TRAVELING WAVES OF A KINETIC TRANSPORT MODEL FOR THE KPP-FISHER EQUATION

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ABSTRACT. A reactive kinetic transport equation whose macroscopic limit is the KPP-Fisher equation is considered. In a scale, where collisions occur at a faster rate than reactions, existence of traveling waves close to those of the KPP-Fisher equation is shown. The method adapts a micro-macro decomposition in the spirit of the work of Caflisch and Nicolaenko for the Boltzmann equation. Stability of these waves is shown for perturbations in a weighted L^2-space, where the weight function is exponential and such that the (macroscopic) linearized operator in the weighted space is self-adjoint and negative definite. Similar approaches to stability of traveling waves are well-known for the KPP-Fisher equation.

1. INTRODUCTION

When the chemical reaction

$$A + B \leftrightarrow 2A$$

takes place in a setting, where the density of species B can be assumed as constant and species A is subject to one-dimensional diffusion, then the dynamics of the density $u(t,x)$ of species A can be described (after non-dimensionalization) by the KPP-Fisher equation

$$\partial_t u = D\partial_x^2 u + u(\bar{\rho} - u)\,, \tag{1.1}$$

with the diffusion coefficient $D > 0$. This equation has two constant equilibrium states, $u \equiv 0$ and $u \equiv \bar{\rho} > 0$, the former linearly unstable and the latter linearly stable. Thus, an initial perturbation of $u \equiv 0$ grows to approach $u \equiv \bar{\rho}$. It is well-known that, in an unbounded domain, this growth may take the asymptotic form of a propagating wave front, i.e. as $t \to +\infty$ the solution approaches the form

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$u(t,x) = u_{TW}(\xi)$ with the traveling wave variable $\xi = x - st$, the constant wave speed $s \in \mathbb{R}$, and u_{TW} satisfying the ordinary differential equation

$$Du_{TW}'' + su_{TW}' + u_{TW}(\bar\rho - u_{TW}) = 0\,. \tag{1.2}$$

We assume throughout that $s \ge 0$. This is no restriction, because (1.2) is invariant under the reflection $s \to -s$, $\xi \to -\xi$. The waves then propagate to the right and satisfy the far-field conditions

$$\lim_{\xi\to-\infty} u_{TW}(\xi) = \bar\rho\,, \qquad \lim_{\xi\to+\infty} u_{TW}(\xi) = 0\,. \tag{1.3}$$

Equation (1.1) has been introduced by Fisher [5] as a model in population genetics that describes the advance of individuals with a favorable gene. At the same time Kolmogorov, Petrovskii and Piskunov [9] investigated (1.1) with a more general nonlinearity. Some results concerning the traveling wave solutions (which have been studied extensively) will be reviewed below.

The subject of this work is a kinetic transport model for the same physical situation. The main modeling difference compared to a reaction-diffusion model is the replacement of the Brownian motion by a velocity jump process. The latter can be thought of being caused by collisions with a (non moving) background medium, which randomize the direction of movement. A kinetic equation for the phase space density $f(t,x,v)$ of particles of species A can be written in the (dimensionless) form

$$\varepsilon^2\partial_t f + \varepsilon v\partial_x f = \mathcal{L}f + \varepsilon^2 Q(f)\,, \tag{1.4}$$

with time $t > 0$, position $x \in \mathbb{R}$ and velocity $v \in V \subset \mathbb{R}$. The left hand side of (1.4) describes the free streaming of particles, and the terms on the right hand side model collisions (described by the operator $\mathcal{L}$) and chemical reactions (described by the operator Q). The dimensionless parameter ε is assumed to satisfy $0 < \varepsilon \ll 1$. Considering its occurrence on the right hand side of (1.4), this means that collisions are much more frequent than reactions. The powers of ε on the left hand side can be achieved by appropriate scalings for time and position.

Collisions are described as instantaneous velocity jumps with an equilibrium distribution $M(v)$, satisfying the moment conditions

$$\int_V M\,dv = 1\,, \qquad \int_V vM\,dv = 0\,, \qquad \int_V v^2 M\,dv = D > 0\,, \qquad \int_V v^3 M\,dv = 0\,.$$

A typical example is the Maxwellian distribution $M(v) = (2\pi D)^{-1/2} e^{-v^2/(2D)}$, $V = \mathbb{R}$. The simplest collision model is the relaxation operator

$$\mathcal{L}f = \int_V [M(v)f(v') - M(v')f(v)]dv' = M\rho_f - f\,,$$

with the macroscopic density $\rho_f(t,x) = \int_V f(t,x,v)dv$. The collision process obviously conserves mass: $\int_V \mathcal{L}f\,dv = 0$. For the chemical reactions, it is assumed that they produce particles with the same equilibrium velocity distribution:

$$Q(f) = \int_V [\bar\rho M(v)f(v') - f(v)f(v')]dv' = \rho_f(M\bar\rho - f)\,.$$

We obtain the kinetic reaction model

$$\varepsilon^2\partial_t f + \varepsilon v\partial_x f = M\rho_f - f + \varepsilon^2\rho_f(M\bar\rho - f)\,. \tag{1.5}$$

A connection between (1.5) and (1.1) can be established by the macroscopic limit $\varepsilon \to 0$. Substitution of the Chapman-Enskog ansatz $f = M\rho_f + \varepsilon f^\perp$ into (1.5) and integration with respect to v leads to the macroscopic equation

$$\partial_t\rho_f + \partial_x \int_V v f^\perp dv = \rho_f(\bar\rho - \rho_f)\,.$$

On the other hand, (1.5) implies

$$f^{\perp} = -vM\partial_x \rho_f + O(\varepsilon)\,.$$

Hence, in the formal limit $\varepsilon \to 0$, ρ_f solves (1.1). This is an example of the derivation of reaction-diffusion equations from kinetic models. Formal asymptotics of this kind for much more general cases, in particular also systems, has been carried out by several authors (see, e.g., [1], [11]). However, a rigorous justification is only known for linear models [1].

It is our aim to study the existence and stability of traveling waves of (1.5). As a preliminary result, in Section 1.3, we prove global existence of solutions of the initial value problem for (1.5) for initial data bounded by a global equilibrium. Our approach for the analysis of traveling waves is based on the fact that, for ε small, (1.5) can be approximated by (1.1). In Section 2 we present a constructive existence proof for traveling waves with speed $s \geq s_0 = 2\sqrt{D\,\bar\rho}$ of (1.5), which shows the asymptotic closeness of the kinetic profiles to the solutions of (1.2) with the same speed. We follow the approach of [4] (that is applied to traveling waves of kinetic BGK models for scalar conservation laws) by first constructing a formal asymptotic approximation, and then showing solvability of the problem for the correction term. For the latter we adapt the micro-macro decomposition introduced by Caflisch and Nicolaenko [2] for the Boltzmann equation. The major difficulty in the current problem is caused by the fact that, in contrast to [4] and [2], the macroscopic problem is not a conservation law. The existence result can be extended to also give strict monotonicity, and therefore positivity for the macroscopic density of the kinetic profile. In Section 3 we show the asymptotic stability of kinetic profiles with $s > s_0$, under perturbations in suitable spaces. Traveling waves for the KPP-Fisher equation are stable under perturbations, which decay faster than (or at least as fast as) the waves. The analogous result is proven here. The required decay properties are built into an appropriately weighted L^2-space. This has the consequence that we can control the macroscopic terms in a similar way as for the KPP-Fisher equation. Concerning the control of the microscopic terms, we have only been successful under the additional assumption that the velocity space V is bounded. In the remainder of this section we recall the stability results for traveling wave profiles of (1.1) and also show, how the stability of these profiles can be proven by using energy estimates. We also carry out the Chapman-Enskog argument for the approximation of kinetic traveling waves.

1.1. **Traveling waves for the KPP-Fisher equation.** Concerning existence of traveling waves of (1.1), the following result is well known.

Theorem 1 ([9])**.** *For $s \geq s_0 := 2\sqrt{D\,\bar\rho}$ there exists a positive solution of (1.2), (1.3), which is unique, up to a shift in ξ, and strictly decreasing.*

Proof. One way of looking at the problem is by writing (1.2) as a planar system and analyzing the (u_{TW}, u'_{TW}) phase-plane. The critical points are clearly given by the zeroes $(0,0)$ and $(\bar\rho, 0)$ of the nonlinearity. Linearization shows that $(\bar\rho, 0)$ is a saddle point, with eigenvalues $(-s \pm \sqrt{s^2 + s_0^2})/(2D)$, and that there is a unique orbit coming out of it in the second quadrant.

The critical point $(0,0)$ has eigenvalues $(-s \pm \sqrt{s^2 - s_0^2})/(2D)$, thus it is a stable node for $s \geq s_0$ and a stable spiral for $s < s_0$. Hence, a positive solution to (1.2) satisfying (1.3) can only exist if $s \geq s_0$. Further, it is easy to see that the triangle

$$0 \leq u_{TW} \leq \bar\rho, \quad 0 \geq u'_{TW} \geq -\frac{s}{2D} u_{TW}\,, \tag{1.6}$$

is an invariant region, so that the unique orbit coming out of the saddle point enters the node, and it does so through the slow manifold when $s > s_0$. This gives existence of traveling waves (unique up to translation in ξ) for every $s \geq s_0$. □

The proof also provides the far-field behavior. On the one hand, we have

$$\bar\rho - u_{TW}(\xi) \sim c\, e^{\alpha_- \xi} \quad \text{as } \xi \to -\infty \quad \text{with} \quad \alpha_- = \frac{\sqrt{s^2 + s_0^2} - s}{2D} > 0\,, \quad c > 0\,. \tag{1.7}$$

On the other hand, for every $s > s_0$,

$$u_{TW}(\xi) \sim c_s\, e^{-\alpha_+ \xi} \quad \text{as } \xi \to +\infty\,, \quad \text{with} \quad \alpha_+ = \frac{s - \sqrt{s^2 - s_0^2}}{2D} > 0\,, \quad c_s > 0\,, \tag{1.8}$$

and, for $s = s_0$,

$$u_{TW}(\xi) \sim c_0 \xi\, e^{-(s_0/2D)\,\xi} \quad \text{as } \xi \to +\infty\,, \quad c_0 > 0\,. \tag{1.9}$$

For the stability analysis below, a comparison principle for the KPP-Fisher equation will be useful.

Lemma 2. *Let $u_1(t,x)$, $u_2(t,x) \geq 0$ be two solutions of (1.1). Let $u_1(0,x) \geq \gamma u_2(0,x)$, for all $x \in \mathbb{R}$, with $\gamma \leq 1$. Then $u_1(t,x) \geq \gamma u_2(t,x)$, for all $t \geq 0$, $x \in \mathbb{R}$.*

Proof. A simple computation shows that $v := u_1 - \gamma u_2$ satisfies

$$\partial_t v - D\partial_x^2 v + v(u_1 + u_2 - \bar\rho) = u_1 u_2 (1-\gamma) \geq 0\,.$$

Since $u_1 + u_2 - \bar\rho \geq -\bar\rho$, we can deduce from standard arguments (see e.g. [9]) that $v \geq 0$. □

In the following, we assume $s > s_0$ and write (1.1) in terms of the moving coordinates t and $\xi = x - st$,

$$\partial_t u - s\partial_\xi u - D\partial_\xi^2 u - u(\bar\rho - u) = 0\,, \tag{1.10}$$

and look for solutions that are small perturbations of u_{TW}. Thus we assume $u = u_{TW} + \rho$, where ρ is small in a sense to be made precise later. The equation for the perturbation reads

$$\partial_t \rho - s\partial_\xi \rho - D\partial_\xi^2 \rho + \rho(2u_{TW} + \rho - \bar\rho) = 0\,. \tag{1.11}$$

It is well known that traveling waves of (1.1) are in general unstable to perturbations; cf. Canosa [3]. In the classical approach to stability, one first studies linear stability by analyzing the spectrum of the linearized operator. In a L^p-setting with $p \geq 2$, the spectrum of the linearized operator about waves having $s > s_0$ extends to the right hand complex plane and always contains 0 as an eigenvalue with eigenfunction u'_{TW} (this eigenfunction is the one generated by perturbations equivalent to small translations in the traveling wave). To overcome this problem one introduces norms with appropriate weights, that push the spectrum into the left hand plane and u'_{TW} out of the space, thus creating a spectral gap. In the seminal work by Sattinger [10] such an analysis is carried out in L^∞ with an exponential weight. We borrow this idea here, but for the kinetic model it is more convenient to use L^2 estimates. In this section, we show how this is done for (1.1) by constructing an appropriate Lyapunov functional. For the control of the nonlinear terms, we use the comparison principle. As an alternative to this, an H^1 based Lyapunov functional can be used, combined with the Sobolev embedding $H^1 \subset L^\infty$ (see [7]).

We define the weight function

$$W(\xi) = e^{\frac{s}{2D}\xi} \tag{1.12}$$

and introduce the Hilbert spaces $L^2_\xi = L^2(\mathbb{R})$ and $L^2_W = L^2(W^2 d\xi)$ with the respective norms

$$\|\rho\|_\xi^2 = \int_{\mathbb{R}} \rho^2 d\xi\,, \quad \|\rho\|_W = \|\rho W\|_\xi\,. \tag{1.13}$$

For the initial data, we assume $\rho(t=0) = u(t=0) - u_{TW} \in L^2_\xi \cap L^2_W$ and $u(t=0) \geq \gamma u_{TW}$ with $0 < \gamma \leq 1$, implying $\rho \geq (\gamma - 1)u_{TW}$ as a consequence of Lemma 2.

Local existence of solutions of (1.11) in $L^2_\xi \cap L^2_W$ (thus in effect the weight acts only as $\xi \to +\infty$) follows by a standard contraction argument. Hence, if we can show the decay of the solution in $L^2_\xi \cap L^2_W$ as time evolves, global existence follows by a continuation principle.

Multiplication of (1.11) by W gives

$$\partial_t(\rho W) - D\partial_\xi^2(\rho W) + (\kappa + 2u_{TW} + \rho)\,\rho W = 0\,, \tag{1.14}$$

with

$$\kappa := \frac{s^2}{4D} - \bar\rho > 0$$

by $s > s_0$. Testing (1.11) with ρ and (1.14) with $\alpha\rho W$ (for some $\alpha > 0$) and adding the resulting equations leads to

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right) + D\left(\|\partial_\xi\rho\|_\xi^2 + \alpha\|\partial_\xi(\rho W)\|_\xi^2\right) \\ &+ \int_{\mathbb{R}} (2u_{TW} + \rho - \bar\rho)\rho^2\, d\xi + \alpha\int_{\mathbb{R}} (2u_{TW} + \rho + \kappa)(\rho W)^2\, d\xi = 0\,. \end{aligned} \tag{1.15}$$

The comparison principle implies

$$\frac{1}{2}\frac{d}{dt}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right) \leq -\kappa\alpha\|\rho\|_W^2 - \int_{\mathbb{R}} \left((1+\gamma)u_{TW} + \alpha(1+\gamma)u_{TW}W^2 - \bar\rho\right)\rho^2 d\xi\,. \tag{1.16}$$

Since u_{TW} decreases from $u_{TW}(-\infty) = \bar\rho$ to $u_{TW}(\infty) = 0$, we can choose $\xi_0 \in \mathbb{R}$, such that $u_{TW}(\xi_0) = \bar\rho(1+\gamma/2)/(1+\gamma)$, implying

$$(1+\gamma)u_{TW}(\xi) \geq (1+\gamma/2)\bar\rho \qquad \text{for } \xi \leq \xi_0\,. \tag{1.17}$$

On the other hand, the decay behavior (1.8) of u_{TW} and the definition of W imply $0 < u_{TW}W^2 \to \infty$ as $\xi \to \infty$. Therefore, α can be chosen large enough, such that

$$\alpha(1+\gamma)u_{TW}(\xi)W(\xi)^2 \geq (1+\gamma/2)\bar\rho \qquad \text{for } \xi \geq \xi_0\,. \tag{1.18}$$

Using (1.17) and (1.18) in (1.16) gives

$$\frac{1}{2}\frac{d}{dt}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right) \leq -\min\left\{\kappa, \frac{\bar\rho\gamma}{2}\right\}\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right)\,. \tag{1.19}$$

An application of the Gronwall lemma concludes the proof of a stability result.

Theorem 3. *Let u_{TW} be a traveling wave solution of the KPP-Fisher equation as in Theorem 1 with wave speed $s > s_0$, and let $u(t,x)$ be a solution of the KPP-Fisher equation (1.1), whose initial values satisfy*

$$\int_{\mathbb{R}} (u(0,x) - u_{TW}(x))^2 (1 + e^{xs/D})dx < \infty\,, \qquad u(0,x) \geq \gamma\, u_{TW}(x)\,,\ x \in \mathbb{R}\,,$$

for a positive $\gamma \leq 1$. Then there exists a positive constant c, such that

$$\int_{\mathbb{R}} (u(t,x) - u_{TW}(x-st))^2 (1 + e^{(x-st)s/D})dx \leq c\, e^{-\lambda t}\,, \quad \forall\, t \geq 0\,,$$

$$\textit{with} \quad \lambda = \bar\rho\, \min\left\{2\left(\frac{s^2}{s_0^2} - 1\right), \gamma\right\}\,.$$

Remark 4. (i) *The weight in the norm implies that the initial perturbation decays faster than the travelling wave as $x \to \infty$, which is known to be necessary for stability. A decay of the perturbation is also required as $x \to -\infty$, which is a weakness of the L^2-approach.*

(ii) *Another weakness of the result is that the exponential decay rate λ depends on the initial data through γ. This could be improved by L^∞-decay of the perturbation, so possibly in the framework of the H^1-approach* [7] *mentioned above.*

(iii) *Obviously, when $s = s_0$, we cannot deduce exponential convergence by this procedure. In fact, the spectrum of the linearized operator in $L^2_\xi \cap L^2_W$ extends to the origin (see* [10]*). A more delicate treatment is needed here, and without further discussion we refer the reader to Kirchgässner* [8].

1.2. **Formal approximation of kinetic traveling waves.** It is instructive to perform the formal limit $\varepsilon \to 0$ before proving existence of traveling waves. We look for traveling waves of (1.5), i.e. solutions of the form $f(t,x,v) = f_{TW}(\xi, v)$ with $\xi = x - st$ and $s > 0$, satisfying

$$\varepsilon(v - \varepsilon s)\partial_\xi f_{TW} = M\rho_{TW} - f_{TW} + \varepsilon^2 \rho_{TW}(M\bar\rho - f_{TW}), \quad \rho_{TW} := \rho_{f_{TW}}, \tag{1.20}$$

subject to the far-field conditions

$$f_{TW}(-\infty, v) = \bar\rho M(v) \quad \text{and} \quad f_{TW}(+\infty, v) = 0 \quad \text{for all } v \in V. \tag{1.21}$$

We make the ansatz

$$f_{TW}(\xi, v) = \rho_{TW}(\xi)M(v) + \varepsilon f^\perp_{TW}(\xi, v) \quad \text{with} \int_V f^\perp_{TW}\, dv = 0. \tag{1.22}$$

Substitution of (1.22) and integration in (1.20) lead to

$$-s\partial_\xi \rho_{TW} + \partial_\xi \int_V v f^\perp_{TW} dv = \rho_{TW}(\bar\rho - \rho_{TW}). \tag{1.23}$$

Substitution of (1.22) into (1.20) gives the asymptotic expansion of $f^\perp_{TW}$ as $\varepsilon \to 0$,

$$\begin{aligned} f^\perp_{TW} &= -vM\partial_\xi \rho_{TW} + \varepsilon[sM\partial_\xi \rho_{TW} - v\partial_\xi f^\perp_{TW} + M\rho_{TW}(\bar\rho - \rho_{TW})] + O(\varepsilon^2) \\ &= -vM\partial_\xi \rho_{TW} + \varepsilon(v^2 - D)M\partial_\xi^2 \rho_{TW} + O(\varepsilon^2), \end{aligned} \tag{1.24}$$

where in the last step we have used (1.23). Substitution of (1.24) into (1.23) shows that ρ_{TW} formally solves (1.2) up to $O(\varepsilon^2)$-terms.

1.3. **Notation and preliminary results.** Next we introduce the underlying spaces of our analysis and establish the global existence of the Cauchy problem and a maximum principle as preliminary results.

We define the weighted inner product in the v-direction by

$$\langle f, g\rangle_v = \int_V \frac{fg}{M}\, dv$$

and denote the induced Hilbert space and norm by $(L^2_v, \|\cdot\|_v)$. With respect to $\langle \cdot,\cdot\rangle_v$, the linear collision operator $\mathcal{L}f = M\rho_f - f$ is symmetric and negative semidefinite, a consequence of

$$\langle \mathcal{L}f, g\rangle_v = -\langle \mathcal{L}f, \mathcal{L}g\rangle_v.$$

The standard norms and spaces of functions of ξ are denoted by $(L^2_\xi, \|\cdot\|_\xi)$, $(H^k_\xi, \|\cdot\|_{H^k_\xi})$, and $(C^b_\xi, \|\cdot\|_\infty)$, and with weight (1.12) by $(L^2_W, \|\cdot\|_W)$ (see (1.13)). The Hilbert space $(L^2_\xi(L^2_v), \|\cdot\|_{\xi,v})$ is then naturally defined by the scalar product

$$\langle f, g\rangle_{\xi,v} = \int_{\mathbb{R}} \langle f, g\rangle_v\, d\xi.$$

For $k \in \mathbb{N} \cup \{0\}$, the space $H^k_\xi(L^2_v)$ of functions whose derivatives up to order k with respect to ξ are in $L^2_{\xi,v}$ is equipped with the norm

$$\|f\|_{H^k_\xi(L^2_v)} = \left(\|f\|^2_{\xi,v} + \cdots + \|\partial^k_\xi f\|^2_{\xi,v}\right)^{1/2} .$$

In a similar way $C^b_\xi(L^2_v)$ is defined by

$$\|f\|_{\infty,v} = \sup_{\xi\in\mathbb{R}} \|f\|_v .$$

Finally, we extend the definition of the norm with weight (1.12) to functions on $\mathbb{R} \times V$, leading to the space $(L^2_W(L^2_v), \|\cdot\|_{W,v})$ with norm

$$\|f\|_{W,v} = \|fW\|_{\xi,v} .$$

For later reference we note that the Cauchy-Schwarz inequality implies

$$\|\rho_f\|_\xi \le \|f\|_{\xi,v} . \tag{1.25}$$

A global existence and uniqueness result for the kinetic Cauchy problem is not hard to prove. We choose a simple setting, where the initial datum is bounded in terms of the equilibrium distribution.

Theorem 5 (Global existence)**.** *Let $0 \le f_0(x,v) \le \hat\rho M(v)$ hold. Then the kinetic equation (1.5) subject to the initial condition $f(t=0) = f_0$ has a unique mild solution $f \in C([0,\infty);\ L^\infty(\mathbb{R}\times V))$, satisfying*

$$0 \le f(t,x,v) \le \max\{\bar\rho, \hat\rho\} M(v) , \quad \forall\, (t,x,v) \in [0,\infty) \times \mathbb{R} \times V . \tag{1.26}$$

Proof. The mild formulation of the initial value problem is given by

$$\begin{aligned} f(t,x,v) &= f_0(x - vt/\varepsilon, v) + M(v)\left(\frac{1}{\varepsilon^2} + \bar\rho\right)\int_0^t \rho_f(\tau, x - v\tau/\varepsilon)d\tau \\ &\quad - \int_0^t \left(\frac{1}{\varepsilon^2} + \rho_f(\tau, x - v\tau/\varepsilon)\right) f(\tau, x - v\tau/\varepsilon, v)d\tau . \end{aligned} \tag{1.27}$$

For $T > 0$, we introduce the Banach space

$$\mathcal{C}_T = \{f \in C([0,T];\ L^\infty(\mathbb{R}\times V)) :\ \|f\|_{\mathcal{C}_T} < \infty\} ,$$
$$\|f\|_{\mathcal{C}_T} = \sup_{(t,x,v)\in[0,T]\times\mathbb{R}\times V} \frac{|f(t,x,v)|}{M(v)} .$$

Using the property $|\rho_f(t, x - vt/\varepsilon)| \le \|f\|_{\mathcal{C}_T}$ for all $(t,x,v) \in [0,T] \times \mathbb{R} \times V$, it is straightforward to uniquely solve (1.27) in $\mathcal{C}_T$ for small enough T by Picard iteration. Global existence will follow from (1.26).

The nonnegativity of f is an obvious consequence of the maximum principle for kinetic equations, after writing (1.5) in the form

$$\varepsilon^2\partial_t f + \varepsilon v\partial_x f + f(1 + \varepsilon^2\rho_f) = \rho_f M(1 + \varepsilon^2\bar\rho) ,$$

and solving by a fixed point iteration, where ρ_f is considered as given and nonnegative. The same argument applies to the function $h(t,x,v) = \max\{\bar\rho,\hat\rho\}M(v) - f(t,x,v)$, that satisfies

$$\varepsilon^2\partial_t h + \varepsilon v\partial_x h + h(1 + \varepsilon^2\rho_f) = \rho_h M + \varepsilon^2\rho_f M(\hat\rho - \bar\rho)_+ , \qquad h(t=0) \ge 0 ,$$

proving $h \ge 0$ and, thus, (1.26). □

A comparison principle, similar to Lemma 2, holds for the kinetic equation and will be used below in the proof of stability of kinetic traveling waves.

Lemma 6. *Let $f_1(t,x,v)$, $f_2(t,x,v) \ge 0$ be two solutions of (1.5). Let $f_2(0,x,v) \le \bar\rho M(v)$ and $f_1(0,x,v) \ge \gamma f_2(0,x,v)$, for all $x \in \mathbb{R}$ and $v \in V$, with $\gamma \le 1$. Then $f_1(t,x,v) \ge \gamma f_2(t,x,v)$, for all $t \ge 0$, $x \in \mathbb{R}$, $v \in V$.*

Proof. A simple computation shows that $g := f_1 - \gamma f_2$ satisfies

$$\varepsilon^2\partial_t g + \varepsilon v\partial_x g + (1+\varepsilon^2\rho_1)g = \rho_g(M + \varepsilon^2(\bar\rho M - f_2)) + \varepsilon^2\rho_1 f_2(1-\gamma)\,.$$

Theorem 5 implies that $f_2 \le \bar\rho M$ for all times, such that the coefficient of ρ_g is nonnegative. Since also the last term is nonnegative by the assumptions, the nonnegativity of g for all times follows as in the proof of Theorem 5. ☐

2. Existence of traveling waves

We prove existence of traveling waves of (1.1) with a given $s \ge s_0$ for $\varepsilon \ll 1$. The proof follows the steps of that in [4], stated in the subsequent sections. Essentially, we make the expansion in Section 2.1 rigorous, but first produce a residual term whose zeroth order moment in v vanishes.

2.1. **The asymptotic approximation.** We start by defining an asymptotic approximation of a traveling wave profile. In view of the computation of Section 1.2 we choose

$$f_{as}(\xi,v) = M(v)u_{TW}(\xi) + \varepsilon f^\perp[u_{TW}](\xi,v)\,,$$

where u_{TW} is a traveling wave of the Fisher equation (i.e. satisfying (1.2), (1.3)), made unique by the requirement

$$u_{TW}(0) = \frac{\bar\rho}{2}\,. \tag{2.1}$$

Recalling the formal expansion (1.24), we set

$$f^\perp[u] = -vMu' + \varepsilon(v^2 - D)Mu''\,.$$

Integration shows that $\int_V f^\perp[u]dv = 0$, implying $\rho_{as} := \rho_{f_{as}} = u_{TW}$. Clearly, f_{as} satisfies (1.21) and the equation (1.20) up to the residual

$$\begin{aligned}\varepsilon^3 h &= \varepsilon(v-\varepsilon s)\partial_\xi f_{as} - M\rho_{as} + f_{as} - \varepsilon^2\rho_{as}(M\bar\rho - f_{as})\\ &= \varepsilon^3(svMu''_{TW} + (v-\varepsilon s)(v^2-D)Mu'''_{TW} + u_{TW}f^\perp[u_{TW}])\,.\end{aligned}$$

It is now not hard to prove that

$$\int_V h\,dv = 0, \qquad \text{and} \qquad \|h\|_{H^k_\xi(L^2_v)} \le C_k \quad \text{for any} \quad k\in\mathbb{N}\,, \tag{2.2}$$

with ε-independent constants C_k.

2.2. **The micro-macro decomposition and the correction term.** In terms of the correction $\varepsilon^2 g = f_{TW} - f_{as}$, the traveling wave equation reads

$$\varepsilon(v-\varepsilon s)\partial_\xi g = \mathcal{L}g + \varepsilon^2 Bg + \varepsilon^4 R[g] - \varepsilon h\,, \tag{2.3}$$

where

$$Bg = \rho_g(M\bar\rho - f_{as}) - \rho_{as}g\,, \qquad R[g] = -\rho_g g\,.$$

On the right hand side of (2.3), we have collected the linear collision operator, a linear term of $O(\varepsilon^2)$, a nonlinear term of $O(\varepsilon^4)$, and the residual. By the properties of f_{as}, a solution g of (2.3) must satisfy the far-field conditions

$$g(\pm\infty, v) = 0 \qquad \text{for all } v\in V\,. \tag{2.4}$$

To prove the existence of such a g, we need some preparation. First, we observe that integration of (2.3) shows that necessarily

$$\partial_\xi \int_V (v-\varepsilon s)g\,dv = \varepsilon\rho_g(\bar\rho - 2\rho_{as}) - \varepsilon^3\rho_g^2\,. \tag{2.5}$$

We now decompose g into a macroscopic term (with separated variables), containing the leading order terms, and a microscopic term of order ε:

$$g(\xi,v) = \Phi(v)z(\xi) + \varepsilon w(\xi,v)\,. \tag{2.6}$$

Here Φ is chosen such that $\mathcal{L}\Phi = -\varepsilon\tau(v-\varepsilon s)\Phi + O(\varepsilon^2)$ for some constant τ, leading to

$$\Phi(v) = \left(1 + \varepsilon\frac{s}{D+\varepsilon^2 s^2}(v-\varepsilon s)\right) M(v)\,,$$

where the coefficient $s/(D+\varepsilon^2 s^2)$ guarantees that

$$\int_V (v-\varepsilon s)\Phi\, dv = 0\,, \tag{2.7}$$

and the decomposition of g is unique by requiring

$$\int_V (v-\varepsilon s)^2 w\, dv = 0\,. \tag{2.8}$$

Integration also shows that $\rho_\Phi = 1 - \varepsilon^2\tau s$ and that

$$D_1 := \int_V (v-\varepsilon s)^2 \Phi dv = D\,\frac{D-\varepsilon^2 s^2}{D+\varepsilon^2 s^2} = D + O(\varepsilon^2)\,,$$

which is positive for ε small enough. We also observe that, due to (2.7), (2.5) is equivalent to

$$\partial_\xi \int_V (v-\varepsilon s) w\, dv = \rho_g(\bar\rho - 2\rho_{as}) - \varepsilon^2\rho_g^2\,. \tag{2.9}$$

The problem we now solve is obtained by substituting (2.6) into (2.3), thus

$$(v-\varepsilon s)\Phi\, z' + \varepsilon(v-\varepsilon s)\partial_\xi w = \frac{1}{\varepsilon} z\,\mathcal{L}\Phi + \mathcal{L}w + \varepsilon B g + \varepsilon^3 R(g) - h\,, \tag{2.10}$$

and, like g, its micro- and macro-components z and w have to satisfy the homogeneous far-field conditions

$$w(\pm\infty, v) \equiv 0\,, \qquad z(\pm\infty) = 0\,. \tag{2.11}$$

The next step consists of writing (2.10) as a system of two equations; one containing only derivatives of z and the other containing only derivatives of w. This is achieved by applying the right macroscopic and microscopic projections. Applying

$$Pf := \int_V (v-\varepsilon s) f\, dv \tag{2.12}$$

to (2.10) we obtain, by (2.8),

$$D_1 z' + s\rho_\Phi z = P\mathcal{L}w + \varepsilon PBg + \varepsilon^3 PR(g) - Ph\,. \tag{2.13}$$

We differentiate (2.13) and use the moment relation (2.9). After multiplying the resulting equation by $D/D_1 = 1 + O(\varepsilon^2)$ and collecting the small linear and nonlinear terms on the right hand side we arrive at

$$Dz'' + sz' + z(\bar\rho - 2\rho_{as}) = \varepsilon B^z(z, z', w, \partial_\xi w) + \varepsilon^2 R^z(g, \partial_\xi g) - \tilde h\,, \tag{2.14}$$

where

$$\begin{aligned} B^z(z,z',w,\partial_\xi w) &= \frac{D}{D_1}\left[-\rho_w(\bar\rho - 2\rho_{as}) - s\rho_w' + \partial_\xi PBg\right] \\ &\quad + \frac{1}{\varepsilon}\left(1 - \frac{D}{D_1}\rho_\Phi\right)(sz' + z(\bar\rho - 2\rho_{as}))\,, \\ R^z(g,\partial_\xi g) &= \frac{D}{D_1}\left[\rho_g^2 + \varepsilon\partial_\xi PR(g)\right]\,, \qquad \tilde h = -\frac{D}{D_1}\partial_\xi Ph\,. \end{aligned}$$

The right hand side of (2.14) is the linearization of the Fisher equation at ρ_{as}.

The microscopic projection

$$\Pi f := f - \frac{(v-\varepsilon s)\Phi}{D_1} Pf \tag{2.15}$$

has the properties $\Pi(v-\varepsilon s)\Phi = 0$ and $\Pi(v-\varepsilon s)w = (v-\varepsilon s)w$, by (2.8). Applying Π to (2.10) we get the following equation for w:

$$\varepsilon(v-\varepsilon s)\partial_\xi w - \mathcal{L}w = \frac{(v-\varepsilon s)\Phi}{D_1}\int_V vw\,dv + \varepsilon\Lambda\, z + \varepsilon\Pi Bg + \varepsilon^3\Pi R(g) - \Pi h\,, \quad (2.16)$$

where

$$\Lambda := \frac{1}{\varepsilon^2}\Pi\mathcal{L}\Phi = s^2\frac{v^2-D}{D^2-\varepsilon^4 s^4}\,M = O(1)\,.$$

Since the symmetric operator $\mathcal{L}$ is only negative semidefinite, we introduce a new symmetric operator $\mathcal{M}$, which is strictly negative and coincides with $\mathcal{L}$ on the set of functions w satisfying (2.8) (this idea is borrowed from [2]):

$$\mathcal{M}w := \mathcal{L}w - (v-\varepsilon s)^2 M\int_V (v-\varepsilon s)^2 w\,dv\,.$$

Lemma 7. *The operator $\mathcal{M}$ is symmetric and negative definite with respect to $\langle\cdot,\cdot\rangle_v$. There exists a constant $\sigma > 0$, such that*

$$-\langle\mathcal{M}w, w\rangle_v \ge \sigma\|w\|_v^2 \qquad \text{for all } w\in L_v^2\,. \quad (2.17)$$

The proof is analogous to that in [4] and we do not repeat it here.

We now replace $\mathcal{L}$ in (2.16) by the operator $\mathcal{M}$:

$$\varepsilon(v-\varepsilon s)\partial_\xi w - \mathcal{M}w = \frac{(v-\varepsilon s)\Phi}{D_1}\int_V vw\,dv + \varepsilon\Lambda\, z + \varepsilon\Pi Bg + \varepsilon^3\Pi Rg - \Pi h. \quad (2.18)$$

The equivalence to the original problem is not obvious:

Lemma 8. *The function $g = \Phi z + \varepsilon w$ is a solution of (2.3), (2.4) if and only if z and w solve (2.14), (2.18) subject to (2.11).*

Proof. We follow the proofs in [2] and [4]. The problem (2.14), (2.18) (2.11) has been derived from (2.3), (2.4) using the properties (2.9), (2.8) of solutions of the latter. In particular (2.8) is not a necessary condition for existence. Hence we have to check that (2.8) also holds for solutions of (2.14), (2.18), (2.11), without requiring it as a side condition. Using

$$\int_V \Pi f\,dv = \int_V f\,dv, \qquad \int_V (v-\varepsilon s)\Pi f\,dv = 0\,,$$

integration of (2.18) implies

$$\begin{aligned}
\varepsilon\partial_\xi\int_V (v-\varepsilon s)w\,dv &= -(D+\varepsilon^2 s^2)\int_V (v-\varepsilon s)^2 w\,dv + \varepsilon(\rho_g(\bar\rho - 2\rho_{as}) - \varepsilon^2\rho_g^2)\,,\\
\varepsilon\partial_\xi\int_V (v-\varepsilon s)^2 w\,dv &= 2\varepsilon s D\int_V (v-\varepsilon s)^2 w\,dv\,.
\end{aligned}$$

The second equation is a linear ODE with constant coefficients for the unknown $\int_V (v-\varepsilon s)^2 w\,dv$. Since $w(\pm\infty, v) = 0$, the only possible solution is

$$\int_V (v-\varepsilon s)^2 w\,dv = 0\,.$$

Knowing this and returning to the first differential equation we also recover (2.9). □

We now eliminate the first term on the right hand side in (2.18) by substituting (2.13):

$$\varepsilon(v-\varepsilon s)\partial_\xi w - \mathcal{M}w = A(z,z') + \varepsilon Bg + \varepsilon^3 Rg - h\,, \quad (2.19)$$

where

$$A(z,z') = -\frac{(v-\varepsilon s)\Phi}{D_1}(D_1 z' + s\rho_\Phi z) + \varepsilon\Lambda\, z\,.$$

Thus we have arrived at our final differential problem (2.14), (2.19), subject to (2.11). In the following sections we show solvability via a fix-point argument.

2.3. **The Linear Problem.** We first analyze the leading order system of (2.14), (2.19), where the given inhomogeneity contains the higher order terms. In particular, we prove the solvability of

$$Dz'' + sz' + z(\bar\rho - 2\rho_{as}) = h_z\,, \qquad \text{with } h_z \in H^1_\xi\,, \tag{2.20}$$

$$\varepsilon(v - \varepsilon s)\partial_\xi w - \mathcal{M}w = A(z,z') + h_w\,, \qquad \text{with } h_w \in H^2_\xi(L^2_v)\,. \tag{2.21}$$

We shall look for solutions in the same spaces as the inhomogeneities. This replaces the homogeneous far-field conditions, and provides uniqueness for the solution of (2.21). This requirement allows, however, a one-parameter set of solutions of (2.20). This reflects the arbitrary shift in the wave and uniqueness will be guaranteed by posing also an initial condition,

$$z(0) = z_0\,, \quad z_0 \in \mathbb{R}\,. \tag{2.22}$$

For (2.20) we obtain

Lemma 9. *Let $h_z \in H^k_\xi$, $k \ge 0$. Then the problem (2.20), (2.22) with $s \ge s_0$ possesses a unique solution $z \in H^{k+2}_\xi$, satisfying (with $C > 0$ independent from z_0 and h_z)*

$$\|z\|_{H^{k+2}_\xi} \le C(|z_0| + \|h_z\|_{H^k_\xi})\,.$$

Proof. Since (2.20) is the linearization of (2.1) at its solution ρ_{as}, the derivative ρ'_{as} is a solution of the homogeneous equation. The standard order reduction procedure then allows to rewrite (2.20) as the first order system

$$z' = \frac{\rho''_{as}}{\rho'_{as}} z + z_1\,, \qquad z'_1 = -\left(\frac{s}{D} + \frac{\rho''_{as}}{\rho'_{as}}\right) z_1 + \frac{h_z}{D}\,. \tag{2.23}$$

Starting with the second equation, (2.1), $\rho'_{as} < 0$, and $0 < \rho_{as} < \bar\rho$ imply

$$-\left(\frac{s}{D} + \frac{\rho''_{as}}{\rho'_{as}}\right) = \frac{\rho_{as}(\bar\rho - \rho_{as})}{D\rho'_{as}} < 0\,.$$

Since, by the asymptotic behavior of ρ_{as}, this coefficient converges to negative values as $\xi \to \pm\infty$, the stronger statement

$$-\left(\frac{s}{D} + \frac{\rho''_{as}}{\rho'_{as}}\right) \le -\gamma < 0\,,$$

holds. By standard ODE methods, a unique decaying solution z_1 of the second equation in (2.23) exists for decaying h_z (using the 'boundary condition' $z_1(-\infty) = 0$). It can be estimated by testing the equation with z_1, giving

$$\|z_1\|_\xi \le \frac{1}{\gamma D}\|h_z\|_\xi\,.$$

Turning to the first equation in (2.23), we observe that

$$\lim_{\xi\to\infty} \frac{\rho''_{as}(\xi)}{\rho'_{as}(\xi)} < 0\,, \qquad \lim_{\xi\to-\infty} \frac{\rho''_{as}(\xi)}{\rho'_{as}(\xi)} > 0\,.$$

This is the situation covered in Lemma 3.5 of [4], implying the existence of a unique solution satisfying

$$\|z\|_\xi \le C'(|z_0| + \|z_1\|_\xi) \le C'\left(|z_0| + \frac{1}{\gamma D}\|h_z\|_\xi\right)\,.$$

Testing (2.20) with z and with z'' we obtain estimates for the first and second derivatives, implying $\|z\|_{H^2_\xi} \le C(|z_0| + \|h_z\|_{L^2_\xi})$. Finally, the same procedure can be applied to differentiated versions of (2.20), completing the proof. □

We remark that the previous proof makes use of the positivity and strict monotonicity of ρ_{as}. The assumption $s \geq s_0$ is therefore crucial.

Now $A(z, z')$ can be considered as a given inhomogeneity in (2.21), and the following result from [4] can be used:

Proposition 10. *Let $\tilde{h}_w \in H^k_\xi(L^2_v)$, $k \geq 0$. Then there exists a unique solution $w \in H^k_\xi(L^2_v)$ of*

$$\varepsilon(v - \varepsilon s)\partial_\xi w - \mathcal{M}w = \tilde{h}_w \,,$$

satisfying

$$\|w\|_{H^k_\xi(L^2_v)} \leq \frac{1}{\sigma}\|\tilde{h}_w\|_{H^k_\xi(L^2_v)} \,,$$

with σ as in Lemma 7.

Sketch of the proof. Uniqueness and the stability estimate are obtained by testing the equation with w and the k-th derivative of the equation with $\partial^k_\xi w$. Existence can be proven in several ways, one of which is the approximation by a discrete velocity system with a finite number of discrete velocities. This reduces the problem to an ODE system. Care has to be taken in order not to destroy the definiteness of $\mathcal{M}$ by the approximation. □

The final result on the linear problem can now be easily proven.

Lemma 11. *Let $h_z \in H^k_\xi$ and $h_w \in H^l_\xi(L^2_v)$, then there exists a unique solution $(z, w) \in H^{k+2}_\xi \times H^m_\xi(L^2_v)$, $m = \min\{k+1, l\}$, of (2.20), (2.21), (2.22), satisfying*

$$\|z\|_{H^{k+2}_\xi(L^2_v)} \leq C(|z_0| + \|h_z\|_{H^k_\xi}) \,, \qquad \|w\|_{H^m_\xi(L^2_v)} \leq C(|z_0| + \|h_z\|_{H^k_\xi} + \|h_w\|_{H^l_\xi(L^2_v)}) \,.$$

Proof. The only thing left to note is the estimate

$$\|A(z, z')\|_{H^{k+1}_\xi(L^2_v)} \leq \|z\|_{H^{k+2}_\xi} \,,$$

whose proof is straightforward by the definition of A. □

2.4. **The Nonlinear Problem.** In this section we prove existence and uniqueness of solutions of the nonlinear problem (2.19), (2.14), subject to $z(0) = z_0$, in the spaces H^3_ξ and $H^2_\xi(L^2_v)$, respectively. After the preparations in the previous sections, the proof is a straightforward contraction argument. We need, however, estimates for the right hand sides of (2.19) and (2.14). In the following, C denotes (possibly different) ε-independent constants.

Lemma 12. (i) *The linear terms B and B^z satisfy the estimate*

$$\|B(\Phi z + \varepsilon w)\|_{H^2_\xi(L^2_v)} + \|B^z(z, z', w, \partial_\xi w)\|_{H^1_\xi} \leq C(\|z\|_{H^2_\xi} + \|w\|_{H^2_\xi(L^2_v)}) \,.$$

(ii) *The nonlinearities R and R^z are quadratic: Let $g_1, g_2 \in H^2_\xi(L^2_v)$, then*

$$\|R(g_1) - R(g_2)\|_{H^2_\xi(L^2_v)} + \|R^z(g_1, \partial_\xi g_1) - R^z(g_2, \partial_\xi g_2)\|_{H^1_\xi}$$
$$\leq C\left(\|g_1\|_{H^2_\xi(L^2_v)} + \|g_2\|_{H^2_\xi(L^2_v)}\right)\|g_1 - g_2\|_{H^2_\xi(L^2_v)} \,.$$

Proof. The proof is straightforward. All that is needed for (ii) is the one-dimensional Sobolev embedding $H^1_\xi \subset C^b_\xi$ and (1.25). □

According to the spaces of the solutions and inhomogeneities of the linear problem we define the norm

$$\|(z, w)\| := \|z\|_{H^3_\xi} + \varepsilon\|w\|_{H^2_\xi(L^2_v)} \tag{2.24}$$

Clearly, $\|g\|_{H^2_\xi(L^2_v)}$ is bounded from above by $\|(z, w)\|$.

Before stating the existence result for traveling waves we note that in terms of the original unknown $f_{TW} = f_{as} + \varepsilon^2 g$, the condition $z(0) = z_0$ reads

$$\int_V (v - \varepsilon s)^2 (f_{TW}(0, v) - f_{as}(0, v)) dv = \varepsilon^2 D_1 z_0 \,. \tag{2.25}$$

Theorem 13. *Let the wave speed satisfy $s \geq s_0$. For every $z_0 \in \mathbb{R}$ and for ε small enough, there exists a solution f_{TW} of (1.20) satisfying (2.25), which is unique in a ball $\{f : \|f - f_{as}\| \leq \delta\}$, where the radius δ can be chosen independently from ε. It satisfies*

$$\|f_{TW} - f_{as}\|_{H^2_\xi(L^2_v)} = O(\varepsilon^2) \,,$$

or, more precisely,

$$f_{TW} = f_{as} + \varepsilon^2 \Phi z + \varepsilon^3 w = M u_{TW} - \varepsilon v M u'_{TW} + \varepsilon^2 (v^2 - D) M u''_{TW} + \varepsilon^2 \Phi z + \varepsilon^3 w \,, \tag{2.26}$$

where u_{TW} satisfies (1.2), (1.3) with (2.1), and $\|z\|_{H^3_\xi}$ and $\|w\|_{H^2_\xi(L^2_v)}$ are uniformly bounded as $\varepsilon \to 0$.

Proof. Let ε be small enough. Then as a consequence of Lemma 12 (i), the solvability results for the above linear problem (2.20), (2.21) can be extended to the full linear problem

$$Dz'' + sz' + z(\bar{\rho} - 2\rho) = \varepsilon B^z(z, z', w, \partial_\xi w) + h_z \,,$$
$$\varepsilon(v - \varepsilon s)\partial_\xi w - \mathcal{M} w = A(z, z') + \varepsilon B(z, w) + h_w \,,$$

with inhomogeneities h_z, h_w and $z(0) = z_0$. Applying the solution operator to the nonlinear problem (2.14), (2.19), we obtain a fixed point problem $(z, w) = \mathcal{G}(z, w)$, where the fix point operator is bounded by

$$\|\mathcal{G}(z, w)\| \leq C_0 (1 + \varepsilon^2 \|(z, w)\|^2) \,.$$

The constant C_0 bounds the initial condition and the residual terms, and the nonlinear terms are of order ε^2. We see that for ε small enough, $\mathcal{G}$ maps both the ball with radius $2C_0$ and the ball with radius $1/(2\varepsilon^2 C_0)$ into themselves. Also, with the property of the nonlinearity, the fixed point operator $\mathcal{G}$ is a contraction on a ball with radius of order $O(\varepsilon^{-2})$.

We can conclude that for ε small enough, the fixed point problem has a solution (z, w) with $\|(z, w)\| \leq 2C_0$, which is unique in a ball with an $O(\varepsilon^{-2})$-radius. Knowing this and returning to the fixed point problem, the boundedness of $\|w\|_{H^2_\xi(L^2_v)}$ follows. □

We remark that the contraction argument above could also be carried out in $H^k_\xi(L^2_v)$ for any $k \in \mathbb{N}$, by using Lemma 11, so the existence result also holds in $H^k_\xi(L^2_v)$ for $k \in \mathbb{N}$.

As in [4], we now deduce the monotonicity of ρ_{TW}.

Lemma 14. *Let the assumptions of Theorem 13 hold and let f_{TW} be a solution of (1.20) as described there. Then ρ_{TW} is strictly decreasing, implying, together with the far-field conditions (1.21), also the inequalities $0 < \rho_{TW} < \bar{\rho}$.*

The proof relies on the fact that the map $z_0 \mapsto \rho_{TW}(0)$ is invertible for ε small, meaning that the traveling wave can also be made locally unique by prescribing the value of $\rho_{TW}(0)$ instead of z_0. This argument can of course be repeated for every $\xi_0 \in \mathbb{R}$ instead of the origin. Now assuming ρ_{TW} is not strictly monotone would lead to the periodicity of f_{TW} as a consequence of the uniqueness result, which contradicts the far-field conditions.

Lemma 14 enables us to apply a comparison argument for the kinetic profile, based on ideas of Golse [6].

Lemma 15. *A traveling wave solution f_{TW} of (1.20), constructed as in Theorem 13, satisfies*

$$0 \le f_{TW}(\xi, v) \le M(v)\bar{\rho}\,, \quad \text{for all } (\xi, v) \in \mathbb{R} \times V\,.$$

Proof. We rearrange terms in (1.20) and write it as

$$\varepsilon(v - \varepsilon s)\partial_\xi f_{TW} + (1 + \varepsilon^2 \rho_{TW}) f_{TW} = (1 + \varepsilon^2 \bar{\rho}) M \rho_{TW} \ge 0\,.$$

We also observe that if we set $\bar{f} = M\bar{\rho} - f_{TW}$, then $\bar{f}$ satisfies

$$\varepsilon(v - \varepsilon s)\partial_\xi \bar{f} + (1 + \varepsilon^2 \rho_{TW})\bar{f} = M(\bar{\rho} - \rho_{TW}) \ge 0\,.$$

Thus, we only need to prove that for a given g, continuous in ξ, such that $g \to M\rho_\pm$ as $x \to \pm\infty$ at an exponential rate, for constants $\rho_\pm \ge 0$, then the inequality

$$\varepsilon(v - \varepsilon s)\partial_\xi g + (1 + \varepsilon^2 \rho_{TW}) g \ge 0 \tag{2.27}$$

implies $g \ge 0$, where ρ_{TW} is the macroscopic profile of a given traveling wave f_{TW}. The key to prove this is to rewrite (2.27) as

$$\varepsilon(v - \varepsilon s) e^{\left(-\xi - \varepsilon^2 \int_{\xi_0}^{\xi} \rho_{TW}(y)\,dy\right)/(\varepsilon(v - \varepsilon s))} \partial_\xi \left(e^{\left(\xi + \varepsilon^2 \int_{\xi_0}^{\xi} \rho_{TW}(y)\,dy\right)/(\varepsilon(v - \varepsilon s))} g \right) \ge 0$$

for a $\xi_0 \in \mathbb{R}$. Then the function

$$\xi \to e^{\left(\xi + \varepsilon^2 \int_{\xi_0}^{\xi} \rho_{TW}(y)\,dy\right)/(\varepsilon(v - \varepsilon s))} g$$

is nondecreasing when $v - \varepsilon s > 0$ and nonincreasing when $v - \varepsilon s < 0$. Now, taking any sequence $\{\xi_n\}_n$ such that $\xi_n \to -\infty$ as $n \to \infty$, when $v - \varepsilon s > 0$, implies that for all $\xi > \xi_n$,

$$e^{\left(\xi + \varepsilon^2 \int_{\xi_0}^{\xi} \rho_{TW}(y)\,dy\right)/(\varepsilon(v - \varepsilon s))} g(\xi, v) \ge e^{\left(\xi_n + \varepsilon^2 \int_{\xi_0}^{\xi_n} \rho_{TW}(y)\,dy\right)/(\varepsilon(v - \varepsilon s))} g(\xi_n, v) \to 0$$

as $n \to \infty$. Taking sequences $\xi_n \to \infty$ when $v - \varepsilon s < 0$, gives that $g(\xi, v) \ge 0$ for all $\xi < \xi_n$, and the result follows. □

3. Dynamic stability of traveling waves

In this section we prove the local asymptotic stability of traveling waves with speed $s > s_0$. For this purpose it is necessary to make the assumption
H1. The set of velocities V is bounded, and we set $v_{max} := \sup_{v \in V} |v|$.

As for the macroscopic equation in Section 1.1, we restrict our attention to nonnegative solutions. This can be done by taking nonnegative initial data, since Theorem 5 guarantees the nonnegativity of the solution.

In terms of the traveling wave variables, (1.5) becomes

$$\varepsilon^2 \partial_t f + \varepsilon(v - \varepsilon s)\partial_\xi f = M\rho_f - f + \varepsilon^2 \rho_f (M\bar{\rho} - f)\,, \tag{3.1}$$

and traveling waves are now stationary solutions of (3.1). We denote by $f_{TW}(\xi, v)$ a traveling wave solution as constructed in Theorem 13. The initial data for (3.1) are assumed to satisfy $f(t = 0) \ge \gamma f_{TW}$ with $0 < \gamma \le 1$. Using the estimate (Lemma 15) for the traveling wave, the comparison principle (Theorem 6) can be applied, such that the perturbation

$$G(t, \xi, v) = f(t, \xi, v) - f_{TW}(\xi, v)\,, \qquad \rho(t, \xi) := \rho_G(t, \xi)\,,$$

satisfies $G \ge (\gamma - 1) f_{TW}$ and, consequently, $\rho \ge (\gamma - 1)\rho_{TW}$. The equation for G is

$$\varepsilon^2 \partial_t G + \varepsilon(v - \varepsilon s)\partial_\xi G = M\rho - G + \varepsilon^2 (M\rho\bar{\rho} - (\rho_{TW} + \rho) G - \rho f_{TW})\,. \tag{3.2}$$

Before proceeding with the energy estimates we apply a micro-macro decomposition to G:

$$G = M\rho + G^\perp\,, \quad \text{i.e.} \int_V G^\perp dv = 0\,, \quad \text{implying} \quad \|G\|_v^2 = \rho^2 + \|G^\perp\|_v^2\,. \tag{3.3}$$

Using (3.3), the scalar product of (3.2) with G gives

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|G\|_{\xi,v}^2 + \frac{1}{\varepsilon^2}\|G^\perp\|_{\xi,v}^2 + \int_{\mathbb{R}}(2\rho_{TW} + \rho - \bar\rho)\rho^2 d\xi \\ &= -\int_{\mathbb{R}}(\rho_{TW} + \rho)\|G^\perp\|_v^2 d\xi - \int_{\mathbb{R}}\rho\langle f_{TW}, G^\perp\rangle_v d\xi \le C(\varepsilon^2\|\rho\|_\xi^2 + \|G^\perp\|_{\xi,v}^2)\,, \end{aligned} \tag{3.4}$$

where we have used $f_{TW} = M\rho_{TW} + O(\varepsilon)$. Here and in the following C denotes ε-independent constants. As in the purely macroscopic case, the integrand of the third term of (3.4) is negative as $\xi \to +\infty$, and we shall control it by combining (3.4) with an estimate in $L^2_{W,v}$.

We rewrite (3.2) in terms of GW,

$$\begin{aligned} &\partial_t(GW) + \frac{1}{\varepsilon}(v - \varepsilon s)\partial_\xi(GW) - \frac{1}{\varepsilon}\frac{s}{2D}(v - \varepsilon s)GW \\ &= -\frac{1}{\varepsilon^2}G^\perp W + (M\bar\rho\rho - (\rho_{TW} + \rho)G - \rho f_{TW})W\,, \end{aligned}$$

and perform the scalar product with GW, which gives the estimate

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\|G\|_{W,v}^2 + \left(\frac{1}{\varepsilon^2} + \frac{s^2}{2D}\right)\|G^\perp\|_{W,v}^2 + \int_{\mathbb{R}}\left(\frac{s^2}{4D} + \kappa + 2\rho_{TW} + \rho\right)(\rho W)^2 d\xi \\ &= -\int_{\mathbb{R}}(\rho_{TW} + \rho)W^2\|G^\perp\|_v^2 d\xi + \frac{s}{\varepsilon D}\int_{\mathbb{R}}\rho W^2\int_V vG^\perp\, dv\, d\xi \\ &\quad + \frac{s}{2D\varepsilon}\int_{\mathbb{R}} W^2\langle vG^\perp, G^\perp\rangle_v d\xi - \int_{\mathbb{R}}\rho W^2\langle f_{TW}, G^\perp\rangle_v d\xi \\ &\le \left(\frac{s^2}{4D} + \frac{\kappa}{2}\right)\|\rho\|_W^2 + \frac{s^2}{(s^2 + 2D\kappa)\varepsilon^2}\|G^\perp\|_{W,v}^2 + \frac{s}{2D\varepsilon}v_{max}\|G^\perp\|_{W,v}^2 \\ &\quad + C(\varepsilon^2\|\rho\|_W^2 + \|G^\perp\|_{W,v}^2)\,. \end{aligned} \tag{3.5}$$

In the last inequality we have used (3.3), the Young inequality, $\left(\int_V vG^\perp\, dv\right)^2 \le D\|G^\perp\|_v^2$, $\kappa > 0$, and **H1**. Now we multiply (3.5) by $\alpha > 0$ and add the result to (3.4):

$$\begin{aligned} &\frac{1}{2}\frac{d}{dt}\left(\|G\|_{\xi,v}^2 + \alpha\|G\|_{W,v}^2\right) + \frac{1 - \varepsilon^2 C}{\varepsilon^2}\|G^\perp\|_{\xi,v}^2 + \frac{\alpha}{\varepsilon^2}(K - \varepsilon C)\|G^\perp\|_{W,v}^2 \\ &+ \int_{\mathbb{R}}(2\rho_{TW} + \rho - \bar\rho - \varepsilon^2 C)\rho^2 d\xi + \alpha\int_{\mathbb{R}}(\kappa/2 + 2\rho_{TW} + \rho - \varepsilon^2 C)\rho^2 W^2 d\xi \le 0\,, \end{aligned} \tag{3.6}$$

with $K := 1 - s^2/(s^2 + 2D\kappa) > 0$. Now we proceed as in the macroscopic case (Section 1.1). Note that in the second line, compared to (1.15), the cost of controlling the microscopic terms is the factor $1/2$ in front of κ and the $O(\varepsilon^2)$-corrections. The comparison principle and Theorem 13 imply

$$2\rho_{TW} + \rho \ge \rho_{TW}(1 + \gamma) \ge u_{TW}(1 + \gamma) - \varepsilon^2 C\,.$$

As in Section 1.1, we choose ξ_0, such that $u_{TW}(\xi_0)(1 + \gamma) = \bar\rho(1 + \gamma/2)$, and α, such that $\alpha(1 + \gamma)u_{TW}(\xi)W(\xi)^2 \ge \bar\rho(1 + \gamma/2)$ for $\xi \ge \xi_0$. As a consequence, the second line of the left hand side of (3.6) can be estimated from below by

$$\left(\frac{\bar\rho\gamma}{2} - \varepsilon^2 C\right)\|\rho\|_\xi^2 + \alpha\left(\frac{\kappa}{2} - \varepsilon^2 C\right)\|\rho\|_W^2 \ge \left(\frac{1}{2}\min\{\bar\rho\gamma, \kappa\} - \varepsilon^2 C\right)\left(\|\rho\|_\xi^2 + \alpha\|\rho\|_W^2\right)\,.$$

Now it is obvious that, for ε small enough, exponential decay with the rate $\min\{\bar\rho\gamma,\kappa\}/2$ can be achieved, completing the proof of the main result of this section.

Theorem 16. *Let $f_{TW}(x-st,v)$ be a traveling wave solution of the kinetic equation as in Theorem 13 with wave speed $s>s_0$, and let $f(t,x,v)$ be a solution of the kinetic equation (1.5), whose initial values satisfy*

$$\int_{\mathbb{R}}\int_V \left(f(0,x,v)-f_{TW}(x,v)\right)^2 (1+e^{xs/D})dv\,dx<\infty\,,$$
$$f(0,x,v)\ge\gamma\, f_{TW}(x,v)\,,\ x\in\mathbb{R}\,,\ v\in V\,,$$

for a positive $\gamma\le 1$. Then, for ε small enough, there exists a positive constant c, such that

$$\int_{\mathbb{R}}\int_V \left(f(t,x,v)-f_{TW}(x-st,v)\right)^2 (1+e^{(x-st)s/D})dv\,dx\le c\,e^{-\lambda t}\,,\quad \forall\, t\ge 0\,,$$
$$\text{with}\quad \lambda=\frac{\bar\rho}{2}\min\left\{\frac{s^2}{s_0^2}-1,\gamma\right\}\,.$$

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