappa \text{ is reflecting}\”$. 

22
Now, for the other direction, assume \( \kappa \) is reflecting. Of course \( \kappa \) is inaccessible. Let \( G \) be generic for \( \text{Col}(\omega, \kappa) \) over \( V \). Then by fact 1.25, \( V[G] \models \kappa = \omega_1 \). We show \( V[G] \models \Sigma^1_3(\text{set-forcing}) \).

So let \( \phi(r) \) be some \( \Sigma^1_3 \) statement with real parameter \( r \in V[G] \) and let \( Q \in V[G] \) be a p.o. such that \( V[G] \models \rightarrow Q \phi(r) \). We must show that already \( V[G] \models \phi(r) \). Choose a nice name \( \check{r} \) for \( r \), that is, a name of the form \( \bigcup_{n \in \omega} \check{n} \times A_n \), where \( A_n \) is an antichain, and observe that the value of \( \check{r} \) is already decided in an initial segment of \( \text{Col}(\omega, \kappa) \): each \( A_n \subseteq \kappa \) and we can choose \( \beta \) regular such that \( \bigcup_{n \in \omega} \{ \text{dom}(q) \mid q \in A_n \} \subset \beta \). By fact 1.28, \( G_0 := G \cap \text{Col}(\omega, \beta) \) is \( \text{Col}(\omega, \beta) \)-generic over \( V \) and \( G_1 := G \cap \text{Col}(\omega, \kappa - \beta) \) is \( \text{Col}(\omega, \kappa - \beta) \)-generic over \( V \). By weak homogeneity of the Lévy-Collapse (corollary 1.30), there exists \( \check{r} \models G[G_0] = r, V[G_0] \models \Rightarrow \text{Col}(\omega, \kappa - \beta) \rightarrow Q \phi(\check{r}) \).

Obviously, \( V[G_0] \models \Rightarrow \exists P \models P \phi(\check{r}) \), so in \( V \), the following holds: there is a name \( \check{P} \) for a p.o. such that \( \models \text{Col}(\omega, \beta) \Rightarrow \models \check{P} \phi(\check{r}) \). Now the fact that \( \kappa \) is reflecting in \( V \) can be used to show that without loss of generality, \( \check{P} \in H_\alpha \), for some \( \alpha < \kappa \). Of course, for \( P := \check{P}[G_0], |P| < \alpha \) in \( V[G_0] \). By universality of the Lévy-Collapse (corollary 1.30), there exists \( i : P \rightarrow Q \), a complete embedding into a p.o. \( Q \) that is equivalent to \( \text{Col}(\omega, \{ \alpha \}) \).

Now we go back to \( V[G] \). Of course, by fact 1.28, \( J_C := G \cap \text{Col}(\omega, \{ \alpha \}) \) is \( \text{Col}(\omega, \{ \alpha \}) \)-generic over \( V[G \cap \text{Col}(\omega, \alpha)] \), and a fortiori over \( V[G_0] \). As the property of being a dense or complete embedding is absolute for transitive models, \( J_C \) can be used to construct a generic \( J_Q \) on \( Q \) (as in 1.8, via dense embeddings) and then (again as in 1.8, via \( i \)) to construct a generic (still over \( V[G_0] \)) for \( P \) - call it \( H \). As \( \models_P \phi(\check{r}) \) over \( V[G_0], V[G_0][H] \models \phi(r) \). But again by completeness of \( i \) and the existence of various dense and complete embeddings, \( V[G_0][H] \subseteq V[G_0][J_Q] = V[G_0][J_C] \subseteq V[G] \), and, being \( \Sigma^1_3 \) in \( r \), \( \phi(r) \) is upward absolute for transitive inner models of \( V[G] \) containing \( r \), whence \( V[G] \models \phi(r) \). This completes the proof that \( V[G] \) is a model of \( \Sigma^1_3 \) absoluteness for set forcing. \( \square \)
3  A lower bound for $\omega_1$-preserving set-forcing

3.1 Theorem. [FB01] Assume $\Sigma^1_3$-absoluteness for all $\omega_1$-preserving set forcings. Then, for $\kappa = \omega_1^V$, $L \models \text{"}\kappa \text{ is inaccessible"}$. 

Proof. Assume that, to the contrary, $L \models \text{"}(\omega_1)^V = \lambda^+\text{"}$. Using the absoluteness assumption, we will obtain a contradiction. As $\lambda < \omega_1$, let $f : \omega \rightarrow \lambda$. Then $f \in HC$ and $\omega^L \subseteq \omega_1$. Further, for each $\xi < \omega_1$, let $g_\xi$ denote the $\prec_L$-least surjection $g_\xi : \omega \rightarrow \xi$. For each $n \in \omega$, $g_\xi(n) < \xi$, so the function $h_n : \omega_1 \rightarrow \omega_1$, defined by $\xi \mapsto g_\xi(n)$ is regressive. Hence, by Fodor’s Lemma, there exists a stationary subset $S_n$ of $\omega_1$ such that $h_n$ is constant on $S_n$.

Clearly, the intersection of all the $S_n$ cannot contain two distinct ordinals: as for all $\alpha, \beta \in S_n$, $g_n(\alpha) = g_n(\beta)$, if $\alpha, \beta \in S_n$ for all $n \in \omega$, we have $g_n = b_\beta$, contradicting the definition of these functions. Of course, the intersection of even only two stationary sets needn’t be stationary. But, using an $\omega_1$-preserving forcing, we can add a cub subset to any one of these stationary sets; and the cub subsets of $\omega_1$ form a $\sigma$-complete filter. The idea of this proof is to use $\Sigma^1_3$-absoluteness to pullback such a cub subset with the desired property into the ground model. While forcing will usually not be able to add cub subsets to the $S_n$ for all $n$ simultaneously, these pullbacks co-exist in the ground model, leading to a contradiction.

So fix $n$ for the moment. We force with $P_{S_n}$, the forcing for adding a cub subset of this particular $S_n$ (see definition 1.20, p. 10). So let $G$ be generic for $P_{S_n}$ and work in $V[G]$, denoting $\bigcup G$, the generic cub subset of $S_n$, by $C$. In this model, clearly

$$\exists C \text{ cub in } \omega_1 \text{ s.t. } \forall \xi, \chi \in C, g_\xi(n) = g_\chi(n).$$

We will now force again so that in the extension, a $\Sigma_2$ strengthening the previous sentence with parameters from $HC^V$ holds. Look at the following function $c : 2^\omega \rightarrow 2^\omega$: if $r \in 2^\omega$ codes a countable ordinal $\alpha$ (in the sense that it codes a relation whose transitive collapse is the $\in$-relation on $\alpha$, see 1.32), define $c(r)$ to be some real that codes the countable ordinal $\text{min}(C - \alpha)$. Now we force with $P_r$, the almost disjoint coding for $c$ (definition 1.15, p. 8). Let $R$ be generic for this forcing and work in $V[G][R]$.

Now we make the following observation: if $\alpha < \omega_1$ is such that $L_\alpha[f][R]$ is a model of $ZF^-$ and all ordinals below $\alpha$ are seen to be countable inside $L_\alpha[f][R]$, then $\alpha \in C$. To see this, let $\eta < \alpha$ be arbitrary. As $L_\alpha[f][R] \models \text{"}\eta \cong \omega\text{"}$, we can easily find $E \in 2^\omega \cap L_\alpha[f][R]$ coding $\eta$ (let $f$ be a bijection from $\omega$ to $\eta$ in that model and let $E$ code the pullback of the relation $\in$ on $\eta$ under that bijection). So as $E \circ R \in L_\alpha[f][R]$ and $L_\alpha[f][R] \models ZF^-$ (in fact
$\Delta_1$-Replacement would suffice), $c(E) = tcoll((\omega, E \cap R)) \in L_\omega[f][R]$. This shows that $C$ is unbounded below $\alpha$, whence $\alpha \in C$.

Let $\theta(\alpha, f, R)$ denote the following:

$$L_\alpha[f][R] \models \text{"ZF} \land \forall \alpha \in On \alpha \cong \omega".$$ 

We have added a real $R$ with the following property:

$$\forall \alpha, \beta < \omega_1 \ ( \theta(\alpha, f, R) \land \theta(\beta, f, R) ) \Rightarrow g_\alpha(n) = g_\beta(n) \tag{8}$$

What is the complexity of this statement? Remember that the $g_\xi$ had a very simple, absolute definition. In fact, whenever $L_\eta[f] \models \text{"} \xi \cong \omega \text{"}$ and $L_\eta[f]$ is a model of, say $\Delta_1$-Replacement (which makes $\prec_L$ definable inside $L_\eta[f]$), $g_\xi \in L_\eta[f]$ and is definable there. So if $\alpha, \beta$ are countable in $L_\eta[f]$, a model of at least $\Delta_1$-Replacement, $L_\eta[f] \models \text{"} g_\alpha(n) = g_\beta(n) \text{"}$ is equivalent to a statement that is $\Delta_1$ in the parameters $\alpha, \beta, n$ and $f$. So (8) can be written as:

$$\forall \alpha, \beta, \gamma < \omega_1 \ [\theta(\alpha, f, R) \land \theta(\beta, f, R) \land \theta(\gamma, f, R) \land \gamma > \alpha, \beta ] \Rightarrow L_\gamma[f] \models \text{"} g_\alpha(n) = g_\beta(n) \text{"}$$

Denote this statement by $\Psi(R, n, f)$. By the arguments of the preceding paragraph this statement is $\Pi_1$ in $R, n$ and $f$ over $HC$, and $\exists R \Psi(R, n, f)$ is $\Sigma_2(f, n)$ over $HC$. Take $r_f \in 2^\omega$ coding $f$; we can find an equivalent $\Sigma_3(r_f)$ sentence $\exists r \Psi_n(r_f)$. As $V[G][R] \models \exists r \Psi_n(r_f)$ (where $r_f \in V$) and $G$ and $R$ where obtained by a finite iteration of $\omega_1$-preserving forcings, the absoluteness assumption yields $V \models \exists r \Psi_n(r_f)$, and hence $V \models \exists R \Psi(R, n, f)$.

We work in $V$ again. We have shown, that for all $n \in \omega$, there exists a witness $R_n$ to (8). So let, for each $n \in \omega$,

$$C_n := \{ \alpha < \kappa \mid L_\alpha[f][R_n] \prec L_\omega[f][R_n] \}$$

By corollary 1.4, these are $\text{cub}$ subsets of $\omega_1$. For any $n \in \omega$ and $\alpha, \beta \in C_n$, $\theta(\alpha, f, R_n)$ holds (by elementarity). So, as (8) holds for each $R_n$, we have

$$\forall n \in \omega \ \forall \alpha, \beta \in C_n \ g_\alpha(n) = g_\beta(n).$$

As $\bigcap_{n \in \omega} C_n$ is $\text{cub}$, we can choose distinct $\alpha, \beta \in \bigcap_{n \in \omega} C_n$. Now as promised, $g_\alpha = g_\beta$, a contradiction.

**3.2 Corollary.** Assume $\Sigma^1_3$-absoluteness for the class of all $\omega_1$-preserving set forcings. Then, $\omega_1$ is inaccessible to reals, i.e. for $\kappa = \omega_1^V$ and any $r \in 2^\omega$, $L[r] \models \text{"} \kappa \text{ is inaccessible} \text{"}.$

**Proof.** Start with an $r$ s.t. $L[r] \models \kappa = \lambda^+$. Repeat the argument above, dragging along $r$ as a parameter, leading to the same contradiction. \qed
4 Coding and reshaping when $\omega_1$ is inaccessible to reals

In this section we give proofs of:

1. [Fri] $\Sigma^1_3$ absoluteness for $\omega_1$-preserving forcing is equiconsistent with the existence of a reflecting cardinal.

2. [Fri] “$\Sigma^1_3$ absoluteness for proper forcing holds and $\omega_1$ is inaccessible to reals” is equiconsistent with a reflecting cardinal.

How to obtain a model for $\Sigma^1_3$ absoluteness for all set forcings from a reflecting has already been shown. It remains to show how to obtain large cardinal strength from the requisite absoluteness assumption. The first claim is almost implicit in [FB01]. In [Sch00a] it is shown that the forcing used in [Fri] to prove the first claim is in fact stationary preserving, leading to the intermediate result for the class of stationary-preserving set forcing. The second claim builds on work in [Sch00b].

4.1 Definition. Let $\alpha < \beta$ be regular cardinals. We say $\beta$ is $\alpha$-reflecting, $\iff$ for any formula $\phi$ with parameters in $H_\alpha$, if there is a cardinal $\lambda$ s.t. $H_\lambda \models \phi$, then there is such a $\lambda$ below $\beta$.

In an earlier attempt to obtain large cardinal consistency strength from $\Sigma^1_3$-absoluteness for $\omega_1$-preserving set forcing, it was proved it implies that $\omega_2^V$ is $\omega_1^V$-reflecting in $L$. But this is consistency-wise not stronger than $\text{ZFC}$.

4.2 Fact. Let $\alpha$ be a cardinal, $\beta$ a regular cardinal $\geq \alpha$. Then $\text{Con}(\text{ZFC} + (\gamma = \beta^+ \Rightarrow L \models \text{“}\gamma \text{ is } \alpha\text{-reflecting”})) \iff \text{Con}(\text{ZFC})$

Proof. For each formula $\phi(x_1, \ldots, x_k)$ and each $\vec{p} = (p_1, \ldots, p_k) \in [L_\alpha]^{<\omega}$, choose $\lambda(\phi, \vec{p})$ s.t. $L_{\lambda(\phi, \vec{p})} \models \phi(p_1, \ldots, p_k)$, if such $\lambda$ exists. Let $\kappa$ be some regular cardinal greater than all the $\lambda(\phi, \vec{p})$. Assuming $\kappa > \beta^+$, force with $\text{Coll}(\beta, \{\kappa\})$, defined in 1.24, p. 11. By fact 1.25, $\text{Coll}(\beta, \{\kappa\})$ preserves $\alpha$ and $\kappa^+$ remains a cardinal. By construction, $\kappa^+$ is $\alpha$-reflecting in $L$, and this still holds in the extension, but there, $\kappa^+ = \beta^+$. \qed

4.1 Preserving $\omega_1$

4.3 Theorem. If $\Sigma^1_3$-absoluteness holds for $\omega_1$-preserving set forcing, $\omega_1^V$ is reflecting in $L$. 

26
Proof. Let $\phi$ be any statement with parameters $\vec{p}$ in $L_{\omega_1} = H_{\omega_1} \cap L = (H_{\omega_1}^V)^L$, and fix $\lambda$ s.t. $L_\lambda \models (H_\lambda)^L = L_\lambda$, as $\lambda$ is an $L$-cardinal. We shall show, using the absoluteness assumption, there is such a $\lambda$ below $\omega_2$. Let me first try to give a sketch of the problems encountered.

We will first collapse $\lambda$ and then we will force to get a real witnessing, in a sense, that there is a $\lambda$ below $\omega_2$ with the desired properties. If we manage to ensure the witnessing via a $\Pi^1_2$ property of the real, by our absoluteness assumption, such a witness will exist in the ground model. When we “decode” $\lambda$ from the real, we obtain $\lambda < \omega_1$ with the desired properties, and this makes crucial use of the previously observed fact that $\omega_1$ is inaccessible to reals.

The first difficulty arises in the “coding” process: of course, after collapsing $\lambda$ to $\omega_1$, we could code it by a set $A \subseteq \omega_1$, and code this by a real using almost disjoint forcing. But we want to be able to decode in the ground model. So we shall use an almost disjoint family recursively constructible from initial segments of $A$. Such a thing exists if we reshape $A$, using Jensen’s classical forcing from [BJW82], but to show the reshaping-forcing preserves $\omega_1$, we will need $A$ to code much more information, in fact, it will need to code all of $H_{\omega_2}$ of the model where $\lambda$ has been collapsed. This problem can be solved as we may assume $0^\sharp$ does not exist.

The second problem is to ensure our real will do its witnessing job via a $\Pi^1_2$-statement, which seems difficult, as it witnesses a property of uncountable sets. A variant of the reshaping-forcing will be able to deal with this. Now let’s get on with the proof.

Collapsing and coding

First of all, we may assume $0^\sharp$ does not exist, as otherwise, any uncountable $V$-cardinal would be rather large in $L$, by all means a true reflecting cardinal. For if $0^\sharp$ exists, for any uncountable cardinals $\alpha < \beta$, we have $L_\alpha < L_\beta$. Using reflection we can see $L_{\omega_1} < L$, which clearly implies $\omega_1$ is reflecting by Fact 1.39.

Take $\kappa$ a strong limit singular above $\lambda$, with uncountable cofinality. As $0^\sharp$ doesn’t exist, we are allowed to use the Covering Lemma (see Fact 1.36, p. 18), whence $2^\kappa = \kappa^+$, and $\kappa^+ = (\kappa^+)^L$, by Fact 1.37.

Now we force with $Col(\omega_1, \{\kappa\})$ (see definition 1.24), temporarily denoted by $Col$. We have $\kappa^\omega = \kappa$, so by fact 1.25, $Col$ has the $\kappa^+ - cc$, whence $\kappa^+ = \omega_2^{V_{Col}} = (\kappa^+)^L$ (so $\kappa$ is just an ordinal between $\omega_1$ and $\omega_2$ in the extension). Moreover, as $Col$ is $\sigma$-closed, $P(\omega)^{V_{Col}} = P(\omega)^V$ and has size less than $\kappa$, so $Col$ forces the continuum hypothesis. Until further notice, we work in $V_{Col}$.

Any $x \in H_{\omega_2}$ can be coded by a well founded relation $E_x$ on $\omega_1$, so we
can find a set $A \subseteq \omega_2$ s.t. $H_{\omega_2} \subseteq L[A]$, as there are only $\omega_2$ subsets of $\omega_1$. Let $S \subseteq \omega_1$. Then $S$ has a name $\dot{S} \in V$, of the form $\bigcup_{\chi < \omega_1} \{ \dot{\chi} \} \times D_\chi$, where each $D_\chi \subseteq Col$. But in $V$, there are at most $\mathcal{P}(Col)^{\omega_1} \approx 2^{\kappa \omega_1} \approx \kappa^+$ such names. So in $V^{Col}$, $2^{\omega_1} = \kappa^+ = \omega_2$. Actually, $H_{\omega_2} \subseteq L_{\omega_2}[A]$. Let $G \subseteq \omega_1$ code a surjection from $\omega_1$ onto $\kappa$. As $\kappa^+ = (\kappa^+)^L$, $L[G] \models \omega_1^+ = \kappa^+$. So we can choose an almost disjoint family $\mathcal{A}$ on $\omega_1$ of size $\kappa^+$, $\mathcal{A} \in L[G]$. Now we force with the almost disjoint coding $P_{A,A}$ (see section 1.2, p. 6). $P_{A,A}$ is $\sigma$-closed. From now on we work in $W := V^{Col*P_{A,A}}$.

Take $A' \subseteq \omega_1$ such that it codes both $G$ and the generic for $P_{A,A}$. Then we have $A \in L[A']$, and so $(H_{\omega_2})^{V^{Col}} \subseteq L_{\omega_2}[A']$. Note that even $(H_{\omega_2})^W \subseteq L_{\omega_2}[A'] = (H_{\omega_2})^{L[A']}$: For $x \in H_{\omega_2}$, let $E_x \subseteq \omega_1$ code $\models \bigcap TC(\{x\})$. This $E_x$ has a “nice name” in $V^{Col}$, of the form $\dot{E}_x = \bigcup_{\chi < \omega_1} \{ \dot{\chi} \} \times A_\chi$, where each $A_\chi$ is an antichain. $P_{A,A}$ has the $\omega_2$-cc, as the set of first components of conditions has size $\omega_1^{<\omega_1} = \omega_1$. Thus $\dot{E}_x \in (H_{\omega_2})^{V^{Col}} \subseteq L[A']$, and so $E_x = \dot{E}_x[A'] \in L[A']$. Also note that by $\sigma$-closedness of $P_{A,A}$, CH still holds in $W$.

**Reshaping**

Now that we have found a sufficiently smart $A' \subseteq \omega_1$, we can set to the task of coding it into a real $R$, again using almost disjoint coding. But when we decode in the ground model $V$, we can only rely on decoding $A' \cap (\omega_1)^L$, having no larger a.d. family easily accessible from both models. It would help if we had $L[A' \cap \xi] \models \xi \cong \omega$, for all $\xi < \omega_1$. This property of $A'$ is called “reshaped”. We use the following forcing:

$$P := \{ s \subseteq \omega_1 \mid (i) \quad \forall \xi \in \text{lim } \forall n, \text{ if } \xi + 2(n + 1) \leq \sup(s), \text{ then } \\
\xi + 2(n + 1) \in s \iff \xi + n \in A' \\
(ii) \quad \forall \xi \leq \sup(s) \quad L[s \cap \xi] \models \text{“} \xi \cong \omega \text{“} \}$$

ordered by end extension. Let’s show this forcing achieves what is promised. For each $\alpha < \omega_1$, the set $D_\alpha := \{ s \in P \mid \alpha \leq \sup(s) \}$ is dense in $P$. Let $p_0$ be a condition with supremum $\delta$, and let $\alpha < \omega_1$. Let $E \subseteq \{ \delta + 2n + 1 \mid n \in \omega \}$ code the epsilon relation on $\alpha$. Now let $p := p_0 \cup E \cup \{ \xi + 2(n + 1) \mid \xi + n \in A \} \cap \alpha + 1$. This is a condition, hits $D_\alpha$ and extends $p_0$. By this density argument, we know that the generic for $P$ will be unbounded in $\omega_1$. Observe that the definition has been set up so that the following, later to be used fact holds:

**4.4 Corollary.** For all $p \in P$ and for any $\alpha < \omega_1$, there is $q \leq p$ with $\alpha \leq \sup(q)$ and $(q - p) \cap \text{Lim} = \emptyset$

**4.5 Lemma.** The Reshaping forcing $P$ is $\omega_1$-distributive.
Proof of lemma. Now we shall use the fact that $A'$ codes all of $H_{\omega_2}$. Let $(D_n)_{n \in \omega}$ be a sequence of dense subsets of $P$, and let $p_0$ be a condition. We will find $p_\omega \in \bigcap_{n \in \omega} D_n$ extending $p_0$. First of all, observe $\{P, (D_n)_{n \in \omega}, p_0\} \subseteq H_{\omega_2} \subseteq L_{\omega_2}[A']$. Let $\mathcal{N}$ denote $\langle L_{\omega_2}[A'], P, (D_n)_{n \in \omega}, p_0 \rangle$ for now. Inductively build a countable chain of elementary submodels:

$$
M_0 := <L[A']>-\text{least countable s.t. } M_0 \prec \mathcal{N}
$$

$$
M_{n+1} := <L[A']>-\text{least countable s.t. } M_{n+1} \prec \mathcal{N}, \text{ and } M_n \in M_{n+1}. \quad (9)
$$

$$
M_\omega := \bigcup_{n \in \omega} M_n \text{ (whence } M_\omega \prec \mathcal{N})
$$

Now, for $i \in \omega+1$, let $\pi_i$ be the inverse of the map collapsing $M_i$ to a transitive set. The collapse of $M_i$ is of the form $\langle L_{\delta_i}[A' \cap \beta_i], P^i, (D^n_i)_{n \in \omega}, p'^i_0 \rangle$, where $\beta_i, \delta_i < \omega_1$, $\{P^i, (D^n_i)_{n \in \omega}, p'^i_0\} \subseteq L_{\delta_i}[A' \cap \beta_i]$ and each of these predicates is mapped by $\pi$ to its corresponding set in $\mathcal{N}$, e.g. $\pi_i(P^i) = P$. Using elementarity, we get that $\pi_i(\beta_i) = \omega_1$, and $\beta_i$ is the least ordinal moved by $\pi_i$. So as $p'^i_0 \subseteq \beta_i$, $p'^i_0 = p_0$ (using the fact that $\pi$ is the identity on $\beta_i$ and elementarity). See the figure for an overview of the situation.

![Figure 1: Traces of a countable chain of elementary submodels](image)

As $\text{ran}\pi_n \subseteq \text{ran}\pi_\omega$,

$$
\langle L_{\delta_n}[A' \cap \beta_n], P^n, (D^m_n)_{m \in \omega}, p^n_0 \rangle \xrightarrow{\pi_\omega^{-1} \circ \pi_n} \langle L_{\delta_\omega}[A' \cap \beta_\omega], P^\omega, (D^m_\omega)_{m \in \omega}, p^\omega_0 \rangle
$$

(10)

for each $n \in \omega$. Call the model on the right hand side of (10) $\mathcal{N}'$, and let $M'_n := \text{ran}(\pi_n \circ \pi_\omega^{-1})$. Then we can write (10) as $M'_n \prec \mathcal{N}'$. Now we can see, using elementarity and absoluteness of $<L[A']$, the output of the definition (9), with $\mathcal{N}$ replaced by $\mathcal{N}'$, is just the chain $(M'_n)_{n \in \omega}$. Thus, the sequences $(\delta_n)_{n \in \omega}$, $(\beta_n)_{n \in \omega}$ are elements of $L_{\omega_2}[A' \cap \beta_\omega]$. 29
Now choose $p_{n+1} \in M_{n+1} \cap D_n$ inductively for all $n \in \omega$, so that $p_{n+1} \leq p_n$ and $\sup(p_{n+1}) > \beta_n = \omega_1 \cap M_n$ (this is possible as $\beta_n$ is countable in $M_{n+1}$). Let
\[ p_\omega := \bigcup_{n \in \omega} p_n. \]
These $p_i$ are not moved by $\pi_j$, for $j \geq i$ in $\omega + 1$. The only thing left to check is that $p_\omega$ is a condition. The part about coding $A'$ is clear. Let $\xi < \sup(p_\omega)$. Then $L[p_\omega \cap \xi] = L[p_n \cap \xi]$, for some $n$, so in this case the desired property holds as $p_n$ is a condition.

Now let $\xi = \sup(p_\omega)$. Then, by construction of $p_\omega$, $\xi = \beta_\omega$. So $L[p_\omega] \supseteq L[A' \cap \beta_\omega]$, and the latter model knows that $\beta_\omega$ is the limit of a countable sequence of countable ordinals, namely $(\beta_n)_{n \in \omega}$, hence $\beta_\omega$ is countable in $L[p_\omega]$.

From now on, we shall work in $W^P$, containing $A''$, the $P$-generic, which is reshaped. Of course $A'$ and thus $(H_{\omega_2})^W$ are constructible from $A''$. By exactly the same argument as for the almost disjoint coding, again $(H_{\omega_2})^{W^P} \subseteq L_{\omega_2}[A'']$, since $P$ consists of hereditarily countable sets while $CH$ holds.

**Killing universes**

We are now almost ready for coding $A''$ into a real. But first we must tackle the second problem mentioned at the beginning. Among other things, we have achieved that
\[ L_{\omega_2}[A''] \models " \exists \lambda \in Card \text{ s.t. } L_\lambda \models \phi(\vec{p})" \]  
(11)
Let " $\exists \lambda \in Card$ s.t. $L_\lambda \models \phi(\vec{p})$" be denoted by $\psi(\vec{p})$. We can w.l.o.g. assume that any transitive $ZF^-$ model containing $A''$ will believe $\psi(\vec{p})$ (by changing $A''$ such that part of it codes the epsilon relation on $\lambda$). Then we have:

for all transitive $M$, $(M \models ZF^- \land \{\omega_1, A''\} \subseteq M) \Rightarrow M \models \psi(\vec{p})$  
(12)
To motivate our next step, let me jump ahead: in order to apply our absoluteness assumption, we shall use a statement very much like the one above, but only mentioning reals. To eventually formulate such a statement, it would be nice if we didn’t have to quantify over all $M$, but only over members of $HC$. In fact we can, by an $\omega_1$ preserving forcing, make a stronger statement than (12) true, where quantification is over $HC$. We will use the following
forcing notion:

\[ P' := \{ p \subseteq \omega_1 \mid \forall \xi \in \text{lim} \forall n, \text{if } \xi + 2(n + 1) \leq \sup(s), \text{ then } \xi + 2(n + 1) \in s \iff \xi + n \in A' \} \]

\[ (ii) \quad \forall M \in HC \]

\[ \text{if } M \text{ is a transitive model of } \"ZF^- + } \exists \omega_1 \" , \]

\[ p \cap \omega_1^M \in M \text{ and } \sup(p) \geq \omega_1^M, \text{ then } M \models \psi(\vec{p}) \}, \]

ordered by end-extension.

Let’s show that the generic for \( P' \) is unbounded in \( \omega_1 \), i.e., that for each \( \alpha \in \omega_1 \), the set \( D_\alpha := \{ s \in P' \mid \alpha \leq \sup(s) \} \) is dense in \( P' \). But this works just like for the Reshaping: Given \( p_0 \) and \( \alpha \), we just append a “short” set \( E \) coding \( \alpha \) to obtain \( p_1 \). Any countable transitive \( ZF^- \)-model \( M \) with \( p_1 \in M \) that has an \( \omega_1 > sup(p_0) \) must contain \( E \), and thus everything up to and including \( \alpha \) is countable in \( M \). We can see that we’re in a way reshaping \( A'' \) again to make ordinals that are \( \omega_1 \) in models containing initial segments of \( A'' \) and believing the wrong things (that is, \( \neg \psi(\vec{p}) \)) increasingly sparse.

Again, we would like to call the following fact to your attention, as we shall re-use it in the next section:

4.6 Corollary. For all \( p \in P' \), and for any \( \alpha < \omega_1 \), there is \( q \leq p \) with \( \alpha \leq \sup(q) \) and \( (q - p) \cap \text{Lim} = \emptyset \)

Now assume \( B \) is the generic for this forcing notion. Then the following improvement of (12) holds:

\[ \text{for all transitive } M \text{ in } HC, \]

\[ (M \models \text{ "ZF^- + } \exists \omega_1 \" \land B \cap \omega_1^M \in M) \Rightarrow M \models \psi(\vec{p}) \]  \( \quad (13) \)

We still have to show that \( P' \) is \( \omega_1 \)-preserving, but the proof is very similar to that of lemma 4.5.

4.7 Lemma. The Killing Universes-forcing \( P' \) is \( \omega_1 \)-distributive.

Proof of lemma. Again, let \( (D_n)_{n \in \omega} \) be a sequence of dense subsets, \( p_0 \in P' \). Construct \( (\delta_i)_{i \in \omega+1}, (\beta_i)_{i \in \omega+1} \) and \( (p_i)_{n \in \omega+1} \) just like in the proof of lemma 4.5, with \( A' \) replaced by \( A'' \) and \( P \) by \( P' \). We show \( p_\omega \) is a condition. We only prove part (ii).

Let \( M \cong \omega, \text{transitive, } M \models \text{ "ZF^- + } \exists \omega_1 \" \). As before, if \( \omega_1^M < sup(p_\omega) = \beta_\omega \), (ii) clearly holds as then for some \( n, \omega_1^M < sup(p_n) \) and \( p_n \) is a condition. What if \( \omega_1^M = sup(p_\omega) = \beta_\omega \)? By (11) and by elementarity,

\[ L_{\delta_\omega}[A'' \cap \beta_\omega] \models \text{ " } \exists \lambda \in \text{Card}^L \ L_\lambda \models \phi \text{" } \]  \( \quad (14) \)
Moreover, we have asked that \( A'' \) code \( \lambda \) in some simple absolute way, so the same is true of \( A'' \cap \beta_\omega \) and \( \lambda_\omega \), whence \( \lambda_\omega \in M \). It only remains to show that \( M \models \lambda_\omega \in \text{Card}^L \). By (14), it remains to show that \( M \cap O_n \leq \delta_\omega \). Suppose not. Using elementarity we see the images of \((D_n)_{n \in \omega}, p_\omega \) and \( P' \) under \( \pi_\omega \) are all constructible from \( A'' \cap \beta_\omega \) by stage \( \delta_\omega \in M \), and thus all the ingredients needed to define \( (\delta_n)_{n \in \omega}, (\beta_n)_{n \in \omega} \) are present in \( M \). But then, once more, \( M \models \beta_\omega \sim \omega \), contradiction. So \( M \cap O_n \leq \delta_\omega \). This shows (ii) holds for \( p_\omega \) and completes the proof of the lemma.

\[
\text{Pulling back by a real}
\]

From now on we work in \( W_{P^*P'} \) where \( B \subseteq \omega_1 \) is generic for \( P' \). \( B \) is still reshaped since \( A'' \) can be recovered via a simple mapping. Now we use the reshapedness to construct an a.d.-family \( B \): Start with the \( <_L \)-least real belonging to an a.d. family in \( L \). Having constructed \( (b_\xi)_{\xi \in \alpha} \in L[A'' \cap \alpha] \), we take

\[
b_\alpha := \text{the } <_{L[B \cap \alpha]} \text{-least set a.d. from } (b_\xi)_{\xi \in \alpha} \text{ in } L[B \cap \alpha],
\]

which exists as \( L[B \cap \alpha] \models \alpha \equiv \omega \). Let \( \mathcal{B} := (b_\xi)_{\xi \in \omega_1} \).

Now we force with \( P_{B,B} \). Call the generic \( R \) and let us from now on work in \( W' := W_{P^*P^*P_{B,B}} \). This real \( R \) will be the witness to the fact that a \( \lambda \) with the desired properties exists. But in order to prove a real exists in the ground model with the same witnessing capacity, we have to express this as a \( \Pi^2_1 \) property of \( R \). If \( \alpha < \omega_1 \) and \( L_\alpha[R] \models "ZF^- + \exists \omega_1" \), we can decode \( B \cap \delta \) in that model, so by (13), the following holds of \( R \):

\[
\forall \alpha < \omega_1 \quad L_\alpha[R] \models "ZF^- + \exists \omega_1" \quad \Rightarrow \quad L_\alpha[R] \models \psi(\vec{p}) \tag{15}
\]

The formula above is clearly \( \Pi^1_1 \) over \( HC \), so by Fact 1.33, it is equivalent to a \( \Pi^1_1(R, r_p) \)-formula \( \Phi(R, r_p) \), for some real \( r_p \) in the ground model \( V \).

So finally, after forcing with a finite iteration of \( \omega_1 \)-preserving forcing notions, there exists a real \( R \) satisfying (15). By \( \Sigma^1_3 \)-absoluteness, there is a real \( R_0 \) with the same property in \( V \). Now we work in \( V \) again. By corollary 3.2, \( \delta := \omega_2^{L[R_0]} < \omega_1^V \). \( L_\delta[R_0] \models "ZF^- + \exists \omega_1" \). But then (15) implies that

\[
L_\delta[R_0] \models \psi(\vec{p})
\]

i.e. \( \exists \lambda \in \text{Card}^L \cap \omega_2 \text{ s.t. } L_\lambda \models \phi(\vec{p}) \). \( \square \)
4.2 Preserving stationary subsets of \( \omega_1 \)

All the forcings in the previous proof where ccc or \( \sigma \)-closed, except for Reshaping and Killing Universes. These were shown to be \( \omega_1 \)-closed. In fact, in [Sch00a] it is shown they preserve stationary subsets of \( \omega \).

4.8 Lemma. Killing Universes and Reshaping are stationary-preserving.

Proof. We shall be working in a context where there is \( A \subseteq \omega_1 \) s.t. \( H_{\omega_2} \subseteq L_{\omega_2}[A] \) and \( CH \) holds. We start with the Reshaping forcing \( P \). W.l.o.g. we can assume that for all \( \alpha \in \text{lim } \omega_1 \), \( L[A \cap \alpha] \models \forall \alpha \geq \alpha \), by padding \( A \) with sets \( E \subseteq [\xi, \xi + \omega) \) which collapse \( \xi \). Let \( S \subseteq \omega_1 \), stationary, and let \( q \in P, \hat{C} \) a \( P \)-name such that \( q \models \text{“} \hat{C} \text{ is cub in } \omega_1 \text{”} \). We need to find a condition \( p \leq q \) s.t. \( p \models \text{“} \hat{C} \cap [\hat{S}] \neq \emptyset \text{”} \).

By replacing \( \hat{C} \) with a “nice name” if necessary, we can assume \( \hat{C} \in L_{\omega_2}[A] \). Let \( N \) denote \( \langle L_{\omega_2}[A], \hat{C}, S, P, q \rangle \). Consider the cub set

\[
C_0 := \{ \alpha < \omega_1 \mid h^N_{\Sigma_\omega}(\alpha) \cap \omega_1 = \alpha \}
\]

Let \( \alpha \in C_0 \cap S \), s.t. \( \alpha > \sup(q) \) and set \( M := h^N_{\Sigma_\omega}(\alpha) \). There is \( \pi : L_\beta[A \cap \alpha] \overset{\sim}{\rightarrow} M \prec L_{\omega_2}[A] \), where \( \pi(\alpha) = \omega_1 \) and \( \alpha \) is the least ordinal moved. We should mention that (as \( P \) is definable and by elementarity) \( \pi^{-1}(P) = P_{L_\beta[A \cap \alpha]} = P \cap L_\alpha[A \cap \alpha] \) and that \( \pi \) is the identity on that set. Observe that \( \langle q \models \pi^{-1}(\hat{C}) \text{ is cub in } \omega_1 \rangle^{L_\beta[A \cap \alpha]} \), by elementarity. Thus (by taking preimages of the corresponding set under \( \pi \)),

\[
\text{for all } \gamma < \alpha, \text{ the set } D_\gamma := \{ p \in \pi^{-1}(P) \mid \exists \xi \in (\gamma, \alpha) \text{ s.t. } p \models \xi \in \hat{C} \}
\]

is dense in \( \pi^{-1}(P) \).

We inductively build a decreasing chain \( (p_i)_{i \in \omega} \), picking conditions from \( P \cap L_\alpha[A \cap \alpha] \), such that \( p_\infty = \bigcup_{i \in \omega} p_i \models \text{“} \alpha \in \hat{C} \text{”} \). We start with \( p_0 = q \) (observe \( q \in L_\alpha[A \cap \alpha] \)).

First, we consider the simple case where \( \alpha \) happens to be countable in \( L[A \cap \alpha] \): we just choose each \( p_{n+1} \) end-extending \( p_n \), such that \( p_{n+1} \models \xi \in \hat{C} \) for some \( \xi \in (\sup(p_n), \alpha) \) (which is possible by (16)), ensuring that \( \bigcup_{n \in \omega} \sup(p_n) = \alpha \). It is easy to see that \( p_\infty \) is a condition, as the required property was assumed to magically hold. Clearly \( p_\infty \models \text{“} \hat{C} \text{ is unbounded in } \alpha \text{”} \), whence \( p_\infty \models \text{“} \alpha \in \hat{C} \text{”} \).

If \( \alpha = \omega_1^{L[A \cap \alpha]} \), we have to choose our sequence of conditions more care-
fully. Using (16), the set
\[ C_1 := \{ \eta < \alpha \mid p_0 \in L_\eta[A \cap \eta] \quad \text{and} \quad \forall p \in L_\eta[A \cap \eta] \cap P \exists q \in L_\eta[A \cap \eta] \cap P \text{ s.t. } q \leq p \quad \text{and} \quad \exists \xi \in (\text{sup}(p), \alpha) \quad q \Vdash \xi \in \check{C} \} \]
is easily seen to be cub in \( \alpha \), and \( C_1 \in L_\beta[A \cap \alpha] \). Moreover, by thinning out, we can assume that \( C_1 \subseteq \text{lim} \) and

for all \( \gamma \in C_1 \), if \( \eta \) is least in \( C_1 \) above \( \gamma \), then \( L_\eta[A \cap \eta] \vDash \gamma \equiv \omega \).  \( 17 \)

Now we pick \( \{ \beta_i \mid i \in \omega \} \), cofinal in \( C_1 \) with order type \( \omega \). Assume we have already chosen \( p_n \in L_{\beta_n}[^A \cap \beta_n] \). Let \( \eta \) be the minimum of \( C_1 - (\beta_n + 1) \).
By corollary 4.4 and (17), we can choose \( p_n' \preceq p_n \) in \( L_\eta[A \cap \eta] \) such that

\[ (p_n' - p_n) \cap \text{Lim} = \{ \beta_n \}. \quad 18 \]

Now extend \( p_n' \) to \( p_{n+1} \in L_\eta[A \cap \eta] \) so that for some \( \xi > \text{sup}(p_n) \), \( p_n' \Vdash \xi \notin \check{C} \). Observe that

\[ (p_{n+1} - p_n') \cap C_1 = \emptyset. \quad 19 \]

Clearly, \( p_{n+1} \in L_{\beta_{n+1}}[^A \cap \beta_{n+1}] \), as \( \eta \leq \beta_{n+1} \). Having built this sequence, if we can show \( p_\infty \) is a condition, we are done: just like in the simpler construction, \( p_\infty \Vdash "\alpha \in \check{C}" \). By construction, \( \text{sup}(p_\infty) = \alpha \). To show \( p_\infty \in P \), we concentrate on the non-trivial case and show \( L[p_\infty] = L[A \cap \alpha, p_\infty] \vDash \alpha \equiv \omega \).
Observe that by (19) and (18), \( C_1 \cap p_\infty \cap \text{Lim} = \{ \beta_n \mid n \in \omega \} \). But \( C_1 \in L_\beta[A \cap \alpha] \), and so \( \{ \beta_n \mid n \in \omega \} \in L_\beta[p_\infty] \).

The proof can be modified to show Killing Universes preserves stationary subsets of \( \omega_1 \), along the lines of 4.7. Having done the same construction as above for \( P' \), we need to make sure \( p_\infty \) is a condition. Again, we only need to check the case of a countable transitive model \( N \vDash "ZFC + \exists \omega_1" \) with \( \omega_1^N = \text{sup}(p_\infty) \). By the same arguments as in the proof of 4.7, there is \( \lambda \in N \) s.t. \( N \vDash \Phi \) and \( L_\beta[A \cap \alpha] = "\lambda \in \text{Card}" \), and we are done if \( N \cap \text{On} < \beta \). But this is clear as \( \alpha \) is collapsed in \( L_\beta[p_\infty] \).

4.9 Corollary. If \( \Sigma^1_3 \)-absoluteness holds for stationary-preserving set forcing and \( \omega_1 \) is inaccessible to reals, \( \omega_1^V \) is reflecting in \( L \).

Proof. We have just shown that all the forcings used in the proof of theorem 4.3 were stationary-preserving. At the end of the proof instead of appealing to the strength provided by absoluteness for \( \omega_1 \)-preserving forcings (corollary 3.2), we use the additional assumption that \( \omega_1 \) is inaccessible to reals.
4.3 Proper forcing

Remarkable cardinals were devised in [Sch00b]. They provide an exact measure for the consistency strength of “reshaping is proper”. We will use the fact that, assuming $\omega_1$ is not remarkable in $L$, Reshaping (and, as it turns out, also Killing Universes) are proper. For the proof, it is convenient to change the definition of Killing Universes a bit:

4.10 Definition. We define the Killing Universes for $A$ to be the following forcing notion:

$$K.U. = \{ p \subseteq \omega_1 | \text{ for all countable transitive } M, \text{ if }$$

$$M \models "ZF^- \land \exists \omega_1" \land \omega_1^M \leq \sup(p),$$

$$\land \{ A \cap \omega_1^M, p \cap \omega_1^M, \vec{p} \} \subseteq M$$

$$\land (L[A \cap \omega_1])^M \models \neg \exists \omega_2,$$

$$\text{then } M \models \exists \lambda \in \text{Card } L \lambda \models \psi(\vec{p}) \},$$

ordered by end-extension.

4.11 Lemma. For $A \subseteq \omega_1$ s.t. $L_{\omega_2}[A] = H_{\omega_2}$, if there is $\theta \in \text{Reg} \cap \omega_2$ such that (\kappa is not $\theta$-remarkable)$^L$, where $\kappa = \omega_1^V$, then Reshaping and Killing Universes for $A$ are proper.

Proof. We shall prove the lemma in two steps. In the first step (following [Sch00b]), we show that there is a cub subset $C$ of $[L_{\omega_2}[A]]^\omega$ such that for all $X \in C$, if $X = \text{ran}(\pi)$ for $\pi : L_\beta[A \cap \alpha] \rightarrow \Sigma_{\omega_2} L_{\omega_2}[A]$, then $\beta$ is not a regular cardinal in $L$.

Look at $C$ equal to the cub set of countable $X \prec \mathcal{N} := \langle L_{\omega_2}[A], A \rangle$ such that $\theta \in X$. Assume that $X \in C^\star$, $\pi$, $\alpha$ and $\beta \in Reg^L$ witness failure of what is claimed above. Set $\bar{\theta} := \pi^{-1}(\theta)$. Clearly, $\alpha = \text{crit}(\pi)$ and $\pi(\alpha) = \kappa$.

By elementarity of $\pi$ and relativizing formulas to $L$, we can see $\bar{\pi} := \pi \upharpoonright L_{\bar{\theta}}$ is an elementary embedding from $L_{\bar{\theta}}$ into $L_{\theta}$, and $\bar{\theta}$ is regular in $L$ (as this holds in $L_\beta$).

Now let $G$ be generic for $Coll(\omega, \kappa)$. Then (as this forcing is constructible, and in fact absolutely definable from $\kappa$), $G$ is also $Coll(\omega, \kappa)$-generic over $L$. As $L[G] \models \bar{\theta} \equiv \omega$, by fact 1.6, the following holds in $L[G]$:

$$\exists \bar{\pi}, \bar{\theta} \in \text{Reg}^L \quad \bar{\pi} : L_{\bar{\theta}} \rightarrow_{\Sigma_\omega} L_{\theta} \text{ s.t. } \bar{\pi}(\text{crit}(\bar{\pi})) = \kappa > \bar{\theta} \quad (20)$$

Let this statement be denoted by $\Phi(\kappa, \theta)$. By homogeneity of the gentle collapse of $\kappa$, we have ($\models_{Coll(\omega, \kappa)} \Phi(\kappa, \theta))^L$. Take a countable elementary submodel of $L_{\omega_2}$ in $L$: let $\gamma < \omega_1^L$ and $\sigma : L_\gamma \rightarrow_{\Sigma_\omega} L_{\omega_2}, \sigma \in L$. Let $\bar{\kappa}, \bar{\theta}$ denote $\sigma^{-1}(\kappa)$, $\sigma^{-1}(\theta)$, respectively. We can choose $\bar{G}$ in $L, Coll(\omega, \bar{\kappa})$-generic
over $L_\gamma$. By elementarity, $L_\gamma[G] \models \Phi(\vec{\kappa}, \vec{\theta})$. So there exists $\vec{\pi} \in L$ and $\vec{\theta}$ such that $\vec{\pi}$ is as in (20) and $\vec{\theta} \in \text{Reg}_{k\gamma}$. We have found elementary embeddings $\vec{\pi}, \sigma \in L$,

$$
\begin{align*}
L_\vec{\theta} & \xrightarrow{\vec{\pi}} L_{\vec{\theta}} \xrightarrow{\vec{\sigma}} L_\vec{\theta} \\
& \downarrow \quad \downarrow
\end{align*}
$$

where $\sigma(\vec{\kappa}) > \vec{\theta} \in \text{Reg}^{L_{\vec{\theta}}}$. So $M := L_{\vec{\theta}}, \sigma, N := L_{\vec{\theta}}$ and $\pi' := \sigma \circ \vec{\pi}$, witness that $\kappa$ is $\theta$-remarkable in $L$ (definition (1.42) on page 20). This completes the first part of the proof.

The second part is a version of the argument for properness of reshaping in [Sch00b], attributed there to [SS92]. It is similar to the arguments that show that reshaping and K.U. are $\omega_1$-distributive or stationary-preserving. The construction is the same for both forcing notions, so let $P$ denote either one of them. It suffices to show that there exists $C$, a cub subset of $[L_{\omega_2}[A]]^\omega$ such that $\forall N \in C \quad \forall p \in N \cap P \quad \exists q \leq p \quad \forall \dot{\alpha} \in N$, if $p \forces \dot{\alpha} \in On$, then $q \forces \dot{\alpha} \in N$. The countable elementary submodels of $\langle [L_{\omega_2}[A]]^\omega, P \rangle$ form a cub set, so by taking the intersection, we can assume $C^*$ consists of such models. Let $N \in C^*, p_0 \in N$ be arbitrary. We show there is an $N$-generic condition below $p_0$.

Let $\pi : L_\beta[A \cap \alpha] \xrightarrow{\pi}= N \prec L_{\omega_2}[A]^\omega$, where $P, p_0 \in N$. We know $\pi(\alpha) = \omega_1$, $\text{crit}(\pi) = \alpha$ and $\beta \notin \text{Reg}^{L[A \cap \alpha]}$, as $N \in C^*$. As there are no cardinals above $\alpha$ in $L_\beta[A \cap \alpha]$, $L[A \cap \alpha] \models \beta \equiv \alpha$. So let $(\dot{\alpha}_\xi)_{\xi<\alpha} \in L[A \cap \alpha]$ be an enumeration of all $\pi^{-1}(P)$-names $\dot{\alpha}$ in $L_\beta[A \cap \alpha]$ such that $L_\beta[A \cap \alpha] \models \forall \dot{\alpha} \in On$. Some easy consequences of elementarity: $(\pi(\dot{\alpha}_\xi))_{\xi<\alpha}$ enumerates all $\dot{\alpha} \in N$ such that $\forall P \dot{\alpha} \in On$. Observe that $\pi^{-1}(P) = L_\alpha[A \cap \alpha] \cap P$. Also, $L_\beta[A \cap \alpha] \models \forall \dot{\alpha} \in \text{On} \quad q \forces_{\pi^{-1}(P)} \dot{\alpha} = \lambda$ exactly if $q \forces_P \pi(\dot{\alpha}) = \pi(\lambda)$. Clearly,

$$
C_\xi := \{ \eta < \alpha \mid \forall p \in L_\eta[A \cap \eta] \cap P, p \leq p_0 \exists q \in L_\eta[A \cap \eta] \cap P, q \leq p \quad L_\beta[A \cap \alpha] \models \exists \lambda \quad q \forces_{\pi^{-1}(P)} \dot{\alpha}_\xi = \lambda \}.
$$

is cub in $\alpha$. Clearly, each $C_\xi \in L_\beta[A \cap \alpha]$. Thus, $C_0 := \bigtriangleup_{\xi<\alpha} C_\xi$ is cub and $C_0 \in L[A \cap \alpha]$. By thinning out $C_0$ if necessary, we can assume

$$
\text{for all } \gamma \in C_0, \text{ if } \eta \text{ is least in } C_0 \text{ above } \lambda, \text{ then } L_\eta[A \cap \eta] \models \gamma \equiv \omega.
$$

Now we build a countable chain of conditions, starting with $p_0$ and working in $L_{\omega_2}[A]$, but picking conditions from $L_\beta[A \cap \alpha]$. For this purpose, choose a
bijection $f : \omega \to \alpha$, and choose \{\beta_j \mid j \in \omega\}, a cofinal subset of $C_0$ of order type $\omega$.

Assume we have chosen $p_n \in L_{\beta_n}[A \cap \beta_n]$. Let $\eta$ be least in $C_0 - (\beta_n + 1)$. By Corollary 4.4 (or 4.6 for K.U.) together with (22), we can pick $p'_n \leq p_n$ in $L_\eta[A \cap \eta]$ such that

$$(p'_n - p_n) \cap Lim = \{\beta_n\}. \quad (23)$$

Now we extend $p'_n$ to $p_{n+1}$, working in $L_\eta[A \cap \eta]$, such that for all $k$ satisfying both $k \leq n$ and $f(k) < \eta$, $p_{n+1} \Vdash_P \pi(\dot{\alpha}_f(k)) = \dot{\lambda}$ for some $\lambda \in N$. This is possible as $\eta \in \bigcap_{\xi < \eta} C_\xi$ and by (21). Of course $p_{n+1} \in L_{\beta_{n+1}}[A \cap \beta_{n+1}]$ (as $\eta \leq \beta_{n+1}$). Observe that

$$(p_{n+1} - p_n) \cap C_0 = \emptyset. \quad (24)$$

Now we finish the construction by setting $p_\infty : = \bigcup_{n \in \omega} p_n$. We claim $p_\infty$ is an $N$-generic condition of $P$.

If $p_\infty$ is a condition at all, it is $N$-generic: for any $\xi < \alpha$ and $m$ such that both $\xi \in f''m$ and $\xi \leq \beta_m$, $p_m \Vdash_{\pi^{-1}(p)} \dot{\alpha}_\xi = \dot{\lambda}$ for some $\lambda \in L_\beta[A]$. Hence $p_m \Vdash \pi(\dot{\alpha}_\xi) = \dot{\lambda}$ for some $\lambda \in N$, and $(\pi(\dot{\alpha}_\xi))_{\xi < \alpha}$ enumerates all ordinal-names that are elements of $N$.

To check that $p_\infty$ is a condition, we must treat Killing Universes and Reshaping separately.

*Case 1: $P = \text{Reshaping}.$*

As usual, we restrict our attention to the non-trivial case and show $L[A \cap \alpha] \succeq \alpha \cong \omega$. By (24) and (23), for each $n \in \omega$,

$$(p_{n+1} - p_n) \cap C_0 \cap Lim = \{\beta_n\},$$

and thus

$$p_\infty \cap C_0 = \{\beta_n \mid n \in \omega\}. $$

As $L[A \cap \alpha] \succeq \alpha \cong \omega$, this model contains the enumeration of the cub sets $C_\xi$ of (21), whence $C_0 \subseteq L[A \cap \alpha]$. So $L[A \cap \alpha, p_\infty] \succeq \alpha \cong \omega$.

*Case 2: $P = \text{Killing Universes}.$*

We need to show: For all countable transitive $M$ which are models of $\models \exists \omega_1 \land L[A \cap \omega_1] \models \neg \exists \omega_2$, if $\omega_1^M \leq \sup(p_\infty)$ and \{\(A \cap \omega_1^M, p_\infty \cap \omega_1^M, \bar{p}\)\} $\subseteq M$, then $M \models \exists \lambda \in Card^L L_\lambda \models P(\bar{p})$. Again, we only consider the case $\omega_1^M = \sup(p_\infty) = \alpha$. As there exists an elementary embedding from $L_\beta[A \cap \alpha]$ into $L_{\omega_1}[A]$, the same arguments as in the proof of lemma 4.7 show there exists $\lambda \in M$ with $L_\lambda \models P(\bar{p})$, and also $\lambda \in Card^{L_\beta[A \cap \alpha]}$. Again we need to show $L_{M \cap On} \models \lambda \in Card$. It suffices to show $M \cap On \leq \beta$.

Assume otherwise. As $L_{M \cap On}[A \cap \alpha] \models \neg \exists \omega_2$, $M \cap On$ must be large enough so that $L_{M \cap On}[A \cap \alpha] \equiv \beta \equiv \alpha$; but then, by the same argument as in case 1, $C_0 \subseteq M$ and $M \equiv \alpha \cong \omega$, a contradiction. \qed
4.12 Corollary. [Fri] If $\Sigma^1_3$-absoluteness holds for stationary-preserving set forcing and $\omega_1$ is inaccessible to reals, $\omega_1^V$ is reflecting in $L$.

Proof. If $\omega_1$ is remarkable in $L$, we are done by fact 1.43. By fact 1.45, we can assume $0^\sharp$ does not exist. The rest is like the proof of theorem 4.3, but when you pick a strong limit singular $\kappa$ to collapse, at the beginning, make sure there is a $\theta < \kappa$ witnessing that $\omega_1$ is not remarkable in $L$ below $\kappa$. Then all the forcing notions used in the proof are proper. In the last step, we need to use the additional assumption that $\omega_1$ is inaccessible to reals. \qed
5 Forcing with the countable chain condition

5.1 Lightface $\Sigma^1_2$-indescribable cardinals

Consider a language with variables of type $k$ for each $k \in \omega$. We say $\Phi(X_0, \ldots X_k)$ is a $\Sigma^m_n$-formula if $\Phi$ starts with a block of quantifiers over variables of type $m$ or less, with $n$ changes of quantifier, the first quantifier being $\exists$, and only quantification over variables of type strictly less than $m$ occurs after that. $\Pi^m_n$ means negation of $\Sigma^m_n$. We define satisfaction for such higher order formulas by induction on the order $m$: for a $\Sigma^m_k$ formula $\Phi$, 

$$\langle M, X_0, \ldots, X_k \rangle \models \exists Y_0 \ldots Q Y_r \Phi(Y_0, \ldots, Y_r, X_0, \ldots, X_r)$$

(where $Q$ denotes $\exists$ or $\forall$) exactly if 

$$\exists Y_0 \in \mathcal{P}^m(M) \ldots Q Y_r \in \mathcal{P}^m(M)$$

s.t. $\langle M, X_0, \ldots, X_k, Y_0, \ldots, Y_r \rangle \models \Phi(Y_0, \ldots, Y_r, X_0, \ldots, X_r)$

($\mathcal{P}^m(M)$ denotes the $m$-th application of the power set operation); satisfaction for $\Sigma^0_n$-formulas is just normal first-order satisfaction (over a structure with additional higher order predicates) extended to atomic formulas $X \in Y$, $X = Y$ over the higher-order predicates so that they hold just if their obvious interpretations hold.

We shall use this definition for the case $m = 1$; variables of type 1 shall be distinguished from those of type 1 by using upper-case for the former, lower-case for the latter. For the structures we consider, the usual translation between sets and sets of ordinals goes through, so we shall assume that second order variables range over sets of ordinals only.

We shall also make essential use of $\Sigma_1$-definable Skolem functions for $L$ (see [Dev84, II, 6]).

5.1 Definition. We say $\kappa$ has the $\Sigma^1_2$ reflection property if whenever $V_\kappa \models \exists X \forall Y \Phi(X, Y, p)$ for $p \in V_\kappa$ (and first-order $\Phi$), then there is $\xi < \kappa$ such that $V_\xi \models \exists X \forall Y \Phi(X, Y, p)$. We say $\kappa$ is (lightface) $\Sigma^1_2$-indescribable if in addition $\kappa$ is inaccessible.

5.2 Fact. 1. If $\kappa$ has said reflection property, it is a limit cardinal and is not equal to $2^\lambda$ for any $\lambda < \kappa$.

2. If there is a Mahlo cardinal, it is consistent that any cub class of ordinals contains a $\Sigma^1_2$-indescribable. The least Mahlo is not $\Sigma^1_2$-indescribable.
3. Reflecting implies $\Sigma^1_2$-indescribable which in turn implies the existence of many inaccessibles.

4. If $P$ is a p.o. of size less than $\kappa$, then forcing with $P$ preserves $\Sigma^1_2$-indescribability of $\kappa$.

**Proof.**

1. Express “there is no cofinal function from $\lambda$ into the ordinals” and “there is a bijection from $\mathcal{P}(\lambda)$ onto the ordinals” as second order statements with parameter $\lambda$ to get sentences that can’t be reflected.

2. Consider the function that assigns to every set $M$ the least $\xi$ such that all formulas of a given class with parameters from $M$ that hold in any $V_\alpha$ already hold in some $V_\alpha$ with $\alpha < \xi$. The closure points under this function form a cub class in the ordinals (these closure points have very strong reflection properties). Carry out this argument inside the initial segment of the universe below a Mahlo to get a stationary set of inaccessibles having the reflection property relativized to that initial segment. To see that the least Mahlo cannot have the reflection property, observe that $\kappa$ being Mahlo is expressible by a $\Pi^1_1$ statement over $V_\kappa$.

3. For the first, observe $V_\alpha \models \exists X \forall Y \Phi(X,Y,p) \iff H_{\alpha^+} \models \exists x \forall y H_\alpha \models \Phi'(x,y,p)$, for some first-order formula $\Phi'$. Secondly, being inaccessible is expressible as a $\Pi^1_1$ statement (f.e. the power set of any set exists and can be mapped injectively into some ordinal and there is no function with a set as domain but unbounded range).

4. Observe that if $\kappa$ is inaccessible and $|P| < \kappa$, the elements of the $(H_\kappa)^P$ and $(H_{\kappa^+})^P$ are precisely the interpretations of the $P$-names in $H_\kappa$ and $H_{\kappa^+}$, respectively. Thus $\Vdash_P "V_\kappa \models \exists X \forall Y \Phi(X,Y,\dot{p})"$ is equivalent to a statement of the form $\exists X \in H_{\kappa^+} \forall Y \in H_{\kappa^+} V_\kappa \models "\forall P \Phi'(X,Y,\dot{p})"$, where $\Phi'$ is a first order expression in the parameter $P \in V_\kappa$, so the whole statement is $\Sigma^1_2$ over $V_\kappa$. So it holds with $\kappa$ replaced by some inaccessible $\xi < \kappa$, and therefore $\Vdash_P "V_\xi \models \exists X \forall Y \Phi(X,Y,\dot{p})"$.

**5.3 Fact.** $\kappa$ is lightface $\Sigma^1_2$-indescribable $\iff \kappa$ is inaccessible and $H_\kappa \prec_{\Sigma^1_2} H_{\kappa^+}$.

**Proof.** First assume indescribability. Let $H_{\kappa^+} \models \exists x \forall y \phi(x,y,p)$, where $p \in H_\kappa$. Pick a witness $x_0$ in $H_{\kappa^+}$. For any transitive $M$ of size $\kappa$ containing $x_0$ and $p$, we have $M \models \forall y \phi(x_0,y,p)$. So $H_{\kappa^+} \models \exists x \forall y \phi(x,y,p)$ is equivalent to a $\Sigma^1_2$ assertion over $H_\kappa$, and thus is reflected by some $H_\xi$, for inaccessible
Thus $H_{\xi^+} \models \exists x \forall y \phi(x, y, p)$, and as $\Sigma_2$ formulas are upwards absolute for members of the $H$-hierarchy, $H_\kappa \models \exists x \forall y \phi(x, y, p)$. For the other direction, let $H_\kappa \models \exists X \forall Y \phi(X, Y, p)$. Then $H_{\kappa^+}$ thinks “there is a cardinal $\theta$ such that $\exists x \subseteq \theta \forall y \subseteq \theta, (H_\theta, x, y) \models (x, y, p)$”. As “$y = H_\theta$” is $\Pi_1$ in $y$ and $\theta$, this is seen to be a $\Sigma_2$ statement in parameter $p$ and so holds in $H_\kappa$.

5.4 Fact. If $\kappa$ is lightface $\Sigma^1_1$-indecomposable, then it is both $\Sigma_1$-Mahlo and $\Pi_1$-Mahlo. If “$x \in C$” is $\Sigma_2$ over $H_{\kappa^+}$ (with parameter in $H_\kappa$) for a cub subset $C$ of $\kappa$, then $C$ contains an inaccessible.

Proof. The first assertion is clearly a consequence of the second; for the latter, since $H_\kappa \prec_{\Sigma_2} H_{\kappa^+}$, the argument given in 1.4.1 for reflecting cardinals goes through.

5.2 Coding using an Aronszajn-tree

We fix the following notation: for a tree $T$, we denote by $<_T$ (or $\leq_T$) the tree order, $T_\alpha$ denotes the $\alpha$-th level of $T$ and $T \upharpoonright \alpha$ denotes the subtree of $T$ consisting of all levels of height less than $\alpha$. By $\text{pred}(t)$ we mean of course \{\mbox{$t' \in T \mid t' <_T t$}\}. The following works for any Aronszajn-tree, that is, a tree of height $\omega_1$ with countable levels and without any cofinal branches. Let $T$ be such a tree. Such a tree can be specialized by a $\text{ccc}$ forcing, in the sense that in the extension, there exists a function $F$ from $T$ into the rationals which is order preserving (such $F$ is called a specializing function). Once a tree is specialized, it is impossible to add a cofinal branch without at the same time collapsing $\omega_1$.

5.5 Lemma. If $T$ is special (i.e. there exists a specializing function) and of height $\omega_1$, $T$ does not have a cofinal branch.

Proof. The image of a cofinal branch under $F$ would be a set of rationals of order type $\omega_1$.

A slight variation of the forcing that specializes $T$ can be used to code a subset of $\omega_1$ very efficiently, in a way that makes reshaping unnecessary. This approach to coding was used in [HS85] to prove that $MA$ together with “$\omega_1$ is inaccessible to reals” implies $\omega_1$ is weakly compact in $L$ (weakly compact cardinals are much stronger than reflecting). Lightface $\Sigma^1_2$-indecomposability is an obvious weakening of weak compactness, which is equivalent to “bold-face” $\Pi^1_1$-indecomposability and closely linked to two-step $\Sigma^1_3$-absoluteness for set forcing.
5.6 Fact. Let \( S = (s_\alpha)_{\alpha < \omega_1} \) be a sequence of reals. There is a ccc forcing \( P \) that adds a real \( r \) such that in the extension the following holds: whenever \( M \) is a transitive model of \( ZF^- \) s.t. \( r \in M \), \( \langle T \leq_T \rangle \in M \), we have \( (s_\alpha)_{\alpha < \omega_1} \in M \).

Proof. To achieve this, we iterate the following notion of forcing: fix \( Q_0, Q_1 \), two disjoint dense sets whose union is all rational numbers. For any sequence \( S = (s_\alpha)_{\alpha < \omega_1} \) consider \( P^S_T \) consisting of all conditions \( f \) s.t.

1. \( f \) is a function with domain a finite subset of \( T \times \omega \)
2. for each \( n \in \omega \), the function \( t \mapsto f(t, n) \) is a partial order preserving mapping from \( (T, \leq_T) \) into the rationals
3. For any \( \alpha < \omega_1, t \) at the \( \omega \cdot \alpha \)-th level of \( T \) and \( n \in \omega \), if \( (t, n) \in \text{dom}(f) \), then \( f(t, n) \in Q_0 \) if and only if \( n \in s_\alpha \).

5.7 Lemma. Let \( F = \bigcup G \), where \( G \) is generic. Then \( F \) is a function from \( T \) into the rationals which is order preserving and continuous at limit nodes of \( T \); moreover, for any \( \alpha < \omega_1 \), and any \( t \in T_{\omega \cdot \alpha} \), \( \{ n \in \omega \mid F(t, n) \in Q_0 \} = s_\alpha \).

Proof. Clearly, \( D_{(t, n)} := \{ p \in P^S_T \mid (t, n) \in \text{dom}(p) \} \) is dense for any \( (t, n) \in T \times \omega \): given a condition \( p \), there is an interval of possible values for \( p \) at \( (t, n) \) (since \( p \) has finite domain), so if \( t \) is at level \( \omega \cdot \alpha \) of \( T \), we can choose a value from \( Q_0 \) or \( Q_1 \), depending on whether \( n \in s_\alpha \) or not. So \( F \) is a total, order preserving function on \( T \), and so the “moreover” clause holds by definition. \( F \) is continuous as \( D_{(t, n), \epsilon} := \{ p \in P^S_T \mid \exists t' \in T \mid |p(t', n) - p(t, n)| < \epsilon \} \) is dense for any \( n \in \omega, \epsilon > 0 \) and \( t \) at a limit level of \( T \) (again, by the finiteness of the domain of any condition). \( \square \)

5.8 Lemma. \( P^S_T \) is ccc.

Proof. Assume \( (p_\alpha)_{\alpha < \omega_1} \) is an uncountable antichain; then \( \{ \text{dom}(p_\alpha) \mid \alpha < \omega_1 \} \) is an uncountable subset of \( [T \times \omega]^\omega \), so we can apply the delta-systems lemma and assume that for each \( \alpha \), \( \text{dom}(p_\alpha) = r \cup d_\alpha \), where \( r, (d_\alpha)_{\alpha < \omega_1} \) are pairwise disjoint. Let us also assume that the \( d_\alpha \) all have the same cardinality \( k \). There are only countably many possibilities for the values of the \( p_\alpha \) on \( r \), so we assume that all the conditions agree on \( r \). So for any \( \alpha, \alpha' < \omega_1 \), there is \( t \in d_\alpha, t' \in d_{\alpha'} \) and \( n \in \omega \) such that \( p_\alpha \cup p_{\alpha'} \) is not order preserving on \( \{(t, n), (t', n)\} \), whence in particular \( t \) and \( t' \) are comparable in the tree order. As any node of the tree has only countably many predecessors in the tree order, by thinning out \( (p_\alpha)_{\alpha < \omega_1} \) we can further assume that for all \( \alpha, \alpha' < \omega_1 \), there are \( t \in d_\alpha, t' \in d_{\alpha'} \) such that \( t \leq_T t' \) (find a function from \( \omega_1 \) to \( \omega_1 \) and consider closure points). Let us now enumerate the \( d_\alpha \) as \( t^{\alpha}_0, \ldots, t^{\alpha}_{k-1} \).
We know that all the conditions in the antichain have comparable nodes in their domain, we will now find a sufficiently coherent subset of conditions to get a branch through $T$. Enlarge (using Zorn’s lemma or AC) the filter of co-initial subsets of $\omega_1$ to an ultrafilter $U$ ($U$ contains only sets of size $\omega_1$, i.e. $U$ is uniform). For any $\alpha < \omega_1$, we have \{ $\beta < \omega_1 \mid \exists i, j \ t^i_\alpha <_T t^j_\beta$ \} $\in U$. So by finite additivity of $U$, for each $\alpha$, there are $i, j$ such that \{ $\beta < \omega_1 \mid t^i_\alpha <_T t^j_\beta$ \} $\in U$. Moreover, there is an uncountable set $I$ and $i, j$ such that the above holds for all $\alpha \in I$. So for any $\alpha, \alpha' \in I$, as elements of $U$ have non-empty (in fact large) intersection, there is $\beta$ such that $t^i_\alpha <_T t^j_\beta$ and $t^i_{\alpha'} <_T t^j_{\beta}$, so $t^i_{\alpha}$ and $t^i_{\alpha'}$ are comparable and $(t^i_\alpha)_{\alpha \in I}$ is an uncountable branch through $T$.

Now we can prove fact 5.6. We build $P$ as the finite support iteration of $(P_\alpha)_{\alpha \in \omega}$. Let $s^0_\alpha = s_\alpha$; $P_0$ is the forcing coding this sequence of reals into a specializing function for $T$. At stage $n$, we have added a specializing function $F_n$; let $s^{n+1}_\alpha$ be a real coding $F_n$ restricted to $T \mathrel{|} \omega \cdot (\alpha + 1) \times \omega$. $P_{n+1}$ is the forcing coding the sequence $(s^{n+1}_\alpha)_{\alpha < \omega_1}$. Let $r$ be a real coding all reals $(s^k_\alpha)_{k \in \omega}$; we check by induction on $\eta \leq \omega_1$ that $r$ has the property promised in 5.6: assume that for all $k$, $s^k_\eta \in M$; then for all $k$, $F_k$ restricted to $T \mathrel{|} \omega \cdot (\eta + 1) \times \omega$ is contained in $M$. Let $t \in T_{\omega \cdot (\eta + 1)} \cap M \neq \emptyset$; by the continuity of the $F_k$ and by lemma 5.7, $n \in s^k_{\eta + 1}$ exactly if $\text{sup}(\{ F(t') \mid t' <_T t \}) \in Q_0$; so $s^k_{\eta + 1} \in M$ for all $k$. For limit $\eta$, using $(s^k_\xi)_{k \in \omega, \xi < \eta}$ we can decode $F_k$ on $T \mathrel{|} \omega \cdot \eta$ inside $M$, and therefore as before $s^k_\eta \in M$ for each $k$.

\[ \square \]

### 5.3 An equiconsistency

**5.9 Theorem.** “$\Sigma^1_3$-(ccc) absoluteness together with $\omega_1$ inaccessible to reals” has the consistency strength of a lightface $\Sigma^1_3$-indescribable.

**Proof.** We work in $L$: let $\kappa$ denote $\omega_1^V$ and observe $\kappa$ is inaccessible. Assume that the reflection property fails: there is $X^* \in L_\delta$, such that for some first order formula $\Phi$ (with parameter in $L_\kappa$, which we suppress), $\langle L_\kappa, X^* \rangle \models \forall \Phi(X^*, A)$, but for all $\xi < \kappa$ there is $A$ such that $\langle L_\xi, X^* \mathrel{|} \xi, A \rangle \models \neg \Phi(X^* \mathrel{|} \xi, A)$. We now define a tree $T$ and its ordering $\leq_T$:

Elements of $T$ are tuples $(\beta, X)$, where $\beta < \kappa$ and $X \in \delta_2$, for some $\delta$, and

1. $L_\beta = h^{L_\beta}_{\Sigma_2}(\|X\| \cup X)$ (in particular, $X \in L_\beta$)
2. $X \mathrel{|} \|X\| = X^* \mathrel{|} \|X\|$
3. for all $\xi \leq \text{dom} X$, there is $A \in L_\beta$ such that $\langle L_\xi, X \upharpoonright \xi, A \rangle \models \neg \Phi(X \upharpoonright \xi, A)$.

Define $(\beta, X) \leq_T (\bar{\beta}, \bar{X}) \iff X \leq_L \bar{X}$ and there is a $\Sigma_1$-elementary embedding $\sigma : L_\beta \rightarrow L_{\bar{\beta}}$ such that $\sigma(X) = \sigma(\bar{X})$ and $\text{crit}(\sigma) \geq |X|$. This can be motivated by observing that branches correspond to a failure of reflection, as will become clear in a moment.

Let’s check $\leq_T$ is a tree order. Clearly, $\leq_T$ is transitive and reflexive. Also, $\leq_T$ is antisymmetric: let $(\beta, X)$ and $(\beta', X')$ be a counterexample. As $X = X'$, the embedding witnessing $(\beta, X) \leq_T (\beta', X')$ shows $L_\beta$ is isomorphic to $h_{\Sigma_1}^{L_\beta'}(|X'| \cup \{X\})$, but the latter is just $L_{\beta'}$, by item 1 above. It remains to check that any two predecessors of a node are comparable: say $(\beta, X)$, $(\beta', X') \leq_T (\bar{\beta}, \bar{X})$, as witnessed by embeddings $\sigma$ and $\sigma'$. Without loss of generality assume $X \leq_L X'$, whence also $|X| \leq |X'|$. So $\text{ran}(\sigma) = h_{\Sigma_1}^{L_\beta}(|X| \cup \{\bar{X}\}) \subseteq h_{\Sigma_1}^{L_{\beta'}}(|X'| \cup \{\bar{X}\}) = \text{ran}(\sigma')$, whence $(\sigma')^{-1} \circ \sigma$ is a well-defined elementary embedding and so $(\beta, X) \leq_T (\beta', X')$.

We now show $T$ is a $\kappa$-Aronszajn tree, that is its level have size less than $\kappa$, it has height $\kappa$ but no cofinal branch (i.e. linearly ordered subset of type $\kappa$). First observe that for a node $(\bar{\beta}, \bar{X})$ of $T$ and a cardinal $\alpha \leq \beta$, there is exactly one predecessor node of cardinality $\alpha$. Existence: look at the transitive collapse $L_\beta$ of $h_{\Sigma_1}^{L_{\bar{\beta}}}(\alpha \cup \{\bar{X}\})$ and let $X$ denote the image of $\bar{X}$ under the collapsing map (let $\sigma$ denote the inverse of this map). Then $|X| = \alpha$, so $L_\beta = h_{\Sigma_1}^{L_{\bar{\beta}}}(\alpha \cup \{\bar{X}\})$. Item 3 holds for $(\bar{\beta}, \bar{X})$, so by a Skolem hull argument, it also holds for $(\beta, X)$. So $(\beta, X) \in T$. If $\alpha < |\beta|$, $X \leq_L \bar{X}$, and $\sigma$ witnesses $(\beta, X) \leq_T (\bar{\beta}, \bar{X})$. If $\alpha = |\beta| = |X|$, by item 1, $X = \bar{X}$ and $\beta = \bar{\beta}$. Uniqueness: say $(\beta, X)$, $(\beta', X') \leq_T (\bar{\beta}, \bar{X})$, and $\alpha = |\beta| = |\beta'|$. Then both $(L_\beta, X)$ and $(L_{\beta'}, X')$ are isomorphic to $h_{\Sigma_1}^{L_{\bar{\beta}}}(\alpha \cup \{\bar{X}\})$, so they are identical. As a corollary we obtain that if $(\beta, X) \in T$ and $|\beta| = \omega_\alpha$, then the height of $(\beta, X)$ in $T$ is exactly $\alpha$. So $T \upharpoonright \alpha \subseteq L_{\omega_\alpha}$ and $T$ is a $\kappa$ tree. For any $\alpha < \kappa$, let $X := X^* \upharpoonright \omega_\alpha$ and let $L_{\beta'}$ be the transitive collapse of $H := h_{\Sigma_1}^{L_\beta}(\omega_\alpha \cup \{X\})$. It is easy to check that $(\beta, X) \in T$ (for item 3, observe that $\text{dom} X = \omega_\alpha \in H$) and we have seen its height is exactly $\alpha$, so $T$ has height $\kappa$.

5.10 Lemma. $T$ does not have a cofinal branch in $V$.

Proof. Else, let $(\beta(\alpha), X(\alpha))_{\alpha < \kappa}$ be such a branch, let $\sigma_\alpha : L_{\beta(\alpha)} \rightarrow L_{\beta(\bar{\alpha})}$ be the embedding witnessing $(\beta(\alpha), X(\alpha)) \leq_T (\beta(\bar{\alpha}), X(\bar{\alpha}))$. A simple argument involving the $\Sigma_1$-definable Skolem function shows that for $\alpha < \alpha' < \bar{\alpha}$, $\sigma_{\alpha'} \circ \sigma_\alpha = \sigma_\alpha$. As $\kappa$ has uncountable cofinality, the direct limit of this chain of models is well-founded and a model of $V = L$, therefore isomorphic to
some \( L_\delta \). Each \( L_\beta(\alpha) \) is \( \Sigma_1 \)-elementarily embeddable into \( L_\delta \) via a map that is the identity on \( [\beta(\alpha)]^L_\delta \), and all the \( X(\alpha) \) are mapped to one \( X_0 \) which must therefore end-extend \( X^* \) (in the sense that \( X_0 \upharpoonright \kappa = X^* \)). So \( \delta > \kappa \) (as \( X_0 \in L_\delta \)). By elementarity (and condition 3 in the definition of \( T \)), there is \( A \in L_\delta \) such that \( (L_\kappa, X_0 \cap \kappa, A) \models \neg \Phi(X_\delta \cap \kappa, A) \), contradiction.

Let’s go back to working in \( L \) again, for yet a little while. \( T \) is not pruned (there are dying branches and branches that don’t split), and \( T \) needn’t even have unique limit nodes (in the sense that for \( t \) and \( t' \) at a limit level \( T_\lambda \), if \( t \) and \( t' \) have the same predecessors, then \( t = t' \)). The latter shortcoming has to be remedied, and this is accomplished easily by replacing \( T \) by \( T' \), where \( T'_\alpha+1 = T_\alpha \) for any \( \alpha < \kappa \), while for limit ordinals \( \lambda \) we set \( T'_\lambda = \{ \text{pred}(t) \mid t \in T_\lambda \} \). \( T' \) carries the obvious order (\( t \leq_T z' \) exactly if either \( t \subseteq t' \) or \( t \in t' \) or \( t \subseteq \text{pred}(t') \) or \( t \leq_T t' \)).

Pick \( E \) such that

1. \( \langle \kappa, E \rangle \cong \langle L_\delta, \in \rangle \) and
2. \( X^*(\xi) = 1 \iff (\xi + 1) E 0 \).

Define \( C := \{ \xi < \kappa \mid \xi \text{ is a cardinal and } \langle L_\xi, X^* \cap \xi, E \cap \xi \rangle \prec \langle L_\kappa, X^* \cap \xi, E \cap \xi \rangle \} \). By inaccessibility of \( \kappa \) this is a \textit{cub} set.

Let \( C \) be enumerated as \( (c_\xi)_{\xi < \kappa} \). Now we work in \( V \): let \( s_\xi \) code (via the transitive collapse) \( (T_{\xi+1}, X^* \upharpoonright c_\xi, E \cap c_\xi) \). Now we apply the forcing just described (fact 5.6) to code the sequence \( S = (s_\xi)_{\xi < \omega_1} \) into a single real \( r \), using \( T \).

Now let \( \beta < \kappa \) and \( L_\beta[r] \) be a model of \( ZF^- \land \exists \omega_1 \), and say \( \alpha = \omega_1^{L_\beta[r]} \). We claim that for some \( \xi \leq \alpha \), there is \( x \in L_\beta \) such that for all \( a \in L_\beta \), \( \langle L_\xi, x, a \rangle \models \Phi(x, a) \), i.e. that from the point of view of \( L_\beta \), reflection occurs before \( \omega_1 \). Assume otherwise; we show how to recursively reconstruct \( (s_\xi)_{\xi \leq \alpha} \) inside \( L_\beta[r] \). This is a contradiction as \( s_\alpha \in L_\beta[r] \) means \( \alpha \) is countable in that model. We construct \( (s_\xi)_{\xi \leq \eta} \) by recursion on \( \eta \leq \alpha \). \( s_0 \in L_\beta[r] \) is immediate. Now say \( \eta = \gamma + 1 \): so \( (s_\xi)_{\xi \leq \gamma} \in L_\beta[r] \), so \( T \upharpoonright \gamma + 2 \in L_\beta[r] \), so using the specializing functions on that tree we get \( s_{\gamma + 1} \in L_\beta[r] \). In the remaining case, where \( \eta \) is limit; by induction hypothesis, \( (s_\xi)_{\xi < \eta} \in L_\beta[r] \). So \( T \upharpoonright \eta = \bigcup_{\xi < \eta} T_{\xi+1} \) is in \( L_\beta \eta ta[r] \), and in fact, all of its elements are countable there. Likewise, \( X^* \upharpoonright c_\eta \cap c_\eta \in L_\beta[r] \). \( E \cap c_\eta \) is of course a well founded relation, and by the definition of \( C \) and elementarity, it is isomorphic to some \( L_\delta \), such that \( X^* \upharpoonright c_\eta \in L_\delta \) and \( \delta^* < \beta \). So since \( c_\eta \geq \omega_\eta \), \( X \upharpoonright \omega_\eta \in L_\beta \) (to be sure we didn’t use any information not accessible in \( L_\beta[r] \), we feel we should mention the triviality that \( (\omega_\eta)^L = (\omega_\eta)^{L_\beta} \)). By assumption, there is \( a \in L_\beta \) such that \( L_\omega \eta \models \neg \Phi(X^* \upharpoonright \omega_\eta ta, a) \). Therefore, by definability of
\[ \Sigma_1\text{-Skolem function and replacement in } L_\beta, \text{ we can look at the collapse } L_\beta^*, \text{ of } h_{\Sigma_1}(\omega, t_a \cup \{X^* \mid \omega, t_a\}), \beta^* < \beta. \text{ Clearly, } (\beta^*, X^* \mid \omega, t_a) \in T \text{ (and its height is at least } \eta; \text{ the only thing of substance to check for membership in } T \text{ is item 3 in the definition of } T). \text{ As all predecessor nodes of } T \text{ are countable in } L_\beta^r \leq_T T \upharpoonright \eta + 1 \text{ (which involves finding an embedding of structures) is absolute for } L_\beta^r \text{ (see lemma 1.6). So } L_\beta^r \text{ can use } (\beta^*, X^* \mid \omega, t_a) \in T \text{ to find a branch though } T \upharpoonright \eta \text{ and by continuity of the ranking functions, } s_\eta \in L_\beta^r. \]

So we have found, after forcing with a ccc partial order, a real } r \text{ with the } \Sigma_3^1 \text{ property}

\[ \forall \beta < \omega_1, \text{ if } L_\beta^r \models ZF^- \land \exists \omega_1, \text{ then } \exists \alpha \leq (\omega_1)^{L_r}[\beta] \text{ such that } (L_\alpha \models \exists X \forall A \Phi(X, A))^{L_r}. \]

So we may assume that } r \text{ is in the ground model. But since } \omega_1 \text{ is inaccessible to reals, we may look at } \beta := (\omega_r)^{L_r}[\beta]. \text{ By the above, for some } \alpha \leq (\omega_r)^{L_r}[\beta], (L_\alpha \models \exists X \forall A \Phi(X, A))^{L_r}, \text{ and therefore } (L_\alpha \models \exists X \forall A \Phi(X, A))^{L_r}. \text{ This completes one direction of the proof.} 

For the other direction, assume } \kappa \text{ is inaccessible and lightface } \Sigma_3^1 \text{-indescribable. We show that after gently collapsing } \kappa \text{ to } \omega_1, \Sigma_3^1(\text{ccc})\text{-absoluteness holds and } \kappa \text{ is inaccessible to reals. The latter is clear, as any real in the extension can be absorbed into an intermediate model where } \kappa \text{ is still inaccessible.} 

In } V^{Coll(\omega, \kappa)}, \text{ let } P \text{ be a ccc (i.e. } \kappa - \text{cc) p.o. that forces a } \Sigma_3^1(r) \text{ statement } \phi(r), r \in V^{Coll(\omega, \{\alpha\})}, \text{ for some } \alpha < \kappa. \text{ Firstly, we can assume that } P \models \kappa: \text{ write } \phi(r) \text{ as } \text{“there is a real } x \text{ such that a certain tree } T(x) \text{ on } \omega_1 \text{ (which is } \Delta_1(x, r, \omega_1) \text{) is well-founded”}. \text{ So } \models P \text{ “}\exists \dot{x} \text{ s.t. } T(\dot{x}) \text{ is well-founded”}. \text{ As } P \text{ has the } \kappa - \text{cc, there is } \xi < \kappa^+ \text{ such that } \models P \text{ rank}(T(\dot{x})) < \xi. \text{ Thus there is a name } \dot{F} \text{ for a ranking function on } T(\dot{x}), \dot{F} \models \kappa. \text{ Now } \models \dot{F} \text{ is a ranking function on } T(\dot{x}) \text{ is a first order statement over the structure } \langle P, \dot{F}, \dot{x} \rangle \text{ and thus also holds for an elementary submodel } P' \text{ of size } \kappa, \text{ containing } \dot{F} \text{ and } \dot{x}. \text{ This proves we can assume } P \models \kappa. \text{ For the moment, we work in } W := V^{Coll(\omega, \{\alpha\})}. \text{ We have that for } Q := Coll(\omega, \kappa) \ast P, Q \models \phi(r), Q \text{ is of size } \kappa \text{ and has the } \kappa\text{-cc. The fact that there exists such a } Q \text{ can be expressed in a } \Sigma_2^1 \text{ way, over } V_\kappa: \text{ as } Q \models \kappa, \text{ w.l.o.g. we may assume } Q \subset V_\kappa. \text{ Let } \Psi(A, Q) \text{ be first order in the predicates } Q, A, \text{ saying “if } A \text{ is an antichain of } Q, A \text{ is equal to some set”, and let } \Theta(Q, r) \text{ be first order in the predicate } Q \text{ and parameter } r \text{ saying “}\models P \phi(r)”. \text{ We have that } V_\kappa \models \exists Q \forall A \Psi(A, Q) \land \Theta(Q, r). \text{ By fact 5.2, } \kappa \text{ still is } \Sigma_2^1\text{-indescribable (in } W) \text{ so there is an inaccessible } \xi < \kappa \text{ reflecting this second order statement. So there is } Q \models \xi \text{ s.t. } V_\xi \models “\models Q \phi(r)”, \text{ but as } V_\xi \models \forall A \Theta(Q, A), Q \text{ has the } \xi\text{-cc and thus } \models Q \phi(r) \text{ is absolute.} \]
between $V_\zeta$ and $V$: \( \phi \) mentions only reals, and all reals of the extension via $Q$ have names in $V_\zeta$; and once all quantification over names is removed from $\models_Q \phi(r)$, only quantifiers ranging over $Q \subset V_\zeta$ remain. Thus, in $W$, there is $Q \cong \xi$ forcing $\phi(r)$. So by corollary 1.30, $\phi(r)$ is forced by the gentle collapse of $\kappa$ over $W$, and this means that, if $\dot{r}$ is some $\text{Coll}(\omega, \kappa)$-name for $r$, $\text{Coll}(\omega, \kappa) \sim \text{Coll}(\omega, \{\alpha\}) \ast \text{Coll}(\omega, \kappa)$ forces $\phi(\dot{r})$ over the ground model. So $\phi(r)$ holds in $V^{\text{Coll}(\omega, \kappa)}$, whence $\Sigma^1_3$-absoluteness holds between this model and any subsequent ccc extension.
References


