

Gauging Noncommutative Theories

In Memoriam of Wolfgang Kummer

H. Grosse¹

University of Vienna, Faculty of Physics,
Boltzmanngasse 5, A-1090 Vienna, Austria

Michael Wohlgenannt²

Dipartimento di Scienze e Tecnologie Avanzate,
Universita' del Piemonte Orientale

and

I.N.F.N. - Gruppo Collegato di Alessandria
Via V. Bellini 25/G, I-15100 Alessandria, Italy

Abstract

We review some recent developments concerning the renormalizability of noncommutative quantum field theories. And present a consequential formulation of noncommutative gauge theories.

¹harald.grosse@univie.ac.at

²miw@mfn.unipmn.it

1 Introduction

One of us (Harald Grosse) remembers very well the early first interactions with Prof. Wolfgang Kummer. Being at the time a very young student I entered the Rochester congress in 1968, which was held in Vienna and a young Professor from the Technical University approached me and asked me to help with the organization.

Later on I learnt several times from contributions of Wolfgang to solve the ultraviolet problem of weak interactions and from his work on the special axial gauge. After Wolfgang turned to two-dimensional gravity, we had more intensive contacts which led to common publications.

Wolfgang tried in his work to attack the unsolved problems of quantum field theory. My recent work on noncommutative models goes exactly in that direction. Therefore, I am sure he would have enjoyed seeing that the Landau ghost problem can be solved this way and noncommutative gauge models can be formulated and lead to promising new directions. In this contribution, we want to present a brief and biased summary of some of these recent developments which lead to an ongoing project on gauge models.

As already indicated, four dimensional quantum field theory still suffers from severe problems: From infrared and ultraviolet divergences as well as from the divergence of the renormalized perturbation expansion. Despite the impressive agreement between theory and experiments and despite many attempts, these problems are not settled and remain a big challenge for theoretical physics. Furthermore, attempts to formulate a quantum theory of gravity have not yet been fully successful. It is astonishing that the two pillars of modern physics, quantum field theory and general relativity, seem to be incompatible. This convinced physicists to look for more general descriptions: After the formulation of supersymmetry and supergravity, string theory was developed, and anomaly cancellation forced the introduction of six additional dimensions. On the other hand, loop gravity was formulated, and led to spin networks and space-time foams. Both approaches are not fully satisfactory. A third impulse came from noncommutative geometry developed by Alain Connes, providing a natural interpretation of the Higgs effect at the classical level. This finally led to noncommutative quantum field theory, which is the subject of this contribution. It allows to incorporate fluctuations of space into quantum field theory. There are of course relations among these three developments. In particular, the field theory limit of string theory leads to certain noncommutative field theory models (NCFT), and some models

defined over fuzzy spaces are related to spin networks.

The argument that space-time should be modified at very short distances can be traced back to Schrödinger and Heisenberg and even to Riemann's habilitation. Noncommutative coordinates appeared already in the work of Peierls for the magnetic field problem, and are obtained after projecting onto a particular Landau level. Pauli communicated this to Oppenheimer, whose student Snyder [1] wrote down the first deformed space-time algebra preserving Lorentz symmetry. After the development of noncommutative geometry by Connes [2], it was first applied in physics to the integer quantum Hall effect. Gauge models on the two-dimensional noncommutative tori were formulated, and the relevant projective modules over this space were classified. Filk [3] developed Feynman rules for canonically deformed four dimensional scalar field theory, and Doplicher, Fredenhagen and Roberts [4] published their work on deformed spaces. The subject experienced a major boost after one realized that string theory leads to noncommutative field theory under certain conditions [5, 6], and the subject developed very rapidly. However, some unexpected features such as IR/UV mixing arise upon quantization. In 2000 Minwalla, van Raamsdonk and Seiberg realized [7] that perturbation theory for field theories defined on the Moyal plane faces a serious problem. The planar regular contributions ($B = 1$) show the standard singularities which can be handled by a renormalization procedure; B is the number of boundary components. The planar nonregular ($B > 2$) one loop contributions are finite for generic momenta, however they become singular at exceptional momenta. The usual UV divergences are then reflected in new singularities in the infrared, which is called IR/UV mixing. This spoils the usual renormalization procedure: Inserting many such loops to a higher order diagram generates singularities of any inverse power. Without imposing a special structure such as supersymmetry, the renormalizability seems lost.

However, progress was made when one of us (H.G.) and R. Wulkenhaar were able to give a solution of this problem for the special case of a scalar four dimensional theory defined on the Moyal-deformed space \mathbb{R}_θ^4 [8],

$$[x_\mu \star x_\nu] = i\theta_{\mu\nu}, \quad (1)$$

where $\theta_{\mu\nu} = -\theta_{\nu\mu} \in \mathbb{R}$ and \star denotes the Moyal star product

$$(a \star b)(x) := \int d^4y \frac{d^4k}{(2\pi)^4} a(x + \frac{1}{2}\theta \cdot k) b(x+y) e^{iky}. \quad (2)$$

In this contribution, we will concentrate on Moyal-deformed Euclidean spaces only. The IR/UV mixing is taken into account through a modification of the free Lagrangian by adding an oscillator term with parameter Ω . The proof follows ideas of Polchinski. There are indications that a constructive procedure might be possible and give a nontrivial ϕ^4 model [9].

Nonperturbative aspects of NCFT have also been studied in recent years. The most significant and surprising result is that the IR/UV mixing can lead to a new phase denoted as “striped phase” [10], where translational symmetry is spontaneously broken. The existence of such a phase has indeed been confirmed in numerical studies [11, 12]. To understand better the properties of this phase and the phase transitions, further work and better analytical techniques are required, combining results from perturbative renormalization with nonperturbative techniques. Here a particular feature of scalar NCFT is very suggestive: The field can be described as a hermitian matrix, and the quantization is defined nonperturbatively by integrating over all such matrices. This provides a natural starting point for nonperturbative studies. In particular, it suggests and allows to apply ideas and techniques from random matrix theory [13, 14].

Then, we will discuss a formulation of gauge theories related to the approach to NCFT presented here. We start with noncommutative ϕ^4 theory with additional oscillator potential. We couple an external gauge field to the scalar field using covariant coordinates. As in the classical case, we extract the dynamics of the gauge field from the divergent contributions to the 1-loop effective action. The effective action is calculated using a heat kernel expansion [15]. The technical details are presented in [16, 17]. The ongoing project on the BRST quantization of a related model [18] will also be discussed. This is a joint project with the group of M. Schweda at the Vienna University of Technology.

2 Renormalization of ϕ^4 -theory

We briefly sketch the methods used in [8] proving the renormalizability for scalar field theory on \mathbb{R}_θ^4 . The IR/UV mixing was taken into account through a modification of the free Lagrangian, by adding an oscillator term which modifies the spectrum of the free Hamiltonian:

$$S = \int d^4x \left(\frac{1}{2} \partial_\mu \phi \star \partial^\mu \phi + 2\Omega^2 (\tilde{x}_\mu \phi) \star (\tilde{x}^\mu \phi) + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) (x) , \quad (3)$$

where $\tilde{x}_\mu = (\theta^{-1})_{\mu\nu} x^\nu$. The model is covariant under the Langmann-Szabo [19] duality relating short and long distance behavior. At $\Omega = 1$, the model becomes self-dual and connected to integrable models.

The renormalization proof proceeds by using a matrix base, which leads to a dynamical matrix model of the type:

$$S[\phi] = (2\pi\theta)^2 \sum_{m,n,k,l \in \mathbb{N}^2} \left(\frac{1}{2} \phi_{mn} \Delta_{mn;kl} \phi_{kl} + \frac{\lambda}{4!} \phi_{mn} \phi_{nk} \phi_{kl} \phi_{lm} \right), \quad (4)$$

where

$$\begin{aligned} \Delta_{\substack{m^1 & n^1 \\ m^2 & n^2}; \substack{k^1 & l^1 \\ k^2 & l^2}} &= \left(\mu^2 + \frac{2+2\Omega^2}{\theta} (m^1+n^1+m^2+n^2+2) \right) \delta_{n^1 k^1} \delta_{m^1 l^1} \delta_{n^2 k^2} \delta_{m^2 l^2} \quad (5) \\ &\quad - \frac{2-2\Omega^2}{\theta} \left(\sqrt{k^1 l^1} \delta_{n^1+1, k^1} \delta_{m^1+1, l^1} + \sqrt{m^1 n^1} \delta_{n^1-1, k^1} \delta_{m^1-1, l^1} \right) \delta_{n^2 k^2} \delta_{m^2 l^2} \\ &\quad - \frac{2-2\Omega^2}{\theta} \left(\sqrt{k^2 l^2} \delta_{n^2+1, k^2} \delta_{m^2+1, l^2} + \sqrt{m^2 n^2} \delta_{n^2-1, k^2} \delta_{m^2-1, l^2} \right) \delta_{n^1 k^1} \delta_{m^1 l^1}. \end{aligned}$$

The interaction part becomes a trace of product of matrices, and no oscillations occur in this basis. The propagator obtained from the free part is quite complicated; in 4 dimensions it is:

$$\begin{aligned} G_{\substack{m^1 & n^1 \\ m^2 & n^2}; \substack{k^1 & l^1 \\ k^2 & l^2}} &= \frac{\theta}{2(1+\Omega)^2} \left(\frac{1-\Omega}{1+\Omega} \right)^{2v^1+2v^2} \quad (6) \\ &\quad \times \sum_{v^1 = \lfloor \frac{m^1-l^1}{2} \rfloor}^{\frac{m^1+l^1}{2}} \sum_{v^2 = \lfloor \frac{m^2-l^2}{2} \rfloor}^{\frac{m^2+l^2}{2}} B \left(1 + \frac{\mu^2 \theta}{8\Omega} + \frac{1}{2} (m^1+k^1+m^2+k^2) - v^1 - v^2, 1+2v^1+2v^2 \right) \\ &\quad \times {}_2F_1 \left(\begin{matrix} 1+2v^1+2v^2, \frac{\mu^2 \theta}{8\Omega} - \frac{1}{2} (m^1+k^1+m^2+k^2) + v^1 + v^2 \\ 2 + \frac{\mu^2 \theta}{8\Omega} + \frac{1}{2} (m^1+k^1+m^2+k^2) + v^1 + v^2 \end{matrix} \middle| \frac{(1-\Omega)^2}{(1+\Omega)^2} \right) \\ &\quad \times \prod_{i=1}^2 \delta_{m^i+k^i, n^i+l^i} \sqrt{\left(\binom{n^i}{v^i + \frac{n^i-k^i}{2}} \right) \left(\binom{k^i}{v^i + \frac{k^i-n^i}{2}} \right) \left(\binom{m^i}{v^i + \frac{m^i-l^i}{2}} \right) \left(\binom{l^i}{v^i + \frac{l^i-m^i}{2}} \right)}. \end{aligned}$$

The propagator shows asymmetric decay properties, cf. Fig. (1). They decay exponentially on particular directions (in l -direction in Fig. (1)), but have power law decay in others (in α -direction). These decay properties are crucial for the perturbative renormalizability of the models.

The proof follows the ideas of Polchinski [20]. The quantum field theory corresponding to the action (4) is defined by the partition function

$$Z[J] = \int \left(\prod_{m,n} d\phi_{mn} \right) \exp \left(-S[\phi] - \sum_{m,n} \phi_{mn} J_{nm} \right). \quad (7)$$

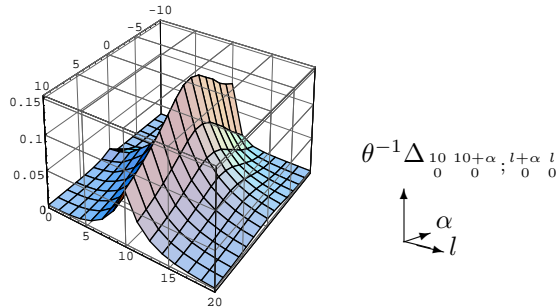


Figure 1: decay properties, $\Omega = 0.1$, $\mu = 0$

$$\Lambda \frac{\partial}{\partial \Lambda} \left(\text{Diagram: a circle with } N \text{ external legs labeled } n_1, m_1, n_2, m_2, \dots, n_N, m_N \right) =$$

$$\frac{1}{2} \sum_{m,n,k,l} \sum_{N_1=1}^{N-1} \left(\text{Diagram: two circles connected by } n, k \text{ and } m, l \text{ lines, with } N_1 \text{ legs on the left and } N-N_1 \text{ on the right} \right) - \frac{1}{4\pi\theta} \sum_{m,n,k,l} \left(\text{Diagram: a circle with } N \text{ legs, some grouped in ovals} \right)$$

Figure 2: Polchinski equation

The strategy due to Wilson consists in integrating in the first step only those field modes ϕ_{mn} which have a matrix index bigger than some scale $\theta\Lambda^2$. The result is an effective action for the remaining field modes which depends on Λ . One can now adopt a smooth transition between integrated and not integrated field modes so that the Λ -dependence of the effective action is given by a certain differential equation, the Polchinski equation. Renormalization amounts to prove that the Polchinski equation admits a regular solution for the effective action which depends on only a finite number of initial data. This requirement is hard to satisfy because the space of effective actions is infinite dimensional and as such develops an infinite dimensional space of singularities when starting from generic initial data.

The Polchinski equation can be solved iteratively in perturbation theory. Graphically, it is depicted in Fig. (2). The graphs are graded by the number of vertices and the number of external legs. Only graphs with a smaller number of vertices and a bigger number of legs contribute to the Λ -variation

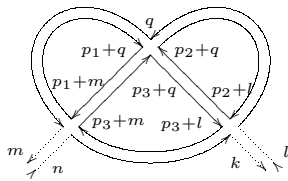


Figure 3: Pretzel

of a graph on the lhs of Fig. (2). A general graph is thus obtained by iteratively adding a propagator to smaller building blocks, starting with the initial ϕ^4 -vertex, and integrating over Λ . The propagators are differentiated cut-off propagators $Q_{mn;kl}(\Lambda)$ which vanish (for an appropriate choice of the cut-off function) unless the maximal index is in the interval $[\theta\Lambda^2, 2\theta\Lambda^2]$. Since fields carry two matrix indices and propagators four, the graphs are ribbon graphs familiar from matrix models.

It can then be shown that the cut-off propagator $Q(\Lambda)$ is bounded by $\frac{C}{\theta\Lambda^2}$. This was achieved numerically in [8] and later confirmed analytically in [21]. A nonvanishing frequency parameter Ω is required for such a decay behavior. As the volume of each two-component index $m \in \mathbb{N}^2$ is bounded by $C'\theta^2\Lambda^4$ in graphs of the above type, the power counting degree of divergence is (at first sight) $\omega = 4S - 2I$, where I is the number of propagators and S the number of summation indices.

It is important to note that given three indices of a propagator $Q_{mn;kl}(\Lambda)$ the fourth one is determined by $m + k = n + l$. For simple planar graphs one finds that $\omega = 4 - N$, where N is the number of external legs. At a closer look, however, one encounters a difficulty concerning completely inner vertices, which require additional index summations. The graph shown in Fig. (3) entails four independent summation indices p_1, p_2, p_3 and q , whereas for the powercounting degree $2 = 4 - N = 4S - 5 \cdot 2$ we should only have $S = 3$ of them. But due to the quasi-locality of the propagator (the exponential decay in l -direction in Fig. (1)), the sum over q for fixed m can be estimated without the need of the volume factor. Remarkably, the quasi-locality of the propagator not only ensures the correct powercounting degree for planar graphs, it also renders all nonplanar graphs superficially convergent. E.g., in the nonplanar graphs in Fig. (4) the summation over q and q, r , resp., is of the same type as over q in Fig. (3) so that the graphs in Fig. (4) can be estimated without any volume factor.

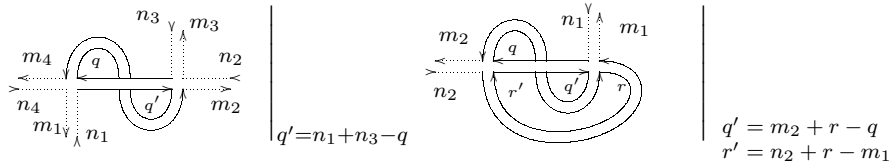


Figure 4: Nonplanar graphs

After all, we have obtained the powercounting degree of divergence

$$\omega = 4 - N - 4(2g + B - 1) \quad (8)$$

for a general ribbon graph, where g is the genus and B the number of holes of the Riemann surface on which the graph is drawn. Both are directly determined by the graph. It should be stressed that although the number (8) follows from counting the required volume factors, its proof in our scheme is not so obvious[22]: The procedure consists of adding a new cut-off propagator to a given graph. The topology (B, g) has many possibilities to arise from the topologies of the smaller parts for which one has estimates by induction. Moreover, the boundary conditions for the integration have to be correctly chosen to confirm (8).

The powercounting behavior is good news because it implies that all planar nonregular graphs are superficially convergent. Which is not true for $\Omega = 0$. However, this does not mean that all problems are solved: Power counting (8) suggests that the remaining planar two- and four-leg graphs which are divergent independent of their matrix indices. An infinite number of adjusted initial data would be necessary in order to remove these divergences. Fortunately, a more careful analysis shows that the powercounting behavior is improved by the index jump along the trajectories of the graph. E.g., the index jump for the graph (3) is defined as $J = \|k - n\|_1 + \|q - l\|_1 + \|m - q\|_1$. Then, the amplitude is suppressed by a factor of order $\left(\frac{\max(m, n \dots)}{\theta\Lambda^2}\right)^{\frac{J}{2}}$ compared with the naive estimate. Thus, only planar four-leg graphs with $J = 0$ and planar two-leg graphs with $J = 0$ or $J = 2$ are divergent (the number of jumps is even). For these cases, a discrete Taylor expansion about the graphs with vanishing indices is employed. Only the leading terms of the expansion, i.e. the reference graphs

with vanishing indices, are divergent whereas the difference between original graph and reference one is convergent. Thus, only the reference graphs must be integrated in a way that involves initial conditions. For example, if the contribution to the rhs of the Polchinski equation, Fig. (2) is given by:

$$\Lambda \frac{\partial}{\partial \Lambda} A_{mn;nk;kl;lm}^{(2)\text{planar},1\text{PI}}[\Lambda] = \sum_{p \in \mathbb{N}^2} \left(\begin{array}{c} \text{graph with } m, n, k, l \text{ legs and } p \text{ internal lines} \end{array} \right) (\Lambda), \quad (9)$$

the Λ -integration is performed as follows:

$$\begin{aligned} A_{mn;nk;kl;lm}^{(2)\text{planar},1\text{PI}}[\Lambda] &= \quad (10) \\ &= - \int_{\Lambda}^{\infty} \frac{d\Lambda'}{\Lambda'} \sum_{p \in \mathbb{N}^2} \left(\begin{array}{c} \text{graph with } m, n, k, l \text{ legs and } p \text{ internal lines} \\ - \begin{array}{c} \text{graph with } m, n, k, l \text{ legs and } 0 \text{ internal lines} \end{array} \end{array} \right) [\Lambda'] \\ &+ \begin{array}{c} m, n, k, l \text{ legs} \\ \text{graph with } 0 \text{ internal lines} \end{array} \left[\int_{\Lambda_R}^{\Lambda} \frac{d\Lambda'}{\Lambda'} \sum_{p \in \mathbb{N}^2} \left(\begin{array}{c} \text{graph with } 0 \text{ legs and } p \text{ internal lines} \end{array} \right) [\Lambda'] + A_{00;00;00;00}^{(2,1,0)1\text{PI}}[\Lambda_R] \right]. \end{aligned}$$

Only one initial condition, $A_{00;00;00;00}^{(2,1,0)1\text{PI}}[\Lambda_R]$, is required for an infinite number of planar four-leg graphs (distinguished by the matrix indices). We need one further initial condition for the two-leg graphs with $J = 2$ and two more initial conditions for the two-leg graphs with $J = 0$ (for the leading quadratic and the subleading logarithmic divergence). This is one condition more than in a commutative ϕ^4 -theory, and this additional condition justifies a posteriori our starting point of adding one new term to the action (3), the oscillator term Ω .

2.1 The Landau ghost

Knowing the relevant/marginal couplings, we can compute Feynman graphs with sharp matrix cut-off \mathcal{N} . The most important question concerns the β -function appearing in the renormalization group equation which describes the cut-off dependence of the expansion coefficients $\Gamma_{m_1 n_1; \dots; m_N n_N}$ of the effective action when imposing normalisation conditions for the relevant and marginal couplings. We have [23]

$$\lim_{\mathcal{N} \rightarrow \infty} \left(\mathcal{N} \frac{\partial}{\partial \mathcal{N}} + N\gamma + \mu_0^2 \beta_{\mu_0} \frac{\partial}{\partial \mu_0^2} + \beta_{\lambda} \frac{\partial}{\partial \lambda} + \beta_{\Omega} \frac{\partial}{\partial \Omega} \right) \Gamma_{m_1 n_1; \dots; m_N n_N}[\mu_0, \lambda, \Omega, \mathcal{N}] = 0, \quad (11)$$

where

$$\begin{aligned}\beta_\lambda &= \mathcal{N} \frac{\partial \lambda}{\partial \mathcal{N}}, \beta_\Omega = \mathcal{N} \frac{\partial \Omega}{\partial \mathcal{N}}, \\ \beta_{\mu_0} &= \frac{\mathcal{N}}{\mu_0^2} \frac{\partial \mu_0^2}{\partial \mathcal{N}}, \gamma = \mathcal{N} \frac{\partial \ln \mathcal{Z}}{\partial \mathcal{N}}.\end{aligned}$$

The couplings depend on the physical parameters $\mu_{\text{phys}}, \lambda_{\text{phys}}, \Omega_{\text{phys}}$ and the cut-off \mathcal{N} . The wavefunction renormalization is denoted by \mathcal{Z} . To one-loop order one finds [23]

$$\begin{aligned}\beta_\lambda &= \frac{\lambda_{\text{phys}}^2}{48\pi^2} \frac{(1-\Omega_{\text{phys}}^2)}{(1+\Omega_{\text{phys}}^2)^3}, \beta_\Omega = \frac{\lambda_{\text{phys}}\Omega_{\text{phys}}}{96\pi^2} \frac{(1-\Omega_{\text{phys}}^2)}{(1+\Omega_{\text{phys}}^2)^3}, \\ \beta_\mu &= -\frac{\lambda_{\text{phys}} \left(4\mathcal{N} \ln(2) + \frac{(8+\theta\mu_{\text{phys}}^2)\Omega_{\text{phys}}^2}{(1+\Omega_{\text{phys}}^2)^2} \right)}{48\pi^2\theta\mu_{\text{phys}}^2(1+\Omega_{\text{phys}}^2)}, \gamma = \frac{\lambda_{\text{phys}}}{96\pi^2} \frac{\Omega_{\text{phys}}^2}{(1+\Omega_{\text{phys}}^2)^3}.\end{aligned}\tag{12}$$

Hence, the ratio of the coupling constants $\frac{\lambda}{\Omega^2}$ remains bounded along the renormalization group flow up to first order. Starting from given small values for Ω_R, λ_R at \mathcal{N}_R , the frequency grows in a small region around $\ln \frac{\mathcal{N}}{\mathcal{N}_R} = \frac{48\pi^2}{\lambda_R}$ to $\Omega \approx 1$. The coupling constant approaches $\lambda_\infty = \frac{\lambda_R}{\Omega_R^2}$, which can be made small for sufficiently small λ_R . This leaves the chance of a nonperturbative construction [9] of the model. In particular, the β -function vanishes at the self-dual point $\Omega = 1$ [24], indicating special and interesting properties of the model.

3 Induced gauge theory

Since elementary particles are most successfully described by gauge theories it is a big challenge to formulate consistent gauge theories on noncommutative spaces. Let u be a unitary element of the algebra such that the scalar fields ϕ transform covariantly:

$$\phi \mapsto u^* \star \phi \star u, \quad u \in \mathcal{G}.\tag{13}$$

The approach employed here makes use of two basic ideas. First, it is well known that the \star -multiplication of a coordinate - and also of a function, of course - with a field is not a covariant process. The product $x^\mu \star \phi$ will

not transform covariantly. The introduction of covariant coordinates $\tilde{X}_\nu = \tilde{x}_\nu + A_\nu$ finds a remedy to this situation. The gauge field A_μ and hence the covariant coordinates transform in the following way:

$$A_\mu \mapsto iu^* \star \partial_\mu u + u^* \star A_\mu \star u, \quad \tilde{X}_\mu \mapsto u^* \star \tilde{X}_\mu \star u. \quad (14)$$

Using covariant coordinates we can construct an action invariant under gauge transformations from (3) - note that $\partial_\mu f = -i[\tilde{x}_\mu, f]_\star$:

$$\begin{aligned} S = \int d^4x & \left(\frac{1}{2} \phi \star [\tilde{X}_\nu, [\tilde{X}^\nu, \phi]_\star]_\star + \frac{\Omega^2}{2} \phi \star \{ \tilde{X}^\nu, \{ \tilde{X}_\nu, \phi \}_\star \}_\star \right. \\ & \left. + \frac{\mu^2}{2} \phi \star \phi + \frac{\lambda}{4!} \phi \star \phi \star \phi \star \phi \right) (x). \end{aligned} \quad (15)$$

Secondly, we apply the heat kernel formalism. The gauge field A_μ is an external, classical gauge field coupled to ϕ . In the classical case, the divergent terms determine the dynamics of the gauge field [25, 26]. There have already been attempts to generalise this approach to the noncommutative realm [27, 28, 29]. However, the results there are not applicable, since we have modified the free action and expand around $-\nabla^2 + \Omega^2 \tilde{x}^2$ rather than $-\nabla^2$.

The regularised one loop effective action for the model defined by the classical action (15) is given by

$$\Gamma_{1l}^\epsilon[\phi] = -\frac{1}{2} \int_\epsilon^\infty \frac{dt}{t} \text{Tr} \left(e^{-tH} - e^{-tH^0} \right). \quad (16)$$

For the effective potential H we have the expression

$$\frac{\theta}{2} \frac{\delta^2 S}{\delta \phi^2} \equiv H = H^0 + \frac{\theta}{2} V. \quad (17)$$

The effective action is calculated as a power series in the potential V up to fourth order using the Duhamel formula:

$$\begin{aligned} \Gamma_{1l}^\epsilon &= \frac{\theta}{4} \int_\epsilon^\infty dt \text{Tr} V e^{-tH^0} - \frac{\theta^2}{8} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' t' \text{Tr} V e^{-t'H^0} V e^{-(t-t')H^0} \\ &+ \frac{\theta^3}{16} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' t'' \text{Tr} V e^{-t''H^0} V e^{-(t'-t'')H^0} V e^{-(t-t')H^0} \\ &- \frac{\theta^4}{32} \int_\epsilon^\infty \frac{dt}{t} \int_0^t dt' \int_0^{t'} dt'' \int_0^{t''} dt''' t''' \text{Tr} V e^{-t'''H^0} V e^{-(t''-t''')H^0} \\ &\quad \times V e^{-(t'-t'')H^0} V e^{-(t-t')H^0}. \end{aligned} \quad (18)$$

Higher order terms are already finite. As before, the calculations are performed in the matrix basis, where the star product is just a matrix product. After a suitable rescaling, all the operators depend, beside on θ , only on the following three parameters:

$$\rho = \frac{1 - \Omega^2}{1 + \Omega^2}, \quad \tilde{\epsilon} = \epsilon(1 + \Omega^2), \quad \tilde{\mu}^2 = \frac{\mu^2 \theta}{1 + \Omega^2}. \quad (19)$$

We concentrate on terms involving only the gauge field and assume $\lambda = 0$. The explicit calculation is very tedious and is given in detail in [16, 17]. Although the method is not manifestly gauge invariant, various terms from different orders add up to a gauge invariant final expression:

$$\begin{aligned} \Gamma_{1l}^\epsilon = & \frac{1}{192\pi^2} \int d^4x \left\{ \frac{24}{\tilde{\epsilon}\theta} (1 - \rho^2) (\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \right. \\ & + \ln \epsilon \left(\frac{12}{\theta} (1 - \rho^2) (\tilde{\mu}^2 - \rho^2) (\tilde{X}_\nu \star \tilde{X}^\nu - \tilde{x}^2) \right. \\ & \left. \left. + 6(1 - \rho^2)^2 ((\tilde{X}_\mu \star \tilde{X}^\mu)^{\star 2} - (\tilde{x}^2)^2) + \rho^4 F_{\mu\nu} F^{\mu\nu} \right) \right\}, \end{aligned} \quad (20)$$

where the field strength is given by $F_{\mu\nu} = [\tilde{x}_\mu, A_\nu]_\star - [\tilde{x}_\nu, A_\mu]_\star + [A_\mu, A_\nu]_\star$. Our main result is summarised in Eqn. (20): The logarithmically divergent part is an interesting candidate for a renormalisable gauge interaction. As far as we know, this action did not appear before in string theory. The sign of the term quadratic in the covariant coordinates may change depending on whether $\tilde{\mu}^2 \leq \rho^2$. This reflects a phase structure. The matrix model in the limit $\Omega = 1$ ($\rho = 0$) is of particular interest. In the limit $\Omega \rightarrow 0$, we obtain just the standard deformed Yang-Mills action. One of the problems of quantising action (20) is connected to the tadpole contribution, which is non-vanishing and hard to eliminate. The Orsay group also considered the 1-loop effective action in the case $\Omega \neq 0$, for a complex model. They calculated the divergent contributions in x -space by evaluating relevant Feynman diagrams and arrived at the same result [30].

An appropriate rescaling $\tilde{X}_\alpha \rightarrow \frac{\sqrt{2\sqrt{3}}}{\sqrt{\theta}} \tilde{X}_\alpha$ and $\tau \equiv -\sqrt{3} \frac{1-\rho^2}{\rho^2}$ leads to the equations of motion

$$D_\nu F^{\sigma\nu} = \tau \tilde{X}^\sigma + \tau^2 \{ \tilde{X}^\sigma, \tilde{X}_\nu \star \tilde{X}^\nu \}_\star, \quad (21)$$

where we have assumed for simplicity $\tilde{\mu} = 0$ and used

$$D_\nu F^{\sigma\nu} = -[\tilde{X}_\nu, [\tilde{X}^\sigma, \tilde{X}^\nu]_\star]_\star.$$

In [31], the matter fields have been included in order to find some solutions. However, the gauge part (21) alone also exhibits a number of solutions currently under investigation, such as $su(2)$.

A similar model has been discussed in [18]. This model includes an oscillator potential for the gauge fields, $\tilde{x}^2 A^2$, and for the ghosts. Other terms occurring here are missing. Hence, the considered action is not gauge invariant, but a BRST invariance could be established. These "missing" terms may nevertheless come into the game through one loop corrections. A brief sketch of this model follows.

3.1 BRST quantization

A BRST invariant model for noncommutative $U(1)$ gauge theory has been introduced in [18]. Its action is given by

$$\Gamma^{(0)} = \int d^4x \left(\frac{1}{4} F_{\mu\nu} \star F_{\mu\nu} + s(\bar{c} \star \partial_\mu A_\mu) - \frac{1}{2} B^2 + \frac{\Omega^2}{8} s(\tilde{c}_\mu \star C_\mu) \right), \quad (22)$$

where $C_\mu = \{ \{ \tilde{x}_\mu, A_\nu \}_\star, A_\nu \}_\star + [\{ \tilde{x}_\mu, \bar{c} \}_\star, c]_\star + [\bar{c}, \{ \tilde{x}_\mu, c \}_\star]_\star$. These terms modify the free theory, the propagators of the involved fields. The action is - by constructed - invariant under the following noncommutative BRST transformations

$$\begin{aligned} sA_\mu &= \partial_\mu c - ig[A_\mu, c]_\star, \quad s\bar{c} = B, \\ sc &= igc \star c, \quad sB = 0, \quad s\tilde{c}_\mu = \tilde{x}_\mu. \end{aligned} \quad (23)$$

Hence, we have $s^2 = 0$. The multiplier field B implements the gauge fixing. In the limit $\tilde{c}_\mu \rightarrow 0$, this reduces to the usual Feynman gauge, $\partial_\mu A^\mu - B = 0$. The field \tilde{c}_μ with mass dimension 1 and ghost number -1 is yet another Lagrange multiplier. It has been introduced in order to rescue the BRST invariance of the x^2 terms for ghosts and gauge field contained in \mathcal{C}_μ . These terms also modify the free theory. All propagators are given by the Mehler kernel (24), where the momentum non-conservation becomes transparent. The index structure for the gauge field propagator is trivial and given by the Kronecker delta $\delta^{\alpha\beta}$. In momentum space, the Mehler Kernel reads

$$\tilde{K}_M(p, q) = \frac{\omega^3}{8\pi^2} \int_\epsilon^\infty \frac{d\alpha}{\sinh^2 \alpha} \exp \left\{ -\frac{\omega}{4} u^2 \coth \frac{\alpha}{2} - \frac{\omega}{4} v^2 \tanh \frac{\alpha}{2} \right\}, \quad (24)$$

where $\omega = \frac{\theta}{\Omega}$, and $u = p - q$, $v = p + q$. We use a UV regulator $\epsilon = 1/\Lambda^2$ in order to cut the integration over the auxiliary Schwinger parameter. The action provides the weights for a three-photon ($\tilde{V}_{\mu\nu\sigma}^{3A}$) and a four-photon vertex ($\tilde{V}_{\mu\nu\sigma\tau}^{4A}$), and also for a ghost – two photon vertex (\tilde{V}_μ^c). The explicit expressions will be presented in a forthcoming publication.

Preliminary results. Let us discuss some preliminary results to one-loop. Remarkably, the tadpole contribution (one external photon) is divergent. After Taylor expansion of the external field, the divergent contributions are given by - summation over all indices is implied:

$$\begin{aligned} T &= \int d^4p \tilde{A}_\rho(p) \int d^4k \int d^4k' \tilde{K}_M(k, k') \delta_{\sigma\tau} (\tilde{V}_{\rho\sigma\tau}^{3A}(p, -k', k) \\ &\quad + \tilde{V}_\rho^c(p, -k', k)) \\ &= -\frac{c}{\epsilon} \int d^4x \tilde{x}_\mu A^\mu - \ln \epsilon \frac{c\Omega\theta}{4 + \Omega^2} \int d^4x \tilde{x}^2 \tilde{x}_\mu A^\mu + \mathcal{O}(\epsilon^0). \end{aligned} \quad (25)$$

This suggests to introduce the counterterms $\tilde{x}_\mu A^\mu$ and $\tilde{x}^2 \tilde{x}_\mu A^\mu$.

The contribution for the gauge loop with two external photon legs and one photon vertex is given by

$$\begin{aligned} D &= \int d^4p \int d^4q \tilde{A}_\rho(p) \tilde{A}_\epsilon(-q) \int d^4k_1 \int d^4k_2 \tilde{K}_M(k_1, k_2) \delta_{\sigma\tau} \\ &\quad \times \tilde{V}_{\rho\sigma\tau\epsilon}^{4A}(p, -k_1, k_2, -q). \end{aligned} \quad (26)$$

To extract the divergent contributions, we perform a Taylor expansion of the integrand depending on q , except for the Mehler kernel, around $p = q$. The divergent contributions are given by

$$D_{(1)} = \frac{c' \omega}{\epsilon} \int d^4p \tilde{A}_\mu(p) \tilde{A}_\mu(-p), \quad (27)$$

$$D_{(2)} = \ln \epsilon \frac{c'}{2\bar{\omega}} \int d^4p \tilde{A}_\mu(p) \tilde{A}_\mu(-p) \tilde{p}^2, \quad (28)$$

$$D_{(3)} = -\ln \epsilon \frac{c'}{2} \left(4 - \frac{\theta^2}{\omega \bar{\omega}} \right) \int d^4p \tilde{A}_\rho(p) (-\partial^{q^2} \tilde{A}_\rho)(-p), \quad (29)$$

with $\bar{\omega} = \omega(1 + \frac{\theta^2}{4\omega^2})$ suggesting further counterterms. Remarkably, all the counterterms are contained in the induced action (20). There are more one-loop diagrams which have to be studied. Then, the exact coefficients of the calculated counterterms can be compared with the induced action.

Of course, it is important to study the behavior at higher loops. Especially the IR behaviour is of interest in order to see whether IR/UV mixing is present in this model. For this aim, we look at a diagram with two gauge loops of type (26). We proceed similar as in the scalar case: To first approximation, momentum is conserved at the vertex, and we obtain

$$D \sim \int d^4 p \tilde{A}_\mu(p) \tilde{A}_\mu(-p) \int d^4 u \int d^4 v \tilde{K}_M(u, v) \left(\cos \frac{p\theta u}{2} - \cos \frac{p\theta v}{2} \right). \quad (30)$$

The first term corresponds to the planar regular part and the second one to the nonregular one. The latter one coincides - except for the sign - with the corresponding contribution in the scalar case (with oscillator term). Therefore, the same estimates, which showed that IR/UV mixing does not occur there, seem to be applicable here. This point is studied at the moment and gives rise to the hope that the discussed gauge model is free of IR/UV divergences.

References

- [1] H. S. Snyder, Quantized Space-Time, *Phys. Rev.* **71**, 38–41, (1947).
- [2] A. Connes, Noncommutative differential geometry, *Inst. Hautes Etudes Sci. Publ. Math.* **62**, 257, (1986).
- [3] T. Filk, Divergencies in a field theory on quantum space, *Phys. Lett.* **B376**, 53–58, (1996).
- [4] S. Doplicher, K. Fredenhagen, and J. E. Roberts, The quantum structure of space-time at the Planck scale and quantum fields, *Commun. Math. Phys.* **172**, 187–220, (1995).
- [5] V. Schomerus, D-branes and deformation quantization, *JHEP.* **06**, 030, (1999).
- [6] N. Seiberg and E. Witten, String theory and noncommutative geometry, *JHEP.* **09**, 032, (1999).
- [7] S. Minwalla, M. Van Raamsdonk, and N. Seiberg, Noncommutative perturbative dynamics, *JHEP.* **02**, 020, (2000).

- [8] H. Grosse and R. Wulkenhaar, Renormalisation of ϕ^4 theory on noncommutative \mathbb{R}^4 in the matrix base, *Commun. Math. Phys.* **256**, 305–374, (2005).
- [9] J. Magnen and V. Rivasseau, Constructive ϕ^4 field theory without tears, *Annales Henri Poincare.* **9**, 403–424, (2008). doi: 10.1007/s00023-008-0360-1.
- [10] S. S. Gubser and S. L. Sondhi, Phase structure of non-commutative scalar field theories, *Nucl. Phys.* **B605**, 395–424, (2001).
- [11] W. Bietenholz, F. Hofheinz, and J. Nishimura, Phase diagram and dispersion relation of the non-commutative lambda ϕ^4 model in $d = 3$, *JHEP.* **06**, 042, (2004).
- [12] X. Martin, A matrix phase for the ϕ^4 scalar field on the fuzzy sphere, *JHEP.* **04**, 077, (2004).
- [13] J. Ambjorn, Y. M. Makeenko, J. Nishimura, and R. J. Szabo, Nonperturbative dynamics of noncommutative gauge theory, *Phys. Lett.* **B480**, 399–408, (2000).
- [14] H. Steinacker, Quantized gauge theory on the fuzzy sphere as random matrix model, *Nucl. Phys.* **B679**, 66–98, (2004).
- [15] P. B. Gilkey, Invariance theory, the heat equation and the Atiyah-Singer index theorem. (1995).
- [16] H. Grosse and M. Wohlgenannt, Induced gauge theory on a noncommutative space, *Eur. Phys. J.* **C52**, 435–450, (2007).
- [17] M. Wohlgenannt, Induced Gauge Theory on a Noncommutative Space, *J. Phys. Conf. Ser.* **103**, 012008, (2008).
- [18] D. N. Blaschke, H. Grosse, and M. Schweda, Non-commutative U(1) gauge theory on \mathbb{R}_θ^4 with oscillator term and BRST symmetry, *Europhys. Lett.* **79**, 61002, (2007). doi: 10.1209/0295-5075/79/61002.
- [19] E. Langmann and R. J. Szabo, Duality in scalar field theory on non-commutative phase spaces, *Phys. Lett.* **B533**, 168–177, (2002).

- [20] J. Polchinski, Renormalization and effective Lagrangians, *Nucl. Phys.* **B231**, 269–295, (1984).
- [21] V. Rivasseau, F. Vignes-Tourneret, and R. Wulkenhaar, Renormalization of noncommutative ϕ^4 -theory by multi-scale analysis, *Commun. Math. Phys.* **262**, 565–594, (2006).
- [22] H. Grosse and R. Wulkenhaar, Power-counting theorem for non-local matrix models and renormalisation, *Commun. Math. Phys.* **254**, 91–127, (2005).
- [23] H. Grosse and R. Wulkenhaar, The beta-function in duality-covariant noncommutative ϕ^4 theory, *Eur. Phys. J.* **C35**, 277–282, (2004).
- [24] M. Disertori, R. Gurau, J. Magnen, and V. Rivasseau, Vanishing of beta function of non commutative $\phi(4)^4$ theory to all orders, *Phys. Lett.* **B649**, 95–102, (2007). doi: 10.1016/j.physletb.2007.04.007.
- [25] A. H. Chamseddine and A. Connes, The spectral action principle, *Commun. Math. Phys.* **186**, 731–750, (1997).
- [26] E. Langmann, Generalized Yang-Mills actions from Dirac operator determinants, *J. Math. Phys.* **42**, 5238–5256, (2001).
- [27] D. V. Vassilevich, Non-commutative heat kernel, *Lett. Math. Phys.* **67**, 185–194, (2004).
- [28] V. Gayral and B. Iochum, The spectral action for Moyal planes, *J. Math. Phys.* **46**, 043503, (2005).
- [29] D. V. Vassilevich, Heat kernel, effective action and anomalies in non-commutative theories, *JHEP.* **08**, 085, (2005).
- [30] A. de Goursac, J.-C. Wallet, and R. Wulkenhaar, Noncommutative induced gauge theory, *Eur. Phys. J.* **C51**, 977–987, (2007).
- [31] H. Grosse and R. Wulkenhaar, 8D-spectral triple on 4D-Moyal space and the vacuum of noncommutative gauge theory. (2007).