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Excess payoff dynamics in games

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Abstract

We present the family of Excess Payoff Dynamics for normal-form games, where the growth of a strategy depends only on its current proportion and the *excess payoff*, i.e., the payoff advantage of the strategy over the average population payoff. Requiring dependence only on the own excess payoff and a natural sign-preserving condition, the class essentially reduces to aggregate monotonic dynamics, a functional generalization of the Replicator Dynamics. However, Excess Payoff Dynamics also include a different subclass which contains the Replicator Dynamics, the Brown-von Neumann-Nash Dynamics, and other interesting examples as, e.g., satisficing dynamics. We also clarify the relation to *excess demand dynamics* from microeconomics.

JEL classification: C73

Keywords: Excess payoff; Evolutionary dynamics; BNN dynamics; Excess demand

1. Introduction

Sandholm (2005) proposed to study a class of well-behaved *excess payoff dynamics* in games, which are evolutionary dynamics where the success of a strategy is measured exclusively by

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its payoff advantage against the overall average population payoff, i.e., the 'excess payoff.' The celebrated Replicator Dynamics is a prominent example fulfilling this basic property (Taylor and Jonker, 1978, Schuster and Sigmund, 1983; see the textbooks of Hofbauer and Sigmund, 1998 or Sandholm, 2010). In this dynamics, the growth rates of strategies are assumed to be identical with their excess payoffs, but of course many alternative functional forms would be possible. The idea of excess payoff is directly related to ideas of relative advantage in evolutionary biology (Garay and Varga, 1999), but also exhibits formal links to the concept of excess demand in economics (Nikaido, 1959).

The class proposed by Sandholm (2005) went beyond the basic idea of dependence on excess payoffs. Specifically, that contribution studied a (reasonable) micro-foundation of evolutionary dynamics which gives rise to excess payoff dynamics in the sense described above *and* also fulfill a number of desiderata. The latter include, e.g., that rest points should exactly coincide with the Nash equilibria of the game. As a result, Sandholm's excess payoff dynamics *exclude* the Replicator Dynamics and many other examples where growth rates are functions of the excess payoffs, although they do include the less-known but interesting BNN (Brown-von Neumann-Nash) dynamics introduced by Brown and von Neumann (1950) (see Hofbauer, 2000 or Sandholm, 2001, p. 94).

In this article, we consider Excess Payoff Dynamics in the more general sense, i.e., all those dynamics formulated only in terms of excess payoffs (and, of course, current population proportions); those naturally include the Replicator Dynamics and the BNN dynamics, and encompass all dynamics considered by Sandholm (2005), but do not necessarily fulfill the desiderata postulated in that work. The purpose of our contribution is to explore this general family and the properties of some natural subclasses. Conceptually, our intention is to explore how far Excess Payoff Dynamics can deviate from the Replicator Dynamics, and illustrate how rich (or not) the class can be, under natural restrictions.

Our results are of two kinds. Initially, we consider two natural restrictions. The first is that the functional dependence on excess payoffs for a given strategy is restricted to the own excess payoff. The second is a natural sign-preserving condition, i.e., that the strategy grows or shrinks depending on the sign of the excess payoff (Ritzberger and Weibull, 1995). Under these conditions, and somewhat surprisingly, we show that the class essentially reduces to aggregate monotonic dynamics as described by Samuelson and Zhang (1992), which are a generalization of the Replicator Dynamics.

Those, however, exclude other interesting dynamics, a prominent example being the BNN (Brown-von Neumann-Nash) dynamics. Hence, we turn to the exploration of a different subclass of Excess Payoff Dynamics, which we call Separable, and that naturally encompass the Replicator and the BNN dynamics. We also illustrate that the class is rich enough to include other interesting examples (e.g., a subclass of satisficing dynamics). We show that the rest points of all such dynamics are either those of the Replicator Dynamics or those of the BNN (which coincide with the set of Nash equilibria). Under an additional condition, these dynamics are evolutionarily well-behaved, meaning that they are myopic adjustment dynamics (the latter ensures, e.g., selection of Nash equilibria in potential games).

The manuscript is structured as follows. Section 2 defines the class of Excess Payoff Dynamics and presents some prominent examples. Section 3 examines the first subclass of interest, Direct and Sign-Preserving dynamics, and shows their equivalence with the aggregate monotonicity condition. Section 4 studies Separable dynamics and shows the examples and results for this class. Section 5 briefly clarifies the relation of Excess Payoff Dynamics to excess demand dynamics from classical microeconomics (Nikaido, 1959). A brief conclusion closes the paper.

2. Excess payoff dynamics

Consider a finite, symmetric 2-person game with n strategies, numbered $i = 1, \ldots, n$ for simplicity, and payoff matrix $A = [a_{ij}]_{i,j=1,\ldots,n}$. Following the standard approach in evolutionary dynamics, this game is played repeatedly in continuous time within a large population through a random matching process. Let $x_i = x_i(t) \in [0, 1]$ denote the population frequency of players adopting strategy i at time t. Population profiles $x = (x_i)_{i=1}^n$ are then elements of the $(n-1)$-dimensional simplex $\Delta = \left\{x \in [0, 1]^n \;\middle|\; \sum_{i=1}^n x_i = 1\right\} \subset \mathbb{R}^n$.

Denote by

$$d_i = (A \cdot x)_i - x \cdot A \cdot x \tag{1}$$

the *excess payoff* of i-strategists, i.e., the difference between the (average) payoff of players using strategy i and the average population payoff, given the population profile x. This corresponds to the concept of relative advantage in evolutionary biology (Garay and Varga, 1999). We obviously abuse notation by not writing d_i as a function of x, but no confusion should arise. Denote also $d = (d_1, \ldots, d_n)$.

Most evolutionary dynamics are based on the idea that x_i should increase whenever d_i is positive. The following is the most general functional form for a dynamics based on excess payoffs.

Definition 1. An *Excess Payoff Dynamics* is a system of differential equations on Δ

$$\dot{x}_i = f_i(d, x) \tag{2}$$

where

(i) $f_i : \mathbb{R}^n \times \Delta \mapsto \mathbb{R}$ are continuous functions, and
(ii) for all $d \in \mathbb{R}^n$ such that $d \cdot x = 0$,

$$\sum_i f_i(d, x) = 0. \tag{3}$$

The crucial aspect of an Excess Payoff Dynamics is that the functions f_i are not allowed to depend on the payoff matrix A. That is, in an Excess Payoff Dynamics, all the dependence of the system on the payoff matrix is channeled through excess payoffs. In other words, the dynamics (2) is defined *independently of the game at hand*.

The minimal requirement (i) is justified because, by the Cauchy-Peano theorem, continuity of the f_i implies the existence of (local, and possibly non-unique) solution paths through any initial condition. If, in addition, the f_i are supposed to be Lipschitz-continuous, solutions would be unique by the Picard-Lindelöf Theorem, and globally defined by the Extension Theorem (if Δ is forward invariant). We do not make this assumption at this point.

Requirement (ii) (Equation (3)) is necessary in order for the hyperplane $\sum_i x_i = 1$ to be invariant. Since the dynamics is defined independently of the game, the requirement must be fulfilled for all x and all possible values d_i derived from x and any payoff matrix A. Condition (ii), though, is not stated for every matrix A, but for every vector of excess payoffs which is orthogonal to x. This is an equivalent formulation. It is easy to see that, given $x \in \operatorname{int}\Delta$, for every vector $u \in \mathbb{R}^n$ such that $\sum_i u_i x_i = 0$, there exists a payoff matrix A such that $d = u$.[3] Hence, condition (ii) must be fulfilled for all d orthogonal to x, as long as x is in the interior of the simplex. If x is not in the interior, a continuity argument on (3) yields the same result.

To the best of our knowledge, the name "excess payoff" is due to Sandholm (2005), but we remark again that our concept of Excess Payoff Dynamics (Definition 1) is more general (and, in the sense given there, also less well-behaved). In particular, unlike the class considered in Sandholm (2005), our Excess Payoff Dynamics do include both the Replicator Dynamics and the BNN dynamics. They also encompass the class in Sandholm (2005).

Example 1. Replicator Dynamics. The best-known evolutionary dynamics is the Replicator Dynamics of Taylor and Jonker (1978), named by Schuster and Sigmund (1983) and widely studied in evolutionary biology and economics. We refer the reader to the textbooks of Weibull (1995), Hofbauer and Sigmund (1998), or Sandholm (2010); for more compact treatments, see also Hofbauer and Sigmund (2003) or Cressman and Tao (2014). This dynamics is given by

$$\dot{x}_i = x_i \cdot d_i \tag{4}$$

i.e., the growth rate of any given strategy is numerically equal to its excess payoff. This corresponds to a particularly-simple Excess Payoff Dynamics, with $f_i(d,x) = x_i \cdot d_i$.

It is well-known (see any of the references above) that the rest points of the Replicator Dynamics are the set of strategies corresponding to symmetric Nash Equilibria of "restricted" games, i.e., games having as strategy set any nonempty subset of the strategy set, and as payoff matrix the appropriate submatrix of A.

Example 2. Brown-von Neumann-Nash Dynamics. A rather different dynamics, inspired by the work of G. W. Brown and J. von Neumann (Brown and von Neumann, 1950) and J. Nash (Nash, 1951; see Weibull, 1996 and Hofbauer, 2000 for details), is the Brown–von Neumann–Nash or BNN dynamics, given by

$$\dot{x}_i = [d_i]_+ - x_i \cdot \sum_{j=1}^{n} [d_j]_+ \tag{5}$$

where $[u]_+ = \max(0, u)$.

This dynamics has been studied by Berger and Hofbauer (2006) and Hofbauer et al. (2009), among others. It is an Excess Payoff Dynamics where the functions f_i are "kinked" in the sense that negative excess payoffs are dropped, creating a failure of differentiability at zero. The dynamics is particularly interesting for many reasons, both historical and formal. For instance, unlike the Replicator Dynamics, its rest points coincide exactly with the Nash Equilibria of the game.

Example 3. Sandholm's Excess Payoff Dynamics are the mean-field approximation of a specific Markov model of evolution based on 'moderation,' where agents "exert moderate levels of effort to find strategies that perform well" instead of relying on imitation or optimization (Sandholm, 2005, p. 150). In our notation, those dynamics are described by the functional form

[3] To see this, just define A to be the diagonal matrix where $a_{ii} = \frac{u_i}{x_i}$. Then, $d_i = (Ax)_i - xAx = a_{ii}x_i - \sum_j a_{jj}x_j^2 = u_i - \sum_j u_j x_j = u_i$.

$$\dot{x}_i = \tilde{\sigma}_i(d) - x_i \cdot \sum_{j=1}^{n} \tilde{\sigma}_j(d) \tag{6}$$

where $\tilde{\sigma} = \left(\tilde{\sigma}_j\right)_{j=1}^{n}$ is a "raw choice function" defined on the set of all possible excess payoff vectors, $D = \mathbb{R}^n \setminus \operatorname{int} \mathbb{R}^n_-$, and mapping them to $\mathbb{R}^n_+$. This function governs revision opportunities in the micro-founding Markovian model, and is assumed to be Lipschitz-continuous and fulfill $\tilde{\sigma}(d) \cdot d \geq 0$ for all excess payoff vectors d in the interior of D.[4] Obviously, the BNN dynamics corresponds to the particular case $\tilde{\sigma}_i = [d_i]_+$, and it is in this sense that Sandholm's Excess Payoff Dynamics are generalizations of the BNN dynamics.

3. Direct excess payoff dynamics and aggregate monotonicity

We start our investigation of Excess Payoff Dynamics by considering two natural properties. The first one concerns the fact that, as formulated, Excess Payoff Dynamics allow the shares of i-strategists to depend on the excess payoffs of *all* strategies. But, since the excess payoff (1) is already a comparison between the payoffs of i-strategists and the payoffs of others, it is natural to consider the subclass of dynamics where the evolution of a strategy i depends only on the excess payoff d_i, and not on the excess payoffs of other strategies. Formally:

Definition 2. An Excess Payoff Dynamics is *Direct* if f_i does not depend on d_j, for each $j \neq i$ and each $i = 1, \ldots, n$.

Abusing notation, the subclass of *Direct* Excess Payoff Dynamics is hence given by

$$\dot{x}_i = f_i\,(d_i, x) \tag{7}$$

Obviously, the Replicator Dynamics (Example 1) is an example of dynamics in this class. However, that the assumption of a direct dynamics is a restriction is illustrated by the BNN dynamics (Example 2), which is clearly not direct (and, in particular, Sandholm's Excess Payoff Dynamics, Example 3, are also not direct in general).

The second condition we consider is sign preservation. Naturally, one would expect that the proportion of i strategists grows or shrinks depending on whether the excess payoff is negative or positive, respectively. Ritzberger and Weibull (1995) studied *Sign-Preserving Selection* (SPS) dynamics, satisfying $\dot{x}_i < 0$ if and only if $d_i < 0$. We consider a slightly stronger condition.

Definition 3. A Direct Excess Payoff Dynamics is *Sign-Preserving* if $f_i(d_i, x) \cdot d_i > 0$ whenever $d_i \neq 0$ and $x \in \operatorname{int} \Delta$.

The next example shows that an entire class of well-known dynamics fulfill both of our conditions.

Example 4. Aggregate Monotonic Selection Dynamics. Samuelson and Zhang (1992) studied Aggregate Monotonic Selection (AMS) Dynamics in the context of asymmetric games. Those can be seen as a direct generalization of the Replicator Dynamics. Specifically, all AMS dynamics can be written as

[4] Sandholm (2005) further considers multiple population models, but we restrict ourselves to the single-population case.

$$\dot{x}_i = \omega(x) \cdot x_i \cdot d_i \tag{8}$$

with $\omega : \Delta \mapsto \mathbb{R}_{++}$ a strictly positive function (see Ritzberger and Weibull, 1995, p. 1376).

AMS dynamics are obviously Direct Excess Payoff Dynamics, and, since $\omega(x) > 0$ for all x, it is obvious that they are also sign-preserving.

Our first main result shows that every Direct and Sign-Preserving Excess Payoff Dynamics coincides with an AMS dynamics in the interior of the simplex, provided there are at least three strategies. Hence, if one restricts attention to the two (natural) conditions described above, the resulting subclass of Excess Payoff Dynamics essentially corresponds to the class of Aggregate Monotonic Selection Dynamics, i.e., a generalization of the Replicator Dynamics. The proof of this result is, however, relatively involved.

Theorem 1. *Let $n \geq 3$. Consider a Direct and Sign-Preserving Excess Payoff Dynamics as in (7). Then, there exists a strictly positive function $\omega : \operatorname{int}\Delta \mapsto \mathbb{R}_{++}$ such that, in $\operatorname{int}\Delta$,*

$$f_i(d_i, x) = \omega(x) \cdot d_i \cdot x_i.$$

Proof. Fix $x \in \operatorname{int}\Delta$. Define the real functions given by $f_i(d_i) = f_i(d_i, x)$. Condition (3) translates into

$$\sum_i f_i(d_i) = 0 \quad \forall d \in \mathbb{R}^n \text{ such that } \sum_i d_i x_i = 0. \tag{9}$$

Notice that $f_i(0) = 0$ because, by the SPS condition, $f_i(d_i) > 0$ for $d_i > 0$, $f_i(d_i) < 0$ for $d_i < 0$, and f_i is continuous.

The proof now proceeds in five steps.

Step 1. Let $f = f_1$. Then, for all $i \neq 1$, $f_i(z) = -f\left(-\frac{x_i}{x_1}z\right)$ for any $z \in \mathbb{R}$.

Let $i \neq 1$. To see the claim, define the vector $d \in \mathbb{R}^n$ given by $d_1 = -\frac{x_i}{x_1}z$, $d_i = z$, and $d_j = 0$ for all $j \neq 1, i$. It follows that $\sum_k d_k x_k = 0$ and, by (9), $\sum_k f_k(d_k) = 0$. For each $j \neq 1, i$, $d_j = 0$ implies $f_j(d_j) = 0$, and hence $f_1(d_1) + f_i(d_i) = 0$, which yields the claim.

Step 2. The function f is odd, i.e. $f(z) = -f(-z)$ for all $z \in \mathbb{R}$.

To see this, let $z \in \mathbb{R}$ and apply the proof of Step 1 with $i = 2$ in the role of $i = 1$ to obtain that $f_3(z) = -f_2\left(-\frac{x_3}{x_2}z\right) = f\left(\frac{x_2}{x_1}\frac{x_3}{x_2}z\right) = f\left(\frac{x_3}{x_1}z\right)$, where the second equality follows from Step 1 (note that this argument requires $n \geq 3$). Then,

$$f(z) = f\left(\frac{x_3}{x_1}\frac{x_1}{x_3}z\right) = f_3\left(\frac{x_1}{x_3}z\right) = -f\left(-\frac{x_3}{x_1}\frac{x_1}{x_3}z\right) = -f(-z).$$

Step 3. f is linear in $\mathbb{R}_+$. I.e., for every $z \in \mathbb{R}_+$, $f(z) = z \cdot f(1)$.

To see this claim, using that $n \geq 3$, we define the vector $d \in \mathbb{R}^n$ given by $d_1 = 2$, $d_2 = -\frac{x_1}{x_2}$, $d_3 = -\frac{x_1}{x_3}$, and $d_i = 0$ for all $i \geq 4$. By construction, $\sum_i d_i x_i = 0$ and it follows from (9) that $\sum_i f_i(d_i) = 0$. For $i \geq 4$, $d_i = 0$ implies $f_i(d_i) = 0$. For $i = 2, 3$, Step 1 implies that $f_i\left(-\frac{x_1}{x_i}\right) = -f(1)$. Thus $\sum_i f_i(d_i) = 0$ reduces to $f(2) = 2f(1)$.

We now proceed by induction (using always only $n \geq 3$). Suppose that $f(m) = m \cdot f(1)$ for some $m \geq 2$. Construct the vector $d \in \mathbb{R}^n$ given by $d_1 = m + 1$, $d_2 = -\frac{x_1}{x_2}m$, $d_3 = -\frac{x_1}{x_3}$, and $d_i = 0$ for all $i \geq 4$. By construction, $\sum_i d_i x_i = 0$ and, by (9), $\sum_i f_i(d_i) = 0$. For $i \geq 4$, $d_i = 0$ implies $f_i(d_i) = 0$. For $i = 3$, $f_3\left(-\frac{x_1}{x_3}\right) = -f(1)$ by Step 1. For $i = 2$, $f_3\left(-\frac{x_1}{x_2}m\right) = -f(m) =$

$-mf(1)$ by Step 1 and the induction hypothesis. Thus $\sum_i f_i(d_i) = 0$ reduces to $f(m) - mf(1) - f(1) = 0$, i.e. $f(m+1) = (m+1) \cdot f(1)$. We conclude that $f(m) = m \cdot f(1)$ for every $m \in \mathbb{N}$.

An analogous argument shows that $f(p/q) = pf(1/q)$ for all $p, q \in \mathbb{N}$. In particular $f(1) = f(q/q) = qf(1/q)$ and hence $f(1/q) = (1/q)f(1)$. Thus $f(p/q) = (p/q)f(1)$ for any positive rational number p/q. For an arbitrary real number $z \in \mathbb{R}_+$, taking a sequence of positive rational numbers approaching z and applying continuity of f shows that $f(z) = z \cdot f(1)$.[5]

Step 3'. For every $z \in \mathbb{R}_-$, $f(z) = -z \cdot f(-1)$.

This follows from Steps 2 and 3 as, for $z < 0$, $f(z) = -f(-z) = zf(1) = -zf(-1)$.

Step 4. There exists $\omega(x) > 0$ such that $f(d_1) = \omega(x)x_1d_1$ for all $d_1 \in \mathbb{R}$.

Suppose first that $d_1 > 0$. Notice that the SPS condition $f_i(d_i)d_i > 0$ implies $f(1) > 0$. Define $\omega(x) = f(1)/x_1 > 0$. Then, for $d_1 > 0$, $f(d_1) = d_1 f(1) = \omega(x)x_1d_1$, where the second equality follows from Step 3 and the third from definition of $\omega(x)$.

Now suppose $d_1 < 0$. Analogously, the SPS condition $f_i(d_i)d_i > 0$ implies $f(-1) < 0$. Then, for $d_1 < 0$, $f(d_1) = -d_1 f(-1) = \omega(x)x_1d_1$, where the second equality follows from Step 3' and the third because, by Step 2, $\omega(x) = \frac{f(1)}{x_1} = -\frac{f(-1)}{x_1}$. Thus the claim follows for all $d_1 \neq 0$. Since $f(0) = 0$, the claim holds for all $d_1 \in \mathbb{R}$.

Step 5. $f_i(d_i, x) = \omega(x) \cdot x_i \cdot d_i$ for all i and all $d_i \in \mathbb{R}$.

For $i = 1$, the claim follows from Step 4. For $i > 1$, $f_i(d_i, x) = f_i(d_i) = -f\left(-\frac{x_i}{x_1}d_i\right) = \omega(x)x_id_i$, where the second equality follows from Step 1 and the third from Step 4.

This completes the proof. □

Remark. Samuelson and Zhang (1992) assume AMS dynamics $\dot{x}_i = f_i(x)$ to be *regular*, which in particular requires Lipschitz continuity and, further, that the limit $\lim_{x_i \to 0} f_i/x_i$ exists and is finite. Since we did not require analogous conditions in our formulation of Excess Payoff Dynamics, our result is restricted to the interior of the simplex. If the function $\frac{f_i(d_i,x)}{d_i \cdot x_i}$ can be extended continuously to the whole simplex, then the function $\omega(x)$ can obviously also be extended. However, it is a priori not guaranteed that $\omega(x) > 0$ on the boundary.

The following example shows why Theorem 1 does not apply to the case of only two strategies.

Example 5. A Direct, Sign-Preserving Excess Payoff Dynamics which is not Aggregate Monotonic, for $n = 2$**.** Consider the Direct Excess Payoff Dynamics given by

$$f_i(d_i, x) = (d_i x_i)^3$$

for $i = 1, 2$. Then, if $d_1x_1 + d_2x_2 = 0$, we have that $d_2x_2 = -d_1x_1$ and

$$f_1(d_1, x) + f_2(d_2, x) = (d_1x_1)^3 + (-d_1x_1)^3 = 0,$$

hence the dynamics is well-defined.

Moreover, $f_i(d_i, x)d_i = d_i^4 x_i^3 > 0$ for all $d_i \neq 0$ and $x_i > 0$, i.e., the dynamics is Sign-Preserving. It is, however, clearly not an AMS dynamics. Notice, though, that $(d_ix_i)^3$ still is an odd function.

[5] This is analogous to the proof that any continuous real function with $f(x+y) = f(x) + f(y)$ must be linear.

The following example exhibits a Sign-Preserving Dynamics which fails to be Aggregate Monotonic, because the remaining assumptions in Theorem 1 do not hold.

Example 6. A piecewise dynamics. Define the dynamics given by

$$\dot{x}_i = \begin{cases} \frac{x_i d_i}{\sum\{x_j [d_j]_+\}} & \text{if } d_i \neq 0 \\ 0 & \text{if } d_i = 0 \end{cases}$$

Clearly, $\sum_i \dot{x}_i = 0$ and $x_i = 0 \Longrightarrow \dot{x}_i = 0$; hence, the simplex is forward-invariant. This piecewise dynamics fulfills the condition $f_i \cdot d_i > 0$ for all $d_i \neq 0$ and $x_i > 0$ and is hence Sign-Preserving. However, it cannot be put in the form $f_i(d_i, x) = \omega(x) d_i x_i$. Theorem 1 does not apply, because, first, f_i is not continuous in d_i at $d = 0$, and, second, f_i depends on all d_j.

4. Separable dynamics

4.1. Definition

While the assumption of a direct dynamics is natural, it excludes some important examples as the BNN dynamics (Example 2). In this section, we take a different route and explore a family of Excess Payoff Dynamics which encompass both the BNN dynamics and the classical Replicator Dynamics (Example 1). We do so by postulating a general, flexible functional form as follows.

Definition 4. An Excess Payoff Dynamics is *separable* if it is of the form

$$\dot{x}_i = c(x_i) h(d_i) - \sum_j \sigma_{ji}(x) c(x_j) h(d_j) \tag{10}$$

where

(h) $h : \mathbb{R} \mapsto \mathbb{R}$ is a continuous function such that $h(u) > 0$ for all $u > 0$ and $h(u) \leq 0$ for all $u \leq 0$.[6]
(c) $c : [0, 1] \mapsto \mathbb{R}$ is a continuous function such that $c(y) > 0$ for all $y > 0$.
(σ) for all j, $\sigma_j : \Delta \mapsto \Delta$ is a continuous function such that $\sigma_{ji}(x) = 0$ if $x_i = 0$ and $\sigma_{ji}(x) > 0$ if $x_i > 0$.

Further, if $\sigma_{ji}(x) = x_i$, then the dynamics is called *uniform*.

4.2. Examples

While equation (10) might appear cumbersome and arbitrary at first glance, we contend that this is a natural generalization of ideas in the evolutionary literature. The intuition for this is best developed by way of examples.

Example 7. The Replicator Dynamics is a Separable Dynamics. For $h(u) = u$, $c(y) = y$, and $\sigma_{ji}(x) = x_i$, equation (10) turns into

[6] Note that continuity implies that $h(0) = 0$, but in general it might be that $h(u) = 0$ for some (or all) $u \leq 0$. The latter is the case in Example 10 below.

$$\dot{x}_i = x_i d_i - \sum_j x_i x_j d_j = x_i \cdot d_i \tag{11}$$

i.e., the Replicator Dynamics. Hence, the Replicator Dynamics is a Separable (and Uniform) Excess Payoff Dynamics.

Example 8. The BNN Dynamics is a Separable Dynamics. Let $c(y) = 1$, $\sigma_{ji}(x) = x_i$, and $h(u) = f([u]_+)$ where $f : \mathbb{R}_+ \mapsto \mathbb{R}_+$ is a continuous function with $f(0) = 0$ and $f(u) > 0$ for $u > 0$, and $[u]_+ = \max(u, 0)$ is the positive part of u. Equation (10) becomes

$$\dot{x}_i = f([d_i]_+) - x_i \cdot \sum_{j=1}^{n} f([d_j]_+) \tag{12}$$

i.e., a "Generalized" Brown-von Neumann-Nash (BNN) dynamics (see Hofbauer, 2000). This dynamics becomes the second example of a Separable (and Uniform) Excess Payoff Dynamics.

Of course, if we take f to be the identity on $\mathbb{R}_+$, we obtain the original BNN dynamics (5). Letting $f(u) = u^\alpha$ for $\alpha > 0$ also lets us recover the continuous time Best-Reply Dynamics in the limit as $\alpha \to \infty$.

Example 9. RD to BNN transformation. Let $c(y) = y^\alpha$, $\sigma_{ji}(x) = x_i$, and $h = f_\alpha$ with $f_\alpha(u) = u$ for $u \geq 0$ and $f_\alpha(u) = \alpha u$ for $u < 0$. Then equation (10) turns into

$$\dot{x}_i = x_i^\alpha f_\alpha(d_i) - x_i \sum_j x_j^\alpha f_\alpha(d_j) \tag{13}$$

which is a homotopy connecting the BNN ($\alpha = 0$) and the Replicator Dynamics ($\alpha = 1$).

The following is a more involved but interesting example.

Example 10. Satisficing Dynamics. Suppose there is a large but finite population of N agents playing the game recurrently so that their (expected) payoff of playing strategy i is $(A \cdot x)_i$. Agents receive revision opportunities according to Poisson processes with rate $\lambda(d_j)$, which verifies that $\lambda(\cdot)$ is continuous, $\lambda(u) = 0$ if $u > 0$, and $\lambda(u) \geq 0$ if $u \leq 0$; the idea is that agents with payoff above average are satisfied and do not change, but those with less-than-average payoffs do. When switching, an agent previously choosing strategy j switches to strategy i with probability $\sigma_{ji}(x)$, so that $\sum_i \sigma_{ji}(x) = 1$, and the $\sigma_j : \Delta \mapsto \Delta$ are required to be continuous. These probabilities are independent of payoffs (e.g. imitate popular choices, imitate a randomly sampled agent, etc.). However, we assume explicitly that, as required by (σ), $\sigma_{ji}(x) > 0$ whenever $x_i > 0$ and $\sigma_{ji}(x) = 0$ whenever $x_i = 0$. That is, unobserved strategies attract no new players, and any existing strategies attract some fraction of players (no matter how small).

With a standard stochastic approximation argument, we approximate the resulting Markov process by the mean field dynamics. The argument is analogous to the one used in Sandholm (2005) (although applied to a different model) and hence we only sketch it here. The expected number of revision opportunities received by agents playing j in an infinitesimal period dt (abusing notation in the usual way) is

$$N \cdot x_j \cdot \lambda(d_j) \cdot dt.$$

Since agents playing j switch to i with probability $\sigma_{ji}(x)$ if given opportunity, the expected number of players who receive revision opportunity and change to i is

$$\sum_j N \cdot x_j \lambda(d_j)\sigma_{ji}(x) \cdot dt$$

and hence the expected number of agents playing i next period is

$$\left(\sum_j N \cdot x_j \lambda(d_j)\sigma_{ji}(x) + (1-\lambda(d_i))N \cdot x_i\right) \cdot dt$$

and the expected change in the number of i players is

$$N\left(\left[\sum_j x_j \lambda(d_j)\sigma_{ji}(x)\right] - \lambda(d_i) \cdot x_i\right) \cdot dt$$

and thus the expected change in the proportion is obtained by dropping N and the dynamics is given by

$$\dot{x}_i = -x_i \cdot \lambda(d_i) + \left(\sum_j \sigma_{ji}(x) x_j \lambda(d_j)\right). \tag{14}$$

Now simply define $h(u) = -\lambda(u)$ (hence satisfying condition (h)) and $c(y) = y$. Hence, the satisficing dynamics is a Separable Excess Payoff Dynamics, but it is only uniform if $\sigma_{ji}(x) = x_i$. This is the intuition behind the name "uniform," for it corresponds to the case where switching agents simply copy a uniformly, randomly sampled agent from the whole population.

The Replicator Dynamics is not a member of this subfamily, since the function λ must be positive. However, there is a dynamics similar in spirit (pure imitation driven by dissatisfaction, Weibull, 1995, p. 153) where revision rates are decreasing functions of $(A \cdot x)_i$. If they are linear functions, then the result is a rescaling of the Replicator Dynamics. Neither is the BNN dynamics a member of this subfamily, because of the multiplicative term x_j in the term in brackets in equation (14).

4.3. Properties

4.3.1. Forward invariance

The general functional form given by (10) does not guarantee that the simplex is forward invariant, although this is the case in all examples in Section 4.2. Actually, forward-invariance can be characterized as follows.

Proposition 1. *Consider a Separable Excess Payoff Dynamics. Then, the simplex is forward-invariant for all possible payoff matrices A if and only if the following condition holds:*

(FI) *Either $c(0) = 0$ or $h(u) = 0$ for all $u \le 0$, or both.*

Proof. First we show that the simplex is forward-invariant under (FI). To see this, we compute:

$$\sum_i \dot{x}_i = \sum_i c(x_i)h(d_i) - \sum_i \sum_j \sigma_{ji}(x)c(x_j)h(d_j) =$$

$$= \sum_i c(x_i)h(d_i) - \sum_j c(x_j)h(d_j)\left(\sum_i \sigma_{ji}(x)\right) = 0$$

where the last equality follows because $\sum_i \sigma_{ji}(x) = 1$. To complete the proof, we need to show that $x_i = 0$ implies $\dot{x}_i \geq 0$. Suppose, then, $x_i = 0$. By (σ), $\sigma_{ji}(x) = 0$ for all j, thus $\dot{x}_i = c(0)h(d_i)$, which is nonnegative under (FI).

To see the converse, we only need to show that, if $c(0) > 0$, then $h(u) \geq 0$ for all u. It will then follow from (h) that $h(u) = 0$ for all $u \leq 0$.

Note that, since $d_i = (Ax)_i - xAx$, for any $u \in \mathbb{R}$ there exists a payoff matrix such that $d_i = u$ for some vector x with $x_i = 0$. Specifically, choose some $k \neq i$ and consider the vector $x = e_k$ all whose entries are 0 but the k-th one. Define a payoff matrix A with $a_{ik} = u$ and $a_{kk} = 0$. Then,

$$d_i = \sum_j a_{ij}x_j - \sum_{\ell,j} a_{\ell,j}x_\ell x_j = u,$$

proving the claim. But, if $c(0) > 0$, whenever $x_i = 0$, the condition $\dot{x}_i = c(0)h(d_i) \geq 0$ implies that $h(d_i) \geq 0$, for all possible payoff matrices A and vectors $x \in \Delta$ such that $x_i = 0$. Thus, $h(u) \geq 0$ for all u. □

The family given in Example 9 satisfies (FI). Consequently, this assumption is also satisfied by the generalized BNN Dynamics (12) and the Replicator Dynamics (4). Consider again the satisficing dynamics in Example 10. The uniform-imitation satisficing dynamics where $\sigma_{ji}(x) = x_i$ is also covered by the previous Proposition; forward-invariance of the simplex for the general equation (14) can be easily established directly, though.

4.3.2. Rest points

That the rest points of a dynamics coincide with the set of Nash equilibria is undoubtedly an appealing property, which led Sandholm (2005) to postulate it as one of his desiderata. However, this has the unappealing consequence of excluding the Replicator Dynamics, undoubtedly the most prominent evolutionary dynamics. Our next main result shows that the rest points of any Separable Excess Payoff Dynamics are *either* exactly the Nash equilibria (as in the case of the BNN dynamics, Example 2) *or* coincide with the rest points of the Replicator Dynamics (Example 1). In this sense, the BNN and the Replicator dynamics can be seen as the canonical examples of separable dynamics.

Theorem 2. *Consider any (forward-invariant) Separable Excess Payoff Dynamics as given by (10). Then,*

(a) *Every Nash equilibrium is a rest point.*
(b) *Every rest point is also a rest point of the Replicator Dynamics (4).*
(c) *If $c(0) > 0$ then the rest points are precisely the Nash Equilibria.*
(d) *If $c(0) = 0$ then the rest points are precisely the rest points of the Replicator dynamics (4).*

Proof. (a) A profile x is a Nash equilibrium if and only if $d_i \leq 0$ for all i, which implies, by definition, that $d_i = 0$ whenever $x_i > 0$.
Under (FI), either $c(0) = 0$ or $h(u) = 0$ for all $u \leq 0$. If $h(u) = 0$ for all $u \leq 0$, then at any Nash equilibrium $c(x_i) \cdot h(d_i) = 0$ for all i, since $d_i \leq 0$. If $c(0) = 0$, then $c(x_i) \cdot h(d_i) = 0$ if $x_i = 0$, but at a Nash equilibrium, if $x_i > 0$, $c(x_i) \cdot h(d_i) = c(x_i) \cdot h(0) = 0$ since $h(0) = 0$.[7]
In both cases, we conclude that $c(x_i) \cdot h(d_i) = 0$ for all i and the result follows from (10).

[7] Recall footnote 6.

(b) At a rest point of (10),

$$c(x_i)h(d_i) = \sum_j \sigma_{ji}(x)c(x_j)h(d_j) \tag{15}$$

Suppose there exists any i such that $x_i > 0$ (hence $c(x_i) > 0$ and all $\sigma_{ji}(x) > 0$)) but $d_i > 0$ (and hence $h(d_i) > 0$ by (h)). Then, from (15) we have that $\sum_j \sigma_{ji}(x)c(x_j)h(d_j) = c(x_i)h(d_i) > 0$. Consider now any other strategy k. Using again (15), we have that $c(x_k)h(d_k) = \sum_j \sigma_{ji}(x)c(x_j)h(d_j) > 0$. Hence, we have proven that $h(d_k) > 0$ for all k with $x_k > 0$. By (h), this implies that $d_k > 0$ for all k with $x_k > 0$ and hence $\sum_k x_k d_k > 0$, a contradiction. The same contradiction obtains if there is any i with $x_i > 0$ but $d_i < 0$. In summary, $d_i = 0$ whenever $x_i > 0$, implying that x is a rest point of the Replicator Dynamics (4).

(c) Suppose $c(0) > 0$. We have to show that all rest points of (10) are Nash equilibria. Any such rest point is a rest point of (4) by (b) and hence $d_i = 0$ whenever $x_i > 0$. If $x_i = 0$, then by (σ) we have that $\sigma_{ji}(x) = 0$ for all j. Then, from (15) and $c(0) > 0$ we observe that $h(d_i) = 0$, i.e., $d_i \leq 0$ (by (h)). Hence, x is a Nash equilibrium. The converse follows from (a).

(d) Suppose now $c(0) = 0$. Let x be a rest point of (4), i.e., $x_j d_j = 0$ for all j. Then, either $x_j = 0$, implying $c(x_j) = 0$, or $d_j = 0$, implying $h(d_j) = 0$ by (h). In any case, $c(x_j)h(d_j) = 0$ for all j and it follows that x is a rest point of (10). The converse follows from (b). □

Since all interior rest points of the Replicator Dynamics are Nash equilibria, we immediately obtain the following.

Corollary 1. *All interior rest points of (forward-invariant) Separable Excess Payoff Dynamics are Nash equilibria.*

Remark. Sandholm (2005, Section 4) made the point that, even though his Excess Payoff Dynamics exclude the Replicator Dynamics, one can consider continuously-perturbed families indexed by a parameter α such that the dynamics is a Sandholm Excess Payoff Dynamics for every $\alpha > 0$, and coincides with the Replicator Dynamics when $\alpha = 0$. Since both the Replicator Dynamics and all Sandholm Excess Payoff Dynamics are Excess Payoff Dynamics in the sense given here, one obtains a family of Excess Payoff Dynamics such that the rest points are exactly the Nash equilibria except in the limit, where they become the rest points of the Replicator Dynamics. Hence, the family exhibited by Sandholm (2005) is akin to a generalization of the BNN which approaches the Replicator Dynamics as $\alpha \to 0$.

The family given by (13), which connects the BNN and the Replicator Dynamics, makes the opposite point. For any $\alpha > 0$, we have that $c(0) = 0^\alpha = 0$ and hence the rest points are exactly those of the Replicator Dynamics (Example 1). In the limit, for $\alpha = 0$, we obtain the BNN dynamics (Example 2), whose rest points are exactly the Nash equilibria. Hence, the family is akin to a generalization of the Replicator Dynamics which approaches the BNN as $\alpha \to 0$.

4.3.3. The uniform case

We now turn to the particular case of Uniform Separable dynamics, i.e., assuming $\sigma_{ji}(x) = x_i$. Our next result shows that, in this case, the paths of the dynamics describe evolutionarily-reasonable adjustments. Specifically, the dynamics becomes a *myopic adjustment dynamics* (Swinkels, 1993; Hofbauer and Sigmund, 1998, Section 8.5), or has *positive correlation* (Sandholm, 2005).

Theorem 3. *Any Separable and Uniform Excess Payoff Dynamics is a myopic adjustment dynamics, meaning that $\dot{x} \cdot Ax \geq 0$ for all x.*

Proof. By direct computation:

$$\dot{x} \cdot Ax = \sum_i \dot{x}_i (Ax)_i = \sum_i c(x_i)h(d_i)(Ax)_i - \sum_i x_i \left(\sum_j c(x_j)h(d_j) \right) (Ax)_i =$$

$$\sum_i c(x_i)h(d_i)(Ax)_i - \left(\sum_j c(x_j)h(d_j) \right) xAx = \sum_i c(x_i)h(d_i)((Ax)_i - xAx) =$$

$$= \sum_i c(x_i)h(d_i)d_i \geq 0$$

The last inequality follows from $c(x) \geq 0$ for all x (by property (c)) and $h(d_i)d_i \geq 0$ for all d_i (by property (h)). □

The following corollary is a direct implication of the last theorem.

Corollary 2. *Consider any Separable and Uniform Excess Payoff Dynamics. For potential games ($A = A^T$), mean payoff increases over time. Every orbit hence converges to the set of equilibria.*

Proof. If $A = A^T$, the time derivative of the mean payoff xAx is $2\dot{x}Ax \geq 0$. □

5. Nikaido and the BNN

The concept of Excess Payoff is conceptually related to the concept of *excess demand* in microeconomics. In the latter field, Nikaido (1959) studied *excess demand dynamics,* and it is natural to examine how they are related to Excess Payoff Dynamics.

This can be done as follows. Consider arbitrary vectors $y \in \mathbb{R}^n_+$ in the positive orthant, and define the simple dynamics

$$\dot{y}_i = [d_i(y)]_+ \tag{16}$$

This dynamics is not confined to the simplex, but the positive orthant is forward invariant. Nikaido (1959) studies a price-adjustment dynamics which amounts to (16), replacing the excess payoff d_i by excess demand functions $E_i(y)$, where the y_i are prices. Unlike excess payoffs, it is proven that excess demand functions are pretty arbitrary and in particular need not be linear. They must, however, satisfy *Walras Law*, which amounts to $\sum_i y_i \cdot E_i(y) = 0$ everywhere, and must be homogeneous of degree zero, which is *not* fulfilled by the excess payoffs.

Suppose that we normalize prices: $x_i = y_i/(\sum_i y_i)$. Then, we can ask what the induced dynamics on x_i would be. Nikaido (1959) shows that the normalized-price adjustment dynamics can be rewritten (through an appropriate time transformation) as the analogue of the BNN equation. Since the E_i are arbitrary (it is not possible to build arguments on the $\dot{E}_i$, as they need not exist), the analysis is quite cumbersome, but Nikaido (1959) established global convergence (to Walrasian equilibria) through a direct, lengthy analysis of the function $\sum_i [E_i(p)]^2_+$.

Following Nikaido (1959), hence, equation (16) can be turned into (12) (with f as the identity). The essence of the argument is as follows. Let $y(t)$ be a solution of (16). Consider the auxiliary differential equation

$$\dot{\alpha} = \sum_{i=1}^{n} y_i(\alpha) \tag{17}$$

with initial condition $\alpha(0) = 0$. Nikaido (1959) uses homogeneity of the E_i to argue that a solution α of (17) exists and is strictly increasing. Suppose it does. Define now $x_i(t) = y_i(\alpha(t))/\sum_i(y_j(\alpha(t)))$. Differentiating $x_i \sum_i y_i(\alpha) = y_i(\alpha)$ and using $\dot{y}_i(\alpha(t)) = [E_i(y_i(\alpha(t))]_+ = [E_i(x_i(\alpha(t))]_+$ shows that $x(t)$ is a solution of the BNN equation. The latter equality, though, uses the zero-homogeneity of the E_i.

More generally, the argument of Nikaido (1959) can be used to derive Separable and Uniform Excess Payoff Dynamics from a less cumbersome functional form. Here we follow this argument only for the BNN.

Let $y_i \in \mathbb{R}_+$ denote the size of the subpopulation playing strategy i. For all $y \neq (0, \dots, 0)$, we can consider the population proportions $z_i = y_i/(\sum_j y_j)$. The usual random matching model gives rise to excess payoffs $d_i(z)$, based on population proportions and not on population sizes. A natural dynamics in $\mathbb{R}^n_+ \setminus \{0\}$ would be given by

$$\dot{y}_i = [d_i(z)]_+ \tag{18}$$

Let $y(t)$ be a forward orbit of (18). Consider the auxiliary Cauchy problem

$$\dot{\alpha} = \sum_j y_j(\alpha) \text{ and } \alpha(0) = 0 \tag{19}$$

From (19), $\dot{\alpha} = \sum_j y_j(\alpha) \geq \sum_j y_j(0) = K > 0$ for $\alpha \geq 0$. That is, α is a transformation of time. Define

$$x_i(t) = z_i(\alpha(t))$$

Then, we claim that $x(t)$ is a forward orbit for the BNN dynamics. To see this, differentiate $x_i \sum_j y_j(\alpha) = y_i(\alpha)$ to get

$$\dot{x}_i \sum_j y_j(\alpha) + x_i \dot{\alpha} \sum_j \dot{y}_j(\alpha) = \dot{\alpha} \dot{y}_i(\alpha)$$

which, simplifying $\dot{\alpha} = \sum_j y_j(\alpha) > 0$, turns into

$$\dot{x}_i + x_i \sum_j \dot{y}_j(\alpha) = \dot{y}_i(\alpha)$$

Replacing (18) now yields

$$\dot{x}_i + x_i \sum_j [d_j]_+ = [d_i]_+$$

i.e., the BNN equation.

6. Conclusion

Prominent evolutionary dynamics as the Replicator Dynamics and the Brown-von Neumann-Nash (BNN) Dynamics are based on the concept of *excess payoffs*, which capture the evolutionary advantage of the subpopulation adopting a given strategy. It is hence natural to study evolutionary dynamics where the dependence of population growth on payoffs is channeled exclusively through excess payoffs. This has caught the attention of previous researchers, including Sandholm (2005), and is conceptually related to *excess demand dynamics* from microeconomics (Nikaido, 1959)

This is, naturally, a large class of dynamics, and many research questions can be asked. This paper is an exploration of the class. We have identified several results of interest.

First, it would be natural to restrict attention to dynamics where the growth of a subpopulation using a given strategy depends only on the excess payoff of that strategy, and where the dynamics points in the right direction, meaning that the sign of the excess payoff determines whether the subpopulation grows or shrinks. It turns out that, essentially, such dynamics are one-to-one with the subclass fulfilling the key condition of the Aggregate Monotonic Selection dynamics if Samuelson and Zhang (1992). They can be seen as a functional generalization of the Replicator Dynamics, but, crucially, exclude the BNN.

Second, a general functional form describes a family of *Separable* Excess Payoff Dynamics which become a simultaneous generalization of the Replicator and the BNN Dynamics, such that the set of rest points is always that of the Replicator Dynamics or the set of all Nash Equilibria (as in the BNN). This family includes other interesting examples, e.g. satisficing dynamics. Under an additional condition (uniformity), these dynamics are myopic adjustment dynamics and hence will, e.g., converge to the set of Nash equilibria in potential games.

Quite clearly, we have but scratched the surface of this class of dynamics, and we hope that future research will deliver further results and insights on this interesting class of dynamics.

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