3 Auctions

3.1. Consider the first-price, sealed-bid auction discussed in class for $n \geq 2$ bidders. Suppose bidders' valuations are identically, independently, and uniformly distributed in the interval $[0, 1]$.

(a) Write the payoffs to bidder $i = 1, \ldots, n$ as a function of i's valuation, denoted ν_i, and the vector of bids given by $(b_1, \ldots, b_n)$.

(b) What is the probability that bidder i wins the auction with bid b_i, if all opponent bidders follow the same bidding function $s(\nu)$? What is the expected payoff to bidder i with bid b_i?

(c) Argue why in equilibrium $s(\nu)$ should be strictly increasing and continuous?

(d) Show that in a symmetric equilibrium with the bidding function is given by

$$s(\nu) = \left(1 - \frac{1}{n}\right) \nu$$

What is the bidding function for the case of $n = 2$ bidders?

(e) Compute the expected revenue to the seller. How do revenues change as the number of bidders increases from $n = 2$ to $n = 3$, and to $n = 4$ bidders? Explain.

3.2. Consider the second-price, sealed-bid auction discussed in class. Assume now $n = 2$ and bidders' valuations are identically, independently, and uniformly distributed in the interval $[0, 1]$.

(a) Write the payoffs to bidder $i = 1, \ldots, n$ as a function of i's valuation, denoted ν_i, and the vector of bids given by $(b_1, \ldots, b_n)$.

(b) Show that it is a weakly dominant strategy for any bidder $i = 1, 2$ to bid according to $s(\nu_i) = \nu_i$ (i. e. bid own valuation).

(c) Argue why it is an equilibrium that all players follow $s(\nu) = \nu$.

(d) Show that the following strategies also constitute an equilibrium: $s_i(\nu_i) = 1$ for all ν_i and $s_j(\nu_j) = 0$ for all ν_j with $i, j = 1, 2$ and $j \neq i$.

(e) Argue carefully why the following is not an equilibrium: $s_i(\nu_i) = b$ for all ν_i with $0 < b < 1$ and $s_j(\nu_j) = 0$ for all ν_j with $i, j = 1, 2$ and $j \neq i$.

(f) Compute expected revenue to the seller and compare to (1e).

3.3. In the so called *all pay auction* $n \geq 2$ bidders make bids $(b_1, \ldots, b_n)$. The object goes to the highest bidder (alternatively to any of the highest bidders with equal probability in case of ties), but all bidders must pay their bids. Assume bidders' valuations are independently and identically distributed with distribution functions $F(\nu)$ and density $f(\nu)$ with support in the interval $[0, 1]$.

(a) Write the payoffs to bidder $i = 1, \ldots, n$ as a function of i's valuation, denoted ν_i, and the vector of bids given by $(b_1, \ldots, b_n)$.

(b) Suppose that all bidders use the same bidding function, $s(\nu)$, assumed continuous and strictly increasing. Compute the expected payoff to any bidder i with bid b_i.

(c) Show that in a symmetric equilibrium the bidding functions is given by

$$s(\nu) = (n - 1) \int_0^{\nu} x f(x) \left(F(x)\right)^{n-2} dx$$

Compare this result with that for the first-price, sealed-bid auction analyzed in class.