2 Bayesian games

2.1. There are five cards on the table. A single card is drawn randomly. Player 1 is told whether it is a (J)ack, a (Q)ueen, or a (K)ing. Player 2 is told whether it is Hearts (♡), Clubs (♣), Spades (♠), or Diamonds (♢).

(a) Show that the following beliefs are consistent.

	♡	♣	♠	♢
$p_1(\cdot\|J)$	1	0	0	0
$p_1(\cdot\|Q)$	0.5	0	0.5	0
$p_1(\cdot\|K)$	0.5	0	0	0.5

	J	Q	K
$p_2(\cdot\|♡)$	1/3	1/3	1/3
$p_2(\cdot\|♣)$	1/3	2/3	0
$p_2(\cdot\|♠)$	0	1	0
$p_2(\cdot\|♢)$	0	0	1

(b) Show that the following beliefs are not consistent.

	♡	♣	♠	♢
$p_1(\cdot\|J)$	1	0	0	0
$p_1(\cdot\|Q)$	0.5	0	0.5	0
$p_1(\cdot\|K)$	0.5	0	0	0.5

	J	Q	K
$p_2(\cdot\|♡)$	0	0.5	0.5
$p_2(\cdot\|♣)$	1	0	0
$p_2(\cdot\|♠)$	0.5	0.5	0
$p_2(\cdot\|♢)$	0	0	1

2.2. Consider the problem of public good provision described in class with two players. Each player must decide whether to (C)ontribute or (N)ot and payoffs are given in the following payoff matrix.

		Player 2: C	Player 2: N
Player 1	C	$1-c_1, 1-c_2$	$1-c_1, 1$
	N	$1, 1-c_2$	$0, 0$

Suppose player 1 may have cost $c_1 \in \{0.75, 1.5\}$ and player 2 may have cost $c_2 \in \{0, 0.75\}$. Each player observes her own cost, but not the opponent's cost. Let $0 < p < 1$ be the probability that player $i = 1, 2$ attaches to player $j \neq i$ having cost $c_j = 0.75$; i. e. type 0.75 occurs with probability p for both players and costs are drawn independently.

(a) Explain and sketch Harsanyi's transformation. What would be Nature's "strategy" in this case?

(b) Define Bayesian equilibrium for this game.

(c) Calculate all pure-strategy Bayesian equilibria and explain each of them intuitively.

2.3. Consider the game with the following payoff matrix, where the value of x is only observable to player 2.

		Player 2: A	Player 2: B
Player 1	A	$0, 0$	$1, x$
	B	$1, 1$	$0, 0$

Calculate all pure-strategy Bayesian equilibria in each of the following two cases.

(a) Player 1 believes that $x = 1$ with probability $0 < p < 1$ and $x = -1$ with probability $(1 - p)$.

(b) Player 1 believes that x is uniformly distributed in the interval $[-1, 1]$.

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2.4. Consider the game with the following payoff matrix. The value of x_i is observable to player $i \in \{1,2\}$ only. Suppose that x_1 and x_2 are independently and uniformly distributed in the interval $[-1,1]$.

		Player 2: A	Player 2: B
Player 1	A	$0,0$	$1,x_2$
	B	$x_1,1$	$0,0$

(a) Show that there is no Bayesian equilibrium where one of the players always plays B; i. e. where, for some $i \in \{1,2\}$, $s_i(x_i) = B$ for all $x_i \in [-1,1]$.

(b) Show that the following pair of strategies is a Bayesian equilibrium, (s_1, s_2) with $s_1(x_1) = A$ and

$$s_2(x_2) = \begin{cases} \text{A} & \text{if } x_2 \leq 0 \\ \text{B} & \text{otherwise} \end{cases}$$

(c) Find at least another Bayesian equilibrium.

2.5. Two players compete for a resource. The value that player $i = 1,2$ attaches to the resource is V_i. Players can behave aggressively as Hawks (H), or they can behave peacefully as Doves (D). If two Doves meet, they share the resource 'equally'. If a Dove meets a Hawk, the former gets nothing and the latter gets the resource. If two Hawks meet, each of them gets the good with probability 1/2; a fight imposes a damage of value -2 on both players. The payoff matrix is given by

		Player 2: H	Player 2: D
Player 1	H	$\frac{V_1}{2}-2, \frac{V_2}{2}-2$	$V_1, 0$
	D	$0, V_2$	$\frac{V_1}{2}, \frac{V_2}{2}$

(a) Suppose first that $V_1 = 2$ and V_2 can take two possible values; $V_2 = 2$ with probability $0 < p < 1$ and $V_2 = 6$ with probability $1-p$. The value of the resource for player 2, V_2, is only observable to player 2.

i. Consider this situation as a Bayesian game. Describe the players, actions, types, beliefs, and payoffs.

ii. Sketch Harsanyi's transformation. What is Nature's strategy in this case?

iii. What are the players' pure strategies in this case?

iv. Calculate all pure-strategy Bayesian equilibria of the game.

(b) Suppose now that V_i is independently and uniformly distributed in the interval $[2,6]$ for both players.

i. Show that at a Bayesian equilibrium players will use threshold strategies; i. e. the optimal strategies will be of the form

$$s_i(V_i) = \begin{cases} D & \text{if } V_i < \overline{V}_i \\ H & \text{if } V_i \geq \overline{V}_i \end{cases}$$

except maybe for the optimal action when $V_i = \overline{V}_i$.

ii. Show that the following pair of strategies is a Bayesian equilibrium: $s_2(V_2) = H$ for all V_2 and

$$s_1(V_1) = \begin{cases} D & \text{if } V_1 < 4 \\ H & \text{if } V_1 \geq 4 \end{cases}$$

iii. Find at least another Bayesian equilibrium.

2.6. Explain and illustrate Harsanyi's purification argument using the *matching pennies* game given by

		Player 2: A	Player 2: B
Player 1	A	$1,-1$	$-1,1$
	B	$-1,1$	$1,-1$