

# Kinetic Models for Chemotaxis and their Drift-Diffusion Limits <sup>\*</sup>

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## Abstract

Kinetic models for chemotaxis, nonlinearly coupled to a Poisson equation for the chemo-attractant density, are considered. Under suitable assumptions on the turning kernel (including models introduced by Othmer, Dunbar and Alt), convergence in the macroscopic limit to a drift-diffusion model is proven. The drift-diffusion models derived in this way include the classical Keller-Segel model. Furthermore, sufficient conditions for kinetic models are given such that finite-time-blow-up does not occur. Examples are given satisfying these conditions, whereas the macroscopic limit problem is known to exhibit finite-time-blow-up. The main analytical tools are entropy techniques for the macroscopic limit as well as results from potential theory for the control of the chemo-attractant density.

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# 1 Introduction

Chemotaxis is a process in which bacteria, or, more generally, cells, change their state of movement, reacting to the presence of a chemical substance, approaching chemically favourable environments and avoiding unfavourable ones.

Generally, the movement of bacteria is composed of two different phases, a “run” phase and a “tumble” phase. The “run” phase consists of a directed movement in a straight line, while the “tumble” phase is the reorientation. In the case of *Escherichia coli*, this “tumble” is accomplished by signal transmission between the receptor complexes (which detect the presence of chemical substances and are located basically in the poles of the cells) and the flagellar-motor complexes (usually 5 to 10 complexes randomly distributed around the cell and embedded within the cell membrane) [3].

In fact, this increases the length of the “run” phase (while the “tumble” phase remains essentially invariant) producing a biased random walk in the direction of the gradient of the chemical substrate [31]. This is the general behavior for flagellated bacteria [1]. On the other hand, in the case of amoebae or leukocytes, the presence of a chemical substance changes the turning pattern, i.e., the “tumble” phase. See [1] and references therein. The “tumble” phase is in general much shorter than the “run” phase [3].

It is important to stress that chemical gradients are not the only factor to influence bacterial movement, a process called *taxis*. Light, pH and oxygen concentration do it as well. In general, bacteria look for optimal conditions for growth, but they can also move looking for aggregation. The last case has been particularly studied for the case of the slime mold amoebae *Dictyostelium discoideum* (see [16]).

Chemotaxis is also important for other biological phenomena, like the embryological development, where cells migrate to form complex organs [8] and in the immunological response. In this case, leukocytes, the cells responsible for the immune response, migrate from the bloodstream to respond to foreign antigens [8, 28].

The mathematical study of chemotaxis started with the work of Patlak [25], and was boosted by the papers of Keller and Segel, where they introduced a model to study the aggregation of *Dictyostelium discoideum* due to an attractive chemical substance [18] and made some further comments and studies [19, 20]. We refer to the reference [17] for a review about the first years of research on the Keller-Segel model.

Their model consists of an advection-diffusion system of two coupled parabolic equations:

$$\partial_t \rho = \nabla \cdot (D \nabla \rho - \chi \rho \nabla S) , \quad (1)$$

$$\partial_t S = D_S \Delta S + \varphi(S, \rho) . \quad (2)$$

In these equations  $\rho = \rho(x, t) \geq 0$  is the cell density at position  $x$  and time  $t$ , and

$S = S(x, t) \geq 0$  is the density of the chemo-attractant. The positive constants  $D_S$  and  $D$  are the diffusivity of the chemo-attractant and of the cells, respectively, and  $\chi \geq 0$  is the chemotactic sensitivity.

In general the substance  $S$  does not only diffuse in the substrate, but it can also be produced by the bacteria themselves. The role of the function  $\varphi(S, \rho)$  is to describe the interaction between both quantities. One typical example is given by

$$\varphi(S, \rho) = \alpha\rho - \frac{S}{\tau_S}, \quad \alpha, \tau_S \geq 0, \quad (3)$$

which describes the production of the chemo-attractant by the bacteria at a constant rate  $\alpha$  as well as chemical decay with relaxation time  $\tau_S$ . Since the bacterial movement is directed toward the higher concentrations of  $S$ , the coupling is attractive.

An important question for the above system is if its solutions blow up in finite time (see, e.g., [4, 10, 11, 21, 24]). In [9] and references therein it is proved for the Keller-Segel model that — when the space dimension  $n$  is equal to 3 — blow up can happen even for small initial conditions. Blow up never holds for  $n = 1$ , and the case  $n = 2$  with spherical symmetry is a borderline case, where blow up may occur or not, depending on the size of the initial conditions (see also [21, 9]). Blow up can be prevented by a modification of the Keller-Segel model, where the chemotactic effect is turned off at a saturation density [13].

The transport equation (1) in the Keller-Segel model can be derived as a macroscopic limit of a stochastic many particle system [30]. The aim of this work is to study kinetic models of chemotaxis and their macroscopic limits. A kinetic equation for the phase space cell density has first been introduced by Alt [1, 2], and further been studied by Othmer, Dunbar and Alt [22]. Here a kinetic model is considered, coupled to an equation for the chemo-attractant, and modeling assumptions are given, such that the Keller-Segel equations (1)–(3) can be derived rigorously as a scaling limit. In the papers [12] and [23] a formal deduction has been presented, but no rigorous proof is available yet.

From now on we consider chemotaxis in 3 dimensions, i.e.  $n = 3$ . The kinetic or “velocity jump” model introduced in [22] is an equation for the phase space cell density  $f = f(x, v, t) \geq 0$ , where  $x$ ,  $v$ , and  $t$  denote, respectively, position, velocity, and time:

$$\frac{\partial f}{\partial t} + v \cdot \nabla_x f = \int_V (T[S]f' - T^*[S]f) dv'. \quad (4)$$

Examples for the dependence of the rate  $T[S](x, v, v', t) \geq 0$  on the density of the chemo-attractant  $S$  can be found in section 5. The abbreviations  $f' = f(x, v', t)$ ,  $T^*[S] = T[S](x, v', v, t)$  are used in (4). In this model it is assumed that the tumble (the reorientation) is a Poisson process with rate

$$\lambda[S] = \int_V T^*[S] dv',$$

and that  $T^*[S]/\lambda[S]$  is the probability density for a change in velocity from  $v$  to  $v'$ , given that a reorientation occurs for a cell at position  $x$ , velocity  $v$ , and time  $t$ .

The set of admissible velocities is denoted by  $V$  and assumed to be compact. We restrict our attention to spherically symmetric  $V$ , with balls, spheres, or spherical shells (with the center in the origin) as typical examples. When  $V$  is a sphere,  $dv$  has to be understood as the surface measure.

In [22], birth-death processes are also considered, adding

$$\left(\frac{\partial f}{\partial t}\right)_{\text{bd}} = r(\rho)f$$

to the right hand side of (4), where  $r(\rho)$  is a function of the cell position density,

$$\rho(x, t) = \int_V f(x, v, t)dv. \quad (5)$$

In this paper we do not consider birth-death processes. This puts a limitation on our model, which is valid only in intervals of time where cell-division is not important.

Equation (4) is an example of a Boltzmann-type integro-differential equation which has originally been introduced for the study of moderately rarefied gases. Macroscopic (or fluid) models, where the velocity distribution is described by a finite set of space-time dependent quantities, can be derived from kinetic models by a classical procedure, the Hilbert expansion. An introduction to the mathematical aspects of Boltzmann-type equations and their diffusion limit can be found in [6].

In [12] and [23], Othmer and Hillen studied the diffusion limit of equation (4), without and with chemotactic effects, respectively. Their analysis is based on the assumption that the chemotactic influence on the tumbling can be seen as a perturbation of a dominating isotropic, i.e., “aimless”, reorientation. We make this assumption specific by postulating a relation  $\tau_0 = \varepsilon\tau_1$ , where  $\tau_0$  and  $\tau_1$  are typical times between aimless and chemotactically oriented turning processes, respectively. The dimensionless parameter  $\varepsilon > 0$  is small. Now we perform a nondimensionalization of the system (2), (3), (4). We scale velocity in the kinetic equation by the maximal speed  $v_0$  occurring in  $V$ . According to the above assumption, the turning kernel is written in the form

$$T[S] = \frac{T_\varepsilon[S/S_0]}{\tau_0 v_0^d},$$

with an appropriately chosen reference value  $S_0$  for the chemo-attractant density and the dimension  $d = 2$  or  $d = 3$  of the velocity set  $V$ . For time and length we use a diffusion scaling with reference values  $t_0 = \tau_0/\varepsilon^2$  and  $x_0 = v_0\tau_0/\varepsilon$ , respectively. Finally, we introduce the reference values  $\rho_0 = S_0 D_S/(\alpha x_0^2)$  for the macroscopic cell

density as well as  $f_0 = \rho_0/v_0^d$  for the distribution function. The nondimensional version of (2), (3), (4) then becomes

$$\delta \frac{\partial S_\varepsilon}{\partial t} = \Delta S_\varepsilon + \rho_\varepsilon - \delta \frac{t_0}{\tau_S} S_\varepsilon, \quad (6)$$

$$\varepsilon^2 \frac{\partial f_\varepsilon}{\partial t} + \varepsilon v \cdot \nabla_x f_\varepsilon = -\mathcal{T}_\varepsilon[S_\varepsilon](f_\varepsilon), \quad (7)$$

with

$$\mathcal{T}_\varepsilon[S](f) = \int_V (T_\varepsilon^*[S]f - T_\varepsilon[S]f') dv',$$

and with the dimensionless parameter  $\delta = v_0^2 \tau_0 / D_S$ . Partially, the same symbols are used for scaled quantities as for their dimensional counterparts.

Our main scaling assumption is the smallness of  $\varepsilon$ . We also assume that the relaxation time  $\tau_S$  of the chemo-attractant is at least of the order of magnitude of the diffusive time scale  $t_0$ . The parameter  $\delta$  measures the strength of the diffusivity of the cells compared to the diffusivity of the chemo-attractant. We assume that  $\delta$  is small and set  $\delta = 0$  as an approximation:

$$-\Delta S_\varepsilon = \rho_\varepsilon = \int_V f_\varepsilon dv. \quad (8)$$

Finally, we assume that the environment for the cells is large compared to the reference length  $x_0$ . As an approximation we consider the whole space problem for (7), (8) subject to the initial conditions

$$f_\varepsilon(x, v, 0) = f^0(x, v), \quad x \in \mathbb{R}^3, \quad v \in V. \quad (9)$$

The behaviour of  $S_\varepsilon$  at infinity is fixed by using the Newtonian potential solution of (8):

$$S_\varepsilon = \rho_\varepsilon * \frac{1}{4\pi|x|}, \quad \text{i.e., } S_\varepsilon(x, t) = \frac{1}{4\pi} \int_{\mathbb{R}^3} \frac{\rho_\varepsilon(y, t)}{|x-y|} dy. \quad (10)$$

We point out that the last two approximations ( $\delta = 0$ , whole space problem) do not make an essential difference from the mathematical point of view. For the problem with  $\delta > 0$  and/or a bounded domain with appropriate boundary conditions, the map  $\rho_\varepsilon \mapsto S_\varepsilon$  would be more regular. All our results can be extended to this case with an adaptation of the proof which takes into account the additional difficulties related to the heat equation (see [5], where the case of a parabolic equation for  $S_\varepsilon$  is treated).

Assuming that  $S_\varepsilon$  is given, smooth enough, and  $\varepsilon$ -independent, Othmer and Hillen [23] derived the linear chemotaxis model (1) from (7) formally by the Hilbert expansion procedure in the limit  $\varepsilon \rightarrow 0$ .

In this paper, we find conditions which guarantee that the coupled nonlinear problem (7)–(10) has Keller-Segel type equations as its macroscopic drift-diffusion limit. Note that the nonlinear coupling is due to the dependence of the turning kernel  $T_\varepsilon$  on the substrate density  $S_\varepsilon$ . From a mathematical point of view, this part of our work is closely related to [27], where the macroscopic limit of the Vlasov-Poisson-Fokker-Planck system is derived. We also show that for suitable turning kernels, blow up can be prevented on the kinetic level, although blow up is known to occur for the corresponding macroscopic limit.

The rest of this paper is organized as follows: In Section 2 the macroscopic equations are derived by formal asymptotic methods. In Section 3 we show that, under appropriate assumptions on the dependence of the turning rates on the chemo-attractant, the kinetic model (with fixed  $\varepsilon > 0$ ) has a global solution and blow up in finite time does not occur. In Section 4 the diffusion limit is carried out rigorously for short enough time intervals. Note that this is the best result to be expected, since blow up occurs in the macroscopic limit problem. Two classes of modeling examples are presented in Section 5. Both examples satisfy the assumptions of the convergence result and one of them the global existence theorem.

## 2 Drift-Diffusion limit: formal computations

In this section the limit  $\varepsilon \rightarrow 0$  is carried out formally in (7), (10). The resulting macroscopic model depends on the properties of the turning operator  $\mathcal{T}_\varepsilon[S]$ . A first basic property is conservation of cells: The integral of  $\mathcal{T}_\varepsilon[S](f)$  with respect to velocity vanishes, leading to the macroscopic conservation equation

$$\frac{\partial \rho_\varepsilon}{\partial t} + \nabla \cdot J_\varepsilon = 0, \quad (11)$$

with the flux density

$$J_\varepsilon(x, t) := \frac{1}{\varepsilon} \int_V v f_\varepsilon(x, v, t) dv.$$

The following analysis is based on the assumption that the turning kernel has an asymptotic expansion of the form

$$T_\varepsilon[S] = T_0[S] + \varepsilon T_1[S] + O(\varepsilon^2). \quad (12)$$

Then the turning operator can be expanded analogously with coefficients

$$\mathcal{T}_k[S](f) = \int_V (T_k^*[S]f - T_k[S]f') dv'.$$

Our aim is the derivation of equations for the leading order terms in the expansions

$$f_\varepsilon = f_0 + \varepsilon f_1 + O(\varepsilon^2), \quad S_\varepsilon = S_0 + \varepsilon S_1 + O(\varepsilon^2).$$

Substitution into (7), (10) gives the leading order equations

$$\mathcal{T}_0[S_0](f_0) = 0, \quad S_0 = \rho_0 * \frac{1}{4\pi|x|} \quad \text{with } \rho_0 = \int_V f_0 dv. \quad (13)$$

Comparing coefficients of  $\varepsilon$  in (7) gives

$$v \cdot \nabla_x f_0 = -\mathcal{T}_0[S_0](f_1) - \mathcal{T}_1[S_0](f_0) - \mathcal{T}_{0S}[S_0, S_1](f_0), \quad (14)$$

where  $\mathcal{T}_{0S}[S_0, S_1]$  is a turning operator whose kernel is the Frechet derivative of  $T_0$  with respect to  $S$ , evaluated at  $S_0$  in the direction  $S_1$ .

Before proceeding further, we need assumptions on the leading order turning operator:

(A0) There exists a bounded velocity distribution  $F(v) > 0$ , independent of  $x$ ,  $t$ , and  $S$ , such that the detailed balance  $T_0^*[S]F = T_0[S]F'$  holds. The flow produced by this equilibrium distribution vanishes, and  $F$  is normalized:

$$\int_V vF(v) dv = 0, \quad \int_V F(v) dv = 1. \quad (15)$$

The turning rate  $T_0[S]$  is bounded, and there exists a constant  $\gamma > 0$  such that  $T_0[S]/F \geq \gamma$ ,  $\forall (v, v') \in V \times V$ ,  $x \in \mathbb{R}^3$ ,  $t > 0$ .

**Remark 1.** *It is a generalization of earlier work [23] that a general equilibrium distribution is allowed instead of only constants with respect to velocity.*

*The aimlessness of the leading order turning processes mentioned above is reflected in the fact that the flux density and, thus, the mean velocity of the equilibrium distribution is zero.*

Before the assumption is used, we state a useful formula.

**Lemma 1.** *Let  $\eta : \mathbb{R} \rightarrow \mathbb{R}$ ,  $g : V \rightarrow \mathbb{R}$ , and let*

$$\phi_\varepsilon^S[S] = \frac{T_\varepsilon[S]F' + T_\varepsilon^*[S]F}{2}, \quad \phi_\varepsilon^A[S] = \frac{T_\varepsilon[S]F' - T_\varepsilon^*[S]F}{2},$$

*denote the symmetric and, respectively, antisymmetric parts of  $T_\varepsilon[S]F'$ . Then*

$$\begin{aligned} \int_V \mathcal{T}_\varepsilon[S](Fg)\eta(g)dv &= \frac{1}{2} \int_V \int_V \phi_\varepsilon^S[S](g - g')(\eta(g) - \eta(g'))dv' dv \\ &+ \frac{1}{2} \int_V \int_V \phi_\varepsilon^A[S](g + g')(\eta(g) - \eta(g'))dv' dv. \end{aligned} \quad (16)$$

*The same holds for  $\mathcal{T}_k[S]$  with analogous definitions of  $\phi_k^S[S]$  and  $\phi_k^A[S]$ .*

*Proof.* The proof is a straightforward computation. First the left hand side is rewritten by interchanging  $v$  and  $v'$  in the double integral. Then the arithmetic mean of both representations is taken, and the (anti)symmetry properties of  $\phi_\varepsilon^S[S]$  and  $\phi_\varepsilon^A[S]$  are used.  $\square$

Note that assuming the expansion (12) and the assumption (A0), the symmetric and antisymmetric terms introduced above have the asymptotic expansions

$$\phi_\varepsilon^S[S] = T_0[S]F' + O(\varepsilon), \quad \phi_\varepsilon^A[S] = \varepsilon \frac{T_1[S]F' - T_1^*[S]F}{2} + O(\varepsilon^2). \quad (17)$$

This shows that for monotone  $\eta$  the leading order contribution to the right hand side of (16) has a sign. This observation is the basis of entropy arguments in the convergence proof in Section 4 and in the following classical result. We give a proof for the sake of completeness.

**Lemma 2.** *Let (A0) hold. Then, the entropy equality*

$$\int_V \mathcal{T}_0[S](f) \frac{f}{F} dv = \frac{1}{2} \int_V \int_V \phi_0^S[S] \left( \frac{f}{F} - \frac{f'}{F'} \right)^2 dv' dv \geq 0$$

*holds. For  $g \in L^2(V; dv/F)$ , the equation  $\mathcal{T}_0[S](f) = g$  has a unique solution  $f \in L^2(V; dv/F)$  satisfying  $\int_V f dv = 0$  if and only if  $\int_V g dv = 0$ .*

*Proof.* The entropy equality is an application of the previous lemma with  $g = f/F$  and  $\eta = \text{id}$ . The detailed balance assumption in (A0) is equivalent to  $\phi_0^A[S] = 0$ .

It is a direct consequence of cell conservation that  $\int_V g dv = 0$  is a necessary condition for solvability of  $\mathcal{T}_0[S](f) = g$ .

For  $\int_V f dv = 0$ , the entropy equality and assumption (A0) lead to the estimate

$$\int_V \mathcal{T}_0[S](f) \frac{f}{F} dv \geq \frac{\gamma}{2} \int_V \int_V FF' \left( \frac{f}{F} - \frac{f'}{F'} \right)^2 dv' dv = \gamma \int_V \frac{f^2}{F} dv, \quad (18)$$

since  $\phi_0^S[S] \geq \gamma FF'$  holds. The statement of the lemma is now a consequence of the Lax-Milgram lemma.  $\square$

It is a consequence of the entropy equality that the kernel of  $\mathcal{T}_0[S]$  is spanned by the distribution  $F$ . Thus, we deduce from the leading order equations (13) that

$$f_0(x, v, t) = \rho_0(x, t)F(v),$$

with  $\rho_0$  (to be determined) being the macroscopic cell density corresponding to  $f_0$  (by the normalization of  $F$ ). Since the equilibrium distribution is independent of  $S$ ,

the term  $\mathcal{T}_{0S}[S_0, S_1](f_0)$  vanishes (by linearization of the detailed balance equation) and the  $O(\varepsilon)$ -equation (14) reads:

$$\mathcal{T}_0[S_0](f_1) = -vF \cdot \nabla \rho_0 - \rho_0 \mathcal{T}_1[S_0](F). \quad (19)$$

When this is seen as an equation for  $f_1$ , the first term on the right hand side satisfies the solvability condition from Lemma 2 by assumption (A0), and the second term by cell conservation. The solution can be written as

$$f_1(x, v, t) = -\kappa(x, v, t) \cdot \nabla \rho_0(x, t) - \Theta(x, v, t) \rho_0(x, t) + \rho_1(x, t) F(v), \quad (20)$$

where  $\kappa = \kappa[S_0]$  and  $\Theta = \Theta[S_0]$  are the solutions of

$$\mathcal{T}_0[S_0](\kappa) = vF, \quad (21)$$

$$\mathcal{T}_0[S_0](\Theta) = \mathcal{T}_1[S_0](F), \quad (22)$$

and  $\rho_1$ , the macroscopic density of  $f_1$ , is a new unknown.

The last step in the asymptotic procedure is passing to the limit  $\varepsilon \rightarrow 0$  in the conservation equation (11). For the flux density, we obtain the asymptotic expansion

$$J_\varepsilon = \int_V v f_1 dv + O(\varepsilon),$$

such that the limit of (11) can be written as the convection-diffusion equation

$$\partial_t \rho_0 - \nabla \cdot (D[S_0] \nabla \rho_0 - \Gamma[S_0] \rho_0) = 0, \quad (23)$$

where the diffusivity tensor and the convection field are given by

$$D[S_0](x, t) = \int_V v \otimes \kappa[S_0](x, v, t) dv,$$

$$\Gamma[S_0](x, t) = - \int_V v \Theta[S_0](x, v, t) dv.$$

Thus, the formal limit of (7), (10) is (23) coupled to the Newtonian potential equation for  $S_0$  in (13).

**Remark 2.** *It is a standard result in the theory of diffusion limits that the matrix  $D$  is symmetric and positive definite. In order to see this, consider  $\eta \in \mathbb{R}^3$ . From (18) we find*

$$(D\eta) \cdot \eta = \int_V (v \cdot \eta)(\kappa \cdot \eta) dv = \int_V \mathcal{T}_0[S_0](\kappa \cdot \eta) \frac{\kappa \cdot \eta}{F} dv \geq \gamma[S_0] \int_V \frac{(\kappa \cdot \eta)^2}{F} dv. \quad (24)$$

*If  $\kappa \cdot \eta$  were identically equal to zero for a  $\eta \neq 0$ , then (by taking the scalar product of (21) with  $\eta$ ) also  $v \cdot \eta$  would be zero for all  $v \in V$ , which is impossible by the spherical symmetry of  $V$ . Thus, the right hand side of (24) is positive for each  $\eta \neq 0$ . The symmetry of  $D$  is an immediate consequence of the fact that  $\mathcal{T}_0[S]$  is selfadjoint with respect to the scalar product in  $L^2(V; dv/F)$ .*

**Remark 3.** *A natural additional assumption is rotational invariance of the leading order turning operator. This means that the turning kernel  $T_0[S]$  depends on  $v$  and  $v'$  only through  $|v|$ ,  $|v'|$ , and  $v \cdot v'$ , and, thus, is invariant under simultaneous rotations of  $v$  and  $v'$ . Then, it can be easily shown that the equilibrium distribution is isotropic, i.e.,  $F = F(|v|)$ , and that the diffusivity tensor is also isotropic, i.e., a multiple of the identity. This is a consequence of the fact that  $\kappa(x, v, t) = \alpha(x, |v|, t)v$  with a scalar function  $\alpha$ . Sufficient conditions for the isotropy of the diffusivity have also been given in [12].*

### 3 Global existence

In this section we show that the solutions of the coupled kinetic system (7)–(10) do not blow up if the turning kernel satisfies a certain structure condition. Without loss of generality, let us set  $\varepsilon = 1$ .

**Theorem 1.** *(Global boundedness) Assume  $f^0 \in L^1_+ \cap L^\infty(\mathbb{R}^3 \times V)$ , and assume:*

*(A1) There exists  $C > 0$  such that  $\forall x \in \mathbb{R}^3$ ,  $v, v' \in V$ ,  $t \in \mathbb{R}^+$  and  $S \in W^{1,\infty}(\mathbb{R}^3)$*

$$0 \leq T[S](x, v, v', t) \leq C(1 + S(x + v, t) + S(x - v', t)) .$$

*Then there exists a global solution  $f \in L^\infty((0, \infty); L^1_+ \cap L^\infty(\mathbb{R}^3 \times V))$ ,  $S \in L^\infty((0, \infty); L^p(\mathbb{R}^3))$  for all  $2 \leq p \leq +\infty$  of the nonlinear system (7)–(10) (with  $\varepsilon = 1$ ).*

**Remark 4.** *The condition on the turning kernel excludes its dependence on the gradient of the concentration of the chemo-attractant. It also provides a dispersion effect by the occurrence of the velocity variable in the arguments of  $S$ . No global existence result is available for a turning kernel with dependence on the gradient of  $S$ , actually not even for models satisfying only*

$$0 \leq T[S](x, v, v', t) \leq C \left( 1 + \|S(\cdot, t)\|_{L^\infty(\mathbb{R}^3)}^\beta \right) ,$$

*with  $\beta \geq 1$ . A global existence result can be shown for  $\beta < 1$ . These remarks are valid for three-dimensional problems considered here.*

*In [14], [15] one-dimensional discrete velocity models have been treated. It is shown that boundedness of the turning rates in terms of the  $W^{1,\infty}$ -norm of the chemo-attractant is sufficient for global existence.*

*An extension of Theorem 1 to the case of a parabolic equation for the density of the chemo-attractant has been shown in [5].*

*Proof.* A local-in-time existence proof of mild solutions can be obtained by using standard methods for nonlinear evolution equations (nonlinear perturbations of  $C^0$ -semigroups, cf. [26]). To obtain a global existence result it is enough to derive the a priori bounds announced in the Theorem (which imply that the local solutions can be extended up to  $t = \infty$ ).

To do so, we decompose  $S$  into long and short range parts

$$\begin{aligned} S &= S^L + S^S, \\ S^L &= \rho * \left( \frac{1}{4\pi|x|} \mathbb{I}_{|x| \geq 1} \right), \\ S^S &= \rho * \left( \frac{1}{4\pi|x|} \mathbb{I}_{|x| \leq 1} \right), \end{aligned}$$

where  $\mathbb{I}_A$  denotes the characteristic function of the set  $A$ . From the mass conservation

$$\|\rho(\cdot, t)\|_{L^1(\mathbb{R}^3)} = \|f(\cdot, \cdot, t)\|_{L^1(\mathbb{R}^3 \times V)} = \|f^0\|_{L^1(\mathbb{R}^3 \times V)} \quad (25)$$

and the Young inequality we have

$$\|S^L(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq \frac{1}{4\pi} \|f^0\|_{L^1(\mathbb{R}^3 \times V)}.$$

Therefore, changing the constant  $C$ , we may replace  $S$  by  $S^S$  in (A1).

In order to estimate  $f$ , we now use that

$$\partial_t f(x, v, t) + v \cdot \nabla_x f(x, v, t) \leq \int_V T[S](x, v, v', t) f(x, v', t) dv',$$

and, thus, using assumption (A1),

$$f(x, v, t) \leq f^0(x - vt, v) + C \int_0^t \rho(x - vs, t - s) ds + C f^1(x, v, t) + C f^2(x, v, t), \quad (26)$$

with

$$\begin{aligned} \partial_t f^1(x, v, t) + v \cdot \nabla_x f^1(x, v, t) &= \int_V S^S(x + v, t) f(x, v', t) dv', \\ \partial_t f^2(x, v, t) + v \cdot \nabla_x f^2(x, v, t) &= \int_V S^S(x - v', t) f(x, v', t) dv', \end{aligned}$$

and

$$f^1(t = 0) = 0, \quad f^2(t = 0) = 0.$$

We estimate these two terms separately. From

$$f^1(x, v, t) = \int_0^t S^S(x - vs + v, t - s) \rho(x - vs, t - s) ds$$

we conclude that

$$\|f^1(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3 \times V)} \leq \sup_{0 < s < t} \|S^S(\cdot, s)\|_{L^p(\mathbb{R}^3)} \int_0^t \|\rho(\cdot, t-s)\|_{L^p(\mathbb{R}^3)} ds.$$

For  $f^2$  we write

$$f^2(x, v, t) = \int_0^t \int_V S^S(x - vs - v', t-s) f(x - vs, v', t-s) dv' ds.$$

Note that the integrand is a convolution (setting  $f(x, v', t) = 0$  for  $v' \notin V$ ). The Young inequality

$$\|g * h\|_{L^r} \leq \|g\|_{L^p} \|h\|_{L^q}, \quad \frac{1}{p} + \frac{1}{q} = 1 + \frac{1}{r},$$

(see, e.g., [7]) thus implies

$$\begin{aligned} |S^S(\cdot, t-s) * f(x - vs, \cdot, t-s)(x - vs)| &\leq \|S^S(\cdot, t-s) * f(x - vs, \cdot, t-s)\|_{L^\infty(\mathbb{R}^3)} \\ &\leq \sup_{0 < s < t} \|S^S(\cdot, s)\|_{L^p(\mathbb{R}^3)} \|f(x - vs, \cdot, t-s)\|_{L^{p'}(V)}, \end{aligned}$$

where  $p$  and  $p'$  are conjugate exponents, i.e.,  $p' = p/(p-1)$ . If  $p \geq 2$ , then  $p' \leq p$  and, from mass conservation (25),

$$\|f(x - vs, \cdot, t-s)\|_{L^{p'}(V)} \leq c_0 \|f(x - vs, \cdot, t-s)\|_{L^p(V)},$$

for a constant  $c_0 = c_0(V)$ . Finally,

$$\|f^2(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3 \times V)} \leq c_0 \sup_{0 < s < t} \|S^S(\cdot, s)\|_{L^p(\mathbb{R}^3)} \int_0^t \|f(\cdot, \cdot, t-s)\|_{L^p(\mathbb{R}^3 \times V)} ds.$$

Coming back to Equation (26) and using

$$\|\rho(\cdot, t)\|_{L^p(\mathbb{R}^3 \times V)} \leq C(V) \|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3 \times V)},$$

we deduce

$$\begin{aligned} \|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3 \times V)} &\leq \|f^0(\cdot, \cdot)\|_{L^p(\mathbb{R}^3 \times V)} \\ &+ C(V) \left(1 + \sup_{0 < s < t} \|S^S(\cdot, s)\|_{L^p(\mathbb{R}^3)}\right) \int_0^t \|f(\cdot, \cdot, s)\|_{L^p(\mathbb{R}^3 \times V)} ds, \quad \forall p \geq 2. \end{aligned} \quad (27)$$

We conclude in two steps. Firstly, we choose  $2 \leq p < 3$  such that

$$\frac{1}{|x|} \mathbb{I}_{|x| \leq 1} \in L^p(\mathbb{R}^3).$$

Then, from the Young inequality,

$$\|S^S(\cdot, t)\|_{L^p(\mathbb{R}^3)} \leq c \|f^0\|_{L^1(\mathbb{R}^3 \times V)} .$$

Using the Gronwall inequality in (27), we conclude the existence of a bound for  $\|f(\cdot, \cdot, t)\|_{L^p(\mathbb{R}^3 \times V)}$ ,  $p \in [2, 3)$  only depending on  $f^0$ ,  $p$  and  $t$ .

Secondly, still from the Young inequality,

$$\|S^S(\cdot, t)\|_{L^\infty(\mathbb{R}^3)} \leq C \|f(\cdot, \cdot, t)\|_{L^2(\mathbb{R}^n \times V)} \leq C(t) ,$$

with  $C \in L^\infty_{loc}([0, \infty))$ . We now set  $p = \infty$  in (27) and apply again the Gronwall inequality.  $\square$

## 4 Drift-diffusion limit: convergence proof

In this section the formal results of section 2 are rigorously justified. We shall use the following result from potential theory (see [29]).

**Lemma 3.** *Assume*

$$S = \rho * \frac{1}{4\pi|x|} \quad \text{with } \rho \in L^1_+(\mathbb{R}^3) \cap L^q(\mathbb{R}^3), \quad q > 3 .$$

Then  $S \in L^p \cap C^{1,\alpha}(\mathbb{R}^3)$  for every  $\alpha < \frac{q-3}{q}$ ,  $3 < p \leq \infty$  and there exists  $c > 0$  such that

$$\|S\|_{L^p(\mathbb{R}^3)} + \|S\|_{C^{1,\alpha}(\mathbb{R}^3)} \leq c (\|\rho\|_{L^1(\mathbb{R}^3)} + \|\rho\|_{L^q(\mathbb{R}^3)}) .$$

Our aim is the derivation of estimates uniform in  $\varepsilon$  as  $\varepsilon \rightarrow 0$ . As a first step, we multiply the transport equation (7) by  $g^{q-1}$ , with  $g = f_\varepsilon/F$  and  $q \geq 1$ , and integrate with respect to  $v$  and  $x$ . With (16), we obtain

$$\begin{aligned} \frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^3} \int_V \frac{f_\varepsilon^q}{F^{q-1}} dv dx &+ \frac{1}{2\varepsilon^2} \int_{\mathbb{R}^3} \int_V \int_V \phi_\varepsilon^S[S_\varepsilon](g - g')(g^{q-1} - (g')^{q-1}) dv' dv dx \\ &= -\frac{1}{2\varepsilon^2} \int_{\mathbb{R}^3} \int_V \int_V \phi_\varepsilon^A[S_\varepsilon](g + g')(g^{q-1} - (g')^{q-1}) dv' dv dx . \end{aligned}$$

The antisymmetric part is now estimated by

$$\begin{aligned} |\phi_\varepsilon^A[S_\varepsilon](g + g')(g^{q-1} - (g')^{q-1})| &\leq \frac{1}{2} \phi_\varepsilon^S[S_\varepsilon](g - g')(g^{q-1} - (g')^{q-1}) \\ &+ \frac{\phi_\varepsilon^A[S_\varepsilon]^2}{2\phi_\varepsilon^S[S_\varepsilon]} \frac{(g + g')^2 (g^{q-1} - (g')^{q-1})}{g - g'} . \end{aligned}$$

For the last term we use the inequality

$$\frac{(g + g')^2(g^{q-1} - (g')^{q-1})}{g - g'} \leq c_q(g^q + (g')^q),$$

and obtain

$$\begin{aligned} \frac{1}{q} \frac{d}{dt} \int_{\mathbb{R}^3} \int_V \frac{f_\varepsilon^q}{F^{q-1}} dv dx &+ \frac{1}{4\varepsilon^2} \int_{\mathbb{R}^3} \int_V \int_V \phi_\varepsilon^S[S_\varepsilon](g - g')(g^{q-1} - (g')^{q-1}) dv' dv dx \\ &\leq \frac{c_q}{2\varepsilon^2} \int_{\mathbb{R}^3} \int_V \int_V \frac{\phi_\varepsilon^A[S_\varepsilon]^2}{F \phi_\varepsilon^S[S_\varepsilon]} \frac{f_\varepsilon^q}{F^{q-1}} dv' dv dx. \end{aligned} \quad (28)$$

This inequality motivates the assumptions of the following result.

**Theorem 2.** *Let  $F \in L^\infty(V)$  be a positive velocity distribution satisfying (15) and let  $\phi_\varepsilon^S[S]$  and  $\phi_\varepsilon^A[S]$  be defined as in lemma 1. Assume that there exist  $q > 3$ ,  $\gamma > 0$ , and a non-decreasing function  $\Lambda \in L_{loc}^\infty([0, \infty))$ , such that*

$$f^0 \in \mathcal{X}_q := L_+^1(\mathbb{R}^3 \times V) \cap L^q\left(\mathbb{R}^3 \times V; \frac{dx dv}{F^{q-1}}\right), \quad (29)$$

$$\phi_\varepsilon^S[S] \geq \gamma(1 - \varepsilon\Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^3)}))FF', \quad (30)$$

$$\int_V \frac{\phi_\varepsilon^A[S]^2}{F \phi_\varepsilon^S[S]} dv' \leq \varepsilon^2 \Lambda(\|S\|_{W^{1,\infty}(\mathbb{R}^3)}). \quad (31)$$

Then there exists  $t^* > 0$ , independent of  $\varepsilon$ , such that the existence time of the local mild solution of (7)–(10) is bigger than  $t^*$ , and the solution satisfies, uniformly in  $\varepsilon$ ,

$$f_\varepsilon \in L^\infty((0, t^*); \mathcal{X}_q),$$

$$S_\varepsilon \in L^\infty((0, t^*); L^p \cap C^{1,\alpha}(\mathbb{R}^3)), \quad \alpha < \frac{q-3}{q}, \quad 3 < p < \infty \quad (32)$$

$$r_\varepsilon = \frac{f_\varepsilon - \rho_\varepsilon F}{\varepsilon} \in L^2\left(\mathbb{R}^3 \times V \times (0, t^*); \frac{dx dv dt}{F}\right).$$

**Remark 5.** *The assumptions (30), (31) correspond to the structural assumptions (12) and (A0) (see also (17)), describing the separation between the leading order turning processes and the chemotactically oriented contributions. In particular, the constant  $\gamma$  has the same role as in Section 2.*

*Proof.* As in the proof of Theorem 1, existence up to time  $t = t^*$  will follow, from the boundedness claimed in the theorem.

Cell conservation implies

$$\|f_\varepsilon(\cdot, \cdot, t)\|_{L^1(\mathbb{R}^3 \times V)} = \|f^0\|_{L^1(\mathbb{R}^3 \times V)}$$

for all  $t > 0$ .

Using assumption (31) in (28) we obtain

$$\frac{d}{dt} \int_{\mathbb{R}^3} \int_V \frac{f_\varepsilon^q}{F^{q-1}} dv dx \leq \frac{qc_q}{2} \Lambda(\|S_\varepsilon(\cdot, t)\|_{W^{1,\infty}(\mathbb{R}^3)}) \int_{\mathbb{R}^3} \int_V \frac{f_\varepsilon^q}{F^{q-1}} dv dx. \quad (33)$$

The next step is to estimate  $S_\varepsilon$ . Cell conservation and lemma 3 imply

$$\|S_\varepsilon(\cdot, t)\|_{C^{1,\alpha}(\mathbb{R}^3)} \leq C(1 + \|\rho_\varepsilon(\cdot, t)\|_{L^q(\mathbb{R}^3)}).$$

By Hölder inequality and the normalization of  $F$ ,

$$\|\rho_\varepsilon(\cdot, t)\|_{L^q(\mathbb{R}^3)} \leq \left( \int_{\mathbb{R}^3} \int_V \frac{f_\varepsilon^q}{F^{q-1}} dv dx \right)^{1/q}.$$

Combining the last two estimates with (33) gives a differential inequality of the form

$$\frac{dP}{dt} \leq \tilde{\Lambda}(P), \quad P = \int_{\mathbb{R}^3} \int_V \frac{f_\varepsilon^q}{F^{q-1}} dv dx,$$

with  $\tilde{\Lambda} \in L_{loc}^\infty([0, \infty))$  (precisely  $\tilde{\Lambda}(P) = \frac{qc_q}{2} \Lambda(C(1 + P^{1/q}))P$ ). This completes the proof of the first two statements in (32).

By interpolation between  $L^1(\mathbb{R}^3 \times V; F dx dv)$  and  $L^q(\mathbb{R}^3 \times V; F dx dv)$  the norm of  $f_\varepsilon(\cdot, \cdot, t)/F$  in  $L^2(\mathbb{R}^3 \times V; F dx dv)$  is bounded uniformly for  $t \in (0, t^*)$ . Thus, setting  $q = 2$  in (28) and integration with respect to  $t$  gives

$$\int_0^{t^*} \int_{\mathbb{R}^3} \int_V \int_V \phi_\varepsilon^S[S_\varepsilon] \left( \frac{f_\varepsilon}{F} - \frac{f'_\varepsilon}{F'} \right)^2 dv' dv dx dt \leq \varepsilon^2 c.$$

With assumption (30), this leads to

$$\gamma \int_0^{t^*} \int_{\mathbb{R}^3} \int_V \int_V F F' \left( \frac{r_\varepsilon}{F} - \frac{r'_\varepsilon}{F'} \right)^2 dv' dv dx dt \leq c.$$

Using  $\int_V r_\varepsilon dv = 0$ , we now deduce the last statement in (32):

$$\int_0^{t^*} \int_{\mathbb{R}^3} \int_V \frac{r_\varepsilon^2}{F} dv dx dt \leq \frac{c}{2\gamma}.$$

□

**Theorem 3.** *Let the assumptions of Theorem 2 hold. Assume further that for families  $S_\varepsilon$ , uniformly bounded (as  $\varepsilon \rightarrow 0$ ) in  $L_{loc}^\infty([0, \infty); C^{1,\alpha}(\mathbb{R}^3))$  for some  $0 < \alpha \leq 1$ , such that  $S_\varepsilon$  and  $\nabla S_\varepsilon$  converge to  $S_0$  and  $\nabla S_0$ , respectively, in  $L_{loc}^p(\mathbb{R}^3 \times [0, \infty))$  for some  $p > 3/2$ , we have the convergence*

$$\begin{aligned} T_\varepsilon[S_\varepsilon] &\rightarrow T_0[S_0] \quad \text{in } L_{loc}^p(\mathbb{R}^3 \times \bar{V} \times \bar{V} \times [0, \infty)), \\ \frac{T_\varepsilon[S_\varepsilon](F)}{\varepsilon} &= \frac{2}{\varepsilon} \int_V \phi_\varepsilon^A[S_\varepsilon] dv' \rightarrow \mathcal{T}_1[S_0](F) \quad \text{in } L_{loc}^p(\mathbb{R}^3 \times \bar{V} \times [0, \infty)). \end{aligned} \quad (34)$$

Then solutions of (7)–(10) satisfy (possibly after extracting subsequences)

$$\begin{aligned} f_\varepsilon &\rightarrow \rho_0 F && \text{in } L^\infty((0, t^*); \mathcal{X}_q) \text{ weak } *, \\ S_\varepsilon &\rightarrow S_0 && \text{in } L^p_{loc}(\mathbb{R}^3 \times (0, t^*)), \quad 3 < p < \infty, \\ \nabla S_\varepsilon &\rightarrow \nabla S_0 && \text{in } L^p_{loc}(\mathbb{R}^3 \times (0, t^*)), \quad \frac{3}{2} < p < \infty. \end{aligned}$$

The limits are weak solutions of (23), (13) subject to the initial condition

$$\rho_0(t=0) = \int_V f^0 dv.$$

*Proof.* The weak convergence of  $f_\varepsilon$  follows from Theorem 2. For  $S_\varepsilon$  and  $\nabla S_\varepsilon$  we still need some compactness in time. For the flux density  $J_\varepsilon$  in the cell conservation equation

$$\partial_t \rho_\varepsilon + \operatorname{div} J_\varepsilon = 0, \quad (35)$$

we have

$$J_\varepsilon = \frac{1}{\varepsilon} \int_V v f_\varepsilon dv = \int_V v r_\varepsilon dv \in L^2((0, t^*); L^2(\mathbb{R}^3)),$$

uniformly in  $\varepsilon$ , by the estimate

$$|J_\varepsilon|^2 \leq \int_V \frac{r_\varepsilon^2}{F} dv \int_V |v|^2 F dv,$$

theorem 2, the boundedness of  $F$ , and the compactness of  $V$ . The gradient of the convolution of (35) with the Newtonian potential  $1/(4\pi|x|)$  is

$$\partial_t(\nabla S_\varepsilon) + \nabla(\nabla \cdot S_{J,\varepsilon}) = 0,$$

with  $S_{J,\varepsilon} = J_\varepsilon * 1/(4\pi|x|)$ . By elliptic regularization we have  $S_{J,\varepsilon} \in L^2((0, t^*); H^2_{loc}(\mathbb{R}^3))$  and, thus,

$$\partial_t(\nabla S_\varepsilon) \in L^2((0, t^*); L^2_{loc}(\mathbb{R}^3)).$$

The derivative of  $S_\varepsilon$  with respect to time is estimated analogously. The strong convergence now follows combining the above with the elliptic regularity for the convolutions defining  $S_\varepsilon$  and  $\nabla S_\varepsilon$  from  $\rho_\varepsilon$ .

After dividing by  $\varepsilon$ , the kinetic equation (7) can be written as

$$\varepsilon \frac{\partial f_\varepsilon}{\partial t} + v \cdot \nabla_x f_\varepsilon = -\rho_\varepsilon \frac{\mathcal{T}_\varepsilon[S_\varepsilon](F)}{\varepsilon} - \mathcal{T}_\varepsilon[S_\varepsilon](r_\varepsilon). \quad (36)$$

Theorem 2 implies weak convergence of  $\rho_\varepsilon$  and  $r_\varepsilon$  to  $\rho_0$  and  $r_0$ . The boundedness of  $r_\varepsilon$  also implies  $f_0 = \rho_0 F$ . With the assumptions (34), we can pass to the limit in (36) and obtain

$$\mathcal{T}_0[S_0](r_0) = -vF \cdot \nabla \rho_0 - \rho_0 \mathcal{T}_1[S_0](F).$$

This equation can be solved for  $r_0$  as (19). The limit of the cell conservation equation is

$$\partial_t \rho_0 + \nabla \cdot J_0 = 0,$$

with the flux  $J_0 = \int_V v r_0 dv$ . □

## 5 Examples

In this section we discuss two specific models for turning kernels and compute explicit formulas for the macroscopic transport coefficients. We also state rigorous results for these models which are applications of the more general results of the previous sections. For additional models and their biological relevance we refer to the work of Othmer and Hillen [23].

The main obstacle to the explicit computation of transport coefficients is the solution of the integral equation (19). This task is straightforward for the relaxation time model

$$\mathcal{T}_0[S](x, v, v', t) = \lambda[S](x, t) F(v), \quad \lambda[S] > 0, \quad (37)$$

which is common to both examples below. In this case, the leading order turning operator becomes

$$\mathcal{T}_0[S](f) = \lambda[S](f - \rho F),$$

and the solution of the problem  $\mathcal{T}_0[S](f) = g$ ,  $\int_V f dv = 0$ , with  $\int_V g dv = 0$ , is given by  $f = g/\lambda[S]$ . So, in particular, we obtain

$$\kappa = \frac{vF}{\lambda[S_0]}, \quad \Theta = \frac{\mathcal{T}_1[S_0](F)}{\lambda[S_0]},$$

for the solutions of (21), (22). Consequently, the macroscopic diffusivity and convection field are given by

$$D[S_0] = \frac{1}{\lambda[S_0]} \int_V v \otimes v F dv, \quad \Gamma[S_0] = -\frac{1}{\lambda[S_0]} \int_V v \mathcal{T}_1[S_0](F) dv.$$

Furthermore we shall assume rotational invariance of the equilibrium distribution, i.e.,  $F = F(|v|)$ , with the consequence

$$D[S_0] = \frac{1}{3\lambda[S_0]} \int_V |v|^2 F dv I.$$

## Model 1

A group of models considered in [23] is of the form

$$T_\varepsilon[S] = T_0[S] + \varepsilon T_1[S], \quad (38)$$

where  $T_0[S]$  is of the form (37), and  $T_1[S]$  depends on pointwise values of  $S$  and  $\nabla S$ . We assume rotational invariance, i.e.,  $\lambda[S](x, t) = \lambda(S(x, t), |\nabla S(x, t)|)$  in (37). In  $T_1$  the dependence on the vectors  $v$ ,  $v'$ , and  $\nabla S$  is only through their Euclidean norms and through the angles between them:

$$T_1[S](x, v, v', t) = T_1(S(x, t), |v|, |v'|, |\nabla S(x, t)|, v \cdot v', v \cdot \nabla S(x, t), v' \cdot \nabla S(x, t)). \quad (39)$$

Then rotational invariance is inherited also by  $\mathcal{T}_1[S_0](F)$  which can be written in the form

$$\mathcal{T}_1[S_0](F) = H(S_0, |v|, |\nabla S_0|, v \cdot \nabla S_0).$$

A symmetry argument shows that the macroscopic convection field is proportional to  $\nabla S_0$ :

$$\Gamma[S_0] = \chi(S_0, |\nabla S_0|) \nabla S_0, \quad (40)$$

with

$$\chi = -\frac{1}{\lambda[S_0]|\nabla S_0|} \int_V v_1 H(S_0, |v|, |\nabla S_0|, v_1 |\nabla S_0|) dv,$$

where  $v_1$  denotes the first (or any) coordinate of  $v$ . More specifically, a model of the form

$$T_1 = a(S)v \cdot \nabla S - b(S)v' \cdot \nabla S, \quad a, b \geq 0, \quad (41)$$

represents the desire of the cell to change both to a favourable direction (by the first term) and away from an unfavorable direction (by the second term). A straightforward computation gives (40) with the chemotactic sensitivity

$$\chi = \frac{1}{3\lambda[S_0]} \left( b(S_0)\mu(V) \int_V |v|^2 F dv + a(S_0) \int_V |v|^2 dv \right).$$

The rigorous convergence analysis of Section 4 can be applied to this class of models. It is easily seen that the assumptions of Theorems 2 and 3 are satisfied under mild conditions.

Note that the macroscopic models obtained include the classical Keller-Segel model exhibiting blow up in finite time. Therefore nothing better than the local-in-time result above can be expected. In general, we also expect blow up in finite time for the kinetic model. However, there is no proof available yet. For the second class of models, on the other hand, global existence for the kinetic model can be proven as well as local-in-time convergence to a Keller-Segel type macroscopic model.

## Model 2

Let  $\psi(S, \tilde{S})$  be a smooth, positive, non-decreasing (in the second argument) function defined on a  $\mathbb{R}^+ \times \mathbb{R}^+$  such that

$$0 < \psi_{\min} \leq \psi(S, \tilde{S}) \leq \alpha_1 \tilde{S} + \alpha_2 ,$$

for real, positive constants  $\alpha_{1,2}$ .

Let the turning kernel be given by

$$T_\varepsilon[S](x, v, v', t) = \alpha_+ \psi(S(x, t), S(x + \varepsilon v, t)) + \alpha_- \psi(S(x, t), S(x - \varepsilon v', t)), \quad (42)$$

where  $\alpha_\pm$  are positive constants. This means that the cell is able to measure the chemo-attractant concentration up to a distance  $\varepsilon v_{max}$  away from its position, where  $v_{max}$  is the maximal speed in  $V$ . Similarly to the model (41) the turning probability is higher for a change to a favorable direction and away from an unfavorable direction.

Expansion of  $T_\varepsilon$  gives

$$\begin{aligned} T_0[S] &= (\alpha_+ + \alpha_-) \psi(S, S) , \\ T_1[S] &= \frac{\partial \psi}{\partial \tilde{S}}(S, S) (\alpha_+ v - \alpha_- v') \cdot \nabla S . \end{aligned}$$

Thus, we find that  $F = 1/\mu(V)$  and

$$\begin{aligned} D[S_0] &= \frac{1}{3(\alpha_+ + \alpha_-) \psi(S_0, S_0) \mu(V)^2} \int_V |v|^2 dv I , \\ \chi[S_0] &= \frac{\partial \psi / \partial \tilde{S}(S_0, S_0)}{3\psi(S_0, S_0) \mu(V)} \int_V |v|^2 dv I , \end{aligned}$$

where  $I$  is the  $3 \times 3$  identity matrix and  $\Gamma[S] = \chi[S] \nabla S$ . The model (42) obviously satisfies the assumptions of the global existence Theorem 1.

In order to check the assumptions of Theorems 2 and 3 we compute

$$\phi_\varepsilon^A[S](x, v, v', t) = \varepsilon \frac{(v - v')}{\mu(V)} \cdot \nabla S(\tilde{x}, t) (\alpha_+ \psi_2(S(x, t), S(\tilde{x}, t)) - \alpha_- \psi_2(S(x, t), S(\tilde{x}, t))) ,$$

where  $\tilde{x}$  lies between  $x - \varepsilon v'$  and  $x + \varepsilon v$ . Thus, the assumptions of Theorems 2 and 3 are satisfied.

If we choose  $\psi(S, \tilde{S}) = \Psi(\tilde{S} - S)$ ,  $\alpha_+ = 1$ ,  $\alpha_- = 0$  with a strictly positive, increasing  $\Psi$  then we reproduce the classical Keller-Segel model with constant macroscopic transport coefficients  $D[S]$  and  $\chi[S]$ . As long as  $\Psi$  is at most linear, we have global existence for the kinetic model.

On the other hand, if we choose  $\psi(S, \tilde{S}) = \Psi(S)\tilde{\Psi}(\tilde{S})$ , with positive  $\Psi$  and  $\tilde{\Psi}$  we can, at least formally, reproduce arbitrary macroscopic transport coefficient  $D[S]$  and  $\chi[S]$  (assuming  $\alpha_+ = 1$  and  $\alpha_- = 0$ ) by setting:

$$\begin{aligned}\tilde{\Psi}(S) &= \exp\left(3\frac{\mu(V)}{\int_V |v|^2 dv} \int_{S_0}^S \chi[S'] dS'\right), \\ \Psi(S) &= \frac{\int_V |v|^2 dv}{3\mu(V)^2 \tilde{\Psi}(S) D[S]},\end{aligned}$$

for a certain reference value  $S_0$ . If  $0 < \Psi_{\min} < \tilde{\Psi}(S) \leq \alpha_1 S + \alpha_2$  and  $\Psi(S)$  is bounded (which is true if  $\chi \geq \chi_{\min} > 0$  and  $D \geq D_{\min} > 0$ , or if  $\chi \geq \chi_0/(S + S_0)$  and  $D \geq D_{\min} > 0$ , for constant and positive  $\chi_0, S_0$ ), then we have global existence for the solutions of the kinetic model.

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