V. Lorentzkovarianz, klassische Feldtheorie und Prinzip der kleinsten Wirkung

1) Lorentzkovarianz

Kontravariante und kovariante Vektoren

$V^\mu \to L^\mu{}_\nu V^\nu$ kontravarianter Vektor

$(V_\mu) \equiv (\eta_{\mu\nu} V^\nu) = \begin{pmatrix} V^0 \\ -\vec{V} \end{pmatrix}$ kovarianter Vektor

$$V_\mu \longrightarrow V^\nu L^\lambda{}_\nu \eta_{\lambda\mu} = V_\sigma \eta^{\sigma\nu} L^\lambda{}_\nu \eta_{\lambda\mu} = V_\sigma (\eta L^T \eta)^\sigma{}_\mu$$

$(\eta^{\sigma\nu}) \equiv (\eta_{\sigma\nu})$, jedoch Indexstellung!

$$\boxed{V_\mu \to V_\sigma (L^{-1})^\sigma{}_\mu}$$

Konsistenz:

$V \cdot W$ invariant: $V \cdot W = V^\mu \eta_{\mu\nu} W^\nu =$

$$= V_\nu W^\nu \longrightarrow V_\sigma (L^{-1})^\sigma{}_\nu L^\nu{}_\lambda W^\lambda = V_\sigma \delta^\sigma{}_\lambda W^\lambda =$$

$$= V_\lambda W^\lambda = V \cdot W$$

Sei $\phi(x)$ skalares Feld

$\partial_\mu \equiv \dfrac{\partial}{\partial x^\mu}$ $\Rightarrow$ $\boxed{(\partial_\mu \phi)(x) \text{ transformiert sich wie kovariantes Vektorfeld}}$

Beweis: $(\partial_\mu \phi)(x) \to \partial'_\mu \big(\phi(L^{-1}x')\big) =$

$L^{-1}x' \equiv y$ $\qquad = \dfrac{\partial \phi}{\partial y^\nu} (L^{-1})^\nu{}_\mu = (\partial_\nu \phi)(L^{-1}x')(L^{-1})^\nu{}_\mu$

$\square$

2) Lorentzinvariante Feldgleichungen

a) Klein-Gordon-Gleichung

$\boxed{(\square + m^2)\,\phi(x) = 0}$ Gl. für skalares Feld

d'Alembert-Operator $\square = \eta^{\mu\nu}\partial_\mu\partial_\nu = \left(\dfrac{\partial}{\partial x^0}\right)^2 - \Delta$

Lorentzinvarianz: $\phi(x)$ Lsg. $\Rightarrow$ $\phi(L^{-1}x)$ Lsg.

Beweis: $\left(\eta^{\mu\nu}\partial_\mu\partial_\nu + m^2\right)\phi(L^{-1}x) =$

$$= \left(\eta^{\mu\nu} \frac{\partial y^\lambda}{\partial x^\mu} \frac{\partial y^\sigma}{\partial x^\nu} \frac{\partial}{\partial y^\lambda} \frac{\partial}{\partial y^\sigma} + m^2\right)\phi(y) =$$

$$\cancel{= \left(\eta^{\mu\nu} (L^{-1})^\lambda{}_\mu (L^{-1})^\sigma{}_\nu \frac{\partial}{\partial x^\lambda}\frac{\partial}{\partial y^\sigma} = \left(\eta^{\lambda\sigma}\frac{\partial}{\partial y^\lambda}\frac{\partial}{\partial y^\sigma} + m^2\right) \phi(y) = 0 \quad \square \quad L^{-1}\eta(L^{-1})^T = \eta\right.}$$

$$= \Big(\underbrace{\eta^{\mu\nu} (L^{-1})^\lambda{}_\mu (L^{-1})^\sigma{}_\nu}_{L^{-1}\eta(L^{-1})^T = \eta} \frac{\partial}{\partial y^\lambda}\frac{\partial}{\partial y^\sigma} + m^2\Big)\phi(y) =$$

$$= \left(\eta^{\lambda\sigma} \frac{\partial}{\partial y^\lambda}\frac{\partial}{\partial y^\sigma} + m^2\right)\phi(y) = 0$$

Ebene Wellen: $(\Box + m^2)\, e^{\pm i k\cdot x} = 0$

$\Rightarrow$ Bed. für Lsg. ist $k^2 = m^2 \rightarrow$ Teilchen mit Masse m

$E = \pm\sqrt{m^2 + \vec{k}^2}$

KG-Gl. beschreibt freie Bosonen mit Spin 0.

b) <u>Proca-Gleichung</u>

$V^\mu(x)$ Vektorfeld $\longrightarrow$ freie Vektorbosonen mit Masse $m \neq 0$

$$\boxed{(\Box + m^2) V^\mu - \partial^\mu \partial_\nu V^\nu = 0}$$

$\Rightarrow (\Box + m^2)\partial_\mu V^\mu - \Box\, \partial_\nu V^\nu = m^2 \partial_\mu V^\mu = 0 \Rightarrow \partial_\mu V^\mu = 0$

Ebene Wellen: $\epsilon^\mu e^{\pm i k \cdot x} \quad \Rightarrow \epsilon \cdot k = 0$

Ruhesystem: $k = \begin{pmatrix} m \\ \vec{0} \end{pmatrix} \Rightarrow \epsilon = \begin{pmatrix} 0 \\ \vec{\epsilon} \end{pmatrix}$

Normierung $\epsilon^2 = -1$

Einsetzen von $\epsilon^\mu e^{\pm i k\cdot x}$ in Proca-Gl. $\Rightarrow k^2 = m^2$

Zu jedem k gibt es drei Polarisationen:

$\epsilon_{1,2} = \begin{pmatrix} 0 \\ \vec{\epsilon}_{\perp 1,2} \end{pmatrix}, \quad \vec{\epsilon}_{\perp 1,2} \cdot \vec{k} = 0$

$\epsilon_3 = \frac{1}{m} \begin{pmatrix} |\vec{k}| \\ E \frac{\vec{k}}{|\vec{k}|} \end{pmatrix} \qquad \epsilon_i \cdot \epsilon_j = -\delta_{ij}, \quad \epsilon_i \cdot k = 0$

Lorentzinvarianz: $V^\mu(x)$ Lsg. $\Rightarrow L^\mu{}_\nu V^\nu(L^{-1}x)$ Lsg.

c) Maxwell-Gleichungen

Heaviside-Lorentz-System

$\alpha = \frac{e^2}{4\pi\hbar c}$ (Hendrik Antoon Lorentz)

$$\boxed{\begin{array}{ll} \vec{\nabla}\cdot\vec{E} = \rho & \vec{\nabla}\cdot\vec{B} = 0 \\ \vec{\nabla}\times\vec{E} + \frac{1}{c}\dot{\vec{B}} = 0 & \vec{\nabla}\times\vec{B} - \frac{1}{c}\frac{\partial\vec{E}}{\partial t} = \frac{1}{c}\vec{j} \end{array}}$$

$\vec{E}$ elektrisches Feld, $\vec{B}$ magn. Feld

ρ Ladungsdichte, $\vec{j}$ Stromdichte

$\boxed{\dot{\rho} + \vec{\nabla}\cdot\vec{j} = 0}$ Kontinuitätsgl. (folgt aus MG!)

Potentiale und Eichinvarianz: Lsg. der hom. MG

$\vec{\nabla}\cdot\vec{B} = 0 \Rightarrow \exists\vec{A}: \vec{B} = \vec{\nabla}\times\vec{A}$

$\Rightarrow \vec{\nabla}\times(\vec{E} + \frac{1}{c}\dot{\vec{A}}) = 0 \Rightarrow \exists\phi: \vec{E} + \frac{1}{c}\dot{\vec{A}} = -\vec{\nabla}\Phi$

$$\boxed{\vec{B} = \vec{\nabla}\times\vec{A}, \quad \vec{E} = -\vec{\nabla}\Phi - \frac{1}{c}\dot{\vec{A}}}$$

Phys. Felder sind $\vec{E}, \vec{B}$

Eichtrafo $\boxed{\vec{A}' = \vec{A} + \vec{\nabla}\Lambda, \quad \Phi' = \Phi - \frac{1}{c}\dot{\Lambda}} \Rightarrow \vec{E}, \vec{B}$ unverändert

beliebige Funktion Λ $\rightarrow$ Eichfreiheit

z.B. kann Λ so gewählt werden, dass $\vec{\nabla}\cdot\vec{A}' + \frac{1}{c}\dot{\Phi}' = 0$
Lorenz-Eichung (Ludvig Lorenz)

Lorentz-kovariante Formulierung der MG:

Inhom. MG:

Def. $$\boxed{(F^{\mu\nu}) = \begin{pmatrix} 0 & -E_x & -E_y & -E_z \\ E_x & 0 & -B_z & B_y \\ E_y & B_z & 0 & -B_x \\ E_z & -B_y & B_x & 0 \end{pmatrix}}$$ $$(A^\mu) = \begin{pmatrix} \Phi \\ \vec{A} \end{pmatrix}$$

$$F^{\mu\nu} = -F^{\nu\mu} = \partial^\mu A^\nu - \partial^\nu A^\mu$$

$$F^{ij} = -\varepsilon_{ijk} B_k \qquad i,j = 1,2,3$$

$$\partial_\mu F^{\mu 0} = \vec{\nabla}\cdot\vec{E}$$

$$\partial_\mu F^{\mu j} = \partial_0 F^{0j} - \varepsilon_{ijk} \nabla_i B_k = -\frac{1}{c}\dot{E}_j + (\vec{\nabla}\times\vec{B})_j$$

$$\Rightarrow \boxed{\partial_\mu F^{\mu\nu} = \frac{1}{c} j^\nu}$$ inhomogene MG

$$j^0 = c\rho$$

Homogene MG:

$$\tilde{F}_{\mu\nu} \equiv \frac{1}{2}\varepsilon_{\mu\nu\rho\sigma} F^{\rho\sigma}$$

$\varepsilon_{\mu\nu\rho\sigma}$ total antisymm., $\varepsilon_{0123} = 1$

$$\boxed{(\tilde{F}^{\mu\nu}) = \begin{pmatrix} 0 & B_x & B_y & B_z \\ -B_x & 0 & -E_z & E_y \\ -B_y & E_z & 0 & -E_x \\ -B_z & -E_y & E_x & 0 \end{pmatrix}}$$

$$\tilde{F}^{ij} = -\frac{1}{2}\varepsilon_{ij0k} E_k \cdot 2 = -\varepsilon_{0ijk} E_k = -\varepsilon_{ijk} E_k$$

$$\partial_\mu \tilde{F}^{\mu 0} = -\vec{\nabla}\cdot\vec{B} = 0$$

$$\partial_\mu \tilde{F}^{\mu j} = \partial_0 \tilde{F}^{0j} - \varepsilon_{ijk}\nabla_i E_k = \frac{1}{c}\dot{B}_j + (\vec{\nabla}\times\vec{E})_j = \vec{0}$$

$$\boxed{\partial_\mu \tilde{F}^{\mu\nu} = 0}$$ homogene MG

Transformationsverhalten des 4-Stroms unter Lorentztransformationen:

Betrachten 4-Strom eines Punktteilchens

$z^\mu(\tau)$ Weltlinie eines Punktteilchens mit Ladung q

τ Eigenzeit

$x^0, \vec{x}$ gegeben $\Rightarrow$ Eigenzeit $\bar{\tau}$ durch $x^0 = z^0(\bar{\tau})$ bestimmt

$$\rho(x^0, \vec{x}) = q\,\delta^{(3)}(\vec{x} - \vec{z}(\bar{\tau}))$$

$$\vec{j}(x^0, \vec{x}) = q\,\vec{v}(\bar{\tau})\,\delta^{(3)}(\vec{x} - \vec{z}(\bar{\tau}))$$

$$\boxed{j^\mu(x) = cq \int d\tau \frac{dz^\mu}{d\tau}\,\delta^{(4)}(x - z(\tau))}$$

Beweis: $\int d\tau \frac{dz^0}{d\tau}\,\delta(x^0 - z^0(\tau)) = \int dz^0\,\delta(x^0 - z^0) = 1$

$\Rightarrow$ für j^0 richtig

$$\frac{1}{cq}\vec{j}(x) = \int d\tau \frac{d\vec{z}}{d\tau}\,\delta(x^0 - z^0(\tau))\,\delta^{(3)}(\vec{x} - \vec{z}(\tau)) =$$

$$= \left.\frac{d\vec{z}}{d\tau}\right|_{\bar{\tau}} \frac{1}{\left.\frac{dz^0}{d\tau}\right|_{\bar{\tau}}}\,\delta^{(3)}(\vec{x} - \vec{z}(\bar{\tau})) = \left.\frac{d\vec{z}}{dz^0}\right|_{\bar{\tau}}\,\delta^{(3)}(\vec{x} - \vec{z}(\bar{\tau}))$$

$$= \frac{\vec{v}(\bar{\tau})}{c}\,\delta^{(3)}(\vec{x} - \vec{z}(\bar{\tau})) \qquad \square$$

Transformation $K \xrightarrow{L} K'$: $\quad z' = Lz$

$$j'^{\mu}(x') = cq \int d\tau \, \frac{dz'^{\mu}}{d\tau} \, \delta^{(4)}(x' - \overbrace{Lz(\tau)}^{z'}) =$$

$$= L^{\mu}{}_{\nu} \, cq \int d\tau \frac{dz^{\nu}}{d\tau} \delta^{(4)}(L(L^{-1}x' - z(\tau)))$$

$$\delta^{(4)}(Ay) = \frac{\delta^{(4)}(y)}{|\det A|}$$

A: Matrix

$$\boxed{j'^{\mu}(x') = L^{\mu}{}_{\nu} \, cq \int d\tau \frac{dz^{\mu}}{d\tau} \delta^{(4)}(L^{-1}x' - z(\tau))}$$

$\Rightarrow \boxed{j'^{\mu}(x') = L^{\mu}{}_{\nu} \, j^{\nu}(L^{-1}x')}$ gilt allgemein

MG Lorentz-invariant:

Lsg. $\vec{E}(x), \vec{B}(x)$ in K für $\rho(x), \vec{j}(x)$

$\Leftrightarrow (F^{\mu\nu}(x))$ für $j^{\mu}(x)$

$\Rightarrow \boxed{L^{\mu}{}_{\lambda} L^{\nu}{}_{\sigma} F^{\lambda\sigma}(L^{-1}x') \text{ Lsg. in } K' \text{ für } j'^{\mu}(x')}$

$$F'^{\mu\nu}(x') = L^{\mu}{}_{\lambda} L^{\nu}{}_{\sigma} F^{\lambda\sigma}(L^{-1}x')$$

bestimmt Transformationsverhalten für $\vec{E}, \vec{B}$ unter Lorentztransformationen

2) Prinzip der kleinsten Wirkung

a) Lagrangeformulierung von Feldtheorien

Wirkung $S = \int d^4x \ \mathcal{L}(\phi_a(x), \partial_\mu \phi_a(x))$
$(c=1)$

$\uparrow$ Lagrangedichte, Felder ϕ_a, $a = 1, \ldots, r$

Wirkung muss Lorentzinvariant sein

Lorentzinv. $\mathcal{L} \longrightarrow S$ Lorentzinv.

Umkehrung nicht zwingend, jedoch $\mathcal{L}$ fast immer Lorentzinvariant.

Analogie zur Mechanik:

$\phi_a(x_1^0, \vec{x})$, $\phi_a(x_2^0, \vec{x})$ vorgegeben, $\phi_a(x)$ so zu wählen, dass S Extremum annimmt $(x_1^0 \leq x^0 \leq x_2^0)$

Variation: $\phi_a(x) \rightarrow \phi_a(x) + \eta_a(x)$ mit $\eta_a(x_1^0, \vec{x}) = \eta_a(x_2^0, \vec{x}) = 0$

$\lim_{|\vec{x}| \to \infty} \eta_a(x^0, \vec{x}) = 0$, η_a infinitesimal

$$\delta S = \int_{x_1^0}^{x_2^0} d^4x \left[\mathcal{L}(\phi_a + \eta_a, \partial_\mu \phi_a + \partial_\mu \eta_a) - \mathcal{L}(\phi_a, \partial_\mu \phi_a) \right]$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} \eta_a + \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \partial_\mu \eta_a \right] =$$

$$= \int d^4x \left[\frac{\partial \mathcal{L}}{\partial \phi_a} - \partial_\mu \left(\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} \right) \right] \eta_a = 0 \quad \forall \eta_a$$

$$\boxed{\frac{\partial}{\partial x^\mu} \frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi_a)} - \frac{\partial \mathcal{L}}{\partial \phi_a} = 0 \quad \forall a = 1, \ldots, r}$$ Euler-Lagrange-Feldgleichungen

Klein-Gordon-Gl. von EL-FGl.:

$$\boxed{\mathcal{L} = \frac{1}{2}\left(\partial_\mu\phi\,\partial^\mu\phi - m^2\phi^2\right)}$$

$$\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi)} = \partial^\mu\phi\,, \quad \frac{\partial\mathcal{L}}{\partial\phi} = -m^2\phi \Rightarrow \partial_\mu\partial^\mu\phi + m^2\phi = 0$$

MG-Gl.: $\boxed{\mathcal{L}_M = -\frac{1}{4}F_{\mu\nu}F^{\mu\nu} - \frac{1}{c}j^\mu A_\mu}$

$$\lambda\neq\sigma: \frac{\partial\mathcal{L}}{\partial(\partial_\lambda A_\sigma)} = -F^{\lambda\sigma}\,, \quad \frac{\partial\mathcal{L}}{\partial A_\sigma} = -\frac{1}{c}j^\sigma$$

$$\Rightarrow -\partial_\lambda F^{\lambda\sigma} + \frac{1}{c}j^\sigma = 0$$

b) <u>Noether-Theorem für innere Symmetrien</u>

Innere Symmetrie: $\phi_a \to \phi_a + \alpha\,\Delta\phi_a$

α kontinuierlicher Parameter!

x unverändert

α infinitesimal, unabh. von x

Lässt $\mathcal{L}$ invariant bis auf 4-Divergenz

$$\alpha\Delta\mathcal{L} = \frac{\partial\mathcal{L}}{\partial\phi_a}\alpha\Delta\phi_a + \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\partial_\mu(\alpha\Delta\phi_a) =$$

$$= \alpha\,\partial_\mu\left(\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta\phi_a\right) + \alpha\underbrace{\left[\frac{\partial\mathcal{L}}{\partial\phi_a} - \partial_\mu\frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\right]}_{=0 \text{ wegen EL}}\Delta\phi_a$$

$$= \alpha\,\partial_\mu \mathcal{J}^\mu$$

4-Divergenz

$$\Rightarrow \boxed{\partial_\mu j^\mu = 0 \quad \text{mit} \quad j^\mu = \sum_a \frac{\partial\mathcal{L}}{\partial(\partial_\mu\phi_a)}\Delta\phi_a - \mathcal{J}^\mu}$$

Beispiel: komplexes Skalarfeld

$$\phi = \frac{\phi_1 + i\phi_2}{\sqrt{2}} \quad \text{mit } \phi_{1,2} \text{ reell}$$

$$\mathcal{L} = \frac{1}{2}\left(\partial_\mu \phi_1 \partial^\mu \phi_1 + \partial_\mu \phi_2 \partial^\mu \phi_2 - m^2(\phi_1^2 + \phi_2^2)\right) =$$

$$= \partial_\mu \phi^* \partial^\mu \phi - m^2 \phi^* \phi$$

$$\frac{\partial \mathcal{L}}{\partial(\partial_\mu \phi^*)} = \partial^\mu \phi, \quad \frac{\partial \mathcal{L}}{\partial \phi^*} = -m^2 \phi \quad \Rightarrow \quad \partial_\mu \partial^\mu \phi + m^2 \phi = 0$$

Selbe FG bei getrennter Variation nach ϕ_1, ϕ_2

$\mathcal{L}$ invariant unter $\phi \to e^{i\alpha}\phi$, $\phi^* \to e^{-i\alpha}\phi^*$

$$\alpha \Delta\phi = i\alpha\phi, \quad \alpha \Delta\phi^* = -i\alpha\phi^*$$

$$\boxed{j^\mu = i(\partial^\mu \phi^*)\phi - i\phi^*(\partial^\mu \phi)}$$

$\partial_\mu j^\mu = 0$ kann nachgeprüft werden durch Verwendung der KG-Gl.

c) Hamiltondichte

$$\boxed{\pi_a \equiv \frac{\partial \mathcal{L}}{\partial \dot{\phi}_a}} \quad \text{zu } \phi_a \text{ konjugierter Impuls}$$

$$\boxed{\mathcal{H} = \sum_a \pi_a \dot{\phi}_a - \mathcal{L}} \qquad H = \int d^3x\, \mathcal{H}(x)$$

Beispiel: reelles Skalarfeld $\quad \frac{\partial \mathcal{L}}{\partial \dot{\phi}} = \dot{\phi} \equiv \pi$

$$\boxed{\mathcal{H} = \pi\dot{\phi} - \mathcal{L} = \frac{1}{2}\left(\dot{\phi}^2 + (\vec{\nabla}\phi)^2 + m^2\phi^2\right)}$$

Allgemeine Lsg. der KG-Gl. durch ebene Wellen:

$$\boxed{\phi(x) = \int \frac{d^3k}{(2\pi)^{3/2}\sqrt{2\omega(\vec{k})}}\left[a(\vec{k})e^{-ik\cdot x} + a^*(\vec{k})e^{ik\cdot x}\right]}$$

$$\omega(\vec{k}) \equiv k^0 = \sqrt{m^2+\vec{k}^2}, \quad k^2 = m^2$$

$$H = \frac{1}{2}\int d^3x \int d^3k \int d^3k' \frac{1}{(2\pi)^3\sqrt{2\omega\cdot 2\omega'}} \cdot$$

$$\cdot \Big\{ (-i\omega)\left[a e^{-ik\cdot x} - a^* e^{ik\cdot x}\right](-i\omega')\left[a' e^{-ik'\cdot x} - a'^* e^{ik'\cdot x}\right]$$

$$+ (i\vec{k})\left[a e^{-ik\cdot x} - a^* e^{ik\cdot x}\right]\cdot(i\vec{k}')\left[a' e^{-ik'\cdot x} - a'^* e^{ik'\cdot x}\right]$$

$$+ m^2\left[a e^{-ik\cdot x} + a^* e^{ik\cdot x}\right]\left[a' e^{-ik'\cdot x} + a'^* e^{ik'\cdot x}\right]\Big\}$$

$$= \frac{1}{4}\int d^3k \int d^3k' \frac{1}{\omega}\Big\{\delta^{(3)}(\vec{k}-\vec{k}')\left[2\omega^2 a^*a + 2\vec{k}^2 a^*a + 2m^2 a^*a\right]$$

$$+ \delta^{(3)}(\vec{k}+\vec{k}')\left[a^2(-\omega^2+\vec{k}^2+m^2) + a^{*2}(-\omega^2+\vec{k}^2+m^2)\right]\Big\}$$

$$\Rightarrow \boxed{H = \int d^3k \;\omega\, a^* a}$$