STRUCTURE TREES, VERTEX CUTS

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Abstract. This paper is based mainly on the work of B. Krön and M. Dunwoody. It can be read as an introduction to the theory of cuts in general, and in particular of vertex cuts. Cut theory is concerned with decompositions of graphs into trees, so called structure strees, carrying with them the intrinsic structure of the underlying graph. This paper presents the whole construction of cut systems and of corresponding structure trees and in the end, some applications.

1. Preliminaries

1.1. **Graphs.** Let VX be a set and EX be a set of two-element subsets of VX. Then, $X = (VX, EX)$ is an *undirected graph*. Elements in VX are called *vertices*, elements in EX edges. If $\{x, y\} \in EX$, then x and y are said to be *adjacent*. For a subset $E \subset VX$, define NE as the set of vertices in $VX \setminus E$ which are adjacent to a vertex in E. A *directed graph* is a graph where the elements of EX are ordered pairs (x, y) of elements of VX. A graph is called $locally\ finite$ if $N\{x\}$ is finite for every $x \in VX$.

A path of length n in a graph X is an $n+1$-tuple of distinct vertices $(x_0, ..., x_{i-1}, x_i, x_{i+1}, ..., x_n)$ where x_i is adjacent to x_{i+1} for all i. Infinite paths (sequences) are called *rays*. Two-way infinite paths are injective functions from $\mathbb{Z}$ into VX where every integer and its successor have adjacent images. These are called *double rays*. A *tail* of a ray $r = (x_0, x_1, ...)$ is a set $t = (x_i, x_{i+1}, ...)$ for any i.

A subset $W \subset VX$ in a graph X is said to be *connected* if for every $a, b \in W$, there is a path $\pi = (x_0, ..., x_n)$ with $a = x_0$ and $b = x_n$. A graph is connected if VX is connected. In this situation, π is said to *join* a and b. A graph is called *n - connected*, if for every distinct vertices a, b there are n paths, such that every vertex in $VX \setminus \{a, b\}$ is in at most one of the paths.A *n-seperator* is an n-element set of vertices which disconnects an n-connected graph. This is also referred to as a *vertex seperator*, whereas an *edge seperator* is a subset of EX which, when removed, leaves a connected graph unconnected. If the *boundary* δE, defined as the set of edges with one vertex in E and the other in $VX \setminus E$, where $E \subset VX$, is finite, then E will be called an *edge cut*. Analogously, C is a *vertex cut* if C is connected and NC is finite.

1.2. **Ends of Graphs.** Let $C \subset VX$ be a vertex (edge) cut. A ray r is said to *lie* in C if there is a tail t such that $t \subset C$. Because all the vertices and therefore also edges between vertices in a ray are distinct, all rays eventually lie in either C or $VX \setminus C$ for every vertex (edge) cut. If for two rays r_1, r_2, there is no vertex (edge) end C such that r_1 lies in C and r_2 lies in $VX \setminus C$, those rays are said to belong to the same vertex (edge) end. This is easily seen to be an equivalence relation. The equivalence classes are called vertex (edge) ends.

If δC for some edge cut C seperates two edge ends, then

$$S = NC \cup N(VX \setminus C)$$

is finite and thus S seperates two vertex ends. There can be, however, distinct vertex ends which are contained in the same edge end.

For example, let K_∞ be a graph where VK_∞ is a countably infinite set and every element is adjacent to every other one. This is the complete countable graph. Let now X be the graph consisting of two copies of K_∞ where a single vertex from each copy is identified. This vertex seperates two vertex ends, but there is no edge cut.

If a graph is locally finite, there is a bijective correspondence between its vertex- and edge ends. This paper will in the first place be concerned with vertex cuts.

2. Axiomatic Cut Systems

The main purpose of the vertex cut theory presented here, is to provide structure tree decompositions of graphs. An axiomatic system, the cut system, will be introduced and investigated, and shall turn out to provide such decompositions.

In a graph X where $C \subset VX$, define the $*$-*complement* to be

$$VX \setminus (C \cup NC) = C^*.$$

When $C, D \subset VX$, we call the intersections $C \cap D, C \cap D^*$, $C^* \cap D$ and $C^* \cap D^*$ the *corners of* C *and* D. We also call $C \cap ND, C^* \cap ND, D \cap NC$ and $D^* \cap NC$ the *links of* C *and* D, and finally $NC \cap ND$ the *center.* Note that all these sets are disjoint. We call corners, links and center adjacent simply if they are connected by an edge in the graph, and we call two corners *adjacent* if they are adjacent to the same link. We call two corners *opposite* if they are not adjacent. We also call links adjacent if they are adjacent to the same corner. Finally, we call links opposite if they are not adjacent. Consider the following sketch of these sets. It indicates the adjacency of the sets and, where lowercase letters are inserted, they indicate the cardinality of the corresponding set, say.

$C \cap D^*$	$D^* \cap NC$, b	$C^* \cap D^*$
$C \cap ND$, a	$NC \cap ND$, m	$C^* \cap ND$, c
$C \cap D$	$D \cap NC$, d	$C^* \cap D$

Let $\mathcal{C}$ be a set of connected subsets of VX of a graph X, such that $(C^*)^* = C$ for all $C \in \mathcal{C}$. The family $\mathcal{C}$ is then a *cut-system*, if $C, D \in \mathcal{C}$ satisfy the following.

- **(A1):** If two opposite corners contain an element of $\mathcal{C}$, then every component of any of these corners which contain an element of $\mathcal{C}$ is in $\mathcal{C}$.
- **(A2):** There are two opposite corners which contain an element of $\mathcal{C}$.

The elements of $\mathcal{C}$ are called *cuts*, and *cut-components* will be just components which are cuts. Unless otherwise stated, $\mathcal{C}$ will from now on denote a cut system and C, D denote cuts in $\mathcal{C}$. We note that for $C = D$, the corners are C, opposite to C^* and two empty ones, so that (A2) implies that C^* has a cut component for every $C \in \mathcal{C}$. Also note that, in general, * is not an involution, for NC^* need not be equal to NC. However it is true that $C^{***} = C^*$ and so * is an involution if we replace $\mathcal{C}$ by $\mathcal{C}^{**} := \{C^{**} \mid C \in \mathcal{C}\}$. Furthermore, we note the following: for a set $E \subset VX$, $E^{**} = E \cup E'$, where E' are the elements of NE which are not adjacent to any element in E^*. Therefore, also $E \subset E^{**}$.

Proposition 2.1. *Axiom (A2) is equivalent to the following:*

(A2') For every $C, D \in \mathcal{C}$*, both* $C \setminus ND$ *and* $C^* \setminus ND$ *contain an element of* $\mathcal{C}$*.*

Proof. Assuming (A2), we can by renaming, if necessary, make both $C \cap D$ and $C^* \cap D^*$ contain an element of $\mathcal{C}$. Since $C \cap D \subset C \setminus ND$, the latter contains a cut. The set $C^* \setminus ND$ contains a cut because of an analog argument, and so (A2') holds.

Assuming (A2') and because $C \setminus ND = (C \cap D) \cup (C \cap D^*)$, we can, again by renaming D and D^* if necessary, make sure that $C \cap D$ contains a cut. If in the first case, also $C \cap D^*$ contains a cut, we are finished, because for $C^* \setminus ND = (C^* \cap D) \cup (C^* \cap D^*)$, one of the latter sets has to contain a cut, being opposite to either $C \cap D$ or $C \cap D^*$. If in the other case, $C \cap D^*$ doesn't contain a cut, then $C^* \cap D^*$ must do so, because for (A2'), also $D^* \setminus NC = (C \cap D^*) \cup (C^* \cap D^*)$ has to contain a cut. □

The definition may seem awkward at first, but it is enlightened by some Examples:

Example 1: In an infinite graph, conveniently chosen connected and with more than one vertex end, define cuts by connected sets C with: (a) C contains a ray, (b) NC is finite, (c) C^* also contains a ray. (A1) obviously holds. Axiom (A2') holds because if there were no opposite corners of C and D containing a ray, there could at most be two corners containing a ray, and they would be adjacent. Without loss of generality, let those corners be $C \cap D$ and $C \cap D^*$. Now C^* can no more contain a ray, contradicting the assumption that C was a cut. In the case where there is only one or no corner containing a ray, the contradiction is trivial.

This can be considered a canonical cut system for infinite graphs. There is a totally analogous edge cut system in infinite graphs, which has edge sets as separators and uses the common set complement instead of the $*$-complement. This edge cut system can easily be translated to a vertex cut system by replacing every edge in every separator by a path of length two and by taking the center vertices of these paths as vertex separators, then. For finite graphs, there is also a canonical structure, which is somewhat more involved. It is subject to the next example.

We define a vertex set B to be *k-inseparable* if it has at least $k+1$ elements and if for every vertex set C where $NC \leq k$, either $B \subset C \cup NC$ or $B \subset C^* \cup NC$. For example, the vertices of a $k+1$-connected graph are $k+1$-inseparable.

Example 2: Let X be a connected finite graph and let κ be the smallest integer such that there are vertex sets B_1, B_2, C where $NC = \kappa$, $B_1 \subset C \cup NC$ and $B_2 \subset C^* \cup NC$. Roughly speaking, we require C to separate two large sets while NC is small. Define $\mathcal{C}$ as the set of all C which separate two κ-inseparable vertex sets as described above. The assertion is that $\mathcal{C}$ is a cut system.

Let C, D be in $\mathcal{C}$. There are two pairs B_1, B_2 and B_3, B_4 of κ-inseparable vertex sets, separated by NC respectively ND. Each of these sets B_i uniquely determines a corner A_i of C, D, as it is κ-inseparable. The mapping assigns to a B_i the corner A_i such that B_i is contained in the union of A_i, its adjacent links and the center. We assert that there are two B_i which are mapped to opposite corners. Suppose the negation, which implies that all B_i lie in one of two adjacent corners (plus links), such that there are no two B_i separated by either NC or ND. Thus, we have, say

$$B_1 \subset (C \cap D) \cup (C \cap ND) \cup (D \cap NC) \cup (NC \cap ND)$$

and

$$B_2 \subset (C^* \cap D^*) \cup (C^* \cap ND) \cup (D^* \cap NC) \cup (NC \cap ND).$$

By considering the figure, these formulae should be clear. In the figure, together with the sets, their cardinalities are displayed in lowercase letters. We note that $|N(C \cap D)| \leq a+m+d$ and $|N(C^* \cap D^*)| \leq b+m+c$. With $\kappa = |NC| = b+m+d =$

$|ND| = a + m + c$, we obviously have $2\kappa = a + b + c + d + 2m$. It follows that either $|N(C \cap D)|$ or $|N(C^* \cap D^*)|$ is lesser than κ or both the numbers equal κ. But by minimality of κ, only the latter case can occur, providing the desired cut (if necessary that cut is not $C \cap D$ or $C^* \cap D^*$ but some of their components). This shows (A2) since C, D were arbitrary, while (A1) holds by the construction of $\mathcal{C}$.

With a given cut system $\mathcal{C}$, we define the set of *separators* $\mathcal{S}$ as elements of the form NC where $C \in \mathcal{C}$. We now say that a cut C *separates sets* A, B if either $A \subset C \cup NC$ and $B \subset C^* \cup NC$ or if these relations hold with A and B interchanged. Also, neither of A, B may be a subset of NC. We say that a separator $s \in \mathcal{S}$ separates A, B if a cut C with $NC = S$ does so. Unless otherwise stated, $\mathcal{S}$ will from now on denote the set of separators whenever a single cut system comes into play.

3. The thin subsystem

The term *thin* is substantial concerning cut systems:

Because the values $|s|$ where $s \in \mathcal{S}$ are all finite, there exist separators with minimal cardinality κ. They are called *thin separators* and the corresponding cuts *thin cuts.*

Lemma 3.1. *If for thin cuts C, D, the corners $C \cap D$ and $C^* \cap D^*$ contain cuts, then $N(C \cap D) = N(C^* \cap D^*) = \kappa$. Furthermore, $NC \cap ND = N(C \cap D) \cap N(C^* \cap D^*)$.*

Proof. Consider the figure. Together with the links, their cardinalities are noted in lowercase letters. The distinct sections of the picture resemble a partition of the graph, elements of it being connected when adjacent in the figure. Both pairs together with the center constitute a partition of either NC or ND, such that $\kappa = a + m + c = b + m + d$, which sums to

$$2\kappa = a + b + c + d + 2m. \tag{1}$$

For the corners, there hold

$$\begin{aligned} |N(C \cap D)| &\leq a + m + d, \\ |N(C^* \cap D^*)| &\leq b + m + c, \\ |N(C^* \cap D)| &\leq c + m + d, \\ |N(C \cap D^*)| &\leq a + m + b. \end{aligned}$$

Because $C \cap D$ and $C^* \cap D^*$ contain cuts, $\kappa \leq a + m + d$ and $\kappa \leq b + m + c$. If one of these inequalities were strict, (1) could no more hold, so this proves the first assertion of the lemma.

Now we see that all elements of the center must be contained in both $N(C \cap D)$ and $N(C^* \cap D^*)$, so that the second assertion follows.

We also need to see now that $a = b$, $c = d$. If $a < b$, then also $d < c$ because $a + c = b + d = \kappa - m$. But thus $a + m + d < b + m + c = \kappa$, a contradiction. $b < a$, $c < d$, $d < c$ lead to a contradiction in an analog way.

□

Corollary 3.2. *In a given cut system on a connected graph, the system of thin cuts forms a cut system.*

Proof. We have just shown (A2) for the thin cut subsystem. Axiom (A1) holds, because if a corner A contains a cut and a component K of that corner also does, then by (A1) on the initial cut system, K is a cut. Finally, K is thin because X was connected, so K was disconnected from the rest surely by NA, and so $NK \subset NA$.

□

Given sets A, B in a graph with an assigned cut system, we call a corner of A and B *isolated*, if it does not contain a cut and if its adjacent links are empty. Two sets are *nested* if they have an isolated corner.

Lemma 3.3. *In a graph X with an assigned cut system $\mathcal{C}$, there hold*

- **(a):** *For sets $A \subset B$, $A \cap B^*$ is always empty and an isolated corner.*
- **(b):** *If for thin cuts C, D one of their corners does not contain a cut and one link adjacent to that corner is empty, then also the other link adjacent to the corner is empty (and hence the corner is isolated).*
- **(c):** *For every two thin cuts: either (i) no link is empty, (ii) exactly two links are empty, and they are adjacent to an isolated corner, or (iii) all links are empty and at least one corner is empty.*

Remark: In (c, ii) and (c, iii), the two cuts are nested, in (c, i) they are not.

Proof. (a) That $A \cap B^*$ is empty is trivial. The set $B^* \cap NA$ is empty because $NA \subset NB$ and $B^* \cap NB = \emptyset$. The set $A \cap NB$ is empty for $A \subset B$ and $B \cap NB = \emptyset$.

(b) A corner A of C, D doesn't contain a cut, and so the corners adjacent to A must do, following (A2). In the proof of Lemma 3.1 we have seen that the links of the corners other than those containing a cut are of the same cardinality, so if one of A's links is empty, so is the other.

(c) If not (i), such that some link is empty, then again after the argument from the end of proof of Lemma 3.1, another link must be empty. The corner enclosed by those links cannot contain a cut, for then after (A1), its components were cuts. But these components would have smaller separators than the initial cuts, contradicting the assumption of a thin system. If now a third link is empty, such that also (ii) doesn't hold, the fourth must be empty too. If every corner contained a cut, then one of the initial cuts would have to be disconnected, again contradicting the minimality of its separator.

□

We define a subset E of VX to be a *pre-cut* if it is a cut or if its $*$-complement is a cut. A pre cut is thin if either E or E^* is thin as a cut.

Lemma 3.4. *For thin pre-cuts E, F, there is a cut component of E which contains $E \cap NF$.*

Proof. Axiom (A2) tells us that there is a corner $A \in \{E \cap F, E \cap F^*\}$ such that it and also its opposite contain a cut. So, there is a cut component C in A and it is clear that $NC \subset NA$ (because the graph we are in is connected), and so $NC = NA$ by considering the cuts are thin. Let E' be the component of E which contains C, then E' contains $E \cap NF$, because every vertex in $E \cap F$ is adjacent to C.

□

A cut C is defined to be a *B-cut* if C^* has exactly one cut component, and a thin cut D is defined to be an *A-cut* if it is nested with all other thin cuts. These concepts are connected through the following

Lemma 3.5. *A thin cut C is either a B-cut or an A-cut.*

Proof. Assume that there is another thin cut D with which C is not nested. According to Lemma 3.4, there is a cut component C_0^* of C^* which contains $C^* \cap ND$. Assume now there was another cut component $C_1^* \subset C^*$. The set NC_1^* is a separator, and hence $NC_1^* = NC^*$ (this was argumented several times already). Because $C^* \cap ND \subset C_0$, we have $C_1^* \cap ND = \emptyset$. We also have C, D nested, such that case

(i) in Lemma 3.3 must occur. All links are hence nonempty and we can choose $a \in D \cap NC^*$ and $b \in D^* \cap NC^*$. The sets NC_1^* and C_1^* are connected, and since there is a path from a to b contained in C_1^* (contained except for a and b themselves). Leading from D to D^*, this path has to intersect ND and we have the contradiction $C_1^* \cap ND \neq \emptyset$. □

With S a separator, a component of $VX \setminus S$ which is not a cut is called *slice*. For example, components of isolated corners are slices.

Lemma 3.6. *Let $\mathcal{C}$ a thin cut system. No slice intersects any separator. Distinct slices are disjoint. For a slice Q, no two elements of NQ are separated by a separator of $\mathcal{C}$.*

Proof. We consider the slices $Q_1 \subset VX \setminus NC$ and $Q_2 \subset VX \setminus ND$ for thin cuts C, D. Following Lemma 3.4, the links $C \cap ND, C^* \cap ND$ are contained in cut components of $VX \setminus C$. Slices cannot be cut components, and so $Q_1 \cap ND = \emptyset$ and D was arbitrary.

In particular, slices also do not intersect NQ for every other slice Q. This implies that $Q_1 \cap NQ_2 = Q_2 \cap NQ_1 = \emptyset$ and so $Q_1 = Q_2$ or $Q_1 \cap Q_2 = \emptyset$.

Suppose that $a, b \in NQ$ are separated by a separator NC to get the contradiction $Q \cap NC \neq \emptyset$ by considering a path from a to b within Q. □

The last lemma essentially enables one to forget about slices given a thin system. In explanation, given a graph X with a thin cut system $\mathcal{C}$, we define $\hat{X}$ as the graph that has $V\hat{X} = \{x \in VX \mid x \text{ is not contained in any slice}\}$ and $x, y \in V\hat{X}$ are adjacent if either they are adjacent in VX or $x, y \in NQ$ for a slice Q. We also define $\hat{\mathcal{C}}$ as the set of intersections $C \cap V\hat{X}$, with C running in $\mathcal{C}$. This procedure eliminates slices, conserving the thin cut system, according to

Theorem 3.7. *Given a connected graph X with a thin subsystem $\mathcal{C}$, the graph $\hat{X}$ is again connected and $\hat{\mathcal{C}}$ is a thin cut system. With respect to $\hat{\mathcal{C}}$, there are no slices in $\hat{X}$.*

Proof. Given $x, y \in V\hat{X}$, there is a path π joining the corresponding elements in X. The path can be lifted to $\hat{X}$: If it intersects a slice Q, remove the intersecting elements from π and connect the remaining pieces through NQ.

By Lemma 3.6, we see that the set of separators $\mathcal{S}$ is in $\hat{X}$. Taking an arbitrary component C_0 of $VX \setminus S$ for an $S \in \mathcal{S}$ and assuming C_0 does not intersect any separator, we see it is either a slice or disjoint from all slices, such that $C_0 = \hat{C}_0$. If on the other hand C_0 intersects a separator T, then by Lemma 3.6 applied to X with the thin system $\mathcal{C}$, C_0 contains T, and so also $\hat{C}_0$ contains $T = \hat{T}$. In particular, $C_0 \neq \emptyset$ and $\hat{C}_0$ is not a slice. To summarize, components containing cuts in X with system $\mathcal{C}$ are also in $\hat{\mathcal{C}}$, such that (A1) holds for the new system. Axiom (A2) holds because of the same reason. Finally the system $\hat{\mathcal{C}}$ is thin because is has the same set of separators as $\mathcal{C}$.

□

We define the *almost-inclusion* relation $\subset^a$ on the set of pre-cuts. For pre-cuts E, F, let $E \subset^a F$ if $E \cap F^*$ is an isolated corner (and hence a union of slices). On the set of pre-cuts of $\hat{X}$ as constructed preceding the last theorem, the almost-inclusion is equivalent to the common set inclusion. We may thus, when concerned with thin cut systems, as well assume that there are no slices and study the properties of the ordered set $(\hat{\mathcal{P}}, \subset)$, where $\hat{\mathcal{P}}$ denotes the set of all pre-cuts of the system $\hat{\mathcal{C}}$. This will in the following sections obtain a certain equivalence relation which enables us to re-structurize X into a tree.

See how cuts are now (in the new system) nested if and only if there is an empty corner.

Lastly, note that in general E, F are nested if for some $A, B \in \{E, F, E^*, F^*\}$ there holds $A \subset^a B$. Now in the graph $\hat{X}$, the same is true, just that $\subset^a$ is replaced by the common inclusion.

Lemma 3.8. *Let C, D, E be thin cuts and D, E not nested. Now if C is nested with D, then it is nested with each cut component of either the adjacent corners $D \cap E, D \cap E^*$ or the adjacent corners $D^* \cap E, D^* \cap E^*$. If C is nested with D and E, then it is nested with every cut component of each corner of D, E.*

Proof. It suffices to prove the case when there are no slices. This is for the fact that all assertions we are concerned with can be stated in terms of the relations $\subset^a, \subset$ in the graphs $X, \hat{X}$ respectively, which are equivalent relations. The problem is transformed, then proven in $\hat{X}$ and then the result is transformed back.

We first know that D, E are B-cuts. Also we know that there are no slices in D^* and E^*, such that these sets are (thin) cuts. Now by renaming D to D^* if necessary (note how this leaves the premiss of the lemma unchanged) we may assume either (a) $C \subset D^*$ or (b) $C^* \subset D^*$. If (a), then also $C \subset D^* \cup E$ and $C \subset D^* \cup E^*$. So, C is nested with every cut component of $(D^* \cup E)^* = D \cap E^*$ and with every component of $(D^* \cup E^*)^* = D \cap E$. If (b), the proof works analogously.

In the case that C is nested with both D and E, we may assume, again by renaming if necessary, that (c) $C \subset D^*$ and $C \subset E^*$. Now by the first assertion of the lemma, C is nested with every corner of D, E except yet for $D^* \cap E^*$. But of course, $C \subset D^* \cap E^*$ by (c) and so C is nested with every corner. □

Concluding the section with a remark, the consideration of slices is not necessary when concerned with thin edge cuts, for a minimal edge separator always has exactly two components in its complement, see [2].

4. Optimally nested cuts and k-separators

In a thin cut system $\mathcal{C}$, let $M(C) = \{D \in \mathcal{C} \mid D, C \text{ not nested}\}$ for a thin cut C. Then let $\mu(C)$ denote $|M(C)|$. Actually, we can extend μ on the set of thin pre-cuts to be $\mu(D) = \mu(C_0)$ where D is a pre-cut and C_0 is an arbitrary cut component of D. This is well defined, for if there are distinct cut components in D, then D^* and cut components in D are not B-cuts but thin, hence they are A-cuts and so μ is zero on all these cut components anyway. By the following results, we will have $\mu(C) < \infty$ for all C, which will enable us to consider minima of μ. However the proofs of these results will be omitted for reasons of space. They can be found in [1].

When given a connected graph X, which may be locally-infinite, call a set of vertices S with cardinality k a *minimal k-vertex separator* if $VX \setminus S$ has at least two components which are adjacent to every $x \in S$. Hence, S is minimal with respect to separating the components of $VX \setminus S$. To note, there can of course be k-separators with distinct k. Call components of $VX \setminus S$ *proper* if they are adjacent to all $x \in S$. Then define $a, b \in VX$ to be *separated properly* if they lie in distinct proper components.

Lemma 4.1. *In a connected graph, any two vertices x, y can only be separated properly by finitely many minimal k-vertex-separators for any k.*

The proof is a short induction argument. Note that there is an analog result in [2] dealing with edge cuts.

Now applying this to separators in thin cut systems, which are easily seen to be minimal k-vertex-separators, we get

Proposition 4.2. *Thin pre-cuts can be not nested with only finitely many thin pre-cuts.*

That is, $\mu(C) < \infty$.

Lemma 4.3. *If the opposite corners $C \cap D$ and $C^* \cap D^*$ of not nested thin cuts C, D contain cuts, then*

$$\mu(C \cap D) + \mu(C^* \cap D^*) < \mu(C) + \mu(D)$$

Proof. First it is to be made clear that the left side is well-defined. Note that C, D are B-cuts for they are not nested. Thus all corners are connected and now Lemma 3.1 and (A1) tells us that the corners specified in the assertion of the lemma are indeed thin cuts.

The inequality with $<$ replaced by $\leq$ is easily seen to be true as straightforward application of Lemma 3.8. It is also strict since C is counted on the right in $\mu(D)$, but not on the left, for each set is obviously nested with every corner of itself and an arbitrary second set. □

As mentioned before, one can take the minimum of $\mu(.)$, which will be called m. A thin cut C with $\mu(C) = m$ is defined to be *optimally nested*.

Theorem 4.4. *Optimally nested thin cuts are nested with all other optimally nested thin cuts. The optimally nested thin cuts are a cut system.*

Proof. Assume the optimally nested cuts C, D are not nested with each other. Without loss of generality there are cut components in $C \cap D$ and $C^* \cap D^*$. Lemma 4.3 implies the contradiction

$$\mu(C \cap D) + \mu(C^* \cap D^*) < 2m$$

Now if given two optimally nested cuts C, D, we see they are nested and hence $C \subset D$ without loss of generality. Thus two opposite corners are the cuts C, D themselves, such that (A2) is proved. Axiom (A1) holds because of the argument given already in the proof of Corollary 3.2. □

A cut system, where every two cuts are nested with each other, will be called a *nested cut system*. The particular nested thin cut system of optimally nested cuts constructed above will be denoted $\mathcal{N}$.

5. Structure trees from cut systems

5.1. **Equivalence of cuts.** To decompose a graph into a tree when given the nested, thin cut system derived in Theorem 4.4, in general we need to regard certain equivalence classes as the vertices of a directed tree.

So, let $\mathcal{C}$ be a nested system of thin cuts on the connected graph X. If there are slices, replace X by $\hat{X}$ and $\mathcal{C}$ by $\hat{\mathcal{C}}$.

Now define the relation $\sim$ on the cuts by $C \sim D$ if

- **(i):** $C = D$, or
- **(ii):** $(NC \neq ND)$ and $(C^* \subset D)$ and $(C^* \subset E \subset D$ for $E \in \mathcal{C} \Rightarrow D = E)$

Proposition 5.1. *The relation $\sim$ is an equivalence relation.*

Proof. Reflexitivity is trivial.

Symmetry is easy to see by $C^* \subset D \Leftrightarrow D^* \subset C$ and $E \subset D \Leftrightarrow D^* \subset E^*$.

When given $C \sim D$ and $D \sim E$ with C, D, E distinct elements of $\mathcal{C}$, we have C, E nested and there are no slices, which gives us four cases:

(a) $C \subset E$. This yields $E^* \subset C^* \subset D$ and the contradiction $E^* = C^*$ and therefore $E = C$ by point (ii) of the definition and symmetry.

(b) $C \subset E^*$. This yields $D^* \subset C \subset E^* \subset D$ and the contradiction $D^* \subset D$.

(c) $C^* \subset E^*$. This yields $D^* \subset E \subset C$ by operating $*$ on the assumption of (c) and considering $E^* \subset D$. Now follows the contradiction $E = C$.

(d) $C^* \subset E$. This means either $C \sim E$ or that we have an $A \in \mathcal{C}$ such that $C^* \subset A \subset E$ and $A \neq E$. If assuming the latter, we again have A, D nested and get the four cases

(w) $D \subset A$. This yields $D \subset A \subset E$ which contradicts $D^* \subset E$.

(x) $D^* \subset A$. This yields $D^* \subset A \subset E$ and hence $A = E$. Since $NC \neq NE$ we have transitivity in this case.

(y) $D \subset A^*$. This yields $C^* \subset A \subset D^*$ which contradicts $C^* \subset D$.

(z) $D^* \subset A^*$. This yields $A = D$ by $C^* \subset A \subset D$, and also the contradiction $D \subset E$. □

As a remark, note that if there are cuts containing C^* properly for a cut C, then one of those cuts is already equivalent to C.

Now let $\mathcal{B}$ be the set of equivalence classes of $\mathcal{C}$ with respect to $\sim$. We obtain a directed graph $T'(\mathcal{C})$, where $VT' = \mathcal{B} \cup \mathcal{S}$ and $ET' = \mathcal{C}$.

This is possible by defining for every $C \in ET'$: $o(C) = NC$ and $t(.)$ is the canonical surjection $\mathcal{C} \to \mathcal{B}$. The mappings o and t are called the original and terminal vertex of an edge, respectively. In the resulting graph, each separator has every adjacent edge pointing from it and each equivalence class has its adjacent edges pointing towards it. Thus in particular, every path in T' will be a sequence of alternating arrows.

We derive the graph T'' from T' by adding the *-complements of cuts as edges. Let $ET'' = \mathcal{C} \cup \mathcal{C}^*$ and for $C \in \mathcal{C}$, let $o(C^*) = t(C) = [C]_\sim$ and $t(C^*) = o(C) = NC$. In order to have this always well-defined we take ET'' to be the disjoint union of $\mathcal{C}, \mathcal{C}^*$ and count twice where necessary.

Note that T'' is bipartite. We now consider the undirected graph $T(\mathcal{C})$ corresponding to T''.

Lemma 5.2. *With $\mathcal{C}$ a nested thin cut system on a connected graph, $T(\mathcal{C})$ is a tree.*

Proof. As noted above, T is bipartite, such that a supposed cycle must have length greater or equal to four. Thus, we have a sequence of edges $(E_i)_{1 \leq i \leq n}$ with $n \geq 4$ and n even. Considering the case $E_1 \in \mathcal{C}$, we have $t(E_1) \in \mathcal{B}$. This implies $E_2 \in \mathcal{C}^*$ because $o(E_2) = t(E_1)$ and by induction, $E_{2k} \in \mathcal{C}^*$ and $E_{2k-1} \in \mathcal{C}$ for $k \leq n/2$. For every $i = 2k-1$ with k between 1 and $n/2$, we get $E_i \sim E_{i+1}$ and $E_i \subset E^*_{i+1}$. For i even, we have $E_i \subset E^*_{i+1}$ also, because the sets are distinct, nested and have the same separator.

All E_i are nested, such that we can build a chain $E_1 \subsetneq \theta_2 E_2 \subsetneq \ldots \subsetneq \theta_n E_n \subsetneq \theta_1 E_1$ where the operator θ_i is either $*$ or the identity. By the above assertions, the value of θ is changed every step and $\theta_2 = *$, such that we get the contradiction $\theta_1 = \mathrm{id}$, considering that all E_i have to be distinct.

Of course, the case that the cycle starts in $\mathcal{C}^*$ is completely analogous. We now have T a forest. Taking arbitrary two vertices x, y of T, we have elements $C, D \in \mathcal{C} \cup \mathcal{C}^*$ such that $o(C) = x, t(D) = y$. Either x, y are both in $\mathcal{B}$ or we find elements $x', y' \in \mathcal{B}$ adjacent to x, y. Now for arbitrary vertices u, v in the original graph which lie in arbitrary cuts of the classes x', y' respectively, there are only finitely many separators that properly separate u, v by Lemma 4.1. The subgraph of T generated by the set of these separators contains a path from x', y' and now obviously there is a path from x to y. □

5.2. **Blocks.** There is a refined way to define the set $\mathcal{B}$ (meaning the vertices of T), as is described in the following results. Essentially these elements are maximal sets which cannot be separated by any NC, C a cut. This approach is more constructive and accessible.

Let X be a graph without slices. We say a set of vertices A is $\mathcal{C}$-*inseparable* if $A \subset C \cup NC$ or $A \subset C^* \cup NC$ for every cut C. Now for every such A there is a maximal $\mathcal{C}$-inseparable set that contains A by Zorn's Lemma, they will be called *blocks*. We define for $b \in \mathcal{C}/\sim$: $B(b) = \bigcap_{C \in b} C \cup NC$. This characterizes the set of blocks, by

Lemma 5.3. *The set of blocks is* $\{B(b) \mid b \in \mathcal{C}/\sim\}$. *For* $b \in \mathcal{C}/\sim$, *we have (i)* $\bigcup_{C \in b} NC \subset B(b)$. *With* $C \in b \in \mathcal{C}/\sim$, *the block* $B(b)$ *is the only block* B *such that* $NC \subset B \subset C \cup NC$. *Here, the set* $B(b)$ NC *is always nonempty.*

Proof. Let $b \in \mathcal{C}/\sim$. Assuming $x, y \in B(b)$ are separated by a separator S, we have $S \subset C \cup NC$ for all $C \in b$ because these cuts are nested, and so $S \subset B(b)$. With $C \in b$ still arbitrary, the set C^* is contained in one of the cuts D which satisfy $ND = S$. Now there is a cut D_0 with $C^* \subset D_0 \subset D$ and $D_0 \sim C$. Now either of the elements x or y is not in D_0 and hence also not in $B(b)$, which contradicts the assumption. $B(b)$ is thus inseparable. Any x which is not contained in $B(b)$, is not contained in $C \cup NC$ for some $C \in b$ and so is separated from $B(b)$ by NC. Thus $B(b)$ contains all elements from which it cannot be separated and is thus a block. Every block has the form $B(b)$ for some b by definition. (We have shown that there is a mapping $B(.)$ from $\mathcal{C}/\sim$ to the set of blocks, and that this mapping is bijective.

Now if $C \sim D \in b$ then, by the definition of $\sim$, $C^* \subset D$, $C^* \cup NC \subset D \cup ND$. Thus, $NC \subset B(b)$ and since C was arbitrary, we get (i). We might consider the other case, were there is only one $C \in b$, as a triviality.

We have $B(b) =: B \subset C \cup NC$ immediately from the definition of $B(b)$.

Suppose there is a block B' distinct from B which also satisfies $NC \subset B' \subset C \cup NC$. Because blocks are inseparable, B and B' must be separated by some NC'. The blocks B, B' both contain NC, and so NC' must be NC. But this is impossible since it implies that one block is contained in $C \cup NC$ and the other in $C^* \cup NC$.

In the case where b only contains one cut we have $B(b) \setminus NC \neq \emptyset$ trivially, but if b contains at least the distinct cuts C, D, then we have $NC \neq ND$ because of the definition of $\sim$. With (i), we also have $B(b) \setminus NC \neq \emptyset$ in this case. □

Corollary 5.4. *From this, it is immediate that every block has at least* $\kappa + 1$ *elements (if* $\mathcal{C}$ *is thin).*

Returning to our tree T, we see that we can define the vertices $\mathcal{B}$ to be just the set of all $\mathcal{C}$-blocks instead of equivalence classes $\mathcal{C}/\sim$, for Lemma 5.3 gives a bijection from $\mathcal{C}/\sim$ to the set of $\mathcal{C}$-blocks. The set $\mathcal{B}$ was the set of terminal edges, and now we have $t(C) = B(b(C))$ where $b(C)$ is the $\sim$-class of C and C is a cut.

Remark: [Ideal edges] Given a nested cut system with the above described block structure, we can join every two vertices in a given block to make blocks complete subgraphs, meaning that we can add these edges to EX without changing the cut system. These added edges are referred to as *ideal edges*.

To conclude, we justify the construction of T. The reader may ask why it is necessary to include both seperators and cuts of a cut system on a graph in the set of vertices of T. As stated before, blocks are not necessarily disjoint, so it could occur that in a single seperators, three blocks have nonempty intersection, meaning that if we excluded the seperators from the vertex-set of T, we would have a nontrivial circle in T, destroying the tree structure.

Considering edge cuts, these precautions are not necessary, see [2].

6. G-trees and Extensions of cut systems

In this section, we will state the main theorem, which deals with groups interacting with structure trees.

A *homomorphism of graphs* X, Y is a mapping f from X to Y which maps EX to EY and VX to VY and which respects the incidental structure, meaning that if $e \in EX$, then $o(f(e)) = f(o(e))$ and similarly $tf = ft$. An *endomorphism of a graph* is just a homomorphism that maps a graph to itself.

An *automorphism of a graph* X is a bijective endomorphism on X. We denote the set of all automorphisms of a graph X as $\mathrm{Aut}(X)$. It is obviously a group.

One can think of graph-automorphisms as geometry preserving, for example rotations about a symmetry axis or, similarly, reflections of the graph, while convniently considering the graph as an analytical, not combinatorial object.

Now if we say that a group G *acts on a graph* X, then there shall be a homomorphism $\phi : G \to \mathrm{Aut}(X)$. It is common to omit the indication of ϕ and write gx where it would be precise to write $(\phi(g))(x)$ where $g \in G, x \in X$. For readers not familiar with group actions, it should be mentioned that from the definition follows that $1_G x = x$ and $g(hx) = (gh)x$ for all $x \in X, g, h \in G$ and where 1_G is the neutrum in G.

It is possible to realize a group action on a graph by considering subgroups of its automorphism group, by just identifying the images of group actions with the corresponding subgroups of $\mathrm{Aut}(X)$. We also consider the important case of G acting *transitively* on X, meaning that for every two $x, y \in VX$, there is a $g \in G$ with $gx = y$.

The triple (T, G, ϕ), where T is a tree and G a group which acts on T by ϕ is called a *G-tree.*

We call a cut system $\mathcal{C}$ on a graph X *G-invariant*, where G is a group, if for every $C \in \mathcal{C}$ and $g \in G$ the set $gC := \{gx \mid x \in C\}$ is again a cut.

We will list a few observations about G-trees, including the main theorem.

Proposition 6.1. *If a group acts transitively on a graph with a cut system invariant under that action, there are no slices.*

Proof. Since the action is transitive, an element of a cut can be mapped to a supposed slice, which is a contradiction since the picture of a cut must again be a cut. □

Proposition 6.2. *When $\mathcal{N}$ is a nested cut system on a graph X invariant under G, where G acts on X, then $T(\mathcal{N})$ is a G-tree.*

Proof. Because $\mathcal{N}$ is G-invariant, the quotient mappings $\phi(g)_q$ defined to take a cut C in $\mathcal{N}$ to the cut in which the image of an arbitrary element $x \in C$ lies, are well defined. Now $g \mapsto \phi(g)_q$ is the desired action of G on $T(\mathcal{N})$. □

Theorem 6.3 (Main Theorem)**.** *Let X be a connected graph with a cut system $\mathcal{C}$ invariant under a group action of a group G on X. In $\mathcal{C}$, there is an optimally nested cut system $\mathcal{N}$ which forms the edges of a G-tree $T(C)$.*

Proof. The existence of $\mathcal{N}$ was proved in section 4. We only need that $\mathcal{N}$ again is G-invariant, so that we can apply Proposition 6.2 to complete the proof. For that, we show that gC, as defined above for $C \in \mathcal{N}$ and $g \in G$, is optimally nested. The preimages of cuts with which gC is not nested are cuts that are not nested with C. Because g is a mapping, the preimages have to be at least be as numerous, so that gC cannot be not nested with more cuts than C is. But it can also not be not nested with less cuts since C already is optimally nested. Hence gC is again optimally nested and we have the invariance of $\mathcal{N}$ under G. □

The actions of groups on trees has been substantially investigated, mainly in [3], and together with this so called Bass-Serre theory, vertex cut theory has a number of applications involving group actions. This will be presented here in a later section. We now turn to a few results on the size of $\mathcal{N}$, roughly speaking, we need to make sure that there exist large enough optimally nested cut systems to yield any results.

Theorem 6.4. *Let X be a connected graph and $G := \mathrm{Aut}(X)$. Let $\mathcal{I}$ be the cut system defined in Example 2 in section 2. There is a nested subsystem $\mathcal{N}$ of the system $\mathcal{I}$ invariant under G which satisfies: Whenever two κ-inseparable sets are separated by a cut in $\mathcal{I}$, they are separated by a cut in $\mathcal{N}$. Furthermore, when we denote by T the structure tree generated by $\mathcal{N}$, then distinct maximal κ-inseparable vertex sets in X correspond to distinct vertices in T.*

Remarks: Note how this guarantees a nested system which has as much information on the structure of X as the original cut system, in the sense of not losing any refinement in the block structure when passing to the nested system. Note also that the theorem does not assert that $\mathcal{N}$ is optimally nested. There are examples where $\mathcal{N}$ is not.

Proof. The Main theorem says that there is an optimally nested subsystem $\mathcal{N}_0$ of $\mathcal{I}$ which is invariant under G. If $\mathcal{N}_0$ does not already have the properties stated above, we have κ-inseparable sets Y_1, Y_2 which are separated by a cut in $\mathcal{I}$, but not by any cut in $\mathcal{N}_0$. We define the integer

$$\mu'(C) = |\{D \mid D \in \mathcal{N}_0,\ C, D \text{ not nested}\}|$$

for any cut $C \in \mathcal{I}$. We show that the minimum of μ' on the set of cuts separating Y_1, Y_2 is 0. Suppose it is not, such that there is at least one cut $D \in \mathcal{N}_0$ which is not nested with C, where C is any cut which takes the minimum of μ'. We have, without loss of generality, that $C \cap D$ and $C^* \cap D^*$ contain cut components. It is analogous to Lemma 4.3 that $\mu'(C \cap D) + \mu'(C^* \cap D^*) \leq \mu'(C)$, because if $K \in \mathcal{N}_0$ is not nested with C (or D in general, which is redundant here because $\mathcal{N}_0$ is nested), then it is not nested with at most one of the corners of C, D and if K is nested with C (and D), it is already nested with every corner. Furthermore, we can replace the $\leq$ in the equation by $<$ because D is not nested with C (as $D \in \mathcal{N}_0$), but D is nested with each of the corners it generates together with C. Since this is a contradiction to the minimality of $\mu'(C)$, there cannot exist such a D and we have some cut C separating Y_1, Y_2 with $\mu'(C) = 0$.

Now our C separates vertices in exactly one $\mathcal{N}_0$-block: it does so in at least one by assumption and can do so in at most one because every $\mathcal{N}_0$-separator is also an $\mathcal{I}$-separator. We define now $\mathcal{I}_B$ to be the set of cuts in $\mathcal{I}$ which separate some vertices in the precise block B and assert that this set is a subsystem in $\mathcal{I}$. Axiom (A2) holds because it does for $\mathcal{I}$ and the cuts in the two opposite corners A, B given by (A2) on $\mathcal{I}$ contain elements of a block and therefore cut components which again separate vertices (some subset of those separated by the original cuts) of the precise block B. As for (A1), it holds because X was connected. Because the separators of cuts in $\mathcal{I}_B$ lie in $\mathcal{N}_0$-blocks, they are nested with cuts in $\mathcal{N}$. Cuts in $\mathcal{I}_B$ are also nested with each other, for μ' vanishes on $\mathcal{I}_B$. The same assertions hold for the systems $\mathcal{I}_{gB}$ where $g \in G$ because automorphisms preserve the incidental structure, and furthermore $\mathcal{I}_{gB}$ is nested with $\mathcal{I}_{hB}$ for $g, h \in G$, because gB and hB are distinct or equal. Note that we used that gB is again a block, which follows because $\mathcal{I}$ is G-invariant, and so is thus the set of blocks.

We can add the subsystems $\mathcal{I}_{gB}$ where $g \in G$ (including $g = 1_G$) to the system $\mathcal{N}$, yielding a nested system, which is invariant under G and which has cuts separating

Y_1, Y_2. We can iterate this process and eventually have a nested, G-invariant system which separates every two κ-inseparable sets separated by $\mathcal{I}$. □

There is an analogous result concerning the canonical cut system in infinite graphs. We omit the proof as it also works analogously, just replacing κ- inseparable sets by rays. The cut system of Example 1 in section 2 is referred to as $\mathcal{E}$.

Theorem 6.5. *In a connected graph X, let $G := \mathrm{Aut}(X)$. There is a nested, G-invariant subsystem $\mathcal{N}$ of $\mathcal{E}$, which satisfies the following. Two rays which are separated by a cut $\mathcal{E}$ are also separated by some cut in $\mathcal{N}$. In the structure tree T corresponding to $\mathcal{N}$, two distinct vertices respectively ends belong to separable rays in X, and inseparable rays in X belong to the same vertex respectively end of $\mathcal{T}$.*

We can state a similar, somewhat weaker theorem which is but true under more general assumptions.

Proposition 6.6. *Every cut system in a connected graph has a nested subsystem which separates as many blocks as the original system.*

Proof. We first choose a nested subsystem $\mathcal{N}$ of the given cut system, $\mathcal{C}$, say, such that $\mathcal{N}$ is a maximal element in the family of subsystems of $\mathcal{C}$ with the partial order by inclusion. We can do so by Zorn's Lemma. Now if there were an $\mathcal{N}$-block B, two vertices of which are separated by a cut in $\mathcal{C}$, consider the function

$$\mu'(C) := |\{K \mid K \in \mathcal{N},\ C, K \text{ not nested}\}|,$$

defined on the set of cuts in $\mathcal{C}$ which separate vertices of some $\mathcal{N}$-block. In particular, the domain of μ' excludes $\mathcal{N}$. Choose a minimum C of μ'. Because of the same argument as in the proof of Theorem 6.4, $\mu'(C) = 0$. This is a contradiction since $\mathcal{N}$ was maximal, but C is nested with all cuts in $\mathcal{N}$. □

7. Applications

The existence of nested, G-invariant (vertex) cut systems has various applications in group theory, one of which will be presented in this section, while in the Appendix we introduce a situation where cut theory as it has evolved to date first seems to yield new results, but fails to apply in the end.

Some group theoretic concepts and results of Bass-Serre theory will be introduced as preliminaries.

If G is a group generated by the subset S, then the *Cayley-Graph* $C = \mathrm{Cay}(G, S)$ is a graph with $VC = G$ and $a, b \in EC$ if there is an $s \in S$ with $as = b$. For example, the Cayley-Graph of the integers generated by 1 is a two way infinite path.

It would be a lengthy detour here to prove that the following is well defined. We can speak of the *ends of a finitely generated group* meaning the ends of its Cayley-Graph. It is nontrivial that the number of ends of the Cayley-Graph does not depend on the way we choose a finite generating subset of the group to construct the Cayley-Graph. Referring our example from above, the integers obviously have two ends.

Here we require the reader to be familiar with free groups and presentations. If for four groups G, H, X, Y we have $X < G$, $Y < H$ and X is isomorphic to Y by $\phi : X \to Y$, then we define the *amalgamated free product of G and H over ϕ* by

$$G *_X H = \langle G, H \mid x = \phi(x),\ x \in X\rangle$$

One can picture this as the free group on the union $G \cup H$ where the isomorphic subgroups have been glued together.

If we have instead two isomorphic subgroups X, Y of H, the isomorphism again called ϕ, we can define the *HNN-extension of H over ϕ* as the group

$$\langle H, t \mid txt^{-1} = \phi(x),\ x \in X\rangle,$$

where t is just some element not in H. This is a group fairly larger than H, but in which the isomorphic subgroups X, Y are conjugate isomorphic.

Now a group is said to *split over a subgroup* if it is either an amalgamated free product or an HNN-extension over that subgroup. Stallings showed the following theorem in [4].

Theorem 7.1 (Stallings' Structure Theorem)**.** *A finitely generated group has more than one end if and only if it splits over a finite subgroup.*

Using edge cuts, a short proof of this can be given, which was worked out in [2]. The key to the proof is the action of G on a structure tree of its own Cayley-Graph. Essentially, one applies the following theorem from Bass-Serre theory.

If a group G acts on a graph X, an *edge inversion* is a $g \in G$ for which $gx = y$, $gy = x$ and x, y are adjacent vertices of X. Define the *stabilizer of an edge* e as the set of group elements which leave both vertices of e fixed.

Theorem 7.2. *If G is a group that acts without edge inversion and transitively on an infinite tree, then G splits over the stabilizer of an edge of that tree.*

The proof can be found in various books, but also in [3].

Stallings' Theorem can be restated in terms of the Cayley-Graph of the considered group, as follows.

Theorem 7.3. *Stallings' Structure Theorem is equivalent to the following. A finitely generated group has a Cayley-Graph with more than one end if and only if it splits over a finite subgroup.*

But vertex cut theory provides the following generalization, which essentially takes advantage of the fact that an infinitely generated, hence locally infinite Cayley-Graph can still be cut by finitely many vertices. The proof can be looked up in [1].

Theorem 7.4. *A group has a Cayley-Graph with more than one end if and only if it splits over a finite subgroup.*

The infinitely generated case has to be stated describing the Cayley-Graphs of the group because the number of ends is no more well defined when considering infinite generating systems. For example, every infinite group G with generating set also G will give a complete graph, which has only one end in every sense.

8. Appendix: Kropholler's Conjecture

Here we will investigate how and why cut theory does (yet) fail to apply to a conjecture of 1991, which arose out of Kropholler's work on the Algebraic Torus Theorem for 3-manifolds.

There are yet another ways to define ends of a group, one of which will be subject to this last section. We say an infinite group G has more than one end if there is some $A \subset G$ such that A as well as $G \setminus A$ are infinite and $Ag \setminus A$ is finite for all $g \in G$. At least in finitely generated groups, this is equivalent to the Cayley-Graph having more than one end.

In a graph, let the *distance* $d(x, y)$ of two vertices x, y be the length of the shortest path from x to y. Then the *diameter* of a connected set of vertices is the maximum of $d(.,.)$ on vertex pairs in this set. We also let ∞ be a valid value for the diameter.

Now with H a subgroup of G, we call a subset A of G *(right-)H-finite* if it is covered by finitely many H-right-cosets. (Otherwise, call A (right-)H-infinite.) Left-H-finite respectively infinite are defined analogously.

Let Γ be the Cayley-Graph of G where we choose any finite generating system, then we can define the *left-coset-graph* $X = \Gamma/H$ by setting $V\Gamma = \{gH \mid g \in G\}$ and two left-cosets N, M are adjacent if they have just two elements which are adjacent in Γ. That is, if one can reach any element in M from any element in N by multiplicating any generator on the right (which is adjacency in Γ). Analogously, define the *right-coset-graph* X' but let the vertices be right-cosets instead of left-cosets

A group is said to be *commensurable* with another group if they both appear as subgroups with finite index in some group.

We now introduce Kropholler's Conjecture.

Theorem 8.1. *In a finitely generated group G with subgroup H, let $A \subset G$ with $AH = A$. Now let A and $G \setminus A$ be right-H-infinite while $Ag \setminus A$ is right-H-finite for all $g \in G$. Then, G splits nontrivially (over a group commensurable with H).*

Let us take a look at the dual statement where we just switch sides, roughly speaking. To be precise, replace right-H-[in]finite by left-H-[in]finite. We then have A and $G \setminus A$ infinite vertex sets in the graph X as defined above, while the sets $Ag \setminus A$ are finite in X. Therefore, there is a vertex cut system consisting of the sets Ag, $g \in G$. Now G acts on X and thus there is a G-tree, the structure tree of the cut system. Theorems similar to those of the previous section will apply to provide a nontrivial splitting of G.

The problematic point in 8.1 is that we cannot use the graph X, but rather X' if we want a similar proof. We again have $Ag \setminus A$ finite in X', but X' can be pathologic and need not be a G-graph. This line of argumentation breaks here. Note that this happens because X is derived from the Cayley-Graph in a way consistent with the adjacency in the Cayley-Graph, while X' is not. But in the assertion of 8.1, we still have (i) $Ag \setminus A$ finite, which prevents us from just considering the dual Cayley-Graph and corresponding right-coset-graph X' (where adjacency is defined by multiplication on the left) because (i) involves multiplication on the right.

If we turn to X again instead, we do not necessarily have $Ag \setminus A$ finite, because the sets are right-H-finite. We only have that $Ag \setminus A$ has finite diameter in X, because it consists of only finitely many right-cosets $\{Hg_1, Hg_2, ..., Hg_n\}$, and therefore there is a maximum path length in X from 1_G to g_i for some i.

Now we have some sort of a cut by a small subset considering diameters. There are results on so-called metric ends, which are defined in terms of being separated by finite-diameter sets. However when trying to construct cut systems where the separators are sets of finite diameter, one runs into many problems. Essentially, Lemma 3.1 does not work, because there seems to be no proper way to measure the size of links and center.

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