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THE HIGH FIELD ASYMPTOTICS FOR A FERMINIOC BOLTZMANN EQUATION: ENTROPY SOLUTIONS AND KINETIC SHOCK PROFILES

NAOUFEL BEN ABDALLAH
Institut de Mathématiques de Toulouse
Unité Mixte de Recherche 5219 (CNRS),Université Paul Sabatier
118, route de Narbonne, 31062 Toulouse Cedex, France
naoufel@mip.ups-tlse.fr

HEDIA CHAKER
Laboratoire de Modélisation et Simulation Numérique
dans les Sciences de l'INgénieurs,
Ecole Nationale d'Ingénieurs de Tunis
BP 37, Campus Universitaire, Le Belvédère
1002, Tunis, Tunisie
Hedia.Chaker@enit.rnu.tn

CHRISTIAN SCHMEISER
Faculty of Mathematics,
University of Vienna
Nordbergstraße 15
1090 Wien, Austria
Christian.Schmeiser@univie.ac.at

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Abstract. The high field approximation of a fermionic Boltzmann equation of semiconductors is performed after the formation of shocks. By employing a new entropy, whose dissipation measures the departure from the high field equilibrium, convergence towards the entropic solution of the limiting conservation law is proven. The entropy is also used to construct kinetic shock profiles for entropic shocks and to prove non-existence of non-entropic shock profiles.

Keywords: : Kinetic shock profiles, semiconductors, fermions, high fields, entropy solutions.

1. Introduction

The high field asymptotics of the Boltzmann equation of semiconductors is a fluid approximation, where both the collision effects and the driving forces dominate the free streaming. It has first been studied by Arlotti and Frosali in [1] and Poupaud in [30] for the linear Boltzmann operator. The limiting equation is a linear convection equation for the macroscopic particle density with a convection proportional to the scaled electric field. It has then been revisited by Cercignani, Gamba, and Levermore in [9,10] where the coupling with the Poisson equation for the electrostatic potential is included. More recently, Nieto, Poupaud, Soler [25], and lately with Goudon [23], the analysis of the high field limit coupled with the Poisson equation has been carried out for the Fokker-Plank equation by means of relative entropy techniques. For the sake of completeness, let us also mention that the high field limit has been studied starting from other macroscopic models instead of the Boltzmann equation, like the energy transport model by Degond and Jüngel [16], and the SHE model by Degond, Markowich, and the first and third authors [5]. Boundary layers appear when the asymptotics is performed in bounded domains and lead to half space problems that have been studied by Gamba, Klar, and the first author in [6] using techniques similar to the ones previously developed by several authors for the gas dynamics Boltzmann equation (e.g. [2,12]) and for diffusion approximations [29]. A similar program was then developed for the fermionic Boltzmann equation by the first two authors [3,4]. In this case, the high field approximation leads to a nonlinear conservation law for the particle density. In [3], the convergence is shown on time intervals on which the limit solution is regular. The techniques are based on the standard Hilbert expansion and on a careful analysis of the transport equation governing the remainder term. Additionally, boundary layers are analyzed in [4] in the spirit of [6]. This allows the rigorous convergence proof for boundary value problems (again for regular limit solutions). Two key ingredients of the method are the supersolution estimates for the Boltzmann equation (see [28]) as well as an entropy inequality satisfied by the fermionic Boltzmann operator.

The aim of the present work is to tackle the problem of convergence of solutions of the singularly perturbed Boltzmann equation towards the entropy solution of the limiting non linear conservation law. We first derive a new entropy inequality satisfied by the sum of the fermionic Boltzmann operator and the acceleration term. The entropy is constructed in the same spirit as in the work of Golse [21] for the Perthame-Tadmor model [27] and which was successfully generalized by the third author and Cuesta [14] for the BGK model. We use this entropy whose dissipation is shown to control the departure from the high field equilibrium in order to pass to the limit in the Boltzmann equation and immediately obtain the entropy solution of the limiting conservation law without an additional hypothesis on its regularity (which means that we pass to the limit even after the formation of shocks). Our proof is however restricted to the case of a constant electrostatic field as will be explained later on. Such a program has been recently performed by Berthelin, Mauser and Poupaud [7] for the high field limit of the Boltzman equation with the BGK operator. By using Kruzkov entropies combined with explicitly given high field equilibria, they prove the convergence to the entropy solution of the limiting conservation law for constant electric fields in the multidimensional case and for general electric fields in the one-dimensional case.

In the present work, the notion of entropy is also used to construct kinetic shock profiles connecting two different high field equilibria. This follows the path of Golse's proof for the Perthame-Tadmor model [21,14], which provides existence of shock profiles even for large data, as opposed to constructive approaches for small amplitude waves [8,15].

The outline of the paper is as follows. In Section 2, we recall the setting of the problem and state the main results: convergence to entropy solutions and the existence of shock profiles. In section 3, we prove the entropy inequalities which are the corner stone of the proofs. Section 4 is devoted to the convergence proof of the high field approximation for regular initial data, while Section 5 concerns the general case. Section 6 contains the proof of the result concerning the kinetic shock profiles: existence of entropic shock profiles as well as their monotonicity and uniqueness (up to translations) and non-existence of non-entropic shock profiles. Section 7 is devoted to the proof of an additional result, namely the dynamic stability of shock profiles.

2. Main results

The starting point is the initial value problem for a scaled Boltzmann equation

$$\partial_t f_\varepsilon + v \cdot \nabla_x f_\varepsilon + \frac{1}{\varepsilon}\left(E \cdot \nabla_v f_\varepsilon - Q(f_\varepsilon)\right) = 0\,, \tag{2.1}$$

$$f_\varepsilon(0,x,v) = f_{ini}(x,v)\,, \qquad \text{for } (x,v) \in \mathbb{R}^d \times \mathbb{R}^d\,, \tag{2.2}$$

with

$$Q(f)(v) = \int_{\mathbb{R}^d} \sigma(v,v')\left\{f(v')(1-f(v))M(v) - f(v)(1-f(v'))M(v')\right\} dv'\,,$$

where $\sigma(v,v')$ is the scattering cross section and $M(v)$ denotes the Maxwellian distribution

$$M(v) = \frac{1}{(2\pi)^{d/2}} \exp(-|v|^2/2)\,.$$

The unknown $f_\varepsilon(t,x,v)$ is the distribution of conduction electrons at time t in the position-velocity phase space $\mathbb{R}^d \times \mathbb{R}^d$. The electric field E is assumed as given and constant, and the Knudsen number ε is a dimensionless parameter. The macroscopic limit $\varepsilon \to 0$ with the above scaling of the electric field (balancing the scattering effects) is called the high field limit. The scattering operator Q models the interaction of electrons with the semiconductor crystal lattice. The factors $(1-f)$ causing the quadratic nonlinearity take into account the Pauli exclusion principle.

Under the hypothesis

$$\textbf{(H1)} \quad \sigma \in W^{2,\infty}(\mathbb{R}^{2d}), \quad 0 < \sigma_0 \le \sigma(v,v') = \sigma(v,v') \le \sigma_1,$$

on the scattering cross section, the following result for the formal limiting equation holds.

Theorem 2.1.
i) For every $E \in \mathbb{R}^d$ and $n \in \mathbb{R}^+$, there exists a unique function $F(n,E) \in L^1(\mathbb{R}^d)$ such that $E \cdot \nabla_v F(n,E) \in L^1(\mathbb{R}^d)$, $0 \le F(n,E) \le 1$, and which satisfies

$$E \cdot \nabla_v F(n,E) - Q(F(n,E)) = 0\,, \qquad \int_{\mathbb{R}^d} F(n,E)(v)dv = n\,. \tag{2.3}$$

Moreover, $(n,E) \mapsto F(n,E)$ is C^2 as a mapping from $\mathbb{R}_+ \times \mathbb{R}^d$ to $L^1(\mathbb{R}^d)$, and it is strictly increasing with respect to n, pointwise in E and v.

ii) For fixed n and E, $F(n,E) \in W^{2,\infty}(\mathbb{R}^d)$ as a function of v. There exist positive constants $\overline{F}(n,E) < 1$, $\gamma(n,E)$, and $C(n,E)$, such that

$$F(n,E)(v) \le \overline{F}(n,E)\,, \quad F(n,E)(v) \le C(n,E)e^{-\gamma(n,E)|v|}\,, \qquad v \in \mathbb{R}^d\,.$$

iii) For any given E, the function $(n,v) \to F(n,E)(v)$ is a C^2 mapping from $\mathbb{R}^+ \times \mathbb{R}^d$ on $[0,1)$.

Proof. Statement i) has been proven in [3]. The results in ii) are obviously satisfied for $E = 0$. For the rest of the proof we therefore assume $E \neq 0$ and, w.l.o.g., $E = (E_1, 0, .., 0)$ and $E_1 > 0$. The smoothness result $F(n,E) \in W^{2,\infty}(\mathbb{R}^d)$ is then an obvious consequence of Hypothesis (H1). Considering $F' = F(v')$ as given, the equation for F,

$$\partial_{v_1} F = \frac{M}{E_1}\int_{\mathbb{R}^d} \sigma F'\, dv' - \frac{F}{E_1}\int_{\mathbb{R}^d} \sigma(MF' + M'(1-F'))dv'\,, \tag{2.4}$$

is a linear ODE, where the coefficient

$$-\frac{1}{E_1}\int_{\mathbb{R}^d} \sigma(MF' + M'(1-F'))dv' \le -\frac{\sigma_0}{E_1}\int_{\mathbb{R}^d} M'(1-F')dv' =: -\gamma < 0\,,$$

is bounded from above by a negative constant. As a function of v_1, the unique bounded solution decays like $e^{-\gamma v_1}$ as $v_1 \to \infty$ (since the inhomogeneity $M/E_1 \int_{\mathbb{R}^d} \sigma F'\, dv' \le M\sigma_1 n/E_1$ decays faster), and it decays like the inhomogeneity (i.e., like the Maxwellian M), as $v_1 \to -\infty$. As a function of $(v_2, \dots, v_d)$, the decay is also like the Maxwellian. Finally, if F would take the value 1 at a point $v_1 = \overline{v}$, then by the differential equation, $\partial_{v_1} F$ would be negative there and, as a consequence, F would assume values larger than 1 for some $v_1 < \overline{v}$, contradicting i). Therefore $F(n,E)(v) < 1$ for all v and, by its smoothness and by the decay properties, it keeps a positive distance from 1.

iii) The equation for F is

$$\partial_{v_1} F(n,E) + \frac{1}{E_1}\lambda(F(n,E))F(n,E) = \frac{M}{E_1}\mu(F(n,E))\,,$$

with

$$\mu(F(n,E))(v) = \int_{\mathbb{R}^2} \sigma(v,v')F(n,v)(v')dv'$$

and

$$\lambda(F(n,E))(v) = \int_{\mathbb{R}^d} \sigma(v,v')(M(v)F(n,E)(v') + M(v')(1 - F(n,E)(v')))dv' .$$

If μ and λ are considered as given, this is a linear ODE, where the coefficients satisfy

$$\frac{1}{E_1}\lambda(F(n,E)) \geq \lambda_0 > 0 ,$$

and

$$\mu(F(n,E))(v) \leq \sigma_1 n^* .$$

The unique bounded solution is given by

$$F(n,E)(v) = \int_{-\infty}^{v_1} \frac{\mu(F(n,E)(u,\mathbf{v_2})}{E_1} \exp(-\frac{1}{E_1}\int_u^{v_1} \lambda(F(n,E)(\tau,\mathbf{v_2})d\tau)du \quad (2.5)$$

where we have denoted $\mathbf{v_2} = (v_2, v_3, \cdots, v_d)$. Since $F(n,E)$ is C^2 as a mapping from $\mathbb{R}_+ \times \mathbb{R}^d$ to $L^1(\mathbb{R}^d)$, the coefficients $\lambda(F(n,E))(v)$ and $\mu(F(n,E))(v)$ are C^2 as mappings from $\mathbb{R}^+ \times \mathbb{R}^d$ on $\mathbb{R}^+$ $[0,1)$. This implies that for a given E, $F(\cdot,E)(\cdot)$ is C^2 as mapping from $\mathbb{R}^+ \times \mathbb{R}^d$ on $[0,1)$. □

Since the electric field E is assumed to be a given constant all along this paper, we shall skip this dependence and use the notation $F(n)$ instead of $F(n,E)$.

Formally, the high field limit $\varepsilon \to 0$ of (2.1) gives $f_\varepsilon(t,x,v) \to F(n(t,x))(v)$, where the macroscopic density $n(t,x)$ satisfies the mass conservation equation

$$\partial_t n + \nabla_x \cdot j(n) = 0 , \quad (2.6)$$

with $j(n) = \displaystyle\int_{\mathbb{R}^d} vF(n)(v)dv$, which is a nonlinear function of n. The existence and uniquenss of an entropy solution of this equation is standard (see [31]).

As mentioned above, the aim of this paper is to prove that the kinetic solution converges toward this entropy solution on arbitrary time intervals. More precisely, we shall show the following result

Theorem 2.2. *Let (H1) hold and let the initial data satisfy*

$$0 \leq f_{ini}(x,v) \leq 1, \quad f_{ini} \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v) \quad \text{and} \quad |v|^2 f_{ini} \in L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v). \quad (2.7)$$

Assume also that $n_{ini} = \int_{\mathbb{R}^d} f_{ini}\, dv \in L^\infty(\mathbb{R}^d_x)$. *Then, the problem (2.1), (2.2) has a unique weak solution* $f_\varepsilon \in C^0(\mathbb{R}^+, L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v))$ *such that* $0 \leq f_\varepsilon \leq 1$. *This solution satisfies* $|v|^2 f_\varepsilon \in L^\infty_{loc}(\mathbb{R}^+, L^1(\mathbb{R}^d_x \times \mathbb{R}^d_v))$. *As* ε *tends to zero, we have*

$$f_\varepsilon \to f \qquad in\ L^\infty((0,T),\ L^1(\mathbb{R}^d_{x,loc} \times \mathbb{R}^d_v)) ,$$

$$\int_{\mathbb{R}^d} f_\varepsilon dv \to n \qquad in\ L^\infty((0,T), L^1_{loc}(\mathbb{R}^d)) ,$$

where the limits satisfy $f = F(n)$, and n satisfies in the distributional sense

$$\partial_t n + \nabla_x \cdot j(n) = 0, \tag{2.8}$$

and

$$\partial_t \mathcal{X}(n) + \nabla_x \cdot G(n) \leq 0, \tag{2.9}$$

where $\mathcal{X}(n) = \int_0^n \chi(\nu)d\nu$, *$\chi$ is any increasing C^1-function, and* $G'(n) = \chi(n)j'(n)$. *Besides, for any positive R*

$$\lim_{T\to 0^+} \frac{1}{T}\int_0^T \int_{|x|\leq R} |n(t,x) - n_{ini}(x)|dxdt = 0, \tag{2.10}$$

so that n is the unique entropy solution of (2.8).

Remark 2.3.
i) The assumption $|v|^2 f_{ini}$ in $L^1(\mathbb{R}_x^d \times \mathbb{R}_v^d)$ is needed to insure the existence of $j_\varepsilon = \int_{\mathbb{R}^d} v f_\varepsilon \, dv$. The fact that $|v|^2 f_\varepsilon(t,.,.)$ stays in$L^1(\mathbb{R}_x^d \times \mathbb{R}_v^d)$ for all time is a consequence of standard entropy. Indeed, multiplying the Boltzmann equation by $\log(f/(1-f)M)$ and integrating with respect to x, v, we obtain

$$\partial_t \int_{\mathbb{R}^d\times\mathbb{R}^d} \left[\frac{|v|^2}{2} f_\varepsilon + f_\varepsilon \log(f_\varepsilon) - (1-f_\varepsilon)\log(1-f_\varepsilon)\right] dxdv - \frac{1}{\varepsilon}\int_{\mathbb{R}^d} E\cdot j_\varepsilon dx \leq 0$$

This leads to $\int_{\mathbb{R}^d\times\mathbb{R}^d} |v|^2 f_\varepsilon dxdv \leq C_\varepsilon$ on $[0,T]$ for all $T>0$ (C_ε blows up when ε tends to 0^+).
ii) The equality (2.10) is needed to obtain the uniqueness of the solution as mentioned in [11,32,26]. The proof of theorem 2.2 is based on compactness arguments. Since the limit is unique, we shall not mention in the sequel extraction of subsequences.

Our further results deal with travelling wave solutions of the one-dimensional verison of (2.1). For the rest of this section we therefore set $d = 1$ and look for solutions of (2.1) of the form $f_\varepsilon(t,x,v) = g(\frac{x-ut}{\varepsilon}, v)$ with wave speed u, connecting two different equilibria $F(n_-)$ and $F(n_+)$. The profile g has to satisfy

$$(v-u)\partial_\eta g + E\partial_v g = Q(g), \qquad \text{for } (\eta,v)\in\mathbb{R}\times\mathbb{R}. \tag{2.11}$$

subject to the far field conditions

$$\lim_{\eta\to-\infty} g(\eta,v) = F(n_-)(v), \qquad \lim_{\eta\to+\infty} g(\eta,v) = F(n_+)(v), \tag{2.12}$$

with positive densities $n_- \neq n_+$.

Remark 2.4. The convergence of (2.12) is meant in $L^1(\mathbb{R}_v)$ weak. It actually holds in the strong topology as the function g will be shown to be monotonous with respect to η. The bound $0 \leq g \leq 1$ yields the convergence in $L^p(\mathbb{R}_v)$ for all $1 \leq p < +\infty$.

In the theory of diffusive regularizations of the conservation law (2.8), the existence of travelling waves connecting different states strongly depends on convexity properties of the flux function. An asymptotic analysis shows that $\partial_n^2 j(0,E) = \kappa E + O(E^2)$ with $\kappa > 0$ holds, and numerical experiments [17] suggest the strict convexity of j with respect to n for $E > 0$ (concavity for $E < 0$), also away from the origin (see figure 1).

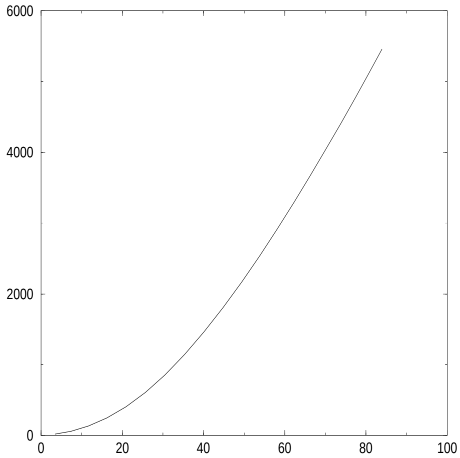

Fig. 1. Numerical simulation of $J(n)$ for $E > 0$ in the relaxation time approximation

However, since we are unable to prove this, we shall pose it as a hypothesis. Since the travelling wave equation (2.11) is invariant under the reflection $(E, u, v, \eta) \to (-E, -u, -v, -\eta)$, we can also w.l.o.g. restrict to one sign of E:

$$\textbf{(H2)} \qquad E > 0 \quad \text{and} \quad \partial_n^2 j(n, E) > 0 \quad \text{for all } n > 0\,. \tag{2.13}$$

For our results below, n only varies within bounded subsets of $\mathbb{R}_+$, and it would be sufficient to require (H2) there.

We shall look for solutions in the set $\mathcal{S}$ defined by

$$\mathcal{S} = \{g \in L^\infty(\mathbb{R}_x \times \mathbb{R}_v), \quad \exists n^* > 0, \quad 0 \le g(x,v) \le F(n^*)(v)\}\,. \tag{2.14}$$

Theorem 2.5. *Let (H1)–(H2) hold. Then*
i) For a solution of (2.11), (2.12) to exist in $\mathcal{S}$, u has to be given by the Rankine-Hugoniot formula

$$u = \frac{j(n_+) - j(n_-)}{n_+ - n_-}. \tag{2.15}$$

ii) If $n_+ > n_-$ (non entropic shock), (2.11), (2.12) has no solution in $\mathcal{S}$.
iii) If $n_- > n_+$ (entropic shock) and (2.15) holds, then (2.11), (2.12) has a solution $g \in \mathcal{S}$ (g is unique up to translations with respect to η). The solution satisfies $F(n_+)(v) \le g(x,v) \le F(n_-)(v)$. The far field conditions (2.12) are sat-

isfied in $L^1(\mathrm{IR}_v)$ *weak. Furthermore,* $g \in C^0(\mathrm{IR}_\eta, L^1(\mathrm{IR}_v) - weak))$, *i.e., for all* $\theta \in L^\infty(\mathrm{IR}_v)$, $\int \Theta(v) g(\eta, v) dv$ *is continous. In particular, the macroscopic density* $\rho(\eta) = \displaystyle\int g(\eta, v) dv$ *is continuous and monotonically decreasing.*

Remark 2.6. Note that the entropy condition derived by constructing kinetic shock profiles is the same as for viscous shock profiles. For small shocks this has to be expected (see, e.g., [14]), but for large amplitude shocks it is not clear a priori.

The last result of this paper concerns the dynamic stability of the travelling waves constructed above. We shall prove that solutions of the Cauchy problem

$$\partial_t f + v\partial_x f + E\partial_v f = Q(f)\,, \tag{2.16}$$

$$f(0,x,v) = f_{ini}(x,v)\,, \qquad \text{for } (x,v) \in \mathrm{IR}\times\mathrm{IR}\,, \tag{2.17}$$

approach travelling waves as $t \to \infty$, if the initial datum has the far field behaviour of an entropic shock, i.e., if $f_{ini}(x,v) - g(x,v) \in L^1(\mathrm{IR}^2_{x,v})$, where g is a travelling wave solution of (2.11), (2.12) with $n_- > n_+$. The monotonicity of shock profiles then implies that there is a unique profile g (i.e., a unique shift in the η-direction) satisfying

$$\int_{\mathrm{IR}}\int_{\mathrm{IR}} (f_{ini}(x,v) - g(x,v))dvdx = 0\,. \tag{2.18}$$

By mass conservation, this property is propagated in time and, thus, g is the only candidate for the asymptotic profile.

Theorem 2.7. *Let* f *be a solution of (2.16), (2.17), where* $0 \le f_{ini} \le 1$ *and (2.18) holds for a solution* g *of (2.11), (2.12). Then, for every sequence* $t_n \to \infty$, $f(t_n + t, \eta + u(t_n + t), v) \to g(\eta, v)$ *in* $L^\infty((0,T)\times\mathrm{IR}^2_{\eta,v}))$ *weak*.*

3. Entropy inequalities

In order to prove the results stated in the previous section, we shall employ two types of entropy inequalities. We first begin by the following definition which immediately follows from Theorem 2.1.

Lemma 3.1. *For any given* v, *the function* $F_v : n \mapsto F(n)(v)$ *is a* C^2 *mapping from* $[0,+\infty)$ *on* $[0,1)$. *It is strictly increasing and satisfies* $\lim_{n\to 0^+} F(n)(v) = 0$ *and* $\lim_{n\to+\infty} F(n)(v) = 1$.

Lemma 3.2.
i) For any given v, *let* $\varphi(\cdot, v)$ *be the inverse function of* F_v. *The function* $\varphi(., v)$ *is a strictly increasing* C^2 *mapping from* $[0,1)$ *on* $[0,+\infty)$. *We have of course* $n = \varphi(f, v)$ *if and only if* $f = F(n)(v)$.
ii) The mapping $(f, v) \mapsto \varphi(f, v)$ *is* C^2 *as a mapping from* $[0,1)\times\mathrm{IR}^d$ *on* $[0,+\infty)$.

A corner stone of our analysis is the following entropy inequality

Proposition 3.3. *Let (H1) hold, let $f = f(v) \in L^1(\mathbb{R}^d)$ and $E \cdot \nabla_v f \in L^1(\mathbb{R}^d)$ satisfy $0 \leq f \leq F(n^*)$ for some positive real number n^*, and denote*

$$n_f = \int_{\mathbb{R}^d} \varphi(f(v), v) M(v) dv \leq n^*.$$

Let χ be C^1 increasing function such that $\chi' \geq \alpha > 0$ on $[0, n^*]$. Then there exists a positive constant $C = C(\alpha, n^*)$ such that

$$D(f) := \int_{\mathbb{R}^d} (Q(f) - E \cdot \nabla_v f) \chi(\varphi(f, v)) dv \leq -C \int_{\mathbb{R}^d} (f - F(n_f))^2 M(v)\, dv\,. \quad (3.1)$$

Proof. In order to prove the property (3.1), we start by regularizing the function f. Consider a family $f^k \in C^1(\mathbb{R}^d)$ converging in L^p to f as $k \to +\infty$ for $1 \leq p \leq \infty$ and $E \cdot \nabla_v f^k$ converging in L^1 to $E \cdot \nabla_v f$ as $k \to +\infty$. The below term can be computed as follows

$$\begin{aligned}\int_{\mathbb{R}^d} E \cdot \nabla_v f^k \chi(\varphi(f^k, v)) dv &= \int_{\mathbb{R}^d} E \cdot \nabla_v \int_0^{f^k} \chi(\varphi(g, v)) dg dv \\ &\quad - \int_{\mathbb{R}^d} \int_0^{f^k} E \cdot \nabla_v \chi(\varphi(g, v)) dg dv\,. \\ &= - \int_{\mathbb{R}^d} \int_0^{f^k} E \cdot \nabla_v \chi(\varphi(g, v)) dg dv\,.\end{aligned}$$

Since $g = F(\varphi(g, v), v)$, we have $\partial_n F(\varphi(g, v), v) \nabla_v (\varphi(g, v) + \nabla_v F(\varphi(g, v), v) = 0$. Therefore, we obain

$$\begin{aligned}\int_{\mathbb{R}^d} \int_0^{f^k} E \cdot \nabla_v \chi(\varphi(g, v)) dg dv &= \int_{\mathbb{R}^d} \int_0^{f^k} E \cdot \nabla_v \varphi(g, v) \chi'(\varphi(g, v)) dg dv \\ &= - \int_{\mathbb{R}^d} \int_0^{f^k} E \cdot \nabla_v F(\varphi(g, v)) \chi'(\varphi(g, v)) \frac{dg dv}{\partial_n F(\varphi(g, v))}\,.\end{aligned}$$

Using $dg = \partial_n F(n, v) dn$ and $E \partial_v F = Q(F)$, we obtain

$$\int_{\mathbb{R}^d} \int_0^{f^k} E \cdot \nabla_v \varphi(g, v) \chi'(\varphi(g, v)) dg dv = - \int_{\mathbb{R}^d} \int_0^{\varphi(f^k, v)} \chi'(n) Q(F(n)) dn dv\,.$$

By passing in the limit in the integral and by using the domined convergence theorem, we obtain

$$\int_{\mathbb{R}^d} E \cdot \nabla_v f \chi(\varphi(f, v)) dv = - \int_{\mathbb{R}^d} \int_0^{\varphi(f, v)} \chi'(n) Q(F(n)) dn dv\,.$$

The entropy dissipation can now be written as

$$\begin{aligned}D(f) &= \int_{\mathbb{R}^d} \int_0^{\varphi(f, v)} [Q(f) - Q(F(n))] \chi'(n) dn dv \\ &= - \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\varphi(f', v')}^{\varphi(f, v)} \sigma \chi'(n) [M(1 - f) + M' F(n)] (F(n)' - f') dn dv dv'\,.\end{aligned}$$

This already shows the nonpositivity of $D(f)$ since $(\varphi(f,v)-\varphi(f',v'))(F(n)'-f') \geq 0$ for n between $\varphi(f,v)$ and $\varphi(f',v')$ and $\chi'(\varphi(g,v)) > 0$.
Moreover, using (H1) and the boundedness assumption on f, we obtain

$$-D(f) \geq C_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} \int_{\varphi(f',v')}^{\varphi(f,v)} M(F(n)' - f') dn dv dv' \,, \tag{3.2}$$

where $C_1 = \sigma_0(1 - \sup_{v \in \mathbb{R}^d} F(n^*)(v)) \inf_{n \in [0,n^*]} \chi'(n)$. The mean value theorem implies

$$|F(n)' - f'| \geq h' |n - \varphi(f',v')| \,,$$

where $h(v) = \inf_{n \in (0,n^*)} \partial_n F(n)(v)$. By Theorem 2.1, h is strictly positive and in L^1.
Using this in (3.2), we obtain

$$\begin{aligned} -D(f) &\geq C_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} M h' (\varphi(f,v) - \varphi(f',v'))^2 dv dv' \\ &= C_1 \int_{\mathbb{R}^d} \int_{\mathbb{R}^d} M h' \left[(\varphi(f',v') - n_f)^2 + (n_f - \varphi(f,v))^2 \right] dv dv' \\ &\geq C_1 \int_{\mathbb{R}^d} M (n_f - \varphi(f,v))^2 dv \,, \end{aligned}$$

A second application of the mean value theorem completes the proof of (3.1) with

$$C = C_1 \left(\sup_{n \in (0,n^*)} \sup_{v \in \mathbb{R}^d} \partial_n F(n)(v) \right)^{-2} > 0 \,.$$

□

Remark 3.4. Inequality (3.1) shows that the entropy production controls a weighted L^2 norm of the departure from equilibrium. However, because the weight is exponentially small for large velocities, the information on large velocities is very poor. This is however compensated by the inequality $f \leq F(n^*)$ which provides this missing control.

The second entropy inequality is known and is recalled here below from the literature.

Proposition 3.5. *i) [29] For any f, $g \in L^1(\mathbb{R}^d)$ such that $0 \leq f, g \leq 1$, we have*

$$-\int_{\mathbb{R}^d} (Q(f) - Q(g)) \operatorname{sgn}(f-g) dv \geq 0 \,. \tag{3.3}$$

Equality holds iff $\operatorname{sgn}(f-g)$ is constant.

ii) [4] Moreover, if $\int_{\mathbb{R}^d} (f-g) dv = 0$, we have

$$-\int_{\mathbb{R}^d} \int_{\mathbb{R}^d} (Q(f) - Q(g)) \operatorname{sgn}(f-g) dv \geq C(g) \int_{\mathbb{R}^d} |f-g| dv \,, \tag{3.4}$$

where $C(g) = \sigma_0 \int_{\mathbb{R}^d} (1-g) M \, dv$.

We end the section by three technical lemmata which follow from the above proposition.

Lemma 3.6. *Let* $f,\, g \in C^0(\mathbb{R}_\eta, L^1(\mathbb{R}_v) - weak)$ *be two solutions of (2.11) with* $F(n_+) \leq f, g \leq F(n_-)$*, such that there exist two sequences* $\{\nu_k\}$ *and* $\{\mu_k\}$ *with* $\lim_{k\to\infty} \nu_k = -\infty$*,* $\lim_{k\to\infty} \mu_k = +\infty$*, and*

$$\lim_{k\to\infty} (f(\nu_k, v) - g(\nu_k, v)) = 0\,, \qquad \lim_{k\to\infty} (f(\mu_k, v) - g(\mu_k, v)) = 0 \text{ in } L^1(\mathbb{R}_v) \text{ weak.}$$

Then the function $sgn(f(\eta, v) - g(\eta, v))$ *is independent of* v*.*

Proof. The difference $H = f - g$ satisfies

$$\begin{aligned} &(v-u)\cdot \frac{\partial H}{\partial \eta} + E \cdot \frac{\partial H}{\partial v} = Q(f) - Q(g)\,, \\ &\lim_{k\to+\infty} H(\nu_k, v) = 0\,, \qquad \lim_{k\to+\infty} H(\mu_k, v) = 0\,. \end{aligned} \tag{3.5}$$

We multiply (3.5) by $sgn(H)$, integrate over $(\nu_k, \mu_k) \times \mathbb{R}$ and let $k \to \infty$ to obtain

$$0 = \int_{\mathbb{R}\times\mathbb{R}} (Q(f) - Q(g)) sgn(f-g) dv dx\,.$$

From Proposition 3.5, the function $sgn(H) = sgn(f-g)$ is independent of v. □

Lemma 3.7. *If* $f,\, g \in C^0(\mathbb{R}_\eta, L^1(\mathbb{R}_v) - weak))$ *are two solutions of (2.11) such that* $F(n_+) \leq f, g \leq F(n_-)$*, and* $sgn(f(\eta, v) - g(\eta, v))$ *is independent of* v*, then for all* $\eta_1, \eta_2 \in \mathbb{R}$*, there exists a constant* $C_{|\eta_2-\eta_1|}$ *only depending on* $\mid \eta_2 - \eta_1 \mid$ *such that*

$$\int_{\mathbb{R}} \mid v-u \mid\mid (f-g)(\eta_2, v) \mid dv \leq C_{|\eta_2-\eta_1|} \int_{\mathbb{R}} \mid v-u \mid\mid (f-g)(\eta_1, v) \mid dv\,.$$

Proof. Multiplying $H = g - f$ by $sgn(H)$ we deduce from (3.5)

$$(v-u)\cdot \frac{\partial |H|}{\partial \eta} + E \cdot \frac{\partial |H|}{\partial v} + \lambda(f)|H| = |S| \tag{3.6}$$

where

$$\lambda(f) = \int_{\mathbb{R}} \sigma(v, v')\,\{f'M + (1-f')M'\}\,dv',$$

$$S = (1-g)\mu(H) + g\gamma(H),$$

$$\mu(f)(v) = M(v)\int_{\mathbb{R}} \sigma(v,v') f' dv', \qquad \gamma(f)(v) = \int_{\mathbb{R}} \sigma(v,v') M' f' dv'.$$

Since $F(n_+) \leq f(\eta, v) \leq F(n_-)$ for all $(\eta, v) \in \mathbb{R}\times\mathbb{R}$, $\lambda(f)$,$\mu(f)$ and $\gamma(f)$ are bounded functions and there exist $0 < \lambda_0 < \lambda_1 < \infty$ such that

$$\lambda_0 \leq \lambda(f) + \frac{\mu(f)(v)}{\int_{\mathbb{R}} f' dv'} + \frac{\gamma(f)(v)}{\int_{\mathbb{R}} M' f' dv'} \leq \lambda_1\,.$$

This implies in particular the inequality

$$\int_{\Omega_v} |S| dv dx \geq C_{\Omega_v} \int_{\mathbb{R}} |S| dv dx \tag{3.7}$$

for any Ω_v with non zero measure.

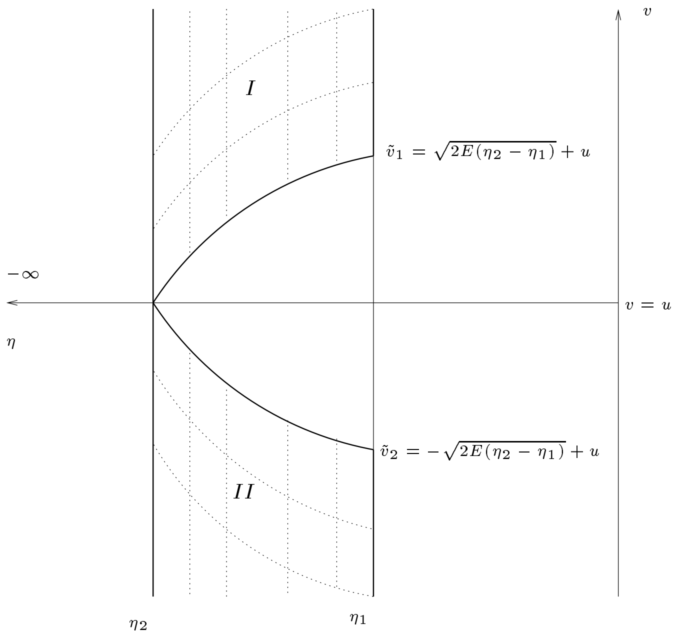

Fig. 2.

Now, let us define the multiplier θ by $\theta(\eta, v) = \exp\left(\lambda_1 (\frac{(v-u)-\sqrt{(v-u)^2 - E(\eta-\eta_2)}}{E}\right)$ for $(\eta, v) \in I$, where $I = \{(\eta, v), \eta \in [\eta_2, \eta_1], v \geq \sqrt{2E(\eta - \eta_2)} + u\}$ (see figure (2)). The function θ is bounded from below and above by two constants

$$0 < \alpha_0 \leq \theta(\eta, v) \leq \alpha_1 < +\infty \tag{3.8}$$

and satisfies the equation

$$-(v-u)\frac{\partial \theta}{\partial \eta} - E\frac{\partial \theta}{\partial v} + \lambda_1 \theta = 0\,.$$

Multiplying the equation (3.6) by θ and integrating over the domain I, we obtain

$$\begin{aligned} &\int_{v>u} (v-u)|H(\eta_2, v)|\theta(\eta_2, v) dv + \int_I |S|\theta \, d\eta dv \\ &= \int_{v>\tilde{v}_1} (v-u)|H(\eta_1, v)|\theta(\eta_1, v) dv + \int_I (\lambda(f) - \lambda_1)|H|\theta \, dv d\eta\,, \end{aligned}$$

where $\tilde{v}_1 = \sqrt{2E(\eta_1 - \eta_2)} + u$. From (3.8), we deduce

$$\alpha_0 \int_{v>u} (v-u)|H(\eta_2, v)|dv + \alpha_0 \int_I |S|dvd\eta \leq \alpha_1 \int_{v>\tilde{v}_1} (v-u)|H(\eta_1, v)|dv \,. \quad (3.9)$$

The integration of the equation (3.6) over the domain $II = \{(\eta, v), \eta \in [\eta_2, \eta_1], v \leq -\sqrt{2E(\eta - \eta_2)} + u\}$ (see figure (2)), leads to

$$\int_{v<u} |v-u|\,|H(\eta_2, v)|dv + \int_{II} \lambda(f)|H|dvd\eta = \int_{v<\tilde{v}_2} |v-u|\,|H(\eta_1, v)|dv + \int_{II} |S|dvd\eta \,,$$

where $\tilde{v}_2 = -\sqrt{2E(\eta_1 - \eta_2)} + u$. From the inequality (3.7), we have

$$\int_{II} |S|dvd\eta \leq C_0 \int_I |S|dvd\eta \,.$$

This implies

$$\int_{v<u} |v-u|\,|H(\eta_2, v)|dv \leq \int_{v<\tilde{v}_2} |v-u|\,|H(\eta_1, v)|dv + C_0 \int_I |S|dvd\eta \,. \quad (3.10)$$

From (3.9) and (3.10), there exists a positive constant $C_{|\eta_1 - \eta_2|}$ such that

$$\int_{\mathbb{R}} |v-u|\,|H(\eta_2, v)|dv \leq C_{|\eta_1-\eta_2|} \int_{\mathbb{R}} |v-u|\,|H(\eta_1, v)|dv \,. \qquad \square$$

Lemma 3.8. *Let f, $g \in C^0(\mathbb{R}_\eta, L^1(\mathbb{R}_v) - weak)$ be two solutions of (2.11), with $F(n_+) \leq f, g \leq F(n_-)$, such that $sgn(f(\eta, v) - g(\eta, v))$ is independent of v. Then $f \equiv g$ if only if there exists $\eta_0 \in \mathbb{R}$ such that $\int_{\mathbb{R}} f(\eta_0, v)dv = \int_{\mathbb{R}} g(\eta_0, v)dv$.*

Proof. Consider the difference $H = f - g$. From Lemma 3.6, the function $sgn(H) = sgn(f - g)$ is independent of v. Then, $\int_{\mathbb{R}} f(\eta_0, v)dv = \int_{\mathbb{R}} g(\eta_0, v)dv$ implies $H(\eta_0, v) = f(\eta_0, v) - g(\eta_0, v) = 0$. By Lemma 3.7, we obtain

$$\int_{\mathbb{R}} |v-u|\,|H(\eta_0, v)|dv \leq 0 \,.$$

Then, $H(\eta, v) = 0$ for all $\eta \in \mathbb{R}$. $\square$

Remark 3.9. The above Lemma immediately implies the uniqueness of the solution of (2.11), (2.12) up to a translation in η.

4. Proof of Theorem 2.2 for regular initial data

In this section we shall assume that $f_{ini} \in C_0^\infty(\mathbb{R}_x^d \times \mathbb{R}_v^d)$ and that there exists a positive number n^* such that

$$0 \leq f_{ini} \leq F(n^*) \,. \quad (4.1)$$

Proposition 4.1. *Under the above assumption, the solution f^ε of (2.1) (2.2) is in $C^1(\mathbb{R}^+ \times \mathbb{R}_x^d \times \mathbb{R}_v^d)$. It satisfies the global estimate $0 \leq f_\varepsilon \leq F(n^*)$. Moreover, $\nabla_x f_\varepsilon$ is in $L^\infty((0,T); L^1(\mathbb{R}_x^d \times \mathbb{R}_v^d))$ and we have the uniform estimates*

$$\|\nabla_x f_\varepsilon(t,\cdot,\cdot)\|_{L^1(\mathbb{R}^{2d}_{x,v})} \leq \|\nabla_x f_{ini}\|_{L^1(\mathbb{R}^{2d}_{x,v})} \,, \quad \|\nabla_x n_\varepsilon(t,\cdot)\|_{L^1(\mathbb{R}^d_x)} \leq \|\nabla_x f_{ini}\|_{L^1(\mathbb{R}^{2d}_{x,v})} \,,$$

where $n_\varepsilon = \int_{\mathrm{IR}^d} f_\varepsilon dv$.

Proof. Existence, uniqueness and regularity are immediate (see for example [24,28]). The fact that $F(n^*)$ is a supersolution implies, thanks to the maximum principle, that

$$0 \le f_\varepsilon \le F(n^*)\,.$$

The estimates on the x gradients are due to the L^1 contractivity of the Boltzmann equation and its linearized version. Indeed, $\nabla_x f_\varepsilon$ solves

$$\begin{aligned} &\partial_t(\nabla_x f_\varepsilon) + v\cdot\nabla_x(\nabla_x f_\varepsilon) + \frac{1}{\varepsilon}\left[E\cdot\nabla_v(\nabla_x f_\varepsilon) - L_{f_\varepsilon}(\nabla_x f_\varepsilon)\right] = 0\,, \\ &\nabla_x f_\varepsilon(t=0) = \nabla_x f_{ini}\,, \end{aligned} \tag{4.2}$$

where

$$L_f(g) = \int_{\mathrm{IR}^d} \sigma(v,v')[fM' + (1-f)M]g'dv' - g\int_{\mathrm{IR}^d} \sigma(v,v')[f'M + (1-f')M']dv'. \tag{4.3}$$

The linearized collision operator L_{f_ε} conserves mass and, since $0 \le f_\varepsilon < 1$, its cross section is nonnegative. Therefore the solution of (4.2) satisfies the maximum principle implying, by the Crandall-Tartar Lemma [13], that the $L^1(\mathrm{IR}^{2d}_{x,v})$-norm of $\nabla_x f_\varepsilon$ is nonincreasing. □

Lemma 4.2.
i) Let the assumptions of Proposition 4.1 hold. Then the solution f_ε *of (2.1), (2.2) satisfies*

$$\partial_t \int_{\mathrm{IR}^d} \Psi(f_\varepsilon, v)dv + \nabla_x\cdot\int_{\mathrm{IR}^d} v\Psi(f_\varepsilon,v)dv \le 0 \tag{4.4}$$

pointwise, with $\Psi(f,v) = \int_0^f \chi(\varphi(g,v))dg$ *and* χ *is an arbitrary nondecreasing function.*
ii) For any $T>0$*, there exists a constant* C *depending on* n^* *but not on* ε *such that*

$$\int_0^T \int_{\mathrm{IR}^d}\int_{\mathrm{IR}^d} (f_\varepsilon - F(n_{f_\varepsilon}))^2\, M(v)dvdxdt \le C\varepsilon\,. \tag{4.5}$$

Proof. Multiplication of (2.1) by $\chi(\varphi(f_\varepsilon,v))$ and integration with respect to v gives

$$\partial_t\int_{\mathrm{IR}^d}\Psi(f_\varepsilon,v)dv + \nabla_x\cdot\int_{\mathrm{IR}^d} v\Psi(f_\varepsilon,v)dv - \frac{1}{\varepsilon}\int_{\mathrm{IR}^d}(Q(f_\varepsilon) - E\cdot\nabla_v f_\varepsilon)\chi(\varphi(f_\varepsilon,v))dv = 0\,.$$

With Proposition 3.3 this immediately implies (4.4) and, with $\chi =$id,

$$\partial_t\int_{\mathrm{IR}^d}\Psi(f_\varepsilon,v)dv + \nabla_x\cdot\int_{\mathrm{IR}^d} v\Psi(f_\varepsilon,v)dv + \frac{C}{\varepsilon}\int_{\mathrm{IR}^d}(f_\varepsilon - F(n_{f_\varepsilon}))^2\, M(v)dv \le 0\,.$$

Integrating with respect to x and t and using the bounds from Proposition 4.1 gives (4.5). □

Lemma 4.3. *Let the assumptions of Proposition 4.1 hold. Then the macroscopic flux* $j_\varepsilon = \int_{\mathbb{R}^d} v f_\varepsilon dv$ *is bounded in* $L^\infty((0,\infty)\times\mathbb{R}^d_x)$ *and* $\partial_t n_\varepsilon$ *is bounded in* $L^\infty((0,\infty), W^{-1,\infty}(\mathbb{R}^d_x))$.

Proof. Since $f_\varepsilon \leq F(n^*)$, we obtain

$$|j_\varepsilon(x,t)| \leq \int_{\mathbb{R}^d} |v| F(n^*) dv < \infty\,.$$

The bound on $\partial_t n_\varepsilon$ comes from the mass conservation

$$\partial_t n_\varepsilon + \nabla_x \cdot j_\varepsilon = 0. \tag{4.6}$$

Let us now end the proof of Theorem 2.2: Since n^ε and $\nabla_x n^\varepsilon$ are bounded in $L^\infty((0,T), L^1(\mathbb{R}^d_x))$ and $\partial_t n^\varepsilon$ bounded in $L^\infty((0,\infty), W^{-1,\infty}(\mathbb{R}^d_x))$, it is clear that up to a the extraction of a subsequence. n^ε converges strongly in $L^1_{loc}(\mathbb{R}^+\times\mathbb{R}^d_x)$ and since it it is bounded in L^∞, the convergence holds in $L^p_{loc}(\mathbb{R}^+\times\mathbb{R}^d_x)$ for any $p\in[1,+\infty)$ and almost everywhere. The entropy estimate (4.5) implies that f^ε converges to $F(n)$ almost everywhere and therefore in $L^p_{loc}(\mathbb{R}^+\times\mathbb{R}^{2d}_{x,v})$. The conservation equation (4.6) obviously leads to the conservation equation $\partial_t n + \nabla_x\cdot j(n) = 0$, while the estimate (4.4) leads to

$$\partial_t \chi(n) + \mathrm{div} G(n) \leq 0$$

where

$$\mathcal{X}(n) = \int_{\mathbb{R}^d} \Psi(F(n),v)dv = \int_{\mathbb{R}^d}\int_0^{F(n)} \chi(\varphi(F(\nu),v)d\nu dv$$

and

$$G(n) = \int_{\mathbb{R}^d}\int_0^{F(n)} v\chi(\varphi(F(\nu),v)d\nu dv\,.$$

Indeed, since $\Psi(f^\varepsilon,v) \leq \|\chi\|_{L^\infty} f^\varepsilon \leq \|\chi\|_{L^\infty} F(n^*)(v)$, we can pass to the limit in $\int \Psi(f^\varepsilon,v)\,dv$ thanks to the Lebesgue dominated convergence theorem. Now, since

$$\mathcal{X}'(n) = \int_{\mathbb{R}^d} \chi(\varphi(F(n),v))\partial_n F(n)dv = \chi(n)\int_{\mathbb{R}^d}\partial_n F(n) dv = \chi(n),$$

and

$$G'(n) = \int_{\mathbb{R}^d} v\chi(\varphi(F(n),v))\partial_n F(n)dv = \chi(n)\int_{\mathbb{R}^d} v\partial_n F(n)dv = \chi(n) j'(n),$$

the entropy inequality for the limiting equation is proved.

The last thing left to be shown is the identity

$$\frac{1}{T}\int_0^T \int_{|x|\leq R} |n(t,x) - n_{ini}(x)| dx dt \to 0.$$

Since our initial data are not well prepared, an initial layer takes place and should be taken into account in the analysis. The starting point is the estimate

$$\frac{1}{T}\int_0^T\int_{|x|\le R}|n(t,x)-n_{ini}(x)|dxdt \le$$

$$\le \frac{1}{T}\int_0^T\int_{\mathbb{R}^d}\int_{|x|\le R}|F(n(t,x))(v)-f_\varepsilon(t,x,v)|dxdvdt$$

$$+\frac{1}{T}\int_0^T\int_{\mathbb{R}^d}\int_{|x|\le R}|f_\varepsilon(t,x,v)-F(n_{ini}(x))(v)|dxdvdt\ .$$

Letting ε tend to zero, we immediately obtain

$$\frac{1}{T}\int_0^T\int_{|x|\le R}|n(t,x)-n_{ini}(x)|dxdt \le$$

$$\le \limsup \tfrac{1}{T}\int_0^T\int_{\mathbb{R}^d}\int_{|x|\le R}|f_\varepsilon(t,x,v)-F(n_{ini}(x)(v)|dxdvdt,$$

where $\limsup$ is the upper limit as ε tends to zero. In order to analyze the right hand side, we denote by $\widehat{f}(\tau,x,v)$ the initial layer profile, solution of

$$\partial_\tau \widehat{f} + E\cdot\nabla_v \widehat{f} = Q(\widehat{f});\quad \widehat{f}(t=0)=f_{ini}.$$

It is proven in [4] that

$$\|(1+|v|)(\widehat{f}(\tau,x,v))-F(n_{ini}(x))(v))\|_{W^{1,1}_{x,v}} \le C_1\mathrm{e}^{-C_2\tau},$$

where C_1 and C_2 are two positive functions depending only on f_{ini}. We then obtain

$$\int_{\mathbb{R}^d}\int_{|x|\le R}|f_\varepsilon(t,x,v)-F(n_{ini}(x)(v)|dxdv \le$$

$$\le \|\widehat{f}(\tfrac{t}{\varepsilon},x,v))-F(n_{ini})\|_{L^1_{x,v}} + \|f_\varepsilon(t,x,v)-\widehat{f}(\tfrac{t}{\varepsilon},x,v))\|_{L^1_{x,v}}$$

$$\le C_1\mathrm{e}^{-C_2t/\varepsilon} + \|f_\varepsilon(t,x,v)-\widehat{f}(\tfrac{t}{\varepsilon},x,v))\|_{L^1_{x,v}}.$$

Now, the function $h^\varepsilon(t,x,v)=f_\varepsilon(t,x,v)-\widehat{f}(\frac{t}{\varepsilon},x,v))$ is a solution of

$$\partial_t h_\varepsilon + v\cdot\nabla_x h_\varepsilon + \frac{1}{\varepsilon}E\cdot\nabla_x h_\varepsilon = \frac{1}{\varepsilon}(Q(f_\varepsilon)-Q(\widehat{f}(\frac{t}{\varepsilon},\cdot,\cdot)) - v\cdot\nabla_x\widehat{f}(\frac{t}{\varepsilon},\cdot,\cdot).$$

Since h_ε vanishes at $t=0$ and $v\cdot\nabla_x\widehat{f}(\frac{t}{\varepsilon},\cdot,\cdot)$ is bounded in $L^1_{x,v}$, we obtain after multipling by $sg(h_\varepsilon)$ and integrating in t, x, v,

$$\|h^\varepsilon(t,x,v)\|_{L^1_{x,v}} \le C_3 t$$

where C_3 is a constant independent from ε. Integrating the above inequalities on $[0,T]$, we finally obtain

$$\frac{1}{T}\int_0^T\int_{|x|\le R}|n(t,x)-n_{ini}(x)|dxdt \le \lim_{\varepsilon\to 0}\frac{\varepsilon C_1}{C_2T}(1-\mathrm{e}^{-C_2T/\varepsilon}) + C_3\frac{T}{2} = C_3\frac{T}{2}.$$

5. Proof of Theorem 2.2 for general initial data

In order to prove the convergence in the general case, we shall use the L^1 contractivity of both the singularly perturbed Boltzmann equation and the limiting conservation law. Let f_{ini}^{δ} be a sequence of compactly supported C^{∞} functions which approach the initial datum f_{ini} in L^1 and such that $0 \leq f_{ini}^{\delta} \leq F(n_*^{\delta})$ for some postive n_*^{δ}. Let n^{δ} be the entropy solution of $\partial_t n + \nabla_x \cdot j(n) = 0$ with the initial datum $n_{ini}^{\delta} = \int_{\mathbb{R}^d} f_{ini}^{\delta}\, dv$. Let f_{ε}^{δ} be the corresponding sequence of kinetic solutions. The last section, result shows that for any fixed δ, we have

$$\lim_{\varepsilon \to 0} f_{\varepsilon}^{\delta} = F(n^{\delta}); \quad \text{in } L^1((0,T) \times \mathbb{R}_{x,loc}^d \times \mathbb{R}_v^d).$$

The L^1 contractivity properties of (2.1) and (2.6) are listed below

$$\|n^{\delta_1}(t,\cdot) - n^{\delta_2}(t,\cdot)\|_{L^1(\mathbb{R}_x^d)} \leq \|n_{ini}^{\delta_1} - n_{ini}^{\delta_2}\|_{L^1(\mathbb{R}_x^d)}$$

and

$$\|f_{\varepsilon}^{\delta_1}(t,\cdot,\cdot) - f_{\varepsilon}^{\delta_2}(t,\cdot,\cdot)\|_{L^1(\mathbb{R}_{x,v}^{2d})} \leq \|f_{ini}^{\delta_1} - f_{ini}^{\delta_2}\|_{L^1(\mathbb{R}_{x,v}^{2d})}.$$

With these results, proving Theorem 2.2 becomes an easy exercice that we leave to the reader.

6. Proof of Theorem 2.5

The Rankine-Hugoniot relation (2.15) is derived by integrating the travelling wave equation (2.11) with respect to v and η. Part ii) of the theorem is a consequence of the following lemma.

Lemma 6.1. *Assume there exists a solution of (2.11), satisfying (2.12) in* $L^1(\mathbb{R}_v)$ *weak. Then* $n_- \geq n_+$ *holds.*

Proof. Multiplication of (2.11) by $\varphi(g,v)$ and integration with respect to v gives

$$\partial_\eta \int_{\mathbb{R}} (v-u)\Phi(g,v)dv \leq 0\,,$$

with $\Phi(g,v) = \displaystyle\int_0^g \varphi(h,v)dh$. Integrating this inequality with respect to η in $\mathbb{R}$, we obtain

$$\int_{\mathbb{R}} (v-u)\Phi(F(n_+,v)dv - \int_{\mathbb{R}} (v-u)\Phi(F(n_-,v)dv \leq 0.$$

The far-field conditions and the Rankine-Hugoniot relations imply

$$\frac{j(n_+) - j(n_-)}{n_+ - n_-}(U(n_+) - U(n_-)) - V(n_+) + V(n_-) \geq 0\,, \tag{6.1}$$

where the macroscopic entropy density and entropy flux are defined by

$$U(n) = \int_{\mathbb{R}} \Phi(F(n),v)dv\,, \qquad V(n) = \int_{\mathbb{R}} v\Phi(F(n),v)dv\,.$$

As seen in the proof of Lemma 4.3, $U(n) = n^2/2$ while $V'(n) = nj'(n)$. Let now $S(n)$ be defined by

$$S(n) := \frac{j(n) - j(n_-)}{n - n_-}(U(n) - U(n_-)) - V(n) + V(n_-)\,,$$

Inserting the above expressions for U and V we obtain $S(n) = (j(n) - j(n_-)(n + n_-)/2 - V(n) + V(n_-)$, and $S'(n) = \frac{1}{2}[j(n) - j(n_-) - j'(n)(n - n_-)]$. Since j is stricly convex, S is strictly decreasing. It satisfies by definition $S(n_-) = 0$. Besides inequality (6.1) implies that $S(n_+) \geq 0$. This clearly implies that $n_+ \leq n_-$. □

From now on we shall assume $n_- > n_+$. Existence of kinetic shock profiles will be shown following the strategy developed in [21] for the Perthame-Tadmor model (see also [14] for BGK models): First an approximating slab problem on a bounded position interval is solved, where the far field conditions are replaced by inflow boundary conditions:

$$\begin{aligned}
&(v-u)\partial_\eta g_L + E\partial_v g_L = Q(g_L)\,, && -L < \eta < L\,,\\
&g_L(-L,v) = F(n_-)(v)\,, && v - u > 0\,,\\
&g_L(L,v) = F(n_+)(v)\,, && v - u < 0\,.
\end{aligned} \tag{6.2}$$

Lemma 6.2. *Let $L > 0$ be fixed. Then,*
i) The problem (6.2) has a unique solution $g_L \in L^1((-L, L) \times \mathbb{R}_v)$. Moreover, we have $F(n_+)(v) \leq g_L(\eta, v) \leq F(n_-)(v)$.
ii)The particle density $\rho_L(\eta) = \int_{\mathbb{R}} g_L(\eta, v)dv$ is a continuous function.

Proof. i) The proof is analogous to corresponding results in [21], [14], and [4] (subsection 4.1), and therefore omitted.
ii) Multiplying (6.2) by a function $\Theta \in W^{1,\infty}(\mathbb{R}_v)$ and integrating with respect to v, we obtain

$$\partial_\eta \int_{\mathbb{R}} (v-u)g_L\Theta\, dv = \int_{\mathbb{R}} Q(g_L)\Theta\, dv + \int_{\mathbb{R}} g_L E\partial_v\Theta\, dv.$$

By using the properties from i) we have $\int_{\mathbb{R}}(v - u)g_L\Theta\, dv \in W^{1,1}(\mathbb{R}_\eta)$ and is therefore continuous. It remains to prove that ρ is continuous. For any $\eta \in \mathbb{R}$, we consider a sequence $\eta_n \to \eta$ as $n \to \infty$. We have

$$\begin{aligned}
|\rho(\eta_n) - \rho(\eta)| &\leq \left|\int_{|v-u|\leq\varepsilon} (g_L(\eta_n, v) - g_L(\eta, v))\, dv\right| + \left|\int_{|v-u|\geq\varepsilon} (g_L(\eta_n, v) - g_L(\eta, v))\, dv\right|\\
&\leq 4\varepsilon + \left|\int_{|v-u|\geq\varepsilon} (v-u)\frac{[g_L(\eta_n, v) - g_L(\eta, v)]}{v-u} dv\right|.
\end{aligned}$$

In particular, we consider a test function $\Theta \in W^{1,\infty}(\mathbb{R}_v)$ such that

$$\Theta(v) = \frac{1}{v-u}, \quad \text{for } |v-u| \geq \varepsilon\,.$$

Using the fact that $\int_{\mathbb{R}}(v-u)g_L\Theta\,dv$ is continuous, there exists $n_0 > 0$ such that for all $n > n_0$

$$|\rho(\eta_n) - \rho(\eta)| \leq 5\varepsilon. \qquad \square$$

Passing to the limit $L \to \infty$. The next step is to pass to the limit $L \to \infty$. Since g_L is bounded by $F(n_-)$ and $F(n_+)$, we obtain by using the boundary condition:

$$\int_{\mathbb{R}} g_L(-L,v)dv \geq \int_{v-u>0} F(n_-)dv + \int_{v-u<0} F(n_+)dv \geq \int_{\mathbb{R}} g_L(L,v)dv.$$

Let now $\eta^L \in [-L, L]$, such that

$$\int_{\mathbb{R}} g_L(\eta^L, v)dv = n^{**} := \int_{v-u>0} F(n_-)dv + \int_{v-u<0} F(n_+)dv\,,$$

with $n_+ < n^{**} < n_-$. Let g_L^1 denote the extension of g_L to the whole η-domain $\mathbb{R}$ defined by

$$g_L^1(\eta, v) = \begin{cases} g_L(-L,v) & \eta < -L\,, \\ g_L(\eta,v) & -L \leq \eta \leq L\,, \\ g_L(L,v) & L < \eta\,. \end{cases}$$

A crucial step in the proof is the normalization of the shift in η. For a sequence $L_k \to \infty$, we denote $\eta_k = \eta^{L_k}$ and $g_k(\eta, v) = g_{L_k}^1(\eta + \eta_k, v)$, such that

$$\int_{\mathbb{R}} g_k(0,v)dv = n^{**}\,. \tag{6.3}$$

Since g_k is bounded, we have (restricting to a subsequence) $g_k \to g$ in $L^\infty(\mathbb{R}^2_{x,v})$ weak*, where $F(n_+) \leq g \leq F(n_-)$ holds. It is also easy to see by reproducing the proof of the previous lemma, that the sequence of densities ρ_L is locally equicontinuous. Even more, the same argument as for the above lemma show that g_L converges strongly as L tends to $+\infty$ in $C^0(I, L^1(\mathbb{R}_v) - weak)$ where I is any bounded closed interval of $\mathbb{R}$, such that $I \subset (-L_k - \eta_k, L_k - \eta_k)$. Therefore, $g \in C^0(I, L^1(\mathbb{R}_v) - weak)$ and

$$\int_{\mathbb{R}} g(0,v)dv = n^{**}\,, \tag{6.4}$$

and the nonlinear term

$$Q(g_k) = (1-g_k)M\int_{\mathbb{R}} \sigma g_k' dv' - g_k \int_{\mathbb{R}} \sigma(1-g_k')M'dv'\,,$$

converge strongly as k tends to $+\infty$ to the same expression with g_k replaced by g. Therefore g satisfies (2.11) on $I \times \mathbb{R}$. We still have to prove that $(-L_k - \eta_k, L_k - \eta_k)$ "tends" to $\mathbb{R}$. This is done in the next lemma.

Lemma 6.3.
i) With the above definitions $-L_k - \eta_k \to -\infty$ *and* $L_k - \eta_k \to \infty$ *as* $k \to \infty$.
ii) There exist two sequences $\mu_k \to \infty$ *and* $\nu_k \to -\infty$ *such that*

$$\lim_{k\to\infty} g(\nu_k, \cdot) = F(n_-, E)\,, \qquad \lim_{k\to\infty} g(\mu_k, \cdot) = F(n_+, E)\,,$$

weakly in $L^1(\mathbb{R}_v)$.

Proof. If we assume $-L_k - \eta_k \to -l > -\infty$ as $k \to \infty$, then by passing to the limit in the differential equation in the distributional sense, g satisfies (2.11) for $\eta > -l$. Since, by velocity averaging, we can pass to the limit in terms of the form $\int g_k(-l, v)\psi(v)dv$ for arbitrary test functions ψ, $g(-l, .) = F(n_-)$ holds for $v > u$. We shall prove that $g(-l, .) = F(n_-)$ for all $v \in \mathbb{R}$.

We multiply (2.11) by $\varphi(g, v)$, integrate with respect to v and η, and use Proposition 3.3 and the bounds on g:

$$\int_{-l}^{\infty} \int_{\mathbb{R}} (g - F(n_g))^2\, M(v) dv d\eta < \infty\,.$$

As a consequence there exists a sequence $\mu_k \to \infty$ and a density $\overline{n}$ such that $n_g(\mu_k) \to \overline{n}$ and $g(\mu_k, \cdot) \to F(\overline{n})$ in $L^2(\mathbb{R}; M\,dv)$. The convergence in $L^1(\mathbb{R}_v) \cap L^2(\mathbb{R}_v)$ now follows from the uniform boundedness of $g(\eta, \cdot)$ in $L^1(\mathbb{R})$.
From the Rankine-Hugoniot condition (2.15), from the convexity of j, and from $n_+ \le \overline{n} \le n_-$ we have

$$j(n_-) - un_- \ge j(\overline{n}) - u\overline{n}\,.$$

Moreover,

$$\int_{\mathbb{R}} (v - u)g(\eta, v)dv = j(\overline{n}) - u\overline{n}\,, \quad \forall \eta \ge -l\,.$$

Since $\int_{\mathbb{R}} (v-u)g(-l, v)dv = \int_{v>u} (v-u)F(n_-)dv + \int_{v<u} (v-u)g(-l, v)dv \ge j(n_-) - un_-$, this implies $\int_{\mathbb{R}} (v - u)g(\eta, v)dv = j(n_-) - un_-$ and, thus,

$$\int_{v<u} (v - u)(F(n_-)(v) - g(-l, v))dv = 0\,.$$

Since g is bounded by $F(n_-)$, this leads to the desired result $g(-l, v) = F(n_-)(v)$, v-a.e.
From Lemma 3.6, we obtain $g(\eta, \cdot) = F(n_-)$ for all $\eta > -l$. This contradicts (6.4), hence $-L_k - \eta_k \to -\infty$ follows.

Analogously $L_k - \eta_k \to \infty$ and the existence of a sequence $\nu_k \to -\infty$ and of a density $\underline{n}$ such that $g(\nu_k, \cdot) \to F(\underline{n})$, in $L^1(\mathbb{R}_v)$ weak, is proven.

By Lemma 6.1, $\underline{n} \ge \overline{n}$ holds. This and the equalities $j(n_-) - un_- = j(\overline{n}) - u\overline{n} = j(\underline{n}) - u\underline{n}$ imply $\underline{n} = n_-$ and $\overline{n} = n_+$, completing the proof. □

Lemma 6.4. *Let* $g \in C^0(\mathbb{R}_\eta, L^1(\mathbb{R}_v - weak))$ *be a solution of (2.11) such that* $F(n_+)(v) \le g(\eta, v) \le F(n_-)(v)$ *and let* $\nu_k \to -\infty$ *be a sequence such that*

$$\lim_{k\to\infty} g(\nu_k, \cdot) = F(n_-)\,, \quad in\, L^1(\mathbb{R}_v)\ weak\,.$$

Then,

$$\lim_{k\to\infty} g(\tilde{\nu_k},\cdot) = F(n_-)\,, \quad in\, L^1(\mathrm{I\!R}_v)\ strong$$

for all $\tilde{\nu_k}$ such that $\tilde{\nu_k} - \nu_k$ is bounded.

Proof. Since $g(\eta,v) \le F(n_-)(v)$, the convergence of $g(\nu_k,\cdot)$ in $L^1(\mathrm{I\!R}_v)$ weak is equivalente to the convergence in $L^1(\mathrm{I\!R}_v)$ strong. Consider the difference $H = F(n_-) - g$. Applying Lemma 3.7, we obtain

$$\int_R |v-u|\,|H(\tilde{\nu_k},v)|dv \le C_{|\tilde{\nu_k}-\nu_k|}\int_R |v-u|\,|H(\nu_k,v)|dv\,.$$

Now, since $H(\nu_k,v)$ converge to zero in $L^1(\mathrm{I\!R}_v)$ strong, the estimate $H(\eta,v) \le F(n_-)(v)$ implies that $\int_R |v-u|\,|H(\nu_k,v)|dv$ tends to as k tends to ∞. Therefore, the same result holds for $H(\tilde{\nu_k},v)$ and this prove the convergence in $L^1(\mathrm{I\!R}_v$ strong of $H(\tilde{\nu_k},v)$. □

Lemma 6.5. *Let $g \in C^0(\mathrm{I\!R}_\eta, L^1(\mathrm{I\!R}_v - weak))$ be the solution of (2.11) constructed above. Then $g(.,v)$ is a monotonically decreasing function of η and*

$$\lim_{\eta\to-\infty} g(\eta,\cdot) = F(n_-)\,, \quad \lim_{\eta\to\infty} g(\eta,\cdot) = F(n_+)\,, \quad in\, L^1(\mathrm{I\!R}_v)\ weak.$$

Proof. We first remark that g cannot be periodic with respect to η. Indeed, if this were the case, we would have by Lemma 3.6

$$\lim_{k\to\infty} g(\nu_k - b,\cdot) = F(n_-)\,, \quad \forall\, b \in \mathrm{I\!R} \quad \text{in}\, L^1(\mathrm{I\!R}_v)\ \text{weak}.$$

This implies that $g(\eta,v) \equiv F(n_-)(v)$. By proceeding analogously with $g(\mu_k - b,\cdot)$, we obtain $g(\eta,v) \equiv F(n_+)(v)$ which is obviously a contradiction.

Let $a > 0$ and define $\hat{g}(\eta,v) = g(\eta - a, v)$. The functions g and $\hat{g}$ verify the assumptions of Lemma 3.6, and, consequently, Lemma 3.7 holds, which implies that

$$\int \hat{g}(\eta,v)dv \ne \int g(\eta,v)dv, \quad \forall \eta \in \mathrm{I\!R}\,.$$

Therefore, $\int g(\eta,v)dv$ is a monotone function, which is decreasing because of its limiting values at $\eta = \pm\infty$. This implies that $g(\eta,v)$ is pointwise in v monotonically decreasing with respect to η. Finally this implies that the limit of $g(\eta,v)$ exists v-a.e. as $\eta \to \pm\infty$ and

$$\lim_{\eta\to\pm\infty} g(\eta,\cdot) = F(n_\pm) \quad \text{in}\, L^1(\mathrm{I\!R}_v)\ \text{weak}.$$ □

This concludes the existence proof for kinetic shock profiles, and it remains to prove uniqueness.

Let f and g be two solutions of (2.11). Since their macroscopic densities vary continuously between the values n_- and n_+, appropriate shifts in the η-direction lead to $\int_{\mathbb{R}} f(0,v)dv = \int_{\mathbb{R}} g(0,v)dv = n^{**}$.
Applying Lemma 3.6 and 3.8 leads to $f \equiv g$ which completes the proof of Theorem 2.5.

7. Proof of Theorem 2.7

The difference $H(t,\eta,v) = f(t,\eta+ut,v) - g(\eta,v)$ is a solution of

$$\partial_t H + (v-u)\partial_\eta H + E\partial_v H = Q(f) - Q(g)\,. \tag{7.1}$$

We multiply (3.5) by $\mathrm{sgn}(H)$ and integrate with respect to v, η, and t:

$$\int_{\mathbb{R}^2} |H(t,\cdot,\cdot)|dv\,d\eta - \int_0^t \int_{\mathbb{R}^2} (Q(f)-Q(g))\mathrm{sgn}(H)dv\,d\eta\,dt = \int_{\mathbb{R}^2} |f_{ini} - g|dv\,d\eta\,.$$

The nonnegativity of the second term provides a uniform-in-time L^1-bound. In particular, defining the sequences $H_n(t,\eta,v) = H(t+t_n,\eta,v)$ and $f_n(t,\eta,v) = f(t+t_n,\eta+u(t+t_n),v)$ with $t_n \to \infty$, both are bounded in $L^\infty(0,T;\ L^1\cap L^\infty(\mathbb{R}^2))$, and (for a subsequence) $H_n = f_n - g \to H_\infty = f_\infty - g$ in $L^\infty((0,T);L^\infty(\mathbb{R}^2))$ weak*.
Moreover,

$$\int_0^\infty \int_{\mathbb{R}^2} (Q(f_n) - Q(g))\mathrm{sgn}(H_n)dv\,d\eta\,dt \le -\int_{t_n}^{+\infty}\int_{-\infty}^{+\infty}\int_{\mathbb{R}} (Q(f)-Q(g))sgn(H)dvd\eta dt$$

tends to 0 as t_n tends to $+\infty$. We have

$$\begin{aligned}\int_{\mathbb{R}} (Q(f_n)-Q(g))\mathrm{sgn}(H_n)(v)dv = -\int_{\mathbb{R}}\int_{\mathbb{R}} &\sigma(v,v')(\mathrm{sgn}(H_n)(v) - \mathrm{sgn}(H_n)(v'))(f_n(v)-g(v))\\ &\{(1-f_n(v'))M(v') + M(v)g(v')\}\,dvdv'\,.\end{aligned} \tag{7.2}$$

Since

$$(\mathrm{sgn}(H_n)(v) - \mathrm{sgn}(H_n)(v'))(f_n(v)-g(v)) = |f_n - g| - sg(f_n' - g')(f_n - g) \ge 0$$

and

$$(1-f_n(v'))M(v') + M(v)g(v') > 0,$$

we have

$$\begin{aligned}-\int_{\mathbb{R}} (Q(f_n)-Q(g))\mathrm{sgn}(H_n)(v)dv \ge \int_0^\infty\int_{\mathbb{R}^2}\int_{\mathbb{R}} &[|f_n - g| - sg(f_n'-g')(f_n-g)]\\ &g'M\,dvdv'd\eta dt \ge 0\end{aligned} \tag{7.3}$$

we obtain

$$\int_0^\infty\int_{\mathbb{R}^2}\int_{\mathbb{R}} [|f_n - g| - sg(f_n'-g')(f_n-g)]\,g'M\,dvdv'd\eta dt \to 0$$

as n tends to $+\infty$. Since $H_n \in L^\infty((0,T)\times \mathbb{R}_x,\ L^1 \cap L^\infty(\mathbb{R}_v))$, we obtain $Q(f_n) - Q(g) \in L^\infty((0,T)\times \mathbb{R}_x, L^1\cap L^\infty(\mathbb{R}_v))$. Applying satndard velocity averaging lemmas (see [19,20,18]) shows that $\int H_n(.,.,v)\Theta(v)dv$ is compact in $L^2_{loc}(\mathbb{R}^+\times\mathbb{R}_x)$. Since $0\le f_n\le F(n^*)$, we obtain $\int H_n(.,.,v)dv$ is compact in $L^2_{loc}(\mathbb{R}^+\times\mathbb{R}_x)$. Let $\tilde H$ the limit of $|f_n-g|$ and s the limit of $\mathrm{sgn}(H_n)$ in $L^\infty((0,T);L^\infty(\mathbb{R}^2_{x,v}))$ weak*. By using the fact that $g\in L^1(\mathbb{R}^2_{x,v})$ and $M\in L^1(\mathbb{R}_v)$, we can pass in the limit in the integral and we obtain

$$\int_0^\infty\int_{\mathbb{R}^2}\int_{\mathbb{R}}\left[\tilde H(v) - s(v')H_\infty(v)\right] g(v')M(v)dvdv'd\eta dt = 0$$

Since $H_\infty(v)\le \tilde H(v)$ and $-1\le s\le 1$ we obtain $\tilde H(v) = s(v')H_\infty(v)$ and $\tilde H(v)\le |H_\infty(v)|$. Then $\tilde H(v) = |H_\infty(v)|$ and $\mathrm{sgn}(H_\infty)$ is independent of v. We can also pass to the limit in (7.1), written as an equation for H_n:

$$\partial_t H_\infty + (v-u)\partial_\eta H_\infty + E\partial_v H_\infty + \lambda H_\infty = S\,,$$

where $\lambda\ge 0$ and $\mathrm{sgn}(S) = \mathrm{sgn}(H_\infty)$. A consequence of the latter is that H_∞ cannot change its sign along characteristics.
Since the property (2.18) is propagated in time, $\int_{\mathbb{R}^2} H_\infty(t,\eta,v)dv\,d\eta = 0$ holds. Assume there exists a time t_0 and a position η_0 such that

$$\int_{\mathbb{R}} H_\infty(t_0,\eta_0,v)dv \neq 0\,.$$

Then there exists another position η_1 ($\eta_0<\eta_1$ w.l.o.g.) such that

$$\int_{\mathbb{R}} H_\infty(t_0,\eta_0,v)dv\int_{\mathbb{R}} H_\infty(t_0,\eta_1,v)dv < 0\,.$$

Moreover, since $\mathrm{sgn}(H_\infty)$ is independent from v, $H_\infty(t_0,\eta_0,v)H_\infty(t_0,\eta_1,v)<0$ for all v. Let $v_0>u$, we denote the characteristic starting at (η_0,v_0) at time t_0 by $(\eta,v) = (X(t;\eta_0,v_0,t_0), V(t;\eta_0,v_0,t_0))$. The characteristics satisfy the ODE

$$\dot X = V-u\,,\qquad \dot V = E\,.$$

Therefore, $X(t;\eta_0,v_0,t_0)$, starting at $X(t_0;\eta_0,v_0,t_0)=\eta_0$ increases and eventually reaches η_1. On the other hand, for $v_1<u$ small enough, $X(t;\eta_1,v_1,t_0)$ decreases from η_1 to η_0. Consequently, there exists a time $t_2>t_0$ such that $X(t_2;\eta_0,v_0,t_0) = X(t_2;\eta_1,v_1,t_0) =: \eta_2$. As mentioned above, H_∞ does not change its sign along characteristics, and therefore

$$H_\infty(t_2,\eta_2,V(t_2;\eta_0,v_0,t_0))H_\infty(t_2,\eta_2,V(t_2;\eta_1,v_1,t_0)) < 0\,,$$

a contradiction to the fact that $\mathrm{sgn}(H_\infty)$ is independent from v. We deduce $H_\infty\equiv 0$ and, equivalently, $f_\infty \equiv g$.

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