ENDS—GROUP THEORETICAL AND TOPOLOGICAL ASPECTS

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Abstract. This is a survey of topological, group theoretical and some graph theoretical aspects of ends. After discussing the notion of ends in topology, we consider ends of graphs and show that the metric end topology of connected graphs is metrizable. The "1–2–Cantor theorem" is proved for graphs whose ends are all limit ends, that is, ends which are accumulation points of an orbit of the group of automorphisms. We also discuss ends of finitely generated groups, Stallings' Structure Theorem and further applications concerning the cycle space of a graph and random walks.

1. Topology

Roughly speaking, an end is a point at infinity. If we remove a "small" set from the ambient space then every "large" component contains at least one end. Hence ends can be separated by removing such small sets. The set of ends which live in these large components form the basic open sets in the space of all ends. The complement of the basic sets in the set of all ends is also open. Hence the basic open sets are open and closed and so the set of ends is always totally disconnected. For locally compact space (or locally finite graphs) the ends yield a compactification of the ambient space.

Several authors have independently introduced notions of ends making this idea precise. In 1913 Constantin Caratheodory studied "Primenden" of subsets of the plane in order to generalist a conformal version of the Schönflies theorem (no assumption on the boundary). This work was further developed for Euclidean space by Stefan Mazurkiewicz in [33]. In 1931 Hans Freudenthal introduced "Enden" in his dissertation (see [17] and also [18, 19]) motivated by topological and group theoretical questions. He generalised his work incorporating the developments of Caratheodory and Mazurkiewicz in [20]. In his dissertation Freudenthal mentioned that the notion of "Randstücke" introduced by Béla von Kerékjártó in 1923 (see [29]) for surfaces is a special case of ends. Simion Stoïlow used Kerékjártó's notion in the theory of analytic functions on surfaces, see [41]. Later the end point compactification became known as Kerékjártó-Stoïlow compactification in potential theory in the context of ideal boundaries of surfaces and the Dirichlet problem see [6]. Finally, in 1964 Rudolf Halin introduced ends of infinite graphs in [22] and studied structural aspects of graph automorphisms in [23].

There are several motivations to study ends. If two spaces have a different number of ends, then they cannot be homeomorphic. If a space has finitely many, but more than two ends, then it cannot be homogeneous. Adding ends to the underlying

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space defines, under mild conditions, a compactification called the end-point compactification or Freudenthal compactification. Ends are a natural extension of the space. This point of view is useful in group theory (see for example [1, 2]), graph theory (see for example [12, 13]) or when considering random walks (see [4, 35]). The book of Hughes and Ranicki [26] is a rich source for ends of manifolds and complexes (including homotopy and homology at infinity).

Freudenthal's first definition of ends [17] is as follows. Let X be a locally connected, connected, and locally compact Hausdorff space. *Ends* of X are equivalence classes of descending sequences $(G_1, G_2, \dots)$ of connected, open subsets with compact boundaries, such that $\overline{G_1} \cap \overline{G_2} \cap \dots = \emptyset$, where two such sequences $(G_1, G_2, \dots)$ and $(H_1, H_2, \dots)$ are equivalent, if, for all n, $H_k \subseteq G_n$ for some k and vice versa. If one is only interested in the number of ends, the following notion is sufficient (see Specker [37], Raymond [36]): X has *at least k ends*, if there is an open subset U with compact closure, such that $X \setminus U$ has at least k non-compact components.

Nowadays the Freudenthal compactification can be constructed via proximities (see [27, 42]). A proximity is a relation on the subsets of a space. If two subsets are in relation then they are called *proximal*. Usually this means that they are close in some sense. For example, the elementary proximity of a normal space is given by the relation

$$A \curlywedge B \iff \overline{A} \cap \overline{B} \neq \emptyset.$$

A proximity $\curlywedge$ defines a topology via the associated Kuratowski closure:

$$\overline{A} = \{x \in X \ : \ \{x\} \curlywedge A\}.$$

Every proximity can be used to construct (via Cauchy completion) a compactification, which is called the Samuel compactification with respect to the proximity. In order to construct the Freudenthal compactification, we start with a locally compact Hausdorff space (in fact it is sufficient to require complete regularity and rim-compactness) and define a proximity as follows: Two subsets A and B are not proximal, if for some compact subset K, $X \setminus K$ decomposes into two open, disjoint subsets G and H with $\overline{A} \subseteq G$ and $\overline{B} \subseteq H$. This proximity is compatible with the ambient topology in the sense, that it induces the topology. The Freudenthal or end point compactification (denoted by εX) is just the Samuel compactification with respect to this proximity. Freudenthal already used similar constructions in [18, 20].

Another construction uses an idea of Stone and Čech: Let X be a locally connected, connected, and locally compact Hausdorff space and denote by $C_{\mathrm{fin}}(X)$ the set of all bounded real-valued continuous functions on X for which there is a compact subset K such that $f(X \setminus K)$ is finite. For every $f \in C_{\mathrm{fin}}(X)$ we choose a compact interval I_f so that $f(X) \subseteq I_f$. By Tychonov's theorem the set

$$Q = \prod_{f \in C_{\mathrm{fin}}(X)} I_f$$

is compact. The evaluation map $e : X \to Q$, $x \mapsto (f(x))_{f \in C_{\mathrm{fin}}(X)}$ is an embedding of X into Q. Thus the closure of $e(X)$ in Q yields a compactification of X which is the Freudenthal compactification. This approach was for instance used in [4, 6].

The Freudenthal compactification has the property that the space of ends $\Omega X = \varepsilon X \setminus X$ is zero-dimensional and εX is the maximal compactification with this property. Furthermore, it is a quotient of the Stone-Čech-compactification βX as follows. Let $\sim$ be the equivalence relation on βX whose equivalence classes are singletons on X and connected components on ΩX. Then $\varepsilon X = \beta X/\sim$. Since the Stone-Čech-compactification of a zero-dimensional space is zero-dimensional, it follows that $\varepsilon X = \beta X$, when X is zero-dimensional. For more details we refer to [27, 42].

Finally we want to mention an elegant definition of an end avoiding equivalence classes (see [34]). In the setting of Freudenthal's first definition we additionally assume that the space X is σ-compact. Then an end is a function f mapping compact subsets K to connected components of $X \setminus K$ satisfying $f(K_1) \supseteq f(K_2)$ whenever $K_1 \subseteq K_2$ are compact sets. This notion of an end was called a *direction* by Diestel and Kühn, see [11]. As a consequence, the set of ends is an inverse limit. Take an ascending sequence $(K_1, K_2, \dots)$ of compact subsets covering X and denote by $\mathcal{C}(X \setminus K_i)$ the set of connected components in $X \setminus K_i$. Then the space of ends is

$$\Omega X = \varprojlim \mathcal{C}(X \setminus K_n),$$

where the inverse system is defined via set inclusion.

2. Ends of graphs

Rudolf Halin defined ends of infinite graphs in [22] via rays. A one-sided infinite path in an infinite graph X is called a *ray*. Two rays are *equivalent*, if one of the following equivalent conditions hold:

(1) There is a third ray which has infinitely many vertices in common with each.
(2) For every finite vertex set F the two rays are eventually contained in the same connected component of $X - F$.
(3) There are infinitely many disjoint paths in X joining the two rays.

An *end* (or *vertex end*) in the graph X is such an equivalence class of rays. Regarding the graph X as a 1-complex ends in the sense of Freudenthal are well defined. For locally finite graphs the topological ends of Freudenthal and the graph-theoretical ends of Halin are in one-to-one correspondence. In general the set of topological ends inject into the set of graph-theoretical ends, see [11]. Starting from Condition (2) one can define further notions of ends in graphs. We say that two rays are *separated* by a set S of vertices or edges, if they are eventually contained in different components of $X - S$. Hence Condition (2) reads as follows: Two rays are equivalent if they cannot be separated by finitely many vertices, hence the name vertex ends. They are the most common type of ends in graph theory, see also Section 6.

If we use finitely many edges for separation we obtain *edge ends*, see [4, 8, 16, 21]. One can also consider vertex sets S with finite diameter with respect to the natural graph metric. Here we have to consider only *metrically transient* rays, i.e. rays without infinite, bounded vertex subsets, or equivalently, rays eventually leaving any bounded vertex sets. The corresponding ends are referred to as *metric ends*, see [24, 30, 31]. One of the reasons to study metric ends is that quasi-isometries extend to homeomorphisms of the metric end boundary, see [30]. A map $\phi : X \to Y$ of two metric spaces (X, d_X) and (Y, d_Y) is called *quasi-isometry* if there is a constant c such that

$$c^{-1}d_X(x_1, x_2) - c \leq d_Y(\phi(x_1), \phi(x_2)) \leq c d_X(x_1, x_2) + c,$$

for $x_1, x_2 \in X$, and if for all $y \in Y$ there is an $x \in X$ such that $d_Y(y, \phi(x)) \leq c$. A graph X is considered as metric space (VX, d), where d is the usual graph metric.

The notions of vertex ends, edge ends, and metric ends coincide for locally finite graphs, but not for non-locally finite graphs, see [30]. The metric end boundary does not provide a compactification in general. As before, we write ΩX for the set of ends (vertex ends, edge ends, or metric ends) and set $\varepsilon X = VX \cup \Omega X$. Depending on the definition of ends we say that a set S is *small*, if it is

- a finite set of vertices (vertex ends),
- a finite set of edges (edge ends), or
- a bounded set of vertices (metric ends).

An end as equivalence class of rays is said to *live* in a vertex set if all its rays are eventually contained in the induced subgraph. For a vertex set C we write ΩC for the set of ends living in C. We call a set of vertices C *connected* if the subgraph spanned by C is connected. A component of a set of vertices is a maximal connected subset. A common topology on εX is given by the basis $\mathcal{B}$ consisting of sets of the form $C \cup \Omega C$, where C is the vertex set of a component in the complement of some small set. Open sets in εX are just the unions of sets in $\mathcal{B}$. Similar topologies on εX for vertex ends which are discrete on VX are defined in [10, Section 8.5] and for edge ends in [4].

For vertex ends and edge ends of non-locally finite graphs, εX is compact but not Hausdorff, see [30, Theorems 1 and 2], even though ΩX is not compact in general. Often the topology on VX is a priori considered as discrete in which case εX will not be compact for the case of vertex ends in non-locally finite graphs. In the case of edge ends we can consider the discrete topology on VX and at the same time consider vertices with infinite degree as degenerated rays. Then we may get ends consisting of vertices of infinite degree but which contain no rays. The result is that εX becomes a compact Hausdorff space, even if X is non-locally finite, see [4, 30].

3. Metrization of end spaces

The end topology of a locally finite graph is metrizable, because any regular T_1 space with a countable base is metrizable. The vertex end topology of non-locally finite graphs is metrizable if and only if the graph contains a normal spanning tree, see [10, Theorem 8.5.2]. P. Sprüssel has proved in [38] that the vertex end topology is normal. In [30] the first author has asked if the metric end topology were normal. We now give a positive answer by showing that it is always metrizable for connected graphs. In the rest of this section we consider metric ends: ΩC is the set of metric ends living in C and $\varepsilon X = VX \cup \Omega X$.

A vertex set S is said to *separate* $a, b \in \varepsilon X$ if either a or b are vertices in S or if there are distinct components C_a and C_b of $X \setminus S$ such that $a \in C_a \cup \Omega C_a$ or $b \in C_b \cup \Omega C_b$. Let $B(o, r)$ denote the ball $\{x \in VX : d(o, x) \leq r\}$, where d is the usual graph metric. The following construction is similar to the well-known metric defined by confluents in trees, see e.g. [43].

Fix a reference vertex o. Dependence on the choice of o will be suppressed. If a and b are distinct elements of εX then there is a least integer $r(a,b) \geq 0$ such that $B(o, r(a,b))$ separates a from b. Let $f : \mathbb{N}_0 \to [0, \infty)$ be a positive, strictly decreasing function such that $f(n) \to 0$ for $n \to \infty$. We set

$$u(a,b) = \begin{cases} 0 & \text{if } a = b, \\ f(r(a,b)) & \text{if } a \neq b. \end{cases}$$

Then, for $a, b, c, d \in \varepsilon X$, $u(a,b) \leq u(c,d)$ if and only if $r(a,b) \geq r(c,d)$.

Lemma 1. *If X is a connected graph then $(\varepsilon X, u)$ is an ultrametric space.*

Proof. Positive definiteness and symmetry follow from the definition of u. We have to show that the strong triangle inequality

$$u(a,b) \leq \max\{u(b,c), u(a,c)\}$$

is satisfied for all a, b and c in εX. We may assume that $a \neq b$ and that

$$u(a,b) = \max\{u(a,b), u(b,c), u(a,c)\},$$

equivalently

$$r(a,b) = \min\{r(a,b), r(b,c), r(a,c)\}.$$

If $r(a,b) = r(a,c)$ then $u(a,b) = u(a,c)$ and we are done. Otherwise $r(a,b) < r(a,c)$. Let C be the component of $VX \setminus B(o, r(a,b))$ such that $c \in C \cup \Omega C$. Since $B(o, r(a,b))$ does not separate a and c, the element a is also in $C \cup \Omega C$. But b is not in C, because $B(o, r(a,b))$ separates a from b. Hence $B(o, r(a,b))$ separates b from c. This implies $r(b,c) \leq r(a,b)$. Thus $r(b,c) = r(a,b)$ and $u(b,c) = u(a,b)$. □

Let $O(a, \rho) = \{b \in \varepsilon X : u(a,b) < \rho\}$ denote the open ball with respect to the ultrametric u with centre a and radius ρ. For an end ω and $r \geq 0$ we write $C(\omega, r)$ to denote the component in the complement of $B(o, r)$ in which ω lives. Then

$$O(\omega, u(\omega, a)) = C(\omega, r(\omega, a)) \cup \Omega C(\omega, r(\omega, a)).$$

The metric end topology is the topology induced by the basis $\mathcal{B}$. Recall that $\mathcal{B}$ consists of the sets $C \cup \Omega C$, where C is a component of in the complement of a bounded set.

Theorem 2. *The ultrametric u of a connected graph X induces the topology of metric ends on εX.*

Proof. A vertex x is separated from any element a of $\varepsilon X \setminus \{x\}$ by $B(o, d(o,x))$, because this ball contains x. In other words, $u(x,a) > f(d(o,x))$, and $O(x, f(d(o,x))) = \{x\}$. Hence the topology which is induced by u is discrete on VX like the topology induced by $\mathcal{B}$. If x is a vertex then note that $\{x\}$ is a basic open set in $\mathcal{B}$, because $\{x\}$ is a component in the complement of the set of neighbours of x, which is a bounded set.

Let the end ω be an inner point of $A \subseteq \varepsilon X$ with respect to the ultrametric u. That is, there is an $\epsilon > 0$ such that $O(\omega, \epsilon) \subseteq A$. Let $(x_0, x_1, x_2, \ldots)$ be a ray in ω. Then

$$\lim_{n\to\infty} r(\omega, x_n) = \infty \qquad \text{which implies} \qquad \lim_{n\to\infty} f(r(\omega, x_n)) = 0.$$

There is an n such that $u(\omega, x_n) < \epsilon$ and hence

$$C(\omega, r(\omega, x_n)) \cup \Omega C(\omega, r(\omega, x_n)) = O(\omega, u(\omega, x_n)) \subseteq O(\omega, \epsilon) \subseteq A.$$

Since $C(\omega, r(\omega, x_n)) \cup \Omega C(\omega, r(\omega, x_n))$ is an element of the basis $\mathcal{B}$ the end ω is an inner point of A with respect to the metric end topology.

Let the end ω be an inner point of $A \subseteq \varepsilon X$ with respect to the end topology. Then there is an element $D \cup \Omega D$ of the basis $\mathcal{B}$ such that $\omega \in D \cup \Omega D \subseteq A$, where D is a component in the complement of some bounded S. We choose r such that $S \subseteq B(o, r)$. Then $C(\omega, r) \subseteq D$ and

$$O(\omega, f(r)) = C(\omega, r) \cup \Omega C(\omega, r) \subseteq D \cup \Omega D \subseteq A$$

which means that ω is an inner point of A with respect to u. □

As a consequence, the topology induced by the ultrametric u does not depend on the choice of o and the metric end topology is strongly zero-dimensional.

4. Limit ends and Cantor sets

A crucial theorem on ends proved by Freudenthal [17, 18, 19] and Hopf [25] states that every locally compact, locally connected, connected Hausdorff space X which admits a group action, such that the translates of a compact subset cover X, has at most 2 or infinitely many ends. In the latter case the set of ends $\varepsilon X \setminus X$ is a Cantor set. Herbert Abels observed in [3] that this assumption can be weakened. A end ω is called a *limit end* if it is accumulation point of an orbit in X, i.e. there is a sequence $g_1, g_2, \ldots$ of group elements and a vertex x, so that $g_n x$ converges to ω, see also [43]. Recall that by definition, a *Cantor set* is a non-empty, compact, totally disconnected, perfect, metrizable set. All Cantor sets are homeomorphic.

Lemma 3. *If a connected graph has more than two metric ends and all ends are limit ends then the set of metric ends is perfect.*

Proof. Suppose there is an isolated point η in the set of metric ends ΩX and there are three distinct metric ends ω_1, ω_2, ω_3. Let F be a bounded set which separates these three ends from each other and let C_i be the component of $VX \setminus F$ in which ω_i is living. Since η is isolated, there is an unbounded set of vertices D which is a component in the complement of a ball $B(o, r)$ such that η is the only end living in D. Let D' be the component in the complement of $B(o, r + \operatorname{diam} F)$ which contains η. Let x be a vertex in F. Since η is a limit end, there a $g \in G$ such that $g(x) \in D$. Now $g(F) \subseteq C$. The set $VX \setminus D$ is connected and disjoint from $g(F)$, hence, for $i = 1, 2, 3$, the set $VX \setminus D$ is either disjoint with $g(C_i)$ or contained in $g(C_i)$. Since $VX \setminus D$ can be contained in at most one of the there sets $g(C_i)$, this means that at least two of the sets $g(C_i)$ are contained in D. Hence ΩD contains at least two of the three ends $g(\omega_i)$ which is a contradiction. □

The metric end boundary is not compact in general, but the end boundary in connected, locally finite graphs always is. Hence we have proved the following.

Theorem 4. *Let G be a group acting on a connected, infinite locally finite graph such that every end is a limit end. Then the graph has* 1 *or* 2 *ends, or ΩX is a Cantor set.*

As above, let X be a connected, locally finite, infinite graph. Equipped with the compact-open topology the automorphism group $\mathrm{Aut}(X)$ is a topological group and the van Dantzig-van der Waerden theorem tells us, that $\mathrm{Aut}(X)$ is locally compact and acts properly on X. That is, the map $\mathrm{Aut}(X) \times X \to X \times X$, $(g, x) \mapsto (gx, x)$ is proper (continuous and preimages of compact subsets are compact), see [7].

Theorem 5. *Let X be a connected, locally finite, infinite graph. Then*

- $\mathrm{Aut}(X)$ *is non-compact, if and only if, there exists at least one limit end. In this case there are at most two or infinitely many ends. If there are infinitely many limit ends then they form a Cantor set.*
- $\mathrm{Aut}(X)$ *is compact, if and only if, there is a finite set F of vertices which is stabilised by* $\mathrm{Aut}(X)$.

The 'if and only if'-statements are simple consequences of the definitions. The implication of the first case was proved by Abels in [2].

5. Ends of groups

Groups were first studied as symmetries of geometric objects and later as fundamental groups of topological space. That is, one starts by considering a geometric or topological object and the group is determined by this object. In the 1960s and 1970s mathematicians began to consider groups as geometric objects themselves and not as something that is defined by another geometric or topological object. Lie groups for instance are considered as differentiable manifolds, and finitely generated groups are considered as Cayley graphs.

Let G be a finitely generated group and let S be a finite set of generators. The *Cayley graph* of G with respect to S is defined as directed graph X with vertex set $VX = G$, where (x, y) is an edge if and and only if $x^{-1}y \in S$. Often X is considered as undirected graph. Note that X is connected, because S generates G, and X is locally finite, because S is finite. Since the definition of a Cayley graph depends on the generating set, one may ask, which properties of a Cayley graph reflect the properties of the group. Cayley graphs of some group with different finite sets of generators are quasi-isometric. Hence properties of graphs which are invariant under quasi-isometries can be considered as properties of the group itself, regardless of the chosen set of generators. Such properties are for instance growth, hyperbolicity, accessibility, the number of ends, and many more. The task of geometric group theory is to relate such geometric properties with algebraic properties of the group.

We know by the Theorem of Freudenthal and Hopf that finitely generated groups have at most two or infinitely many ends. The question from the perspective of geometric group theory is, how can groups with one, two or infinitely many ends be classified algebraically? Before giving the answer, let us have a look back at history.

The Seifert-van-Kampen Theorem (see [28]) says that if a topological space can be decomposed into two open path connected spaces C and D then its fundamental group is a free product of the fundamental groups of the two subspaces with amalgamation over the fundamental group of their intersection $C \cap D$. In Seifert-van-Kampen's Theorem we obtain an algebraic decomposition of the fundamental group by a geometric decomposition of the space. Stalling's Theorem (see below) is the analogue of Seifert-van-Kampen's Theorem in geometric group theory.

Let G_1 and G_2 be two groups with presentations $G_i = \langle S_i \mid R_i \rangle$. Let $\phi : H \to K$ be an isomorphism of subgroups of G_1 and let t be not an element of G, then the *HNN-extension* of G_1 over H with respect to ϕ is defined by

$$G_1 *_\phi = \left\langle S_1 \cup \{t\} \,\middle|\, R_1 \cup \{t^{-1}h^{-1}t\phi(h) \,:\, h \in H\} \right\rangle .$$

Let H_1 and H_2 be isomorphic subgroups of G_1 and G_2, respectively, with isomorphism $\psi : H_1 \to H_2$. The *free product with amalgamation* of G_1 and G_2 over H_1 with respect to ψ is defined by

$$G_1 *_\psi G_2 = \left\langle S_1 \cup S_2 \,\middle|\, R_1 \cup R_2 \cup \{h^{-1}\psi(h) \,:\, h \in H_1\} \right\rangle .$$

A finitely generated group G *splits over a subgroup* H if it is a free product with amalgamation or an HNN-extension over H.

Theorem 6 ([39, 40])**.** *A finitely generated group has more than one end if and only if it splits over some finite subgroup.*

Today Stallings' theorem is considered as one of the most important contributions to geometric group theory. A proof of Stallings' theorem which is based on graph-theoretic considerations and elementary Bass-Serre theory was published by Dunwoody in [16]. There are still open problems in this area. For instance Kropholler's conjecture is a purely algebraic version of Stallings' theorem, see [32] and the exposition of Niblo and Sageev in [5, Chapter 50].

6. Cycle space

Recently, the notion of cycle space was adapted for locally finite, infinite graphs by Diestel and Kühn [12, 13], see also the expository paper [9]. Denote by S^1 the topological circle. Let X be a locally finite graph regarded as a 1-complex. A homeomorphic image of S^1 in εX is called a *circle* in εX. Such a circle contains an edge completely or touches at most its end vertices. Since $\varepsilon X \setminus X$ is totally disconnected, a circle in εX is determined by the edges that it contains. Hence we may identify a circle in εX with its edge set. A family of edge sets is called *thin*, if no edge is contained in an infinite number of edge sets. The (thin) *sum* of a thin family consists of those edges that lie in an odd number of edge sets. The (topological) *cycle space* of X is the set of all sums of thin families of circles in εX. Note that for finite graphs this definition agrees with the usual one. Many well known theorems concerning the cycle space of finite graphs extend to the infinite case—with appropriate adjustments. For example (see [14]):

Theorem 7. *The fundamental circles of a topological spanning tree generate the cycle space.*

Here a *topological spanning tree* is a subgraph whose closure in εX is a "tree", i.e. path-connected and free of circles.

Diestel and Sprüssel showed that the topological cycle space of a locally finite graph is a quotient of the first singular homology group of its end point compactification, see [15]. These new results must be seen as a major achievement in generalising notions and theorems for finite graphs to infinite ones.

7. Random walks

For a transient random walk Z_n on an infinite graph X two questions arise naturally in connection with boundary notions. Let some compactification κX of X be given. Firstly, we may ask whether there is a random variable Z_∞ with values in $\kappa X \setminus X$, so that $Z_n \to Z_\infty$ almost surely in κX. Secondly, we may consider the Dirichlet problem for the boundary: Given a real-valued, continuous function on $\kappa X \setminus X$, is there a continuous extension to κX which is harmonic on X? Under some assumptions on the random walk both questions were studied for the end point compactification. For details and further references we refer to Woess' book [43].

The Green function $G : X \times X \to \mathbb{R}$ of a transient random walk is defined by

$$G(x, y) = \sum_n p_n(x, y),$$

where $p_n(x, y)$ denotes the n-step transition probability. Fixing a reference vertex $o \in X$, the Martin kernel $K : X \times X \to \mathbb{R}$ is given by $K(x, y) = G(x, y)/G(o, y)$. The Martin compactification mX is constructed using the set of functions $\{K(x, \cdot) : x \in X\}$ following the idea of Stone and Čech outlined in Section 1. The Martin boundary $mX \setminus X$ was studied from different viewpoints. Its identification is often a difficult task. For trees it was shown that the Martin boundary coincides with the space of ends, see [35]. In [4] it was shown that the Martin boundary of trees coincides with the set of edge ends in absence of local finiteness.

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