# Groups acting on locally finite graphs
# - a survey of the infinitely ended case -

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## Introduction

The study of infinite graphs has many aspects and various connections with other fields. There are the classical graph theoretic problems in infinite settings (see the survey by Thomassen [49]); there are special graph theoretical questions which have no direct analogues for finite graphs, such as questions about ends (see [7], [44] and the monograph [6]); Ramsey graph theory with its connections to set theory; the study of spectra of infinite graphs and random walks on infinite graphs (see the surveys [32] and [58]); the study of group actions on infinite graphs.

This survey is on the last subject, or rather on a small corner of the last subject. As is usual one concentrates on the case where the automorphism group acts transitively on the graph. The study of group actions can then be spilt up into three cases according to whether the graph under investigation has one, two or infinitely many ends. A graph has one end if there is always just one infinite component when finitely many vertices are removed from the graph. (‘‘Component’’ will always mean a connected component in the graph theoretical sense.) The case of graphs with only one end is the hardest one, but in the special case of graphs with polynomial growth there are some very nice results (see [23]). The two ended case is the easiest one: roughly speaking these graphs all look like fat lines and one can say that they are very well understood (see [29] and [22]). Then there is the infinitely ended case, which is the one that this paper is all about.

Some of the motivation behind the study of graphs with infinitely many ends comes from group theory, in particular the Bass–Serre theory of group actions on trees and Stallings' Ends Theorem. In general these graphs can be said to resemble trees, and it is precisely that resemblance that one tries to extract and use when studying them. The aim of this paper is twofold: firstly to give a survey of known results and secondly to give an exposition of a new and powerful technique to capture the ‘‘treeness’’. It is hoped that the treatment is more or less self-contained. We will be concentrating on locally finite graphs but will also mention results for graphs with infinite valencies.

In Section 1 we define the ends of a graph and go over the basic properties concerning group actions. The new technique mentioned above is then discussed in the second section and in the third section we show how it can be used in practice by giving new proofs of some results of Woess [54] relating amenability to group actions on graphs. In Section 4 we look at the graph theoretical version of the group theoretical concept of accessibility. And, finally, in Section 5 we consider the effect of placing extra conditions either on the graph or on the action of the automorphism group.

# 1 Graphs and ends

We think of a graph $X$ as a pair $(VX, EX)$ where $VX$ is the vertex set and $EX$ is the set of edges. The graphs considered will be without multiple edges and loops. Unless otherwise stated our graphs are undirected. So the set of edges can be viewed as a set of two element subsets of $VX$. Our notation is fairly standard, but note that "$\subset$" is used to denote strict inclusion.

Let $G$ be a group acting on a set $Y$. If $x \in Y$ then let $G_x$ denote the *stabilizer of* $x$ in $G$; that is, $G_x$ is the subgroup of all elements in $G$ that fix $x$. For a subset $A$ in $Y$ we define $G_{\{A\}}$ as the subgroup of $G$ consisting of all elements $g \in G$ such that $gA = A$. We call the group $G_{\{A\}}$ the *setwise stabilizer of* $A$. The automorphism group of $X$ is denoted by $\mathrm{Aut}(X)$, and we think of $\mathrm{Aut}(X)$ primarily as a permutation group on $VX$. If $\mathrm{Aut}(X)$ acts transitively on $VX$ then the graph $X$ is said to be *transitive*.

A graph is said to be *locally finite* if all its vertices have finite valency. Note that a connected locally finite graph has a countable vertex set. A *ray* (also called a *half-line*) in a graph $X$ is a sequence $\{v_i\}_{i\in\mathbf{N}}$ of distinct vertices such that $v_i$ is adjacent to $v_{i+1}$ for all $i \in \mathbf{N}$. A *line* in $X$ is a sequence $\{v_i\}_{i\in\mathbf{Z}}$ of distinct vertices such that $v_i$ is adjacent to $v_{i+1}$ for all $i \in \mathbf{Z}$. We say that a path is *simple* if all its vertices are distinct. Let $d(u, v)$ denote the minimum length of a path in $X$ between the vertices $u$ and $v$. If $X$ is connected then $d$ is a metric on $VX$.

For the rest of this section $X$ will denote a locally finite connected graph.

## 1.1 Ends

There are various ways of defining the ends of a graph. The graph theoretic approach is to define the ends as equivalence classes of rays.

**Definition 1** ([18]) *Two rays $R_1$ and $R_2$ are said to be in the same* end *if there is a ray $R_3$ in $X$ which contains infinitely many vertices from both $R_1$ and $R_2$.*

This definition becomes very simple in the special case where $X$ is a tree: then two rays are in the same end if and only if their intersection is a ray.

There are several ways to rephrase this definition. Clearly $R_1$ and $R_2$ are in the same end if and only if there are infinitely many disjoint paths connecting vertices in $R_1$ to vertices in $R_2$. Now it is easy to check that *being in the same end* is an equivalence relation. (The idea for the proof of transitivity is indicated on Figure 1.) The equivalence classes are called the *ends* of $X$ and the set of ends is denoted by $\Omega X$ (in [5] and [33] the set of ends is denoted by $\mathcal{E}X$).

Another way to rephrase the definition is to say that $R_1$ and $R_2$ are in the same end if and only if for every finite set $F \subseteq VX$ there is a path in $X \setminus F$ connecting a vertex in $R_1$ to a vertex in $R_2$. This in turn leads to yet another reformulation of the definition: two rays $R_1$ and $R_2$ are not in the same end if and only if one can find a finite set $F$ of vertices and distinct components $C_1$ and $C_2$ of $X \setminus F$ such that $C_1$ contains infinitely many vertices of $R_1$ and $C_2$ contains infinitely many vertices of $R_2$. It is clear that a locally finite connected graph $X$ has more than one end if and only if there is a finite set of vertices $F$ such that $X \setminus F$ has more than one infinite component.

For a set $C \subseteq VX$ we define the *boundary* $\partial C$ as the set of vertices in $VX \setminus C$ that are adjacent to a vertex in $C$. The *co-boundary* $\delta C$ is defined as the set of edges that have one end vertex in $C$ and the other one in $VX \setminus C$.

From the above definition of an end of a graph it is evident that if $C \subseteq VX$ with finite boundary and $C$ contains infinitely many vertices from some ray $R$ then $C$ also contains infinitely many vertices from every ray in the same end as $R$. Thus it is reasonable to say that $C$ contains the end that $R$ is in. Let $\Omega C$ denote the set of ends that are contained in $C$. If $F \subseteq VX$ is finite and two ends $\omega$ and $\omega'$ are in different components of $X \setminus F$ then we say that $F$ *separates* the ends $\omega$ and $\omega'$.

In general one can say that this definition of the set of ends suggests that the ends describe how the graph "branches". Each end somehow represents one way of going to infinity.

*Example.* The infinite grid $\mathbf{Z} \times \mathbf{Z}$ does not "branch" at all and has only one end. (It is easy to find a ray in $\mathbf{Z} \times \mathbf{Z}$ that contains all the vertices. Every other ray must be in the same end as that ray.) The infinite line $\mathbf{Z}$ has two ends. On the other hand, the 3-valent regular tree $T_3$ clearly has a lot of "branching". It is one of the special properties of trees that given a vertex $v$ and an end $\omega$ there is precisely one ray in $\omega$ that starts at $v$. Hence $T_3$ has $2^{\aleph_0}$ ends.

Ends come in various shapes and sizes. The main distinction is between *thick* and *thin* ends: an end is said to be thick if it contains infinitely many disjoint rays, and thin otherwise. The end of $\mathbf{Z} \times \mathbf{Z}$ is thick but the ends of $\mathbf{Z}$ and $T_3$ are all thin. For a thin end $\omega$ we define the *thickness* or *size* of $\omega$, denoted by $m_1(\omega)$, as the maximum number of disjoint rays contained in $\omega$. Halin [19] proved that if $\omega$ is thin then $m_1(\omega)$ is finite.

We say that a sequence $\{C_i\}_{i\in\mathbf{N}}$ with $\bigcap_{i\in\mathbf{N}} C_i = \emptyset$ *converges to an end* $\omega$ if all the sets in the sequence are connected and have finite boundary, and $\omega \in \Omega C_i$ for all $i$. The size of $\omega$ is the same as the lowest number $m$ such that there is a sequence $\{C_i\}_{i\in\mathbf{N}}$ converging to $\omega$ such that $|\partial C_i| \leq m$ for all $i$.

One can also think of the ends as a boundary of the graph. This becomes clearer if we give a topological definition. This definition can be traced back to papers of Hopf and Freudenthal in the thirties and fourties (e.g. [15]).

Let $\mathcal{F}$ denote the set of all finite subsets of $VX$. For $F \in \mathcal{F}$ define $\mathcal{C}_F$ as the set of all infinite components of $X \setminus F$. If $F_1$ and $F_2$ are two elements of $\mathcal{F}$ such that $F_1 \subseteq F_2$ then there is a natural projection $\mathcal{C}_{F_2} \to \mathcal{C}_{F_1}$: a component of $X \setminus F_2$ being mapped to the component of $X \setminus F_1$ that contains it. So we have an inverse system in our hands. Let $\Omega$ denote its inverse limit. Now we want to identify $\Omega$ and $\Omega X$. An element of $\Omega$ can be represented as a family $(C_F)_{F\in\mathcal{F}}$ such that if $F_1 \subseteq F_2$ then $C_{F_2} \subseteq C_{F_1}$. Given an end $\omega \in \Omega X$ it is easy to find the corresponding element in $\Omega$: for $F \in \mathcal{F}$ we just let $C_F$ denote the component of $X \setminus F$ that $\omega$ belongs to and then $(C_F)_{F\in\mathcal{F}}$ does the trick. Clearly the element constructed is the only element in $\Omega$ such that each of its components contains $\omega$.

The next step is to show how we find the end corresponding to an element $\omega$ in $\Omega$. Let $(C_F)_{F\in\mathcal{F}}$ be an element in $\Omega$. Take a strictly increasing sequence $F_1 \subset F_2 \subset \ldots$ of finite subsets of $VX$ such that $VX = \bigcup_{i\in\mathbf{N}} F_i$. Then $\{C_{F_i}\}_{i\in\mathbf{N}}$ is an decreasing sequence. First of all it is clear that any two ends in $X$ are separated by some set $F_i$. However, one can find a ray that includes at least one vertex from $\partial C_{F_i}$ for all $i \in \mathbf{N}$. Then there is precisely one end $\omega$ that belongs to all of the sets $C_{F_i}$. For any $F \in \mathcal{F}$ we can find $i \in \mathbf{N}$ such that $F \subseteq F_i$. Then $C_{F_i} \subseteq C_F$ and therefore $\omega$ is in $\Omega C_F$.

The inverse limit construction gives a topology on $\Omega X$. A basis of open set for this topology is given by sets $\Omega C$ where $C \subseteq VX$ and $C$ has finite boundary. It is easy to see that $\Omega X$ with this topology is compact. Indeed, if one puts the discrete topology on $VX$ then one can view $VX \cup \Omega X$ as a compactification of $VX$. We will have occasion later on in this paper to make use of the topology introduced above, but the inverse limit construction will not be needed.

*Remark.* The assumption of local finiteness is not essential; ends can be defined in exactly the same manner for non locally finite graphs. There is one important difference: the space of ends with the topology given above will not, in general, be compact if the graph is not locally finite.

## 1.2 Ends and automorphisms

It is clear from the definition of an end that an automorphism of $X$ has a natural action on $\Omega X$. As shown by Halin in his fundamental paper [20] the action on the ends gives vital clues to how the automorphism acts. The same is also evident from Tits' paper [51], where group actions on infinite trees are studied. Halin shows how automorphisms of $X$ can be split up into three disjoint classes. Let $g \in \mathrm{Aut}(X)$. Then one of the following holds:

**(i)** $g$ leaves invariant some non-empty finite subset of $VX$;

**(ii)** $g$ fixes precisely one thick end and does not satisfy (i);

**(iii)** $g$ fixes precisely two thin ends and does not satisfy (i).

Automorphisms that satisfy (ii) or (iii) are often collectively known as *translations*. For a translation $g$ it is possible to find a line in $X$ and some power of $g$ that acts like a translation on the line. If $X$ is a tree then $g$ will act like a translation on the line. Those automorphisms that satisfy (i) are called *elliptic*, those that satisfy (ii) are called *parabolic* and those that satisfy (iii) are called *hyperbolic* (or *proper translations*). The above classification resembles the classification of automorphisms of hyperbolic space.

It is indeed quite easy to describe how one is to find an invariant line in cases (ii) and (iii). Suppose that $g$ does not satisfy (i). Put $n = \min d(g^k v, v)$, where $k$ ranges over $\mathbf{Z} \setminus \{0\}$ and $v$ ranges over $VX$. Find $k_0 \in \mathbf{N}$ and $v_0 \in VX$ such that $n = d(g^{k_0} v_0, v_0)$, then take a path $P$ of length $n$ between $v_0$ and $g^{k_0} v_0$ and set $L = \bigcup_{i \in \mathbf{Z}} g^{ik_0} P$. It is left to the reader to show that $L$ is a line and that $g^{k_0}$ acts like a translation on $L$, (see [20, Theorem 7]).

Now one can ask about the existence of translations in $\mathrm{Aut}(X)$. The following theorem, which is a strengthened version for locally finite graphs of Theorem 1 in [25], gives the answer. First let us identify a simple fundamental property of a group acting on a graph. Let $G \leq \mathrm{Aut}(X)$. We say that $G$ *shuffles* $X$ if for every infinite set $C \subseteq VX$ with finite boundary and every finite set $F \subseteq VX$ there is an automorphism $g \in G$ such that $gF \subseteq C$. If $X$ is transitive and locally finite then it is easy to see that $G$ shuffles $X$. We take a vertex $v$ in $C$ such that the distance from $v$ to $\partial C$ is greater than the diameter of $F$ as a subset of the metric space $(VX, d)$. Then just find an automorphism $g \in G$ that maps some vertex of $F$ to $v$. It is clear that $gF \subseteq C$.

**Theorem 1** *Let $X$ be a connected locally finite transitive graph. Suppose $C$ is an infinite subset of $VX$ with infinite complement and finite boundary. Then there is an element $g \in \mathrm{Aut}(X)$ such that $gC \subset C$ and $g$ is of type* (iii).

*Proof.* Set $G = \mathrm{Aut}(X)$. (In fact our argument works for any subgroup $G$ of $\mathrm{Aut}(X)$ that shuffles $X$.) Let $F$ be a finite connected set of vertices containing $\partial C$ and put $C' = VX \setminus (C \cup F)$. Because $G$ shuffles $X$ we can find $h \in G$ such that $h$ maps $F$ into $C$. If we are lucky then we can take $g = h$, but let us suppose that this does not work. Now find $h'$ such that $h'F \subseteq C'$. We can now ask if $h'C' \subseteq C'$ because then $h'^{-1}$ would work, but again let us suppose not. Then $hC \not\subseteq C$ and $h'C' \not\subseteq C'$. Suppose that $F \cap hC = \emptyset$. Then, as $h(F \cup C)$ is connected, it follows that $h(F \cup C) \subseteq C$, contrary to our assumptions. (Figure 2 gives a schematic view of how things must lie.) We can apply the same argument with $h'$ and $C'$ replacing $h$ and $C$. Whence we see that $F \subseteq hC \cap h'C'$. Now $F \cup C'$ is a connected subset of $X \setminus hF$ and $F \cup C'$ meets $hC$ so $F \cup C' \subseteq hC$. Similarly $F \cup C \subseteq h'C'$. Then $h^{-1}(F \cup C) \subseteq C'$ and $h^{-1}C' \subseteq C$. Now we see that $g = h^{-1}h'^{-1}$ is as desired. It is clear that $g$ cannot satisfy (i). For the details of how to prove the last statement in the Theorem we refer to [8, Proposition 2]. □

From this result the following corollary can be deduced.

**Corollary 1** *An infinite connected locally finite transitive graph has either* $1, 2$ *or* $2^{\aleph_0}$ *ends.*

It is natural to ask if the action of $\mathrm{Aut}(X)$ on $\Omega X$ is faithful and, if not then try to identify the kernel. An automorphism $g$ of a graph $X$ is said to be *bounded* if there is a natural number $n$ such that $d(v, gv) \leq n$ for all $v \in VX$. The bounded automorphisms form a subgroup of $\mathrm{Aut}(X)$, which we denote by $B(X)$.

**Theorem 2** *Let $X$ be a connected locally finite transitive graph with infinitely many ends. Then*

- **(i)** ([16, Theorem 5]) $B(X)$ *is a locally finite group (every finitely generated subgroup is finite);*
- **(ii)** ([33, Theorem 6]) $B(X)$ *is the kernel of the action of* $\mathrm{Aut}(X)$ *on* $\Omega X$.

*Remark.* The situation as regards automorphisms of graphs that are not locally finite is similar. Often the proofs are more delicate and there are also some subtle variations from what is valid in the locally finite case. For detailed treatment see [26].

Transitive graphs of infinite valency with infinitely many ends resemble locally finite graphs strongly. This is because if $X$ is such a graph and $G$ acts transitively on $X$ then $G$ shuffles $X$ (see [8, Theorem 3]). For example Theorem 1 and Theorem 2 remain valid for connected transitive graphs with infinitely many ends without assuming local finiteness (see [8, Theorem 4 and Theorem 6]). Corollary 1 also has an analogue: an infinite connected transitive graph of infinite valency has either 1 or at least $2^{\aleph_0}$ ends (see [8, Corollary 4 and Theorem 7]).

## 1.3 Ends of groups

Let $G$ be a finitely generated group. The number of ends of $G$ is defined as the number of ends of the Cayley graph of $G$ with respect to some finite generating set of $G$ (it does not depend on the choice of a generating set). Group theorists like to have an algebraic way of expressing the number of ends of a group. It was proved by Specker [47] that for a finitely generated infinite group $G$ with finitely many ends the number of ends equals $1 + \dim H^1(G, \mathbf{Z}_2 G)$ and $\dim H^1(G, \mathbf{Z}_2 G)$ is infinite if the group has infinitely many ends (for a proof see [9]).

The structure of groups with 2 or infinitely many ends is described in the following theorems.

**Theorem 3** *Let $G$ be a finitely generated group. Then the following are equivalent:*

**(i)** *G has precisely two ends;*

**(ii)** *G has an infinite cyclic subgroup of finite index;*

**(iii)** *G has a finite normal subgroup N such that G/N is either isomorphic to the infinite cyclic group or to the infinite dihedral group.*

Next we state an extension of Stallings' Ends Theorem. This extension can be proved by combining the results in the next section with the Bass–Serre theory of group actions on trees.

**Theorem 4** ([48]) *Suppose $G$ is a finitely generated group with infinitely many ends. Then $G$ can be written as a non-trivial free product with amalgamation $B *_C D$ where $C$ is finite, or $G$ can be written as a non-trivial HNN-extension $B *_C x$ where $C$ is finite.*

# 2 Structure trees

As mentioned above, it is the "treeness" that one tries to extract and use when studying graphs with infinitely many ends. The "structure tree" approach has its roots in the proof of Stallings' Ends Theorem [48] (see also [4]).

The plot is to find a family of subsets of $VX$ that is invariant under the action of $\mathrm{Aut}(X)$ and then represent them as the edge set of a tree, which $\mathrm{Aut}(X)$ acts on. First we discuss the properties of that family of sets, then how to construct the tree and the connections between the tree and the original graph. In the last part of this section we look at some simple examples of structure trees.

From now on suppose that $X$ is a connected graph.

## 2.1 Tree sets

We define $\mathcal{B}X$ as the Boolean ring of all subsets of $VX$ that have finite co-boundary. The elements of $\mathcal{B}X$ will be called *cuts*. For $n \in \mathbf{N}$ define $\mathcal{B}_nX$ as the subring of $\mathcal{B}X$ generated by those $C \subseteq VX$ with $|\delta C| \leq n$. If $C \subseteq VX$ then set $C^* = VX \setminus C$. A cut $C$ is said to be *tight* if both $C$ and $C^*$ are connected.

**Definition 2** *We say that $E \subseteq \mathcal{B}X$ is a* tree set *if*

**(i)** *for all $e, f \in E$ we have that one of*

$$e \cap f,\ e \cap f^*,\ e^* \cap f,\ e^* \cap f^*$$

*is empty, (i.e. $e \subseteq f, e \subseteq f^*, e^* \subseteq f$ or $e^* \subseteq f^*$);*

**(ii)** *for all $e, f \in E$ there are only finitely many sets $g \in E$ such that $e \subset g \subset f$;*

**(iii)** *neither $\emptyset$ nor $VX$ is in $E$.*

*If in addition the following holds then we say that $E$ is an* undirected *tree set*

**(iv)** *if $e \in E$ then $e^* \in E$.*

*We will only be interested in undirected tree sets that consist of tight cuts $C$ all of which with $|\delta C| \leq n$, for some natural number $n$. Let us call such a tree set* tight.

*Remark.* Of course the definition of tree sets need not be restricted to subsets of $\mathcal{B}X$, but those are the only cases that we are interested in. Note also that if $E$ is a tree set then we can always add to $E$ the complements of the sets in $E$ and the resulting set will also be a tree set. Thus it is not restrictive to consider only undirected tree sets.

*Examples.* Let $X$ be a locally finite graph. The set of all one-vertex subsets of $VX$ and their complements is an undirected tree set. Another obvious example of a tree set is the set of all subsets of $VX$ that have a co-boundary consisting of only one edge. In both these examples the tree sets are invariant under $\mathrm{Aut}(X)$.

In the next section it will be convenient to think of a tree set as a partially ordered set, the ordering being given by inclusion. The $^*$-operation is then an order reversing involution.

The difficult bit is to prove the existence of ''nice'' tree sets. The main result of Chapter II in [5] gives us almost all the tree sets that one could hope for and ties them nicely up with the action of the automorphisms group and the separating properties of the graph. The proof is long and technical, and will not be discussed here.

**Theorem 5** ([5, Theorem II.2.20]) *Let $X$ be a connected infinite graph and let $G \leq \mathrm{Aut}(X)$. There is a chain of $G$-invariant undirected tree sets $E_1 \subseteq E_2 \subseteq \ldots$ in $\mathcal{B}X$ such that all elements in $E_n$ are tight and $E_n$ generates $\mathcal{B}_nX$ for all $n$.*

Suppose that some two vertices (edges, ends) in $X$ have the property that if an element in $E_n$ contains one then it must contain both. Then clearly any element of the boolean ring $\mathcal{B}_nX$, which is generated by $E_n$, that contains one of the vertices must also contain the other. If for some two vertices (edges, ends) there is an element in $\mathcal{B}_nX$ that contains one and not the other, then there is an element in $E_n$ that contains only one of the two vertices (edges, ends).

In applications it is often enough to find an infinite tight cut $e$ with infinite complement such that $Ge \cup Ge^*$ is a tree set. Such a cut is called a D-*cut*. The existence of D-cuts follows from Theorem 1.1 in [11]. To end this section we state the following lemma which will come in very handy.

**Lemma 1** ([50, Proposition 4.1] and [5, Lemma II.2.5]) *Let $n$ be a natural number. For any given edge $e$ in $X$ there are only finitely many tight cuts $C$ with $|\delta C| = n$ such that $e \in \delta C$. Now let $E$ be a tight tree set, that is $E$ consists of tight cuts, all of which have co-boundary with $n$ or fewer edges. If $u, v \in VX$ then every descending chain in $\{e \in E \mid u \in e\}$ is finite and every chain in $\{e \in E \mid u \in e, v \notin e\}$ is finite.*

## 2.2 Trees, ends and automorphisms

In this section $E$ will always denote a tight undirected tree set. For some of the things discussed here these assumptions are unnecessarily restrictive, but they are necessary to get the properties that are useful in applications to graph theory. (Note that in [33, §3] one needs to add the assumption that the tree sets discussed are tight.)

Let $T$ be a tree with directed edges that come in pairs $e, e^*$ with opposite directions. Thinking about $T$ as a directed graph is purely a formal device that eases the presentation. Define a partial ordering on the edges of $T$ such that $e \geq f$ if and only if there is a directed edge path $e = e_0, e_1, \ldots, e_n = f$ in $T$ such that $e_i^* \neq e_{i+1}$ for all $i$. The $^*$-operation acts like an order reversing involution on $ET$. It is easy to check that $ET$ with this ordering and the $^*$-operation satisfies the conditions in the definition of a tree set if "$\subseteq$" is replaced with "$\leq$".

What is wanted now is a tree $T$ such that $ET$ and our tree set $E$, ordered by inclusion, can be identified and that this identification is an order isomorphism that commutes with the $^*$-operation.

Let $Y$ be a directed graph such that each component of $Y$ is isomorphic to the graph and such that there is an identification $EY \leftrightarrow E$ such that if $e \in E$ is identified with $(u, v)$ then $e^*$ is identified with $(v, u)$. From now on we will not distinguish between elements of $EY$ and $E$. We want to glue the components of $Y$ together so that the resulting graph is the tree we are after and so that the ordering by inclusion of $E$ is the same as the edge path ordering of the edge set of that tree. Let $e = (u, v)$ and $f = (x, y)$ be edges in $Y$ (having a dual existence as elements in $E$). Then we identify $v$ and $x$, and write $v \sim x$, if and only if $v = x$ or $f \subseteq e$ and there is no element $g \in E$ such that $f \subset g \subset e$. Let us write $f \ll e$ if $f \subset e$ and there is no $g \in E$ such that $f \subset g \subset e$. The relation $\sim$ is clearly reflexive and symmetric. We want it to be an equivalence relation so we have to prove that it is transitive. The proof is copied from [10, Theorem 2.1].

Let $e = (u, v), f = (x, y)$ and $g = (w, z)$. Suppose that $v \sim x$ and $x \sim z$. We want to show that $v \sim z$, that is $e^* \ll g$. If $e = g, e = f^*$ or $g = f^*$, then there is nothing to show, so let us assume that $f \ll e$ and $f \ll g$. We know that one of the following holds $e \subseteq g, e \subseteq g^*, e^* \subseteq g, e^* \subseteq g^*$. It is also known that $f \subset e$ and $f \subset g$, so $e \subseteq g^*$ is impossible. If $e \subseteq g$ then $f \subseteq e \subseteq g$ and because $f \ll g$ we must have either $f = e$ or $e = g$ and neither is allowed. Now suppose that $e^* \subseteq g^*$. Then $g \subseteq e$, and as in the case $e \subseteq g$ we get that $g = e$ or $g = f$, which is a contradiction. Finally we have to deal with the case $e^* \subseteq g$. We have to show that $e^* \ll g$, and then we have $v \sim z$. Suppose $e^* \subseteq h \subseteq g$. Again there are four cases. First if $f \subseteq h$ then $f \subseteq h \subseteq g$, so by assumption $f = h$ or $g = h$. If $f = h$ then we have $e^* \subseteq f \subseteq e$ which is clearly out of the question because $e \neq VX$. Then it is the possibility that $f \subseteq h^*$. Whence $f \subseteq h^* \subseteq e$ so $f = h^*$ or $e = h^*$. Now $f = h^*$ would imply that $f \subseteq g$ and $f^* \subseteq g$, which is impossible. So we must have the latter possibility, that is $e^* = h$. If $f^* \subseteq h$ then again both $f$ and $f^*$ are contained in $g$. Finally it is the possibility $f^* \subseteq h^*$. But then $h \subseteq f \subseteq e$ so $e^* \subseteq h \subseteq e$ which is impossible. Thus we must have $e^* \ll g$.

It is also clear that the ordering of $E$ by inclusion is the same as the edge path ordering of $E$ when $E$ is considered as the edge set of $T$.

This process leaves us with a graph which we call $T = T(E)$. Now forget all about the graph $Y$ and identify $E$ and $ET$ (from the construction it is obvious that $ET = EY$). Suppose that $u$ and $v$ are distinct vertices in $T$. Because one of $e \subseteq f, e \subseteq f^*, e^* \subseteq f$ or $e^* \subseteq f^*$ holds for all $e, f \in E$ we can find two edges $e, f$ with $f \subseteq e$ such that $v$ is an end vertex of $e$ and $u$ is an end vertex of $f$. By condition (ii) in the definition of a tree set there is a finite chain in $E$ such that $e = e_1 \gg e_2 \gg \ldots \gg e_n = f$. This chain defines a directed path in $T$ and both $u$ and $v$ lie on that path. So $T$ is connected.

A simple cycle in $T$ of length greater than 2 would give us a directed edge cycle $e_1, \ldots, e_n = e_1$. That would imply $e_1 \gg e_2 \gg \ldots \gg e_n = e_1$, which is impossible. So the undirected graph corresponding to $T$ has no cycles of length greater than 2. Thus it is reasonable to call $T$ a tree.

If $E$ is invariant under $G \leq \mathrm{Aut}(X)$ then $G$ has a natural action as a group of automorphisms of $T$.

**Definition 3** *If $E$ is a tight undirected $\mathrm{Aut}(X)$-invariant tree set then we call $T = T(E)$ a* structure tree *of $X$.*

To relate $X$ and $T$ more closely we have two maps $\phi : VX \to VT$ and $\Phi : \Omega X \to VT \cup \Omega T$. When $T$ is a structure tree, the action of $\mathrm{Aut}(X)$ commutes with both $\phi$ and $\Phi$.

The fundamental principle is that the edges in $T$ should point towards what they contain, that is if $v \in e$ then $e$ (as an edge in $T$) should point towards $\phi(v)$. So, if $v_0 \in VX$ and $e = (x, y)$ is an minimal element in $E$ subject to containing $v_0$ then set $\phi(v_0) = y$. Of course one has to show that the choice of the minimal element $e$ does not matter. Suppose that $f = (z, w)$ is another such element. We need to show that $e^* \ll f$, which implies $y = w$. Now recall the possibilities in condition (i) in the definition of a tree set. The condition that $v_0$ is in both $e$ and $f$ and that they are both minimal subject to this condition gives us immediately that $e^* \subseteq f$. Suppose $e^* \subset g \subset f$. By the minimality of $f$ one sees that $v_0 \notin g$. But $g^* \subset e$, so $v_0 \notin g^*$. Here is a contradiction and therefore $e^* \ll f$. We also want to be sure that if $f$ is some element in $E$ containing $v_0$ that then $f$ points towards $\phi(v_0) = y$. Looking over the possibilities in condition (i) in the definition of a tree set we see that either $e \subseteq f$ or $e^* \subseteq f$, and in both cases $f$ points towards $y$.

It is also clear from this definition that if $u$ and $v$ are vertices in $X$ and there is an element $e \in E$ that contains one and not the other then $\phi(u) \neq \phi(v)$. It should also be noted that if $u$ and $v$ are adjacent then $\{u, v\} \in \delta e$.

Now we define $\Phi$ in a similar manner. The situation is more involved here: it might happen that for some end $\omega$ of $X$ that there is an infinite descending chain of elements in $E$ all of which contain $\omega$. But this chain will then define an end $\epsilon$ in $T$ and we set $\Phi(\omega) = \epsilon$. If there is no such chain then we define $\Phi(\omega)$ in the same way as $\phi$ was defined. By the same argument as above we get that every edge in $E$ that contains $\omega$ points towards $\Phi(\omega)$. (We say that an edge $(u, v)$ in $T$ points towards the end $\omega$ if $v$ lies on the ray in $\omega$ that starts at $u$.) Two ends $\omega$ and $\omega'$ have different images under $\Phi$ if and only if there is an element in $E$ that contains one but not the other.

It is often useful when studying $\Phi$ (and can be used to define $\Phi$) to take some ray $R$ in an end $\omega$ and consider $\phi(R)$. In general $\phi(R)$ need not be a ray but by adding in the unique simple paths in $T$ between the images of successive vertices in $R$ one gets a path $P$. Note that $P$ need not be simple. It can be shown that if $\Phi(\omega)$ is an end then all the rays in the subgraph spanned by $P$ are in that end. It can also be shown that if $\Phi(\omega)$ is a vertex then that vertex is the only vertex that one will go through infinitely often as one goes along the path $P$ (see [50, §6]).

The following lemma gives further information about $\Phi$.

**Lemma 2** *The map $\Phi : \Omega X \to VT \cup \Omega T$ has the following properties.*

**(i)** ([33, Lemma 2]) *The restriction of $\Phi$ to $\Phi^{-1}(\Omega T)$ is bijective.*

**(ii)** ([33, Lemma 4]) *A vertex $v$ in $T$ is in the image of $\Phi$ if and only if $\Phi^{-1}(v)$ is infinite or $v$ has infinite valency.*

**(iii)** ([50, Lemma 8.2]) *Let $E_1, E_2, \ldots$ be as in Theorem 5. If $\omega$ is a thin end of $X$ then there is $n$ such that if $T = T(E_n)$ then $\omega$ is mapped by $\Phi$ to an end of $T$.*

Let us just look briefly at why (i) is true. For full proofs see the references.

First injectivity. Let $\omega$ be an end of $X$ and suppose that $\Phi(\omega)$ is an end. Take an infinite descending sequence $e_1 \gg e_2 \gg \ldots$ in $E$ such that all the elements in the sequence contain $\omega$. Then Lemma 1 implies that $\bigcap e_i = \emptyset$. Suppose $\omega'$ is some other end of $X$ and $F$ is some finite set of vertices separating $\omega$ and $\omega'$. Then we find $i$ such that $F \cap e_i = \emptyset$. Because $e_i$ is connected we have that $e_i$ is contained in some component of $X \setminus F$ and thus $e_i$ cannot contain both $\omega$ and $\omega'$. So $\Phi(\omega) \neq \Phi(\omega')$. To prove that $\Phi$ is surjective we take an end $\epsilon$ in $T$ and some ray $\{v_i\}_{i \in \mathbf{N}}$ in that end. Set $e_i = (v_i, v_{i+1})$. Then the chain $e_1 \gg e_2 \gg \ldots$ is a directed edge path in $T$. Now we find a ray in $X$ that includes at least one vertex from $\partial e_i$ for all $i$. The end $\omega$ of $X$ that contains this ray will clearly belong to all of the sets $e_i$ and thus $\Phi(\omega) = \epsilon$.

The structure tree approach really comes into its own when one wants to study the action of the automorphism group on $X$. The following lemma relates the action of $\mathrm{Aut}(X)$ on $X$ and $\Omega X$ to the action of $\mathrm{Aut}(X)$ on $T$.

**Lemma 3** ([33, Corollary 1]) *Let $X$ be a connected locally finite graph and $T = T(E)$ be some structure tree of $X$, where $E$ is a tight undirected tree set.*

**(i)** *If $g \in \mathrm{Aut}(X)$ acts like a translation on $T$ then $g$ acts like a translation on $X$ and $g$ is hyperbolic.*

**(ii)** *If $g \in \mathrm{Aut}(X)$ is a translation then either $g$ acts as a translation on $T$ or there is an unique vertex of $T$ fixed by $g$ and that vertex has infinite valency.*

**(iii)** *If $g \in \mathrm{Aut}(X)$ is hyperbolic then there is a tight undirected tree set $E_g$ such that $g$ acts as a translation on $T(E_g)$.*

Let us look at the proof of (i), the others are similar. Let $L$ be the line in $T$ that is invariant under $g$. Suppose that $e$ is an edge in $L$. Then $\delta e$ has an infinite orbit under $g$, and therefore $g$ cannot fix any nonempty finite subgraph. There are two ends of $T$ that are fixed by $g$, so there are also two ends of $X$ fixed by $g$, which means that $g$ is hyperbolic.

## 2.3 Examples

**1.** First let us consider a partial portrait of some structure tree and consider what can be read directly from it.

**Figure 3.**

Let us now look at few examples of what we can read of from Figure 3:

**(i)** $e_1 \gg e_3 \gg e_5$;

**(ii)** $e_3 = e_4^*$;

**(iii)** $e_3 \not\subset e_6$;

**(iv)** if $v \in VX$ and $v_3 = \phi(v)$ then $v$ is contained in $e_1, e_2, e_4$ and $e_6$ but not contained in $e_3$ and $e_5$;

**2.** Let $X$ be a connected regular locally finite graph. The simplest example of a tree set is the set $E$ of all one element subsets of $VX$ and their complements. Then $T = T(E)$ is a structure tree. In this case the structure tree looks like a star; the edges corresponding to the one element subsets pointing away from the center. It is obvious that $\phi$ is injective and the image of $\phi$ is the set of leaves of $T$. All the ends of $X$ are mapped by $\Phi$ to the central vertex of $T$. This example shows that a structure tree of a locally finite graph need not be locally finite.

**3.** Now let $X$ be a tree and let $E$ be the set of all subsets of $VX$ that have exactly one edge in their co-boundary. Apart from the fact that formally we consider $T(E)$ to be a directed graph the trees $X$ and $T(E)$ are identical.

**4.** Let $X$ be a graph as depicted on Figure 4; that is, $X$ is a kind of an infinite comb with infinite teeth. Let $e_1$ and $e_2$ be cuts as shown on Figure 4.

**Figure 4.**

Set $G = \mathrm{Aut}(X)$. Define

$$E_1 = Ge_1 \cup Ge_1^*,$$
$$E_2 = Ge_2 \cup Ge_2^*,$$

and

$$E_3 = E_1 \cup E_2.$$

These are all tight $G$-invariant tree sets. The structure tree $T(E_1)$ is a star. The edges in $Ge_1$ point away from the center and the ends of $X$ that correspond to the teeths of the comb are mapped to the leaves of $T(E_1)$. The two ends corresponding to the baseline are mapped by $\Phi$ to the central vertex. The tree $T(E_2)$ is a line. The maps $\phi$ and $\Phi$ have the same effect as contracting the teeth down to the baseline.

**Figure 5.**

On Figure 5 we see a part of $T(E_3)$, which already looks more like $X$ than the other two. As we take bigger tree sets we get a better description of our graph.

**5.** Let $X$ be the natural Cayley-graph of $\mathbf{Z}_3 * \mathbf{Z}_3$ as shown on Figure 6. Take a vertex $v$ in $X$. Denote by $C_v$ and $C_v'$ the two components of $X \setminus \{v\}$. Set $\mathcal{C} = \{C_v, C_v' \mid v \in VX\}$ and let $\mathcal{C}^*$ denote the set of the complements of the sets in $\mathcal{C}$. Then set $E = \mathcal{C} \cup \mathcal{C}^*$, and it is easy to see that $E$ is a tight undirected tree set. The vertices in $T(E)$ all have valencies either 2 or 3. The ones with valency 2 are in the image of $\phi$ and one can think of the vertices of valency 3 as corresponding to the triangles in $X$. The map $\Phi$ gives a bijection between $\Omega X$ and $\Omega T$. In Section 5.1 we see that this is a part of a more general phenomenon.

**Figure 6.**

# 3 Fixed point properties

Since the advent of the Bass–Serre theory of groups acting on trees and the paper [51] by Tits several papers have appeared discussing generalized "fixed point properties" of group actions on trees. Recently, around the same time and independently, Nebbia [38], Pays and Valette [42], and Woess [54] arrived at what appears to be the final and fundamental truth in this matter. Of these the results of Woess are the most general because he not only solves the problem for trees but also for locally finite graphs in general. But, they are also the most difficult to prove; Woess' proofs are a real combinatorial *tour de force*. In this section we give short proofs of Woess' theorems, using the theory of structure trees.

First we do some preliminary work relating fixed point properties of group actions on trees to fixed point properties of general graphs. Then we discuss briefly the concept of amenability, which turns out to be the key to our fixed point properties. Finally we prove Woess' theorems.

## 3.1 Serre's property (FA)

Let $X$ be a locally finite graph, and let $H \leq \mathrm{Aut}(X)$. The fixed point properties mentioned above are of the following types:

**(a)** there is a non-empty finite subgraph of $X$ invariant under $H$;

**(b)** there is an end of $X$ fixed by $H$;

**(c)** there is a pair of ends of $X$ invariant under $H$.

First we prove a generalization of a result of Tits [51, Proposition 3.4] (see also [42, Proposition 1]). Tits proved the special case where $X$ is a tree. We then get a sharper result: either $H$ fixes some vertex or leaves some edge invariant or $H$ fixes some end of $T$.

**Theorem 6** ([54, Proposition 1]) *Let $X$ be a connected locally finite graph, and let $H \leq \mathrm{Aut}(X)$, and suppose $H$ contains no hyperbolic elements. Then* (a) *or* (b) *holds and, furthermore, if* (b) *holds then $H$ fixes exactly one end.*

*Proof.* Let $T$ be some undirected structure tree of $X$. We know that no element of $H$ acts like a translation on $T$ and we can thus apply Tits' result to $H$ acting on $T$. If $H$ fixes an end of $T$ or has a finite orbit on $ET$ then we are clearly finished. This is because, if $e$ is an edge in $T$ and $He$ is finite then $H(\delta e)$ spans a finite subgraph of $X$, which is invariant under $H$. We can thus assume, without loss of generality, that for all structure trees $T$ of $X$ that $H$ fixes precisely one vertex of $T$ and that this vertex has infinite valency in $T$. (If $H$ would fix two vertices in $T$ then the edges in the path between those two vertices would also be fixed and we would not have to do more. And, if $H$ would fix a vertex of finite valency then H would have a finite orbit on $ET$.) Let $E_1 \subseteq E_2 \subseteq \ldots$ be a sequence of undirected tree sets as in Theorem 5. In each of the trees $T(E_j)$ there is a unique vertex $v_j$ fixed by $H$. Since this vertex has infinite valency then, as noted before, there is a non-empty set $I_j$ of ends of $X$ corresponding to $v_j$. Clearly the sets $I_j$ form a decreasing chain. Put $I = \bigcap I_j$. Because $\bigcup E_j$ generates the whole of $\mathcal{B}X$, we know that $I$ is either empty or contains precisely one element. (For any two ends $\omega$ and $\omega'$ in $X$ it is always possible to find $n$ such that there is an element $C \in \mathcal{B}_n X$ with $\omega \in \Omega C$ and $\omega' \notin \Omega C$. Hence there is an element $e \in E$ with $\omega \in \Omega e$ and $\omega' \notin \Omega e$. Then we cannot have both $\omega$ and $\omega'$ in $I_n$.)

Now we use the topology on $\Omega X$. The elements of $\mathcal{B}X$ define closed sets in the end space of $X$. The ends in $I_j$ are precisely those ends that are contained in every element of $E_j$ that points towards $v_j$ in $T(E_j)$. Hence the set $I_j$ is closed, since it is equal to the intersection of a family of closed sets. By assumption all the sets $I_j$ are non-empty and $\Omega X$ is compact, so, by a standard fact about compact sets, the set $I$ is non-empty. □

A group $H$ acts on a tree $T$ *without inversion* if no element of $H$ transposes some pair of adjacent vertices. If there exists $g \in H$ and a pair of adjacent vertices $u, v \in VX$ such that $gu = v$ and $gv = u$, then we say that $H$ acts *with inversion*.

Following Serre [45] we say that a group $H$ has property (FA) if whenever $H$ acts on a tree without inversion then there is some vertex of the tree fixed by $H$. If $H$ acts on a tree $T$ with inversion then we can take the barycentric subdivision $T'$ of $T$ and $H$ will act on $T'$ without inversion. Allowing actions with inversion then property (FA) is equivalent to there either being a vertex fixed by $H$ or there being an edge invariant under $H$. Bass [3] studied an analogous property (FA′). If we allow actions with inversion then property (FA′) is equivalent to that every element of $H$ either fixes a vertex or leaves some edge invariant. We can now deduce the following from the proof of Theorem 6.

**Corollary 2** **(i)** *If a group $H$ with property* (FA) *acts on a connected locally finite graph $X$ then either* (a) *or* (b) *holds and, if* (b) *holds then $H$ fixes exactly one end, which is thick.*

**(ii)** *If a group $H$ with property* (FA′) *acts on a connected locally finite graph then either* (a) *or* (b) *holds.*

There are various results on which groups do have these properties: for example, countable groups having Kazhdan's property (T) have property (FA) (see [2] and [53]) and pro-finite groups have property (FA′) (see [3]).

## 3.2 Amenability

Now we turn to the concept of amenability. For further information concerning amenability the reader is referred to the book by Wagon [52].

**Definition 4** *A group $G$ acts* amenably *on a set $Y$ if there is a non-negative function $\mu$ defined on the power set of $Y$ such that*

**(i)** $\mu(Y) = 1$;

**(ii)** *$\mu$ is finitely additive;*

**(iii)** *$\mu$ is $G$-invariant.*

*We say that $\mu$ is a $G$-*invariant measure *on $Y$. If the left regular action of $G$ on itself is amenable then we say that $G$ is* amenable.

An important variation of this definition is when $Y$ is a topological space and instead of requiring the measure $\mu$ to be defined on all subsets we only ask for it to be defined on the Borel subsets of $Y$. Then we say that the action is *topologically amenable*, or just amenable if there is no danger of confusion. A topological group $G$ is *topologically amenable* (or just, amenable) if we can find a measure $\mu$ defined on the Borel subsets of $G$ that satisfies (i)-(iii) with respect to the left regular action of $G$ on itself. If the topology on $Y$ is the discrete one then topological amenability is the same as amenability. For completeness we list here the facts about amenability that we will be using.

**(i)** A compact topological group is amenable.

**(ii)** Let $G$ be a topological group and $H$ a closed normal subgroup of $G$. If both $H$ and $G/H$ are amenable then $G$ is amenable.

**(iii)** The direct union of a directed system of amenable groups is amenable.

**(iv)** Abelian and soluble groups are amenable.

**(v)** A topological group containing a discrete non-abelian free group is not amenable. A non-abelian free group cannot act freely and amenably on a set.

To prove the first item it suffices to take $\mu$ as the Haar-measure. The second, third and fourth items are similar to [52, Theorem 10.4], and the fifth one follows also because closed subgroups of amenable groups are amenable and a non-abelian free group is not amenable.

## 3.3 The theorems of Woess

First we have to define a topology on $\mathrm{Aut}(X)$. We take as a basis of neighbourhoods around the identity the pointwise stabilizers of finite subsets of $VX$. This is indeed just the topology of pointwise convergence on $\mathrm{Aut}(X)$. Because we assume that $X$ is locally finite we can easily show that the setwise stabilizers in $\mathrm{Aut}(X)$ of non-empty finite subsets of $VX$ are compact. Whence $\mathrm{Aut}(X)$, endowed with this topology, is a locally compact group. For more information on this topology consult Woess' survey paper [56].

**Theorem 7** ([54, Theorem 1]) *Let $X$ be a locally finite connected graph and $H$ a subgroup of* Aut$(X)$*. If $H$ acts amenably on $X$ then one of* (a), (b), (c) *holds.*

*Proof.* Let $\mu$ be an $H$-invariant measure on $VX$. Let $T = T(E)$ be some structure tree of $X$ and let $\phi$ be the map described in Section 2.2. Define a function $\nu$ on the power set of $VT$ by

$$\nu(\Delta) = \mu(\phi^{-1}(\Delta)),$$

for $\Delta \subseteq VT$. Clearly $\nu$ is a $H$-invariant measure on $VT$, so $H$ acts amenably on the tree $T$. By Theorem 6 we can assume that $H$ contains a hyperbolic element $h$ and, by Lemma 3, we may also assume that $h$ acts like a translation on $T$.

Now we see that if the action of $H$ on $T$ satisfies none of (a), (b), (c) then the action on $X$ will also have none of these properties. Under these conditions one can use the same method as used by Nebbia in [38], and Pays and Valette in [42] to prove that if $H$ satisfies none of (a), (b), (c) then $H$ would contain a free group $F$ on two generators acting freely on $T$. The first step is to show that if none of (a), (b) and (c) is satisfied the we can find two elements $g, h \in H$ that act like translations on $T$ such that the line invariant under $g$ does not intersect the line that is invariant under $h$. Then one applies the "ping-pong lemma" [42, Lemma 7] to show that $\langle g, h\rangle$ is a free group freely generated by $g$ and $h$ and it acts freely on $VT$. Hence $H$ could not act amenably, contradicting our assumption. □

From this we can deduce the following corollary which is stated in [33, Theorem 8], but had been proved earlier by H.A. Jung.

**Corollary 3** *Let $X$ be a connected locally finite graph and let $H \leq$* Aut$(X)$*. If $H$ satisfies none of* (a)*,* (b) *or* (c) *then $H$ contains a free group on two generators.*

*Remarks.* 1. For another recent proof of Nebbia's theorem and Theorem 7 see [1, Theorem 4.1 and Theorem 4.2]. That proof uses analytic properties related to amenability.

2. Theorem 6 and Corollary 3 are also valid without the assumption of local finiteness (see [27, Corollary 1.3 and Theorem 1.4]).

3. Woess [57] has proved results similar to Theorem 6 and Corollary 3 for compactifications of metric spaces that satisfy certain natural conditions concerning compatibility with the action of the isometry group. The results for ends of locally finite graphs are a special case of his results.

The converse to Theorem 7 is not true, the reason being the existence of finitely generated non-amenable groups having a Cayley-graph with only one end, see Example 2, part D in [54]. But, it is very close to being true because if we replace condition (b) with the stronger condition

(b′) there is a thin end of $X$ fixed by $H$,

then we get the following theorem. Note that in the special case of trees (b′) is the same as (b).

**Theorem 8** ([54, Theorem 2]) *Let $X$ be a connected locally finite connected graph and $H$ a subgroup of* $\mathrm{Aut}(X)$*. If one of* (a), (b′), (c) *holds then $H$ acts amenably on $X$.*

*Proof.* First we notice that we may assume that $H$ is a closed subgroup of $\mathrm{Aut}(X)$. Because, if the closure of $H$ acts amenably then surely $H$ will act amenably and also because if $H$ satisfies one of (a), (b′), (c) then the closure of $H$ will also satisfy one of them. From now on we will assume that $H$ is closed.

We then only need to show that $H$ is (topologically) amenable because every continuous action by $H$ on a locally compact space (in our case $VX$ with the discrete topology) is then necessarily amenable (see [43, pp. 362-363]). It is indeed quite easy to find an $H$-invariant measure on $VX$. From the definition of the topology on $H$ it is clear that $\{h \in H \mid hv_0 \in A\}$ is open in $H$. Let $\mu$ be a measure on $H$ as described in Section 3.2, and let $v_0$ be a vertex in $X$. For $A \subseteq VX$ we set $\nu(A) = \mu(\{h \in H \mid hv_0 \in A\})$. It is easily checked that $\nu$ is an $H$-invariant measure on $VX$.

The first case arises when $H$ satisfies (a). But then $H$ is compact and thus amenable.

If $H$ satisfies (b′) then, by Lemma 2, we can assume that the end $\omega$ fixed by $H$ appears as an end in some structure tree $T$ of $X$, that is $\Phi(\omega)$ is an end of $T$. Then $H$ fixes the end $\Phi(\omega)$. Let $\{v_i\}_{i\in\mathbf{N}}$ be a ray in $\Phi(\omega)$ and set $e_i = (v_i, v_{i+1})$. An element of $H$ that fixes some $e_i$ will also fix $e_j$ for all $j \geq i$. Then $H_{\{e_1\}} \subseteq H_{\{e_2\}} \subseteq \dots$ and all these groups are amenable because they are compact. Denote the direct union of the chain $H_{\{e_1\}} \subseteq H_{\{e_2\}} \subseteq \dots$ by $H_0$. Now $H_0$ is amenable because $H_0$ is the direct union of amenable groups. For every $h \in H$ there is an integer $\alpha(h)$ such that for all $i$ large enough we have that $hv_i = v_{i+\alpha(h)}$. The map $\alpha : H \to \mathbf{Z}$ is a homomorphism with kernel $H_0$. So, if $H$ does not contain any hyperbolic elements then clearly $H$ is equal to $H_0$, otherwise $H_0$ is a closed normal subgroup of $H$ and $H/H_0$ is infinite cyclic. Indeed, if $h \in H$ is a translation and $h$ is chosen such that $\min_{v\in VT} d(v, hv)$ is as small as possible then $H$ is a semi-direct product of $H_0$ and $\langle h\rangle$. In both cases we have that $H$ is amenable.

If $H$ satisfies (c) then find a structure tree $T$ such that both ends $\omega$ and $\omega'$ in the pair of ends that $H$ leaves invariant appear as ends in $T$. Let $L$ be the line in $T$ with ends $\omega$ and $\omega'$. Then $H_{(L)}$ is a closed normal subgroup of $H$ and it is easy to see that $H/H_{(L)}$ is either trivial, cyclic with two elements, infinite cyclic or infinite dihedral (because $H/H_{(L)}$ acts faithfully on the line $L$ and must therefore be a subgroup of $\mathrm{Aut}(L)$, and $\mathrm{Aut}(L)$ is the infinite dihedral group). In all cases it follows that $H$ is amenable, because $H_{(L)}$ is compact and therefore amenable, and also $H/H_{(L)}$ is amenable. □

# 4 Accessibility

One of the main motivations behind studying infinite graphs with more than one end is the connection with group theory that is epitomized by Stallings' Ends Theorem.

In particular the so called Accessibility Conjecture has graph theoretic relevance. A counterexample to the conjecture has been found by Dunwoody [13], but the graph theoretic interpretion of accessibility is still of much interest.

**Definition 5** *Let $X$ be a locally finite graph. If there is a natural number $k$ such that any two ends in $X$ can be separated by a set containing $k$ or fewer vertices, then we say that $X$ is* accessible.

In their paper [50] Thomassen and Woess show that accessible graphs are "tree-like". This is indicated by the next result stated here, but more precise descriptions of what "tree-like" means are given in [7, §4] and [55].

**Theorem 9** ([50, Theorem 7.3]) *Let $X$ be a connected locally finite transitive graph. Then $X$ is accessible if and only if there is a natural number $n$ such that $\mathcal{B}_n X = \mathcal{B}X$.*

*Suppose $X$ is accessible and $n$ is a number such that $\mathcal{B}_n X = \mathcal{B}X$. Let $E_n$ be a tight undirected tree set generating $\mathcal{B}_n X$ and let $T = T(E)$. Then the map $\Phi$ described in Section 2.2 is injective and a thin end of $X$ is mapped to an end of $T$.*

So for accessible graphs it is possible to find a structure tree that represents the whole end structure of $X$. From this certain properties of the ends are obvious.

**Corollary 4** ([50, Corollary 7.4 and Corollary 7.5]) *A connected locally finite transitive accessible graph has only countably many thick ends and there is a finite upper bound on the sizes of thin ends.*

For inaccessible graphs all this breaks down.

**Theorem 10** ([50, Theorem 5.4 and Theorem 8.4]) *Let $X$ be a connected locally finite transitive inaccessible graph.*

**(i)** *There is a thick end $\omega$ in $X$ such that for every natural number $n$ there is an end $\omega'$ that cannot be separated from $\omega$ by fewer than $n$ vertices.*

**(ii)** *For every natural number $n$ there is a thin end in $X$ that has size bigger than $n$.*

We can now get a new characterization of inaccessible graphs.

**Theorem 11** [37] *Let $X$ be a connected locally finite transitive graph. Then $X$ is inaccessibel if and only if $X$ has uncountably many thick ends.*

A finitely generated group is said to be accessible if its Cayley-graph, with respect to some finite generating set, is accessible. In group theory accessibility is really about decomposing groups as free products with amalgamation or as HNN-extensions, and is best understood within the framework of the Bass–Serre theory of groups acting on trees.

**Definition 6** *Let $G$ be a finitely generated group. If there exists a tree $T$ such that $G$ acts on $T$ with finitely many orbits and for each $e \in ET$ the group $G_e$ is finite and for each $v \in VT$ the group $G_v$ has at most one end, then $G$ is said to be* accessible.

**Theorem 12** ([12]) *A finitely presented group is accessible.*

It is possible to prove accessiblity under weaker, but more technical, assumptions (see [12] and [17]).

It was generally believed for twenty years that every finitely generated group was accessible. But in 1991 Dunwoody [13] found an example of a finitely generated inaccessible group which has uncountably many thick ends. In [14] Dunwoody has also given an example of a transitive 4-valent inaccessible graph such that the automorphism group acts transitively on the edges.

Several of the results in the next section can be interpreted as saying that graphs satisfying certain extra conditions are accessible. It would be interesting to have more results along those lines. Dunwoody has in [14] extended the notion of structure trees to gain a suitable tool to study inaccessible graphs. For inaccessible graphs ordinary structure trees can only reflect a part of the end structure, but Dunwoody constructs what he calls *protrees*, which can be thought of as limits of structure trees. Thus he gets hold of the whole end structure, but protrees are not graphs and are more difficult to handle.

# 5 Extra conditions on the graph or the group

From Dunwoody's examples of inaccessible graphs it is evident that transitive graphs with infinitely many ends can be very badly behaved, but, as the results in this section show, it needs only relatively innocent looking extra conditions to bring them into line.

Before getting on with the job at hand we must introduce another new concept. Let $G$ be a group acting on a set $Y$. For $x \in Y$ let $G_x$ denote the stabilizer of $x$ in $G$; that is, the subgroup of $G$ consisting of all elements in $G$ that fix $x$. The orbits of $G_x$ on $Y$ are called *suborbits*. If $G$ acts transitively on $Y$ then there is a 1-1 correspondence between the $G_x$-orbits and the $G$-orbits on $Y \times Y$: the suborbit $G_x y$ corresponding to the orbit $G(x, y)$. The orbits on $Y \times Y$ are called *orbitals*. When studying suborbits and orbitals it is often convenient to consider directed graphs of the form $(Y, G(x, y))$, which are called *orbital graphs*. A related idea can be used to construct new graphs on the basis of old graphs and also to clarify the structure of the graphs that we are studying. A graph $(Y, G\{x_1, y_1\} \cup \ldots \cup G\{x_n, y_n\})$ is called a *poly-orbital graph*. If $X$ is a graph then we say that a poly orbital graph $X'$ is *based* on $X$ if $X' = (VX, \mathrm{Aut}(X)\{x_1, y_1\} \cup \ldots \cup \mathrm{Aut}(X)\{x_n, y_n\})$. It follows from [34, Proposition 1] that a graph $X$ and a connected poly-orbital graph based on it have identical end structure (the end spaces are homeomorphic). The reader that wants to know more about orbital graphs and their uses in permutation group theory is refered to [39].

## 5.1 Graphs with connectivity 1

The connectivity of a graph is the least number of vertices one has to remove in order for the rest of the graph to become disconnected. That a graph $X$ has connectivity 1 means that $X$ is connected but it is possible to find a vertex $v$ in $X$ such that $X \setminus \{v\}$ is not connected. Such a vertex is called a *cut-vertex*. A *block* in $X$ is a maximal connected subgraph that has connectivity higher than 1. In most books on graph theory (e.g. [21, Chapter 3]) it is explained how a graph $X$ with connectivity 1 is built up out of its blocks and how this is described by the block-cut-vertex tree of $X$. The vertex set of the block-cut-vertex tree is the union of the set of cut-vertices in $X$ and the set of blocks of $X$. A block $B$ is adjacent in the block-cut-vertex tree to a cut-vertex $v$ if and only if $v$ is in $B$. (Note that in [28] and [30] the blocks are called "lobes".) If a graph with connectivity 1 is transitive then every vertex is a cut-vertex.

Let $L$ denote the set of blocks in $X$ and let $\{L_i \mid i \in I\}$ be the partition of $L$ into isomorphism types. For a block $\Lambda$ in $L_i$ denote by $\{L_i^{(j)} \mid j \in J_i\}$ the partition of the vertex set of $\Lambda$ into orbits of $\mathrm{Aut}(\Lambda)$. We can of course choose this labeling of the orbits so that it is the same for all blocks in $L_i$. Then define $m(v, L_i^{(j)})$ as the cardinality of the set of orbits of type $L_i^{(j)}$ that contain $v$.

**Theorem 13** ([28, Theorem 3.2]) *A graph $X$ with connectivity 1 is transitive if and only if for all $L_i^{(j)}$ the cardinal number $m(v, L_i^{(j)})$ is the same for all $v \in VX$.*

Structure trees can be used in a very straightforward manner to describe locally finite transitive graphs $X$ with connectivity 1. Let $\mathcal{C}_v$ denote the set of all components of $X \setminus \{v\}$, where $v$ is a vertex in $X$, and let $\mathcal{C}_v^*$ denote the set of the complements of the sets in $\mathcal{C}_v$. Let

$$E = (\bigcup_{v \in VX} \mathcal{C}_v) \cup (\bigcup_{v \in VX} \mathcal{C}_v^*).$$

Then $E$ is a tight tree set (that $E$ is tight is ensured by the transitivity of $X$) and $E$ is invariant under $\mathrm{Aut}(X)$. It is easy to convince oneself that the structure tree $T(E)$ is isomorphic to the block cut-vertex tree.

## 5.2 Primitivity

The group $G$ is said to act *primitively* on a set $Y$ if $G$ acts transitively and the only $G$-invariant equivalence relations on $Y$ are the trivial one, where all classes have size one, and the equivalence relation which has only one class, $Y$. The following is well known.

**Proposition 1** *Let $G$ be a group acting transitively on a set $Y$. Then the following are equivalent:*

**(i)** *$G$ acts primitively on $Y$;*

**(ii)** *for all $x \in Y$ the stabilizer $G_x$ is a maximal proper subgroup of $G$;*

**(iii)** *for any pair* $x, y \in Y$ *the graph* $(Y, G\{x, y\})$ *is connected.*

If $X$ is a graph then we say that $X$ is *primitive* if $G = \mathrm{Aut}(X)$ acts primitively on $VX$. Condition (iii) above is very useful: it allows us to choose a different edge set without loosing connectivity. Thus we are able to bring to the surface properties that might have been hiding behind over-abundance of edges.

First we should mention a special case of Theorem 13 which gives us a complete characterisation of primitive graphs with connectivity 1.

**Theorem 14** ([28, Theorem 4.2]) *For a graph* $X$ *with connectivity 1 the following are equivalent:*

**(i)** $X$ *is primitive;*

**(ii)** *the blocks of* $X$ *are primitive, pairwise isomorphic and each one has at least three vertices.*

It turns out that behind every connected locally finite primitive graph there is one with connectivity 1. The following theorem is proved with the aid of structure trees.

**Theorem 15** ([36, Theorem 2 and Theorem 3]) *Suppose* $X$ *is a connected locally finite primitive graph. Then there are vertices* $u, v$ *in* $X$ *such that the graph* $X' = (VX, G\{u, v\})$ *has connectivity 1 and each block of* $X'$ *has at most one end. In particular, every locally finite primitive graph is accessible.*

Thus the end structure of primitive graphs is very simple. The following result underlines further how special primitive graphs are.

**Theorem 16** ([30, Proposition 1.2 and Theorem 1.3]) *For every* $n = 0, 1, 3, 4, 5, \ldots$ *there exists an infinite locally finite primitive graph with connectivity* $n$*, but there are no infinite locally finite primitive graphs with connectivity* 2.

Infinite graphs with no edges at all provide examples of locally finite primitive graphs with connectivity 0 and Theorem 14 tells us how to construct infinite locally finite primitive graphs with connectivity 1. Let now $n \geq 3$. Let $X$ be the graph with connectivity 1 such that each block of $X$ is a complete graph with $n$ vertices and each vertex belongs to precisely 2 blocks (the graph in the fifth example in Section 2.3 is the graph you get with $n = 3$). From Theorem 14 it follows that $X$ is primitive. Set $F = \{\{u, v\} \mid u, v \in VX \text{ and } d(u, v) = 2\}$. It is now left to the reader to show that the graph $X' = (VX, EX \cup F)$ has connectivity $n$. It should also be pointed out that $X'$ is a poly-orbital graph.

The difficult part of the proof of the above theorem is to prove that no infinite locally finite primitive graph has connectivity 2. This is proved by elementary arguments but the proof is long and involved. Given a graph with connectivity 2 it would be nice to be able to find some kind of a description of an $\mathrm{Aut}(X)$-invariant equivalence relation on $VX$. It would also be nice to have some understanding of why inaccessible graphs cannot be primitive.

## 5.3 Transitivity conditions

First let us consider the concept of distance-transitivity.

**Definition 7** *A graph $X$ is* $k$-distance-transitive *if for any four vertices $v, v', u, u'$ in $X$ with $d(v, v') = d(u, u') \leq k$ there is an automorphism $g \in \mathrm{Aut}(X)$ such that $gv = u$ and $gv' = u'$. If $X$ is $k$-distance-transitive for all $k$ then we say that $X$ is* distance-transitive*.*

Infinite locally finite connected distance-transitive graphs were classified by Macpherson [31] and, independently, by Ivanov [24]. Macpherson's proof uses D-cuts (a streamlined proof can be found in [5, Chapter II Section 3]). But Ivanov's proof uses techniques from the theory of finite distance-transitive graphs. It is difficult to see further applications of his techniques to infinite graphs.

The obvious examples of distance-transitive graphs are regular trees, but there is also a bigger closely related class of such graphs. Let $k, l$ be natural numbers with $k \geq 1$ and $l \geq 2$. We let $X_{k,l}$ denote the infinite transitive graph with connectivity 1 where each block is a complete graph with $k+1$ vertices and each vertex belongs to $l$ blocks. If $k = 1$ then $X_{k,l}$ is just the $l$-regular tree. In the fifth example in Section 2.3 we came across $X_{2,2}$ and $X_{2,3}$ is shown on Figure 7. The graphs $X_{k,l}$ with $k \geq 1$ and $l \geq 2$ are all clearly distance-transitive.

**Theorem 17** ([31, Theorem 1.2] and [24, Theorem 4]) *If $X$ is an infinite connected locally finite distance-transitive graph then $X$ is isomorphic to $X_{k,l}$ for some $k \geq 1$ and $l \geq 2$.*

Note that in particular $X$ must have more than one end. Graphs with more than one end are very sensitive to transitivity conditions of this type.

**Theorem 18** ([35]) *Let $X$ be a locally finite connected graph with more than one end. If $X$ is 2-distance-transitive then $X$ is distance-transitive. In particular $X$ must be isomorphic to $X_{k,l}$ for some $k \geq 1$ and $l \geq 2$.*

This result was conjectured by Thomassen and Woess on the basis of the following two results. The first one is an easy corollary to the above theorem. The proof given here of the latter theorem employs many of the ideas that go into the proof of Theorem 18. We say that a graph $X$ is *$k$-arc-transitive* if $\mathrm{Aut}(X)$ acts transitively on the set of simple paths of length $k$.

**Theorem 19** ([50, Theorem 3.2]) *If $X$ is a connected locally finite 2-arc-transitive graph with more than one end, then $X$ is a regular tree.*

**Theorem 20** ([50, Theorem 3.3]) *Let $X$ be a connected locally finite 1-arc-transitive $r$-regular graph, where $r$ is a prime. If $X$ has more than one end then $X$ is an $r$-regular tree.*

*Proof of Theorem 20.* Set $G = \mathrm{Aut}(X)$. Find a D-cut $e_0$ in $X$ and set $E = Ge_0 \cup Ge_0^*$. Take a vertex $v$ in $X$ and let $N(v)$ denote the set of vertices that are adjacent to $v$. Now recall the definition of the map $\phi$ from Section 2.2. Of particular importance is the fact that the action of $G$ commutes with $\phi$. Set $u = \phi(v)$. Suppose that $\phi(N(v))$ is contained in some component $C$ of $T \setminus \{u\}$. Let $e$ be the edge in $T$ with origin in $u$ and pointing towards $C$. Then all the edges in $X$ that have $v$ as an end vertex are in $\delta e$ but $v$ is not in $e$. Now we have that $\{v\}$ would be a component of $e^*$, contradicting our original assumption that $e$ is a D-cut. So $\phi(N(v))$ is dispersed over more than one component of $T \setminus \{u\}$.

From 1-arc-transitivity it follows that $G_v$ acts transitively on $N(v)$. Then $G_v$, which is a subgroup of $G_u$, acts transitively on the components of $T \setminus \{u\}$ that contain elements from $\phi(N(v))$. Hence for every two such components $C$ and $C'$ we have that the number of vertices of $N(v)$ mapped by $\phi$ to $C$ and $C'$ is the same. Because the valency of $v$ is a prime this number must be 1.

Now we have to prove that $X$ does not have any cycles. Let us look what happens to a simple path $v_0, v_1, \ldots, v_n$ when it is mapped to $T$ by $\phi$. Because $X$ is 1-arc-transitive there is a number $c$ such that for every pair of adjacent vertices $v$ and $v'$ in $T$ we have that $d_T(\phi(v), \phi(v')) = c$. First of all it is clear that $\phi(v_2)$ cannot be in the same component of $T \setminus \{\phi(v_1)\}$ as $\phi(v_0)$. So

$$d_T(\phi(v_0), \phi(v_2)) = d_T(\phi(v_0), \phi(v_1)) + d_T(\phi(v_1), \phi(v_2)) = 2 \cdot c.$$

Note that $\phi(v_0), \ldots, \phi(v_{k-1})$ all belong to the same component of $T \setminus \{\phi(v_k)\}$. It is now easily proved by induction that $d_T(\phi(v_0), \phi(v_k)) = k \cdot c$. From this it follows that $X$ has no cycles because if $k > 1$ then $v_0$ and $v_k$ cannot be adjacent in $X$. □

It is tempting to think that some kind of a classification or description can be found for 1-arc-transitive graphs, but Dunwoody has given an example of an inaccessible 4-regular 1-arc-transitive graph so there is probably not much hope for any such results.

It is also natural to ask about what happens if we put extra conditions on the action of the automorphism group on the ends, and in [56] Woess asks for a classification of connected locally finite graphs with more than one end where the automorphism group acts transitively on the ends. His question was answered independently by Möller [33] and Nevo [40]. In both cases the proofs use structure trees. Examples of such graphs are distance-transitive graphs and, indeed, all other examples are very similar to them. Before we state the result in full details we must first define some notation. A tree $T$ is a bipartite graph, that is the vertex set can be partitioned into two disjoint sets so that no two vertices in the same set are adjacent. The automorphism group respects this partition. Let $\mathrm{Aut}^+(T)$ denote the subgroup of $\mathrm{Aut}(T)$ that stabilizes this partition. A tree is *semi-regular* if the vertices in each part of the partition all have the same valency.

**Theorem 21** *Let $X$ be a connected locally finite graph with infinitely many ends. Suppose $X$ is end-transitive. Then there is an* $\mathrm{Aut}(X)$*-congruence $\pi$ with finite classes such that $X/\pi$ is isomorphic to a connected component of a poly-orbital graph arising from the action of* $\mathrm{Aut}^+(T)$ *on $VT$, where $T$ is a semi-regular locally finite tree, or there is a non-empty finite subset of $VX$ invariant under* $\mathrm{Aut}(X)$*. In particular, if there is no non-empty* $\mathrm{Aut}(X)$*-invariant finite set of vertices then* $\mathrm{Aut}(X)$ *acts with finitely many orbits on $VX$.*

This result has been applied by Nevo in [41] to study random walks and harmonic analysis on such graphs.

Woess [56] also asks for information about graphs $X$ where there is an end $\omega$ such that the stabilizer in $\mathrm{Aut}(X)$ of $\omega$ acts transitively on $VX$. Let us now look at an example of such a graph [46, Example 2 and Lemma 9]. Let $T$ be a regular tree, and let $\omega$ be an end of $T$. Now add edges to $T$ such that $\{u, v\}$ becomes an edge if the distance between $u$ and $v$ in $T$ is precisely 2, and $v$ lies on the ray starting at $u$ and belonging to $\omega$ or *vice versa*. The resulting graph, $X$, will have the desired properties and the end $\omega$ is fixed by $\mathrm{Aut}(X)$. This example is fairly descriptive of the general situation.

**Theorem 22** *Let $X$ be a locally finite connected graph with infinitely many ends. Put $G = \mathrm{Aut}(X)$. Assume there is an end $\omega$ of $X$ such that $G_\omega$ acts transitively on $VX$.*

**(i)** ([34, Theorem 2]) *Then $G$ acts transitively on $\Omega X \setminus \{\omega\}$. In particular, if $\omega$ is not fixed by $G$ then $G$ acts transitively on $\Omega X$.*

**(ii)** ([34, Theorem 4]) *Then there is a $G$-congruence $\pi$ with finite classes on $VX$ such that $X/\pi$ is a poly-orbital graph arrising from the action of a transitive group $H$ on a locally finite distance transitive graph $X_0$. There is also an end $\omega_0$ in $X_0$ such that $H_{\omega_0}$ is transitive on $VX_0$.*

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