BACHELORARBEIT

Graphs of Groups
and their
Fundamental Groups

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Abstract:
We discuss graphs of groups and their fundamental groups. Before defining them we take a look at the special cases of amalgamated free products and HNN extensions. We also describe their action on graphs and the structure of the rising factor graphs.

We follow the book [1], Chapters 2.10 - 2.18.

1 Free products

Let A and B be groups such that $A \cap B = \{1\}$. A *normal form* is a sequence $g_1 g_2 \cdots g_n$, where $n \geq 0$, $g_i \in (A \cup B) \setminus \{1\}$ for $1 \leq i \leq n$, and the adjacent factors g_i, g_{i+1} do not lie in the same group A or B. The number $n \in \mathbb{N}$ is called the *length* of the normal form and the normal form of length zero is identified with the identity element. We define a multiplication on the set of normal forms:

Let $x = g_1 g_2 \cdots g_n$, $n \geq 1$, and $y = h_1 h_2 \cdots h_m$, $h \geq 1$, be normal forms. Then

$$x \cdot y := \begin{cases} g_1 \cdots g_n h_1 \cdots h_m & \text{if } g_n \in A, h_1 \in B \text{ or } g_n \in B, h_1 \in A, \\ g_1 \cdots g_{n-1} z h_2 \cdots h_m & \text{if } g_n, h_1 \in A \text{ or} \\ & g_n, h_1 \in B \text{ and } z = g_n h_1 \neq 1, \\ g_1 \cdots g_{n=1} h_2 \cdots h_m & \text{if } g_n, h_1 \in A \text{ or} \\ & g_n, h_1 \in B \text{ and } g_n h_1 = 1. \end{cases}$$

The set of normal forms with this multiplication forms a group, called the *free product* of the groups A and B, denoted by $A * B$.

Proposition 1.1. *Let A and B be subgroups of a group G such that any nontrivial element $g \in G$ can be represented in a unique way as a product $g = g_1 g_2 \cdots g_n$, where $g_i \in (A \cup B) \setminus \{1\}$ for $1 \leq i \leq n$, and the adjacent factors g_i, g_{i+1} do not lie in the same group A or B. Then $G \cong A * B$.*

Theorem 1.2. *Let $A = \langle X \mid R_1 \rangle$, $B = \langle Y \mid R_2 \rangle$ and $X \cap Y = \emptyset$. Then $A * B = \langle X \cup Y \mid R_1 \cup R_2 \rangle$.*

Proof. Denote by $\overline{R_1}$, $\overline{R_2}$ and $\overline{R_1 \cup R_2}$ the normal closures of the sets R_1, R_2 and $R_1 \cup R_2$ in the free groups F_X, F_Y and $F_{X \cup Y}$.

Let $\varphi : F_X \to A$ and $\psi : F_Y \to B$ be homomorphisms with $\ker \varphi = \overline{R_1}$ and $\ker \psi = \overline{R_2}$. Furthermore let $\theta : F_{X \cup Y} \to A * B$ be the homomorphism with $\theta_{|F_X} = \varphi$ and $\theta_{|F_Y} = \psi$. It is sufficient to prove that $\ker \theta = \overline{R_1 \cup R_2}$.

Since θ maps every element of $\overline{R_1 \cup R_2}$ to the identity we only have to show that $\ker \theta \subseteq \overline{R_1 \cup R_2}$.

Let $g = g_1 g_2 \cdots g_n \in \ker \theta$ with $g_i \in (F_X \cup F_Y) \setminus \{1\}$ and the adjacent factors g_i, g_{i+1} do not lie in the same group F_X or F_Y. Since g is an element of the kernel of θ we have $\theta(g) = 1 \in A * B$. To conclude that g lies in the normal closure of $R_1 \cup R_2$ in $F_{X \cup Y}$ we will use induction on the length n of the element g.

For $n = 1$ we see that since $\theta(g) = \theta(g_1) = 1 \in A * B$ either $g_1 \in \overline{R_1}$ or $g_1 \in \overline{R_2}$ and so $g_1 \in \overline{R_1 \cup R_2}$. From $\theta(g) = \theta(g_1)\theta(g_2) \cdots \theta(g_n) = 1 \in A * B$ it follows the existence of an $i \in \{1, \ldots, n\}$ such that $\theta(g_i) = 1$ and as in the case of $n = 1$ we see that $g_i \in \overline{R_1 \cup R_2}$. Now we have

$$\theta(g) = \theta(g_1) \cdots \theta(g_{n-1}\theta(g_{n+1}) \cdots \theta(g_n) = \theta(g_1 \cdots g_{n-1} g_{n+1} \cdots g_n) = 1$$

and conclude that g is an element of $\overline{R_1 \cup R_2}$. □

Remark 1.3. One can also define the free product of groups via universal property, similarly to the definition of free groups.

Let A and B be groups. The group G together with homomorphisms $\varphi_1 : A \to G$ and $\varphi_2 : B \to G$ is called the *free product of A and B* if for all groups H and for the homomorphisms $\psi_1 : A \to H$ and $\psi_2 : B \to H$ there exists an unique homomorphism $\psi : G \to H$ such that $\psi \circ \varphi_i = \psi_i$ for $i \in \{1, 2\}$.

In other words, the following diagram commutes:

$$\begin{tikzcd}
A \arrow[r, "\varphi_1"] \arrow[dr, "\psi_1"'] & G \arrow[d, dashed, "\psi"] & B \arrow[l, "\varphi_2"'] \arrow[dl, "\psi_2"] \\
& H &
\end{tikzcd}$$

Example 1.4.

(i) In Figure 1 we see the free product

$$V_4 * V_4 = \left\langle a, b, c, d \mid a^2 = b^2 = c^2 = d^2 = (ab)^2 = (cd)^2 = 1 \right\rangle,$$

where $V_4 = \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z} = \langle x, y \mid x^2 = y^2 = (xy)^2 \rangle$ denotes the Klein four-group.

(ii) Figure 2 shows the free product $\mathbb{Z}^2 * \mathbb{Z}/2\mathbb{Z}$.

2 Amalgamated free products

Let G and H be groups with $A \leq G$ and $B \leq H$ distinguished isomorphic subgroups and $\varphi : A \to B$ an isomorphism. The *free product of G and H with amalgamation of A and B by the isomorphism φ* is the factor group of the free product $G * H$ by the normal closure of the set $\{\varphi(a)a^{-1} \mid a \in A\}$, which is $\langle G * H \mid a = \varphi(a), a \in A \rangle$, $G *_{A=B} H$ or $G *_A H$.

Remark 2.1. Note that the second and third notation for the amalgamated free product are ambiguous.

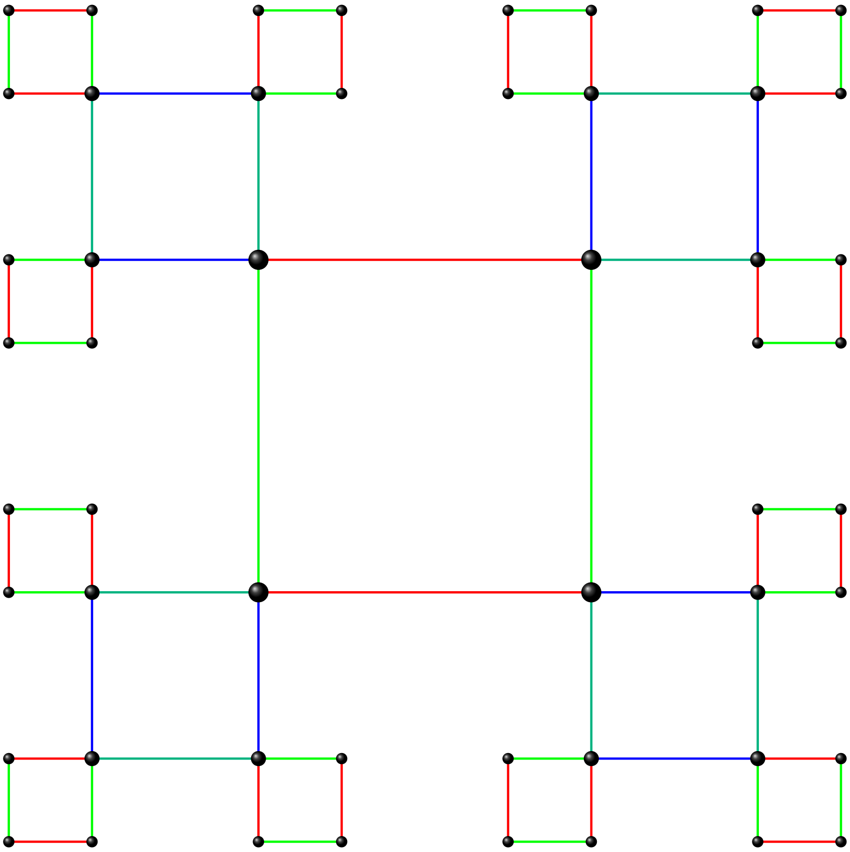


Figure 1: The free product $V_4 * V_4$.

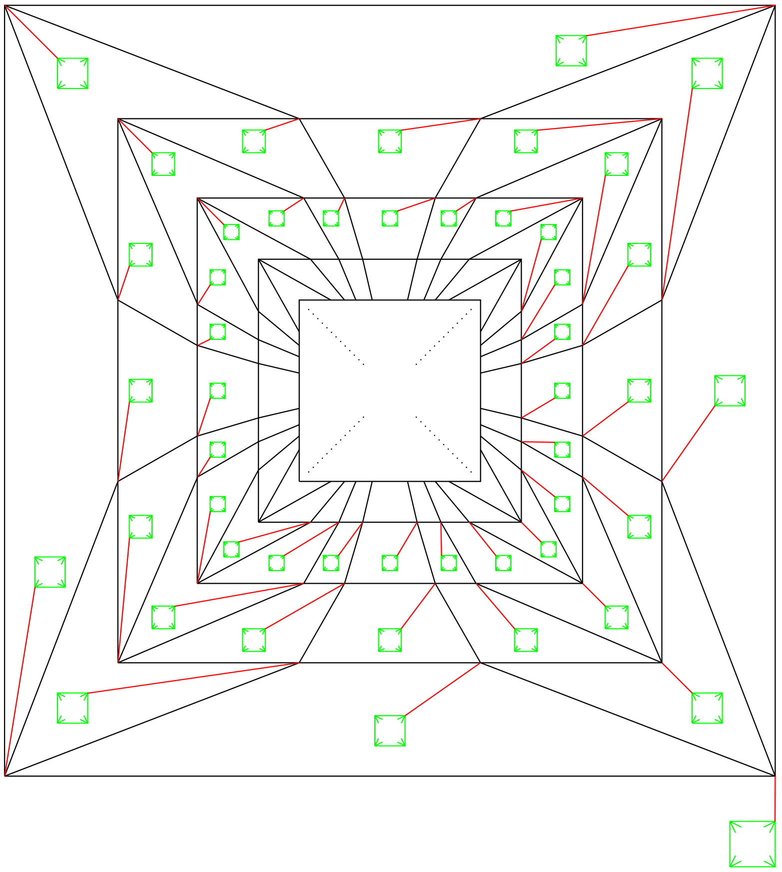


Figure 2: The free product $\mathbb{Z}^2 * \mathbb{Z}_2$.

Example 2.2.

(i) In Figure 3 we see the amalgamated free product

$$V_4 *_{\mathbb{Z}_2} V_4 = \left\langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^2 = 1 \right\rangle .$$

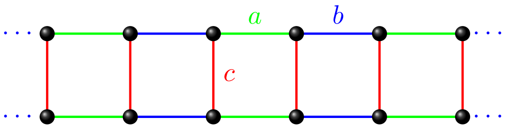


Figure 3: The amalgamated free product $V_4 *_{\mathbb{Z}_2} V_4$.

(ii) Figure 4 shows the amalgamated free product

$$\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z} .$$

Let F be a group and $i : G * H \to F$ be the canonical homomorphism, then every element $f \in F$ can be written as $f = i(x_0)i(x_1)\cdots i(x_n)$, where $x_i \in G \cup H$. In the following we will write this as $f = x_0 x_1 \cdots x_n$.

Now choose systems of representatives T_A of right cosets of A in G and T_B of right cosets of B in H. Then any $g \in G$ can be uniquely written in the from $g = \underline{x}\overline{x}$, where $\underline{x} \in A$, $\overline{x} \in T_A$.

Definition 2.3. An *A-normal form* of an element $f \in F$ is a sequence $(x_0, x_1, \ldots, x_n)$ such that

(i) $x_0 \in A$;

(ii) $x_i \in T_A \setminus \{1\}$ or $x_i \in T_B \setminus \{1\}$ for $i \geq 1$, and the consecutive terms x_i, x_{i+1} lie in distinct systems of representatives.

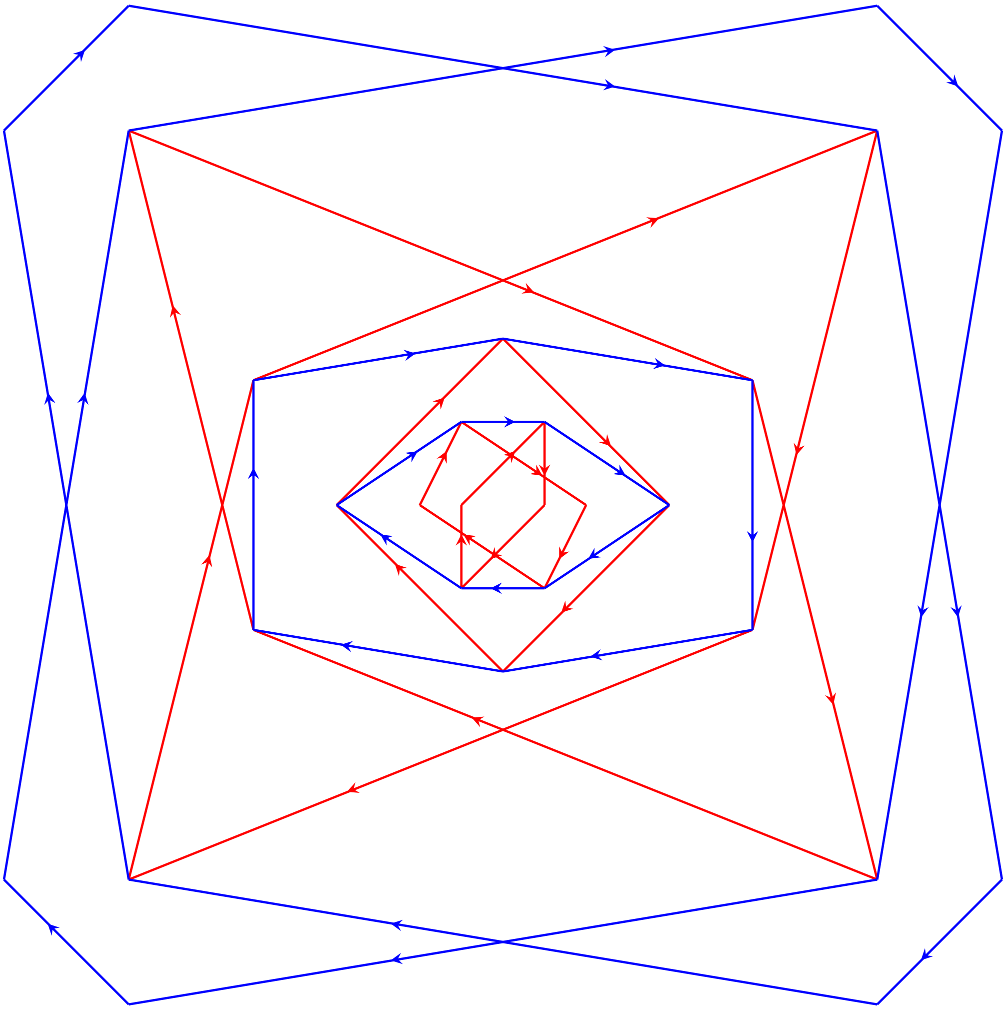


Figure 4: The amalgamated free product $\mathbb{Z}_4 *_{\mathbb{Z}_2} \mathbb{Z}_6$.

Analogously, one can define a *B-normal form.*

Example 2.4. Let $G := \langle a, b \mid a^4 = 1, b^6 = 1, a^2 = b^3 \rangle$ be the free amalgamated product $\mathbb{Z}/4\mathbb{Z} *_{\mathbb{Z}/2\mathbb{Z}} \mathbb{Z}/6\mathbb{Z}$, as in Example 2.2 [(ii)]. Let $T_A = \{1, a\}$ and $T_B = \{1, b, b^2\}$ be systems of representatives of right cosets of A in $\mathbb{Z}_4$ and of B in $\mathbb{Z}_6$, respectively. Then we rewrite the element $f = a^3b^2ab^5 \in G$ as a product of factors which form an A-normal form:

$$f = a^3b^2ab^5 = a^3b^2a^3 \cdot b^2 = a^3b^5 \cdot ab^2 = a^5 \cdot b^2ab^2 = ab^2ab^2.$$

Thus (a, b^2, a, b^2) is an A-normal form for f.

Theorem 2.5. *Any element $f \in G *_{A=B} H$ can be uniquely written in the form $f = x_0x_1 \cdots x_n$, where $(x_0, x_1, \ldots, x_n)$ is an A-normal form.*

Proof. Denote $G *_{A=B} H$ by F. The existence of an A-normal form for an element $f = y_0y_1 \cdots y_n \in F$ follows by induction on n. We rewrite f from the right to the left by forming coset representatives and replacing some elements in A by corresponding elements in B, see Example 2.4.

Now we will prove the uniqueness of the A-normal form of an element. Let W_A be the set of all A-normal forms and W_B be the set of all B-normal froms. Let $\varphi_* : W_A \to W_B$ be the bijection given by $(x_0, x_1, \ldots, x_n) \mapsto (\varphi(x_0), x_1, \ldots, x_n)$.

Define a left action of G on W_A:
For g in G and $\lambda = (x_0, x_1, \ldots, x_n)$ in W_A, where $n \geq 1$, we set

$$g \cdot \lambda := \begin{cases} (gx_0, x_1, \ldots, x_n) & \text{if } g \in A, \\ (\underline{gx_0}, \overline{gx_0}, x_1, \ldots, x_n) & \text{if } g \notin A, x_1 \in H, \\ (gx_0x_1, x_2, \ldots, x_n) & \text{if } g \notin A, x_1 \in G, gx_0x_1 \in A, \\ (\underline{gx_0x_1}, \overline{gx_0x_1}, x_2, \ldots, x_n) & \text{if } g \notin A, x_1 \in G, gx_0x_1 \notin A. \end{cases}$$

Also we define

$$g \cdot (x_0) = \begin{cases} (gx_0) & \text{if } g \in A, \\ (\underline{gx_0}, \overline{gx_0}) & \text{if } g \notin A. \end{cases}$$

Analogously, we can define an action of H on W_B and extend this action to W_A by $h \cdot \lambda = \varphi_*^{-1}(h \cdot \varphi_*(\lambda))$, where $\lambda \in W_A$, $h \in H$. Now we have actions of the groups G and H on W_A. We can extend these to an action of the free product $G * H$ on W_A and since all elements $\varphi(a)a^{-1}$, $a \in A \subseteq G$, lie in the kernel of this action, we have a natural action of the group $F = G *_{A=B} H$ on W_A.

For an element $f = x_0 x_1 \cdots x_n$ in F, where $(x_0, x_1, \ldots, x_n)$ is an A-normal form, we compute the image of $(1) \in W_A$ under the action of f.

$$\begin{aligned} f \cdot (1) &= x_0 x_1 \cdots x_{n-1} \cdot (1, x_n) \\ &= x_0 x_1 \cdots x_{n-2} \cdot (1, x_{n-1}, x_n) \\ &= \ldots \\ &= x_0 \cdot (1, x_1, \ldots, x_{n-1}, x_n) \\ &= (x_0, x_1, \ldots, x_{n-1}, x_n). \end{aligned}$$

So to every element f of the amalgamated free product of G and H by φ corresponds a unique A-normal form.

□

Remark 2.6. We call the expression $x_0 x_1 \cdots x_n$ the *normal form of the element f.*

Corollary 2.7. *Let* $F = G *_{A=B} H$. *Then the canonical homomorphism* $i : G * H \to F$ *induces embeddings of the groups* G *and* H *into* F. *The subgroups* $i(G)$ *and* $i(H)$ *generate the group* F *and their intersection is* $i(A) = i(B)$.

Corollary 2.8. *Let* $F = G *_{A=B} H$. *If* $f \in F$ *and* $f = x_0 x_1 \cdots x_n$, *where $n \geq 1$, and $x_i \in G \setminus A$ or $x_i \in H \setminus A$, depending on the parity of i, then $f \neq 1$.*

3 HNN extensions

Let G be a group and let A and B be subgroups of G with $\varphi : A \to B$ an isomorphism. Also let $\langle t \rangle$ be the infinite cyclic group generated by a new element t. The *HNN extension of G relative to A, B* and φ is the factor group G^* of $G * \langle t \rangle$ by the normal closure of the set $\{t^{-1}at(\varphi(a))^{-1} \mid a \in A\}$.

The group G is called the *base* of G^*, t is the *stable letter*, and A and B are the *associated subgroups*. We can write

$$G^* = \left\langle G, t \mid t^{-1}at = \varphi(a), a \in A \right\rangle .$$

Example 3.1.

(i) Figure 5 shows the HNN extension

$$\mathcal{C}_2^* = \left\langle a, t \mid a^2 = 1, t^{-1}at = a \right\rangle .$$

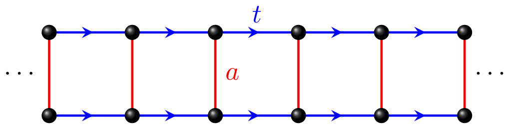


Figure 5: The HNN extension $\mathcal{C}_2^*$.

(ii) The HNN extension of the Klein four-group is

$$V^* = \left\langle a, b, t \mid a^2 = b^2 = 1, t^{-1}at = b \right\rangle .$$

Let $i : G * \langle t \rangle \to G^*$ be the canonical homomorphism, then any element $x \in G^*$ can be written as $x = i(g_0)i(t)^{\epsilon_1}i(g_1) \cdots i(t)^{\epsilon_n}i(g_n)$, where $g_i \in G$ and $\epsilon_j \in \{\pm 1\}$. We write this as $x = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n$.

Now choose systems of representatives T_A of right cosets of A in G and T_B of right cosets of B in G. For an element $g \in G$ we denote by $\overline{g}$ the representative of the coset Ag and by $\underline{g}$ the representative of the coset Bg.

Definition 3.2. A *normal form* of an element $x \in G^*$ is a sequence $(g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n)$ such that

(i) g_0 is an arbitrary element of G

(ii) if $\epsilon_i = -1$, then $g_i \in T_A$

(iii) if $\epsilon_i = 1$, then $g_i \in T_B$

(iv) there is no consecutive subsequence $(t^{\epsilon}, 1, t^{-\epsilon})$.

Example 3.3. Let $G^* = \langle a, b, t \mid t^{-1}a^3t = b^4 \rangle$ be the HNN extension with base $G = F_2 = F(a, b)$ and associated subgroups $A = \langle a^3 \rangle$ and $B = \langle b^4 \rangle$. As systems of representatives T_A of right cosets of A in G we choose the set of all reduced words in F_2 which do not begin with a power of a except possibly a and a^2, and for T_B, the system of representatives of right cosets of B in G, the set of all reduced words in F_2 which do not begin with a power of b except possibly b, b^2 and b^3.

Then we compute the normal form of the element

$$g = b^4t^{-1}a^{-6}tb^5abt^{-1}a^4b^3a \in G$$

from the right: Since $\overline{a^4b^3a} = ab^3a$ and $t^{-1}a^3 = b^4t^{-1}$, we can write

$$g = b^4t^{-1}a^{-6}tb^5ab^5t^{-1}ab^3a.$$

Also $\underline{b^5ab^5} = bab^5$ and $tb^4 = a^3t$, so

$$g = b^4t^{-1}a^{-3}tbab^5t^{-1}ab^3a.$$

Finally with $t^{-1}a^{-3} = b^{-4}t^{-1}$ follows

$$\begin{aligned} g &= t^{-1}tbab^5t^{-1}ab^3a \\ &= bab^5t^{-1}ab^3a. \end{aligned}$$

Thus the sequence (bab^5, t^{-1}, ab^3a) is a normal form of g.

Theorem 3.4. *Let* $G^* = \langle G, t \mid t^{-1}at = \varphi(a), a \in A\rangle$ *be the HNN extension of the group* G *with associated subgroups* A *and* B. *Then the following statements hold:*

(i) Every element x *of* G^* *has a unique representation*

$$x = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n,$$

where $(g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n)$ *is a normal form.*

(ii) The group G *is embedded in* G^* *by the map* $g \mapsto g$. *If*

$$y = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n,\ n \geq 1,$$

and this expression does not contain subwords of the form $t^{-1}g_it$ *with* $g_i \in A$, *or* tg_jt^{-1} *with* $g_j \in B$, *then* $y \neq 1$ *in* G^*.

Remark 3.5. *Ad (i)* This statement was proven by G. Higman, B.H. Neumann and H. Neumann in 1949. Therefore this construction is called an HNN extension. *Ad (ii)* The second part of this statement is known as *Britton's Lemma*, because it was proved by J.L. Britton.

Proof. First we prove the existence of the desired representation of x. We take an arbitrary expression for x as a word in elements of G and t and rewrite it from right to left using the following operations:

(i) forming coset representatives

(ii) replacing $t^{-1}a$ by $\varphi(a)t^{-1}$ if $a \in A$

(iii) replacing tb by $\varphi^{-1}(b)t$ if $b \in B$

Now we prove the uniqueness of such a representation. Let W be the set of all normal froms. We define an action of G^* on W such that the image of $(1) \in W$ under the action of the element x will be equal to its normal form.

Let $\lambda = (g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n) \in W$. We define the actions of the elements $g \in G$, t and t^{-1} on λ by:

$$g \cdot \lambda = (gg_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n);$$

$$t \cdot \lambda = \begin{cases} (\varphi^{-1}(g_0)g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) & \text{if } \epsilon_1 = -1, g_0 \in B, \\ (\varphi^{-1}(b), t, \underline{g_0}, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n) & \text{otherwise,} \end{cases}$$

where b is the element of B such that $g_0 = b\underline{g_0}$;

$$t^{-1} \cdot \lambda = \begin{cases} (\varphi(g_0)g_1, t^{\epsilon_2}, g_2, \ldots, t^{\epsilon_n}, g_n) & \text{if } \epsilon_1 = 1, g_0 \in A, \\ (\varphi(a), t^{-1}, \overline{g_0}, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n) & \text{otherwise,} \end{cases}$$

where a is the element of A such that $g_0 = b\overline{g_0}$.

So we have actions of G on W and $\langle t \rangle$ on W. We can extend these actions to an action of the free product $G * \langle t \rangle$ on W. Let N be the normal closure of the set $\{t^{-1}at\varphi(a)^{-1} \mid a \in A\}$ in $G * \langle t \rangle$, then N acts trivially on W. Hence $G \cap N = \{1\}$ and G is embedded in $G^* = (G * \langle t \rangle)/N$.

Since N lies in the kernel of the action of G on W, the group G^* also acts on W.

Let $x = g_0 t^{\epsilon_1} g_1 \cdots t^{\epsilon_n} g_n \in G^*$, where $(g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n) \in W$ is a normal form. We compute the image of the normal form $(1) \in W$ under the action of x, using the formulas defined above. Since the result is the form $(g_0, t^{\epsilon_1}, g_1, \ldots, t^{\epsilon_n}, g_n) \in W$, the uniqueness of the normal form follows.

To prove the last claim of the second statement we apply the process of rewriting $y \in G^*$ to the normal from. Using the assumptions of y one sees that the corresponding normal from has $2n + 1 \geq 3$ terms. The normal form of the identity element is (1), thus has one term. So we conclude that $y \neq 1 \in G^*$, using the first statement of the theorem.

$\square$

Corollary 3.6. *Let* $G^* = \langle G, t \mid t^{-1}at = \varphi(a), a \in A\rangle$ *be the HNN extension of the group* G *with associated subgroups* A *and* B*. Then the canonical homomorphism* $i : G * \langle t\rangle \to G^*$ *induces embeddings of the groups* G *and* $\langle t\rangle$ *into* G^**. By identifying the subgroups* A *and* B *with their images in* G^* *these groups are conjugate in* G^* *by the element* t *and the restriction to* A *of conjugation by* t *coincides with the isomorphism* φ*.*

$$\begin{tikzcd}
A \arrow[dr, hook] & & B \arrow[dl, hook'] & & & \\
& G \arrow[rr, hook] \arrow[drr, hook, dashed] & & G * \langle t\rangle \arrow[d, "i"] & & \langle t\rangle \arrow[ll, hook'] \arrow[dll, hook', dashed] \\
& & & G* & &
\end{tikzcd}$$

4 Structure Theorem

In this section we investigate the following statement: Let a group G act without inversion of edges on a tree that has no vertices of degree one and suppose G acts transitively on the set of (undirected) edges. If G acts transitively on the tree then G is an HNN extension of the stabilizer of a vertex over the pointwise stabilizer of an edge. If there are two orbits on the vertices of the tree, then G is a free product of the stabilizer of two adjecent vertices with amalgamation over the pointwise stabilizer of an edge.

Let H be a subgroup of a group G. We denote by G/H the set of left cosets of H in G, even if H is not normal in G.

Theorem 4.1. *Let* $G = G_1 *_A G_2$ *be a amalgamated free product. Then there exists a tree* X*, on which* G *acts without inversion of* *edges such that the factor graph* $G \setminus X$ *is a segment. Moreover this segment can be lifted to a segment in* X *with the property that the stabilizers of* G *of its vertices and edges are equal to* G_1, G_2 *and* A *respectively.*

Proof. First we construct a graph X on wich G acts without inversion of edges. Let $VX = G/G_1 \cup G/G_2$ and $EX_+ = G/A$. We define $\iota(gA) = gG_1$, $\tau(gA) = gG_2$. Let T' be the segment in X with vertices G_1, G_2 and positively oriented edge A. The group G acts in X by left multiplication.

We have to show that X is a tree. First we prove that the graph X is connected. It is sufficent to prove that any vertex $gG_1 \in VX$ is connected by a path to the vertex G_1. We write the element $g \in G$ in the from $g = g_1 g_2 \cdots g_n$, where $g_i \in G_1$ or $g_i \in G_2$ depending on the parity of i. The vertices $g_1 \cdots g_{i-1} G_1$ and $g_1 \cdots g_i G_1$ coincide if $g_i \in G_1$ and are connected by edges to the vertex $g_1 \cdots g_{i-1} G_2 (= g_1 \cdots g_i G_2)$ if $g_i \in G_2$. By induction on n we conclude that X is connected.

It is left to prove that X has no circuit. Suppose there exists a closed reduced path $p = e_1 e_2 \cdots e_n$ in X. Applying an appropriate element $g \in G$ to p, we may assume without loss of generality that $\iota(e_1) = G_1$. Since adjacent vertices v_i, v_{i+1} are cosets of different subgroups G_1 or G_2 and p is closed, we conclude that n is even and there exist elements $g_i \in G_1 \setminus A$, $h_i \in G_2 \setminus A$ such that $\iota(e_2) = g_1 G_2$, $\iota(e_3) = g_1 f_1 G_1, \ldots$, $\iota(e_n) = g_1 f_1 \cdots g_{n/2} G_2$, $\tau(e_n) = g_1 f_1 \cdots g_{n/2} f_{n/2} G_1$. But $\tau(e_n) = \iota(e_1) = G_1$ and we obtain a contradiction to the uniqueness of the normal form of an element in $G = G_1 *_A G_2$, Theorem 2.5. □

Theorem 4.2. *Let the group* G *act without inversion of edges on a tree* X *and suppose that the factor graph* $G \backslash X$ *is a segment in* X*. Let* T' *be an arbitrary lift of this segment in* X *and denote its vertices by* v, u *and the edge by* e*. Let* G_v, G_u *and* G_e *be the stabilizers in* G *of these vertices and the edge. Then the homomorphism* $\varphi : G_v *_{G_e} G_u \to G$*, which is the identity mapping on* G_v *and* G_u*, is an* *isomorphism.*

Proof. First we prove that $G = \langle G_v, G_u \rangle$. We define $G' = \langle G_v, G_u \rangle$ and suppose $G' \subsetneqq G$. Since the vertices v and u are not equivalent under the action of G, i.e. $\nexists\, g \in G$ with $v = gu$, we have $g'v \neq gu$, where $g' \in G'$, $g \in G \setminus G'$. Also we have $g'u \neq gv$ and $g'w \neq gw$, where $w \in \{v, u\}$,$g' \in G'$, $g \in G \setminus G'$. So the graphs $G' \cdot T'$ and $(G \setminus G') \cdot T'$ are disjoint. But $X = G \cdot T'$ is a connected graph, and therefore it cannot be represented as the union of two nonempty subgraphs, which is a contradiction to $G' \subsetneqq G$. It is left to prove that the homomorphism φ is injective. Let $G' = G_v *_{G_e} G_u$ and let X' be the tree constructed from G' as in the proof of Theorem 4.1. We define a morphism $\psi : X' \to X$ by $gG_x \mapsto \varphi(g) \cdot x$, where $x \in \{u, v, e\}$, $g \in G'$. As established in the first part of the prove $X = G \cdot T'$ and $G = \langle G_v, G_u \rangle$ so ψ is surjectivity. From the construction of X' follows that all edges e in X' with initial vertex $\iota(e) = gG_1 \in VX'$ have the form gg_1A, where g_1 is an element of the set of representatives of left cosets of A in G_1. Hence the degree of the vertex gG_1, i.e. the number of edges attached to it, is equal to the index $[G_1 : A]$ of A in G_1. The stabilizer of the vertex gG_1 equals gG_1g^{-1}. We can make analogous claims about any edge e in X' with $\iota(e) = gG_2 \in VX'$. The restrictions $\varphi_{|G_v}$ and $\varphi_{|G_u}$ are the identity by definition and so injective. A locally injective morphism from a connected graph to a tree is injective. So ψ is an isomorphism.

Let $g \in G' \setminus G_v$. Then $G_v \in VX'$ and $gG_v \in VX'$ are distinct and therefore $v \in VX$ and $\varphi(g) \cdot v \in VX$ are distinct. Hence $\varphi(g) \neq 1$ and the injectivity of φ is shown. □

Example 4.3. Let $G = \langle a, b, c \mid a^2 = b^2 = c^2 = (ac)^2 = (bc)^2 = 1 \rangle$ be a group.

In Figure 6 we see that the group G acts on the tree X without inversion of edges. The factor graph $G \setminus X$ is a segment. So we see G is an amalgamated free product $V_4 *_{\mathbb{Z}_2} V_4$.

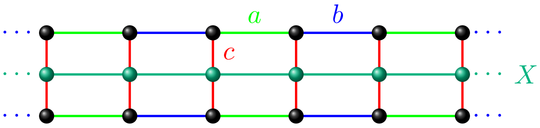


Figure 6: The structure tree X of $V_4 *_{\mathbb{Z}_2} V_4$.

Theorem 4.4. *Let* $G = \langle H, t \mid t^{-1}at = \varphi(a), a \in A\rangle$ *be an HNN extension of the group* H *with the associated subgroups* A *and* $\varphi(A)$*. Then there exists a tree* X *on which* G *acts without inversion of edges such that the factor graph* $G \setminus X$ *is a loop. Moreover, there is a segment* Γ' *in* X *such that the stabilizers of its vertices and edges in the group* G *are equal to* H*,* tHt^{-1} *and* A *respectively.*

Proof. First we contruct a graph X on which G acts by left multiplication. Therefore we set $VX = G/H$, $EX_+ = G/A$, $\iota(gA) = gH$ and $\tau(gA) = gtH$. We define Γ' to be the segment in X with vertices H and tH and positively oriented edge A.

Analogously to the proof of Theorem 4.1 it is left to show that the graph X is a tree, i.e. connected and without circuits. □

Theorem 4.5. *Let a group* G *act without inversion of edges on a tree* X *and let the factor graph* $\Gamma = G \setminus X$ *be a loop. Let* Γ' *be an arbitrary segment in* X*, let* v*,* u*, and* e*,* e^{-1} *be the vertices and the edges of this segment, and let* G_v*,* G_u *and* $G_e = G_{e^{-1}}$ *be the stabilizers of these vertices and edges in the group* G*. Let* $g \in G$ *be an arbitrary element such that* $u = gv$*. Put* $G'_e = g^{-1}G_eg$ *and let* $\varphi : G_e \to G'_e$ *be an isomorphism induced by the conjugation by* g*. Then* G'_e *is a subgroup in* G_v *and the homomorphism*

$$\left\langle G_v, t \mid t^{-1}at = \varphi(a), a \in G_e \right\rangle \to G,$$

which is the identity mapping on the group G_v *and maps* t *to* g*, is an isomorphism.*

The proof is analogous to the proof of Theorem 4.2.

Example 4.6. Let $G = \langle a, b \mid a^2 = abab^{-1} = 1\rangle$ be a group.

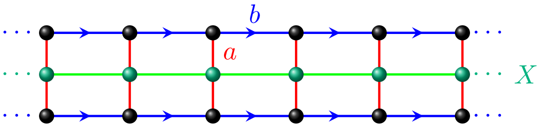


Figure 7: The structure tree X of $\mathcal{C}_2^*$.

In Figure 7 we see that the group G acts on the tree X without inversion of edges. The factor graph $G \setminus X$ is a loop. So we see G is a HNN extension H^* with $H = \mathcal{C}^2$, stable letter b and $\varphi : \mathcal{C}^2 \to \mathcal{C}^2$, mapping $x \mapsto x$.

5 Fundamental groups

5.1 The fundamental group of a graph

Let Γ be a connected graph with a distinguished vertex x. Consider the set $P(\Gamma, x)$ of all path in Γ which beginn and end at x. For any two paths $p = e_1 e_2 \cdots e_n$ and $q = e'_1 e'_2 \cdots e'_m$ in $P(\Gamma, x)$ their product $pq = e_1 e_2 \cdots e_n e'_1 e'_2 \cdots e'_m$ also lies in $P(\Gamma, x)$. We identify the degenerated path x, the empty expression as a product of edges in Γ, with the identity element and define an equivalence relation on $P(\Gamma, x)$ by considering the paths $e_1 \cdots e_i e e^{-1} e_{i+1} \cdots e_n$ and $e_1 \cdots e_i e_{i+1} \cdots e_n$ as equivalent.

More precisely, we say that the paths $p, q \in P(\Gamma, x)$ are *homotopic*, if q can be obtained from p by a finite number of insertions and deletions of subpath of the form ee^{-1}. The set of all paths in $P(\Gamma, x)$ homotopic to $p \in P(\Gamma, x)$, denoted by $[p]$, is called the *homotopy class* of the path p.

By the equivalence relation defined above, the set $P(\Gamma, x)$ is partitioned into homotopy classes. The product of two classes $[p], [q]$ is defined by [p][q]=[pq]. The product of two homotopy classes is well-defined, i.e.does not depend on the choice of represenatives. In each homotopy class there exists only one reduced path.

The set of homotopy classes of paths in $P(\Gamma, x)$ with respect to this multiplication forms a group, called the *fundamental group of the graph* Γ *with respect to the vertex* x, denoted by $\pi_1(\Gamma, x)$.

Remark 5.1. For another vertex $x' \neq x \in V\Gamma$ we have $\pi_1(\Gamma, x) \cong \pi_1(\Gamma, x')$:
The isomorphism

$$f : \pi_1(\Gamma, x) \to \pi_1(\Gamma, x')$$

is given by

$$[p] \mapsto [qpq^{-1}],$$

where q is a fixed path from x to x'.

Choose a maximal subtree T in Γ. For any vertex $v \in V\Gamma$ there exists a unique reduced path in T form x to v, denoted by p_v. Now we associate with each edge $e \in E\Gamma$ the path $p_e = p_{\iota(e)} e p_{\tau(e)}^{-1}$. It follows directly from the definition that $[p_{e^{-1}}] = [p_e]^{-1}$.

Theorem 5.2. *Let* Γ *be a connected graph,* $x \in V\Gamma$ *a vertex in* Γ *and let* T *be a maximal subtree in* Γ*. Then the fundamental group* $\pi_1(\Gamma, x)$ *is a free group with basis* $S = \{[p_e] \mid e \in E\Gamma_+ \setminus ET\}$*, where* $E\Gamma_+$ *is an orientation on* Γ*.*

Proof. Let $p = e_1 e_2 \cdots e_n$ be a closed path in Γ beginning at $x \in V\Gamma$, then $[p] = [p_{e_1}][p_{e_2}] \cdots [p_{e_n}]$ is an representant of the homotopy class of p. Since $[p_e] = 1$ for $e \in ET$, the group $\pi_1(\Gamma, x)$ is generated by the set S.

It is left to prove that the reduced form of elements $[p] \in \pi_1(\Gamma, s)$ with respect to the generating set S is unique:
Let $[p] = [p_{e_1}][p_{e_2}] \cdots [p_{e_n}]$ be an arbitrary reduced form of the element $[p]$ with respect to S. Then $e_i \in E\Gamma \setminus ET$ and $e_{i+1} \neq e_i^{-1}$ for all $i \in \{1, \ldots, n\}$. The path p_{e_i} goes first inside the tree T, then along the edge e_i outside the tree and finally inside the tree T again. So cancellations in the path $p = p_{e_1} p_{e_2} \cdots p_{e_n}$ do not concern the edge e_i of p_{e_i}. Hence the path p is homotopic to a reduced path of the from $t_1 e_1 t_2 e_2 \cdots t_n e_n t_{n+1}$, where the paths t_i are inside the tree T. Since each homotopy class contains only one reduced path, the se-

quence $e_1, e_2, \dots, e_n$ is uniquelly determined by $[p]$. So we conclude that the reduced form of $[p]$ is unique. $\square$

Let $f : \Gamma_1 \to \Gamma_2$ be a morphism of graphs. For any path $p = e_1 e_2 \cdots e_n \in \Gamma_1$, we define the path $f(p) \in \Gamma_2$ by $f(p) = f(e_1) f(e_2) \cdots f(e_n)$.

Lemma 5.3. *Let Γ_1 and Γ_2 be connected graphs and let $f : (\Gamma_1, x_1) \to (\Gamma_2, x_2)$ be a morphism. Then the mapping $f_* : \pi_1(\Gamma_1, x_1) \to \pi_1(\Gamma_2, x_2)$, given by $f_*([p]) = [f(p)]$, is a homomorphism.*

5.2 Graphs of groups and their fundamental groups

Definition 5.4. A *graph of groups* $(\mathbb{G}, \Gamma)$ consists of a connected graph Γ, a *vertex group* G_v for each vertex $v \in V\Gamma$, an *edge group* G_e for each edge $e \in E\Gamma$, and monomorphisms $\{\iota_e : G_e \to G_{\iota(e)} \mid e \in E\Gamma\}$. Additionally we require that $G_e = G_{e^{-1}}$.

Remark 5.5. Instead of $\{\iota_e : G_e \to G_{\iota(e)} \mid e \in E\Gamma\}$ one can use the monomorphisms $\{\tau_e : G_e \to G_{\tau(e)} \mid e \in E\Gamma\}$, defined by $\tau(e) = \iota(e^{-1})$.

Let $F(\mathbb{G}, \Gamma)$ be the factor group of the free product of all vertex groups G_v, $v \in V\Gamma$, and the free group with basis $\{t_e \mid e \in E\Gamma\}$ by the normal closure of the set of elements $t_e^{-1} \iota_e(g) t_e \cdot (\iota_{e^{-1}}(g))^{-1}$ and $t_e t_{e^{-1}}$, where $e \in E\Gamma$, $g \in G_e$.
We will define the fundamental group of a graph of groups in two ways:

(i) with respect to a vertex of the graph Γ;

(ii) with respect to a maximal subtree of the graph Γ;

and show that these definitions yield isomorphic groups.

Definition 5.6. Let $(\mathbb{G}, \Gamma)$ be a graph of groups and let $x \in V\Gamma$

be a vertex of the graph Γ. The *fundamental group* $\pi_1(\mathbb{G}, \Gamma, x)$ *of the graph of groups* $(\mathbb{G}, \Gamma)$ *with respect to the vertex* x is the subgroup of the group $F(\mathbb{G}, \Gamma)$ consisting of all elements of the form $g_0 t_{e_1} g_1 t_{e_2} \cdots t_{e_n} g_n$, where $e_1 e_2 \cdots e_n$ is a closed path in Γ with initial vertex x, $g_0 \in G_x$, $g_i \in G_{\tau(e_i)}$, where $1 \leq i \leq n$.

Definition 5.7. Let $(\mathbb{G}, \Gamma)$ be a graph of groups and let T be the maximal subtree of the graph Γ. The *fundamental group* $\pi_1(\mathbb{G}, \Gamma, T)$ *of the graph of groups* $(\mathbb{G}, \Gamma)$ *with respect to the subtree* T is the factor group of the group $F(\mathbb{G}, \Gamma)$ by the normal closure of the set of elements t_e, where $e \in ET$.

Theorem 5.8. *Let* $(\mathbb{G}, \Gamma)$ *be a graph of groups,* $v \in V\Gamma$ *a vertex of the graph* Γ *and let* T *be a maximal subtree of* Γ. *Let* $p : F(\mathbb{G}, \Gamma) \to \pi_1(\mathbb{G}, \Gamma, T)$ *be the canonical homomorphism. Then* $p_{|_{\pi_1(\mathbb{G},\Gamma,x)}}$, *the restriction of* p *to the subgroup* $\pi_1(\mathbb{G}, \Gamma, x)$*, is an isomorphism onto* $\pi_1(\mathbb{G}, \Gamma, T)$.

Proof. For any vertex $u \in V\Gamma \setminus VT$, there exists a unique reduced path $e_1 e_2 \cdots e_n$ in the tree T from v to u. We denote the corresponding element $t_{e_1} t_{e_2} \cdots t_{e_n}$ of the group $\pi_1(\mathbb{G}, \Gamma)$ by γ_u. According to our definition we set $\gamma_v = 1$.

Let S_T be the set of generators of the group $\pi(\mathbb{G}, \Gamma, T)$ and S_v the set of generators of the group $\pi(\mathbb{G}, \Gamma, v)$. Now we definie a map $q' : S_T \to S_v$ by

$$g \mapsto \gamma_u g \gamma_u^{-1}, \; u \in V\Gamma,$$

$$t_e \mapsto \gamma_{\iota(e)} t_e \gamma_{\tau(e)}^{-1}, \; e \in E\Gamma.$$

We can extend q' to a homomorphism $q : \pi(\mathbb{G}, \Gamma, T) \to \pi(\mathbb{G}, \Gamma, v)$. Also $q \circ p$ and $p \circ q$ are the identities in $\pi(\mathbb{G}, \Gamma, v)$ and $\pi(\mathbb{G}, \Gamma, T)$ respectively . □

Corollary 5.9. *The fundamental groups* $\pi_1(\mathbb{G}, \Gamma, x)$ *and* $\pi_1(\mathbb{G}, \Gamma, T)$ *are isomorphic for any choice of the vertex* x *and any choice of the* *maximal subtree T in the graph Γ.*

We denote the isomorphism class of there groups by $\pi_1(\mathbb{G},\Gamma)$.

Example 5.10.

(i) If the vertex group $G_v = \{1\}$ for all $v \in V\Gamma$, then $\pi_1(\mathbb{G},\Gamma,x) \cong \pi_1(\Gamma,x)$, where $\pi_1(\Gamma,x)$ is the fundamental group of the graph Γ with respect to the vertex $x \in V\Gamma$.

(ii) If Γ is a segment, then the group $\pi_1(\mathbb{G},\Gamma,\Gamma)$ is isomorphic to the free product of the vertex groups G_v and G_u amalgamated over the subgroups $\iota_e(G_e)$ and $\iota_{e^{-1}}(G_e)$.

(iii) If Γ is a loop, then the group $\pi_1(\mathbb{G},\Gamma,v)$ is isomorphic to the HNN extension with base group G_v and associated subgroups $\iota_e(G_e)$ and $\iota_{e^{-1}}(G_e)$.

(iv) Now we determine how to obtain the fundamental group $\pi_1(\mathbb{G},\Gamma,T)$ for an arbitrary graph of groups:

    (a) construct the group $\pi_1(\mathbb{G},T,T)$ from the fundamental group of a segment, for $\mid VT \mid \geq 1$, by successive applications of the construction of an amalgamated product;

    (b) construct the group $\pi_1(\mathbb{G},\Gamma,T)$ from the group $\pi_1(\mathbb{G},T,T)$ by consecutive applications of HNN extensions.

Definition 5.11. Let $(\mathbb{G},\Gamma)$ be a graph of groups and T a fixed maximal subtree in Γ. Also let $g \in G_v$ and $q \in G_u$ for $u,v \in V\Gamma$. Then we call the elements g and q equivalent (with respect to T) if $q = \tau_{e_n}\iota_{e_n}^{-1}\cdots\tau_{e_1}\iota_{e_1}^{-1}(g)$, where $e_1\cdots e_n$ is a path in T from v to u. Note that g is equivalent to g.

If we fix an orientation $E\Gamma_+$ of Γ, then any element $x \in \pi_1(\mathbb{G},\Gamma,T)$ can be written as a product $x = g_1g_2\cdots g_n$, where each g_i belongs to a vertex group of $(\mathbb{G},\Gamma)$ or is equal to $t_e^{\pm 1}$ for $e \in E\Gamma_+ \setminus ET$. Such a product is called *reduced* if

(i) the adjacent elements g_i, g_{i+1} do not lie in the same vertex group;

(ii) it does not contain subwords of the form $t_e t_e^{-1}$ or $t_e^{-1} t_e$;

(iii) it does not contain subwords of the form $t_e^{-1} g t_e$, where g is an element of a vertex group of $(\mathbb{G}, \Gamma)$ equivalent to an element from $\iota_e(G_e)$;

(iv) it does not contain subwords of the form $t_e g t_e^{-1}$, where g is an element of a vertex group of $(\mathbb{G}, \Gamma)$ equivalent to an element from $\tau_e(G_e)$;

Note that if an expression $g_1 g_2 \cdots g_n$ is not reduced, one can shorten it by using the relations in $\pi_1(\mathbb{G}, \Gamma, T)$. So their exists a reduced expression for any element $x \in \pi_1(\mathbb{G}, \Gamma, T)$, but this reduced expression is not unique. In particular an element can have several reduced expressions.

Theorem 5.12. *If an element g of the fundamental group $\pi_1(\mathbb{G}, \Gamma)$ has a reduced expression different from 1, then $g \neq 1$. In particular the groups G_v, $v \in V\Gamma$, can be canonically embedded into the group $\pi_1(\mathbb{G}, \Gamma)$.*

Proof. To prove the statement we use induction on the number of edges of the graph Γ and Example 5.10. The base of induction follows from Corollaries 2.7, 2.8 and Theorem 3.4. ☐

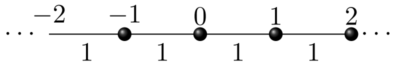
5.3 The relationship between amalgamated free products and HNN extensions

Let $G = \langle H, t \mid t^{-1} a t = \varphi(a), a \in A \rangle$ be an HNN extension and let

$$\begin{array}{rcl} \theta : G & \to & \langle t \rangle \\ t & \mapsto & t, \\ h & \mapsto & 1, \text{ where } h \in H, \end{array}$$

be an epimorphism.

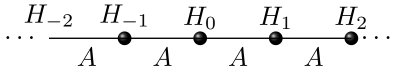
Let $\mathcal{C}^\infty$ be the graph $\mathrm{Cay}(\mathcal{C}^\infty, \{1\})$ of the infinite cyclic group with respect to the generator 1.



To each vertex $n \in V\mathcal{C}^{\infty}$ we associate the group $H_n := \{h_n \mid n \in H\}$, which is the nth copy of the group H, and to each edge we associate the group A.

Next we define embeddings of A, corresponding to an edge $e_n \in E\mathcal{C}^{\infty}$, into the vertex groups H_n and H_{n+1} by

$$\begin{aligned} a &\mapsto (\varphi(a))_n, \\ a &\mapsto a_{n+1}. \end{aligned}$$



The fundamental group $F := \pi_1(\mathbb{G}, \mathcal{C}^{\infty})$ of the graph of groups $(\mathbb{G}, \mathcal{C}^{\infty})$, defined above, has the presentation

$$\langle *_{i\in\mathbb{Z}} H_i \mid a_{n+1} = (\varphi(a))_n, a \in A, n \in \mathbb{Z} \rangle .$$

Let $\langle t \rangle$ be the infinite cyclic group generated by a new element t. We define the semidirect product $F \rtimes \langle t \rangle$ by $t^{-1}h_i t \mapsto h_{i+1}$, $h_i \in H_i$, $i \in \mathbb{Z}$.

Theorem 5.13. *$F \rtimes \langle t \rangle \cong G$.*

Proof. The semidirect product $F \rtimes \langle t \rangle$ is generated by H_0 and t and its relators are the relators of H_0 and $t^{-1}a_0 t = (\varphi(a))_0$, $a \in A$. □

One of the characteristic properties of the semidirect product gives us the desired result $\ker \theta \cong F$, so we conclude that the amalgamated free product F is a normal subgroup of the HNN extension G.

6 The structure of a group acting on a tree

Definition 6.1. Let $p : X \to \Gamma$ be a morphism from a tree X to a connected graph Γ and let T be a maximal subtree in Γ. A pair (T', Γ') of subtrees in X is called a *lift of the pair* of graphs (T, Γ) if $T' \subseteq \Gamma'$ and

(i) each dege $e \in E\Gamma' \setminus ET'$ has the initial or terminal vertex in T';

(ii) the morphism p maps T' isomorphically onto T and $E\Gamma' \setminus ET'$ bijectively onto $E\Gamma \setminus ET$.

For any vertex $v \in V\Gamma$ $(= VT)$ let v' denote its preimage in VT' and for any edge $e \in E\Gamma$ denote by e' its preimage in $E\Gamma$.

We can identify the vertex groups of the graph of groups $(\mathbb{G}, \Gamma)$ with their canonical images in the fundamental group $\pi_1(\mathbb{G}, \Gamma, T)$ - see Theorem 5.12.

Theorem 6.2. *Let $G = \pi_1(\mathbb{G}, \Gamma, T)$ be the fundamental group of a graph of groups $(\mathbb{G}, \Gamma)$ with respect to a maximal subtree T. Then the group G acts without inversion of edges on a tree X such that the factor graph $G \backslash X$ is isomorphic to the graph Γ and the stabilizer of the vertices and edges of the tree X are conjugate to the canonical images in G of the groups G_v, $v \in V\Gamma$, and $\iota_e(G_e)$, $e \in E\Gamma'$, respectively.*

Moreover, for the projection $p : X \to \Gamma$ corresponding to this action, there exists a lift (T', Γ') of the pair (T, Γ) such that

(i) the stabilizer of any vertex $v' \in VT'$ in the group G is equal to the group G_v;

(ii) if the terminal vertex of an edge $e' \in E\Gamma'$ does not lie in VT', then the element t_e^{-1} carries this vertex into VT'.

Proof. First we construct the graph X. We choose an arbitrary orientation of the graph Γ and identify for any vertex $v \in V\Gamma$ the group G_v with its canonical image in G and for any edge $e \in E\Gamma_+$ the group G_e with the canonical image of the subgroup $\iota_e(G_e)$ in the group G. The element $t_e = 1$ in G if and only if $e \in ET$. In the following all unions are disjoint and all cosets are left. We define the graph X by

$$\begin{aligned} VX &= \bigcup_{v \in V\Gamma} G/G_v, \\ EX_+ &= \bigcup_{e \in E\Gamma_+} G/G_e, \\ \iota(gG_e) &= gG_{\iota(e)}, \\ \tau(gG_e) &= gt_eG_{\tau(e)}, \end{aligned}$$

where $g \in G$, $e \in E\Gamma_+$. The group G acts on X by left multiplication. The degree of the vertex gG_v is

$$deg(gG_v) = \sum_{e \in E\Gamma with \iota(e)=v} \mid G_v : \iota_e(G_e) \mid .$$

We define the lift T' of the tree T by

$$\begin{aligned} VT' &= \bigcup_{v \in VT} \{G_v\}, \\ ET'_+ &= \bigcup_{e \in ET_+} \{G_e\}. \end{aligned}$$

The graph Γ' consists additionally to the vertices and edges of the graph T' also of the vertices of the form $t_eG_{\tau(e)}$ and edges G_e, where $e \in E\Gamma_+ \setminus ET_+$, together with their inverses. In order to establish that X is a tree, we have to show that X is connected and that there are no circuits in X. After [1], Chapter 2.18., Theorem 18.2., the proof is analogous to the proof of Theorem 4.1 and Theorem 4.4. $\square$

Lemma 6.3. *Any finite subgroup of the fundamental group* $\pi_1(\mathbb{G}, \Gamma, T)$ *is conjugate to a subgroup of its vertex group.*

Proof. ia Let H be a finite subgroup of the fundamental group $\pi_1(\mathbb{G}, \Gamma, T)$. Then H acts on X and there exists a vertex $v \in X$ with $St_H(v) = H$.
By Theorem 6.2 $St_G(v)$ is conjugate to an image in G of the group G_v. Since $St_H(v)$ is a subgroup of $St_G(v)$ the statement follows. □

Definition 6.4. Let G be a group acting on a tree X without inversion of edges and let $\Gamma = G \setminus X$ be a factor graph, $p : X \to \Gamma$ the canonical projection, T a maximal subtree of Γ and (T', Γ') a lift of the pair (T, Γ).

We define a graph of groups $(\mathbb{G}, \Gamma)$ in the following way. For each vertex and each edge $v \in \Gamma$ we set $G_v = St_G(v')$, where v' is the lift of v. For each edge $e \in E\Gamma \setminus ET$ with $\tau(e') \notin VT'$ we choose an arbitrary element $t_e \in G$ such that $\tau(e') = t_e \tau(e)'$, $\tau(e) \in VT'$. Note that $t_{e^{-1}} = t_e^{-1}$.

For each edge $e \in E\Gamma$ we define an embedding $\tau(e) : G_e \to G_e$ by

$$\tau(e) := \begin{cases} g & \text{if } \tau(e') \in VT', \\ t_e^{-1} g t_e & \text{if } \tau(e') \in V\Gamma' \setminus VT'. \end{cases}$$

Theorem 6.5. *Let a group G act without inversion of edges on a tree X. Then there exists a canonical isomorphism from G onto the group $\pi_1(\mathbb{G}, \Gamma, T)$, defined above in Definition 6.4. This isomorphism extends the identity isomorphism $St\,\mathbb{G}(v') \to G_v$, $v \in V\Gamma$, and carries t_e to t_e, $e \in E\Gamma' \setminus ET$.*

Proof. After [1], Chapter 2.18., Theorem 18.5., the proof is analogous to the proof of Theorem 4.2 and Theorem 4.5. □

Remark 6.6. Let $(\mathbb{G}, \Gamma)$ be a graph of groups and let Γ' be the tree constructed from this graph of groups as in the proof of Theorem 6.2. Any subgroup H of the fundamental group $\pi_1(\mathbb{G}, \Gamma, T)$ acts on

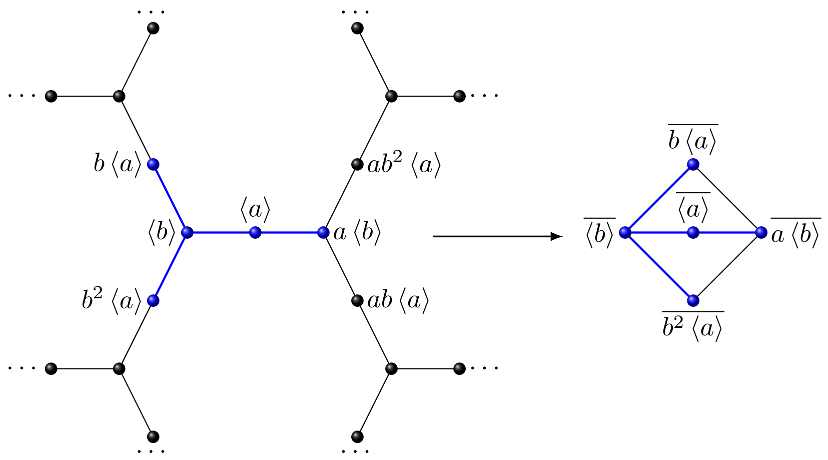


Figure 8: The factor graph $\Gamma = H \setminus X$.

X, and by Theorem 6.5 the subgroup H itself is the fundamental group of a graph of groups.

Example 6.7. Let $G = \langle a, b \mid a^2 = b^3 \rangle$ be the trefoil knot and let

$$\begin{array}{rcl} \varphi : \pi_1(G, \Gamma) & \to & S_3 \\ a & \mapsto & (12), \\ b & \mapsto & (123), \end{array}$$

be a homomorphism from the fuiandamental group of the trefoil knot to the group S_3. Let H denote the kernel of φ.

The group G is the fundamental group of a sement of groups with vertices $\langle a \rangle$, $\langle b \rangle$ and edge $\langle a^2 \rangle = \langle b^3 \rangle$. By Theorem 6.2 we can draw the corresponding tree X. The vertices of X are the left cosets of the cyclic subgroups $\langle a \rangle$ and $\langle b \rangle$ in G, the positively oriented edges are the left cosets of the subgroup $\langle a^2 \rangle$ $(= \langle b^3 \rangle)$ in G and the vertices $g \langle a \rangle$ and $g \langle b \rangle$ are connected by the positively oriented edge $g \langle a^2 \rangle$.

The group H acts on the tree X by left multiplication. The corresponding factor graph $\Gamma = H \setminus X$ is drawn in Figure 8.

The set $\{1, b, b^2, a, ba, b^2a\}$ is a system of representatives of left cosets of H in G. Therefore any vertex of the form $g\langle a\rangle$ is H-equivalent to one of the vertices $\langle a\rangle$, $b\langle a\rangle$ or $b^2\langle a\rangle$ and these vertices are not H-equivalent. Analogously, since $\{1, b, b^2, a, ab, ab^2\}$ is also a system of representatives of left cosets of H in G, any vertex of the form $g\langle b\rangle$ is H-equivalent to $\langle b\rangle$ or $a\langle b\rangle$ and these vertices are not H-equivalent.

So we have 5 euqivalence classes of vertices of the tree X with representatives $\langle a\rangle$, $b\langle a\rangle$, $b^2\langle a\rangle$, $\langle b\rangle$, $a\langle b\rangle$.
Also there are 6 equivalence classes of positively oriented edges of X which are represented by the positively oriented edges from the minimal subtree Γ' containing the vertices $b\langle a\rangle$, $b^2\langle a\rangle$, $ab\langle a\rangle$,$ab^2\langle a\rangle$. We compute

$$\begin{aligned} ab^2a^{-1}b^{-1}\cdot b\langle a\rangle &= ab^2\langle a\rangle, \\ aba^{-1}b^{-1}\cdot b^2\langle a\rangle &= ab\langle a\rangle. \end{aligned}$$

So the vertices $b\langle a\rangle$ and $ab^2\langle a\rangle$ are H-equivalent, as are the vertices $b^2\langle a\rangle$ and $ab\langle a\rangle$. Hence $b\langle a\rangle$ and $ab^2\langle a\rangle$ are projected to the same vertex $\overline{b\langle a\rangle}$ in Γ, and $b^2\langle a\rangle$ and $ab\langle a\rangle$ are projected to the same vertex $\overline{b^2\langle a\rangle}$ in Γ.

Let T be the maximal subtree of the graph Γ containing all vertices and edges of Γ except the edges $(\overline{a\langle b\rangle}, \overline{b\langle a\rangle})$, $(\overline{a\langle b\rangle}, \overline{b^2\langle a\rangle})$ and their inverses. As its lift T' in X we choose the minimal subtree containing the vertices $b\langle a\rangle$, $b^2\langle a\rangle$ and $a\langle b\rangle$. Hence the pair (T', Γ') is a lift of (T, Γ).

The stabilizers of all vertices of T' and all edges of Γ' in the group H are equal to $\langle a^2\rangle$. Since $\langle a^2\rangle$ is the center of the group G we see that all embeddings of edge groups into the corresponding vertex groups are identities.

The fundamental group of the above constructed graph of groups $(\mathbb{G}, \Gamma)$ with respect to the maximal subtree T is isomorphic to H.

Therefore we conclude that H has the presentation

$$\left\langle a^2, t_1, t_2 \mid t_1^{-1} a^2 t_1 = a^2, t_2^{-1} a^2 t_2 = a^2 \right\rangle,$$

where t_1 corresponds to $ab^2a^{-1}b^{-1}$ and t_2 corresponds to $aba^{-1}b^{-1}$. As we saw in the computations above the element $ab^2a^{-1}b^{-1}$ maps the vertex $b\langle a\rangle \in VT'$ to $ab^2\langle a\rangle \in V\Gamma'$ and the element $aba^{-1}b^{-1}$ maps the vertex $b^2\langle a\rangle \in VT'$ to $ab\langle a\rangle \in V\Gamma'$.

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