Flat Semantics

Christian Damböck
Moritz Schlick Project
Institute Vienna Circle
Museumstraße 5/2/19
1070 Wien, Austria
christian.damboeck@univie.ac.at

Flat Semantics we will call first order languages where $\zeta \Vdash \phi$ is a kind of first order formula. There ζ is an *interpretation* in the sense defined below and ϕ is any formula of the language. $\Vdash$ is a relation equivalent to the usual second order relation $\vDash$ telling 'ϕ is satisfied in ζ'.

The idea is quite simple. We just have to build our first order language as a *many sorted* one, including the set of all interpretations as a sort into the language:

Let $\mathcal{S}$ be a (usually finite) set – the *domain of the first category.* We will have individual constants and variables and a finite number of n-place predicate and function symbols for some $n > 0$. For simplification every element of $\mathcal{S}$ should be denoted by exactly one individual constant. This construction we will call the *first category* of our language.

Now an *interpretation* $\zeta = (\mathcal{S}_\exists, \tau)$ over the first category is defined as a set $\mathcal{S}_\exists \subseteq \mathcal{S}$ of *existing individuals* and a function τ, assigning to the predicate and function symbols of the first category relations and functions over $\mathcal{S}_\exists$. With $\mathcal{A}$ we denote the set of all interpretations. $\mathcal{A}$ is called *the domain of the second category.*

There are again individual constants, variables and a finite number of predicate and function symbols for the second category. A *modal interpretation* o assigns to the predicate and function symbols of the second category relations and functions over $\mathcal{A}$. (Roughly speaking, o defines such things like relations of 'accessibility of possible worlds'.)

That way our language is given by a *domain structure* $\mathfrak{A} = (\mathcal{S}, o)$, where $\mathcal{S}$ and o are defined as mentioned. It is a two sorted language with the sorts $\mathcal{S}$ and $\mathcal{A}$.

We introduce atomic formulas and a syntax like in first order logic but with addition of the clause:

If ϕ is a formula and ζ is an interpretation then $\zeta \Vdash \phi$ is also a formula.

We define a value for each interpretation $\zeta = (\mathcal{S}_\exists, \tau)$ and each valuation of a function $f(c_1, \ldots, c_i)$ of the first category, where $c_1, \ldots, c_i$ are individual constants. If the entities denoted by $c_1, \ldots, c_i$ are contained in $\mathcal{S}_\exists$, then the value is given by τ. Otherwise the value is an arbitrary constant **null** not denoting any element of $\mathcal{S}$.

Identity and quantification will also be defined relative to the set $\mathcal{S}_\exists$ of existing individuals. The remaining semantical definitions are done like in first order logic. Additionally, for any formula ϕ and any interpretations ζ, ζ' there applies:

(F) $\zeta \models \zeta' \Vdash \phi$ iff $\zeta' \models \phi$.

Because $\Vdash$ is reduced to the second order relation $\models$, there cannot be any problem with paradox. In the case of a finite $\mathcal{S}$ the set $\mathcal{A}$ is also finite and the language, then, is decidable (say, for any formula $z \Vdash \phi$ we can decide if it is satisfied in finitely many steps).

The last important element we need is the *meta-constant* $\aleph$ which denotes in any formula the 'recently active' interpretation ζ. In the formula $\zeta \Vdash \forall z R(\aleph, z)$ for example the constant $\aleph$ denotes ζ. We can define the modal operator $\Box$:

$\Box\phi$ iff $\forall z : R(\aleph, z) \to z \Vdash \phi$

Here R is a predicate over $\mathcal{A}$, defining a modal system S5 if it is an equivalence relation and so on. Necessity turns out as a simple first order operator in our language.

Flat semantics gives us some languages, where Tarski's convention (T) has axiomatic validity.[1]—From definition (F) it follows immediately that

$\zeta \models \phi \leftrightarrow \zeta \Vdash \phi$.

It is clear that Tarski-Semantics is unable to implement convention (T) as an axiom, because it simply gives us a definition of being true in an interpretation (or rather in a structure). Therefore, it is impossible in Tarski-Semantics to say something like

ϕ is true in ζ, iff ϕ in ζ,

because there is no way to formulate 'ϕ in ζ' *in the object language*. In flat semantics, on the other hand, we can define

$T(\Phi) := \zeta \models \phi$
$\Phi := \zeta \Vdash \phi$

and will get as an Axiom:

(T) $T(\Phi) \leftrightarrow \Phi$.

[1]Alfred Tarski: 'Der Wahrheitsbegriff in den formalisierten Sprachen', *Studia Philosophica Commentarii Societatis philosophicae Polonorum, Vol I, Leopoli, 261-405.*