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Deterministic Evolutionary Game Dynamics

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Abstract. This is a survey about continuous time deterministic evolutionary dynamics for finite games. In particular, six basic dynamics are described: the replicator dynamics, the best response dynamics, the Brown–von Neumann–Nash dynamics, the Smith dynamics, and the payoff projection dynamics. Special classes of games, such as stable games, supermodular games and partnership games are discussed. Finally a general nonconvergence result is presented.

1. Introduction: Evolutionary Games

We consider a large population of players, with a finite set of pure strategies $\{1, \ldots, n\}$. x_i denotes the frequency of strategy i. $\Delta_n = \{x \in \mathbb{R}^n : x_i \geq 0, \sum_{i=1}^n x_i = 1\}$ is the $(n-1)$–dimensional simplex which will often be denoted by Δ if there is no confusion.

The payoff to strategy i in a population x is $a_i(x)$, with $a_i : \Delta \to \mathbb{R}$ a continuous function (population game). The most important special case is that of a symmetric two person game with $n \times n$ payoff matrix $A = (a_{ij})$; with random matching this leads to the linear payoff function $a_i(x) = \sum_j a_{ij} x_j = (Ax)_i$.

$\hat{x} \in \Delta$ is a Nash equilibrium (NE) iff

$$\hat{x} \cdot a(\hat{x}) \geq x \cdot a(\hat{x}) \quad \forall x \in \Delta. \tag{1.1}$$

Occasionally I will also look at bimatrix games (played between two player populations), with $n \times m$ payoff matrices A, B, or at N person games.

Evolutionarily stable strategies. According to Maynard Smith [**36**], a mixed strategy $\hat{x} \in \Delta$ is an *evolutionarily stable strategy* (ESS) if

$$\begin{array}{llcll} \text{(i)} & x \cdot A\hat{x} & \leq & \hat{x} \cdot A\hat{x} & \forall x \in \Delta, \quad \text{and} \\ \text{(ii)} & x \cdot Ax & < & \hat{x} \cdot Ax & \text{for } x \neq \hat{x}, \text{ if there is equality in (i).} \end{array}$$

The first condition (i) is simply Nash's definition (1.1) for an equilibrium. It is easy to see that $\hat{x}$ is an ESS, iff $\hat{x} \cdot Ax > x \cdot Ax$ holds for all $x \neq \hat{x}$ in a neighbourhood of $\hat{x}$. This property is called *locally superior* in [**61**]. For an interior equilibrium

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$\hat{x}$, the equilibrium condition $\hat{x}\cdot A\hat{x} = x\cdot A\hat{x}$ for all $x \in \Delta$ together with (ii) implies $(\hat{x} - x)\cdot A(x - \hat{x}) > 0$ for all x and hence

$$(1.2) \qquad z\cdot Az < 0 \qquad \forall z \in \mathbb{R}^n_0 = \{z \in \mathbb{R}^n : \sum_i z_i = 0\} \quad \text{with} \quad z \neq 0.$$

Condition (1.2) says that the mean payoff $x\cdot Ax$ is a strictly concave function on Δ. Conversely, games satisfying (1.2) have a unique ESS (possibly on the boundary) which is also the unique Nash equilibrium of the game. The slightly weaker condition

$$(1.3) \qquad z\cdot Az \leq 0 \qquad \forall z \in \mathbb{R}^n_0$$

includes also the limit cases of zero–sum games and games with an interior equilibrium that is a 'neutrally stable' strategy (i.e., equality is allowed in (ii)). Games satisfying (1.3) need no longer have a unique equilibrium, but the set of equilibria is still a nonempty convex subset of Δ.

For the *rock–scissors–paper* game with (a cyclic symmetric) pay-off matrix

$$(1.4) \qquad A = \begin{pmatrix} 0 & -b & a \\ a & 0 & -b \\ -b & a & 0 \end{pmatrix} \quad \text{with } a, b > 0$$

with the unique Nash equilibrium $E = (\frac{1}{3}, \frac{1}{3}, \frac{1}{3})$ we obtain the following: for $z \in \mathbb{R}^3_0$, $z_1 + z_2 + z_3 = 0$,

$$z\cdot Az = (a - b)(z_1z_2 + z_2z_3 + z_1z_3) = \frac{b-a}{2}[z_1^2 + z_2^2 + z_3^2].$$

Hence for $0 < b < a$, the game is negative definite, and E is an ESS. On the other hand, if $0 < a < b$, the game is positive definite:

$$(1.5) \qquad z\cdot Az > 0 \quad \forall z \in \mathbb{R}^n_0 \setminus \{0\},$$

the equilibrium E is not evolutionarily stable, indeed the opposite, and might be called an '*anti–ESS*'.

For a classical game theorist, all RPS games are the same. There is a unique Nash equilibrium, even a unique correlated equilibrium [**60**], for any $a, b > 0$. In evolutionary game theory the dichotomy $a < b$ versus $a > b$ is crucial, as we will see in the next sections, in particular in the figures 1–6.

2. Game Dynamics

In this section I present 6 special (families of) game dynamics. As we will see they enjoy a particularly nice property: Interior ESS are globally asymptotically stable.

The presentation follows largely [**22, 24, 28**].

1. Replicator dynamics
2. Best response dynamics
3. Logit dynamics (and other smoothed best reply dynamics)
4. Brown–von Neumann–Nash dynamics
5. Payoff comparison dynamics
6. Payoff projection dynamics

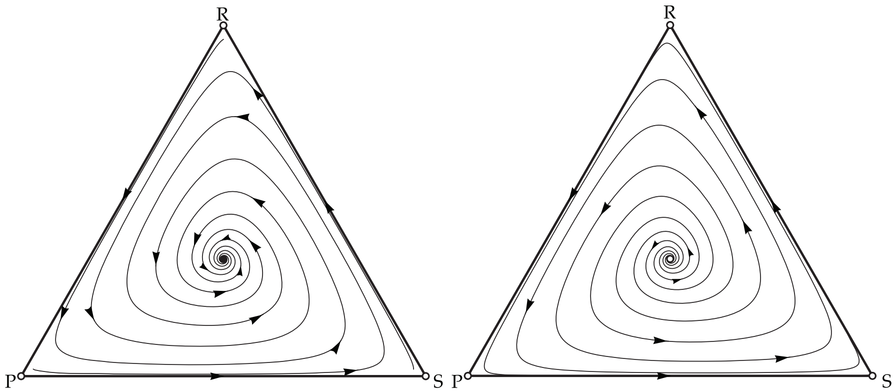

FIGURE 1. Replicator dynamics for Rock-Paper-Scissors games: $a > b$ versus $a < b$

Replicator dynamics.

$$\dot{x}_i = x_i \left(a_i(x) - x{\cdot}a(x)\right), \quad i = 1, \ldots, n \qquad \text{(REP)} \tag{2.1}$$

In the zero-sum version $a = b$ of the RSP game, all interior orbits are closed, circling around the interior equilibrium E, with $x_1x_2x_3$ as a constant of motion.

THEOREM 2.1. *In a negative definite game satisfying* (1.2)*, the unique Nash equilibrium $p \in \Delta$ is globally asymptotically stable for (REP). In particular, an interior ESS is globally asymptotically stable.*
On the other hand, in a positive definite game satisfying (1.5) *with an interior equilibrium p, i.e., an anti-ESS, p is a global repellor. All orbits except p converge to the boundary $bd\,\Delta$.*

The proof uses $V(x) = \prod x_i^{p_i}$ as a Lyapunov function.
For this and further results on (REP) see Sigmund's chapter [**53**], and [**9, 26, 27, 48, 61**].

Best response dynamics. In the best response dynamics[1] [**14, 35, 19**] one assumes that in a large population, a small fraction of the players revise their strategy, choosing best replies[2] $BR(x)$ to the current population distribution x.

$$\dot{x} \in BR(x) - x. \tag{2.2}$$

Since best replies are in general not unique, this is a differential *inclusion* rather than a differential equation. For continuous payoff functions $a_i(x)$ the right hand side is a non-empty convex, compact subset of Δ which is upper semi-continuous in x. Hence solutions exist, and they are Lipschitz functions $x(t)$ satisfying (2.2) for almost all $t \geq 0$, see [**1**].

For games with linear payoff, solutions can be explicitely constructed as piecewise linear functions, see [**9, 19, 27, 53**].

For interior NE of linear games we have the following stability result [**19**].

[1]For bimatrix games, this dynamics is closely related to the 'fictitious play' by Brown [**6**], see Sorin's chapter [**56**].

[2]Recall the set of best replies $\mathrm{BR}(x) = \mathrm{Argmax}_{y\in\Delta}\; y{\cdot}a(x) = \{y \in \Delta : y{\cdot}a(x) \geq z{\cdot}a(x) \forall z \in \Delta\} \subseteq \Delta$.

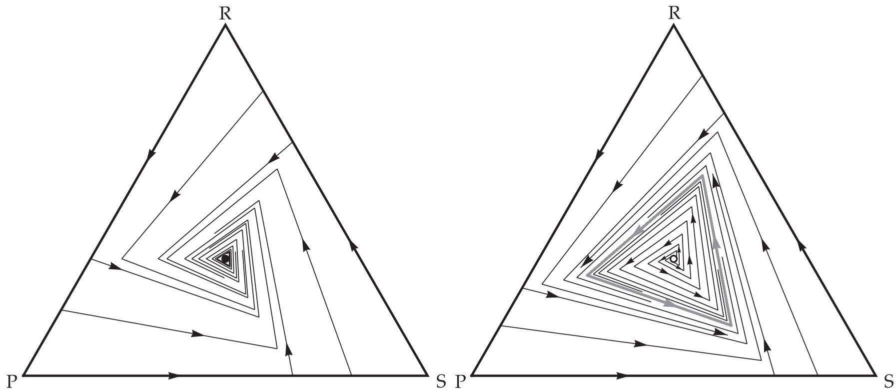

FIGURE 2. Best response dynamics for Rock-Paper-Scissors games, left one: $a \geq b$, right one: $a < b$

Let $\mathfrak{B} = \{b \in \text{bd}\Delta_n : (Ab)_i = (Ab)_j \text{ for all } i, j \in \text{supp}(b)\}$ denote the set of all rest points of (REP) on the boundary. Then the function[3]

$$w(x) = \max \left\{ \sum_{b \in \mathfrak{B}} b \cdot Ab\, u(b) : u(b) \geq 0, \sum_{b \in \mathfrak{B}} u(b) = 1, \sum_{b \in \mathfrak{B}} u(b)b = x \right\} \tag{2.3}$$

can be interpreted in the following way. Imagine the population in state x being decomposed into subpopulations of size $u(b)$ which are in states $b \in \mathfrak{B}$, and call this a $\mathfrak{B}$*–segregation of* b. Then $w(x)$ is the maximum mean payoff the population x can obtain by such a $\mathfrak{B}$–segregation. It is the smallest concave function satisfying $w(b) \geq b \cdot Ab$ for all $b \in \mathfrak{B}$.

THEOREM 2.2. *The following three conditions are equivalent:*
(a) There is a vector $p \in \Delta_n$*, such that* $p \cdot Ab > b \cdot Ab$ *holds for all* $b \in \mathfrak{B}$.
(b) $V(x) = \max_i (Ax)_i - w(x) > 0$ *for all* $x \in \Delta_n$.
(c) There exist a unique interior equilibrium $\hat{x}$*, and* $\hat{x} \cdot A\hat{x} > w(\hat{x})$.
These conditions imply:
The equilibrium $\hat{x}$ *is reached in finite and bounded time by any BR path.*

The proof consists in showing that the function V from (b) decreases along the solutions of the BR dynamics (2.2).

In the rock–scissors–paper game the set $\mathfrak{B}$ reduces to the set of pure strategies, the Lyapunov function is simply $V(x) = \max_i (Ax)_i$ and satisfies $\dot{V} = -V$ (except at the NE), see [**13**]. Since $\min_{x \in \Delta} V(x) = V(\hat{x}) = \hat{x} \cdot A\hat{x} = \frac{a-b}{3} > 0$ the exponentially decreasing $V(x(t))$ reaches this minimum value after a finite time. So all orbits reach the NE in finite time.

If $p \in \text{int}\,\Delta$ is an interior ESS then condition (a) holds not only for all $b \in \mathfrak{B}$ but for all $b \neq p$. In this case the Lyapunov function $V(x) = \max_i (Ax)_i - x \cdot Ax \geq 0$ can also be used. This leads to

THEOREM 2.3. [**22**] *For a negative semidefinite game* (1.3) *the convex set of its equilibria is globally asymptotically stable for the best–response dynamics.*[4]

[3]If $\mathfrak{B}$ is infinite it is sufficient to take the finitely many extreme points of its convex pieces.

[4]Using the tools in Sorin's chapter [**56**, section 1] this implies also global convergence of (discrete time) fictitious play. A similar result holds also for nonlinear payoff functions, see [**24**].

PROOF. The Lyapunov function $V(x) = \max_i (Ax)_i - x\cdot Ax \geq 0$ satisfies $\dot V = \dot x\cdot A\dot x - \dot x\cdot Ax < 0$ along piecewise linear solutions outside the set of NE. □

Note that for zero–sum games, $\dot V = -V$, so $V(x(t)) = e^{-t}V(x(0)) \to 0$ as $t \to \infty$, so $x(t)$ converges to the set of NE. For negative definite games, $\dot V < -c - V$ for some $c > 0$ and hence $x(t)$ reaches the NE in finite time.

For positive definite RSP games ($b > a$), $V(x) = \max_i (Ax)_i$ still satisfies $\dot V = -V$. Hence the NE is a repeller and all orbits (except the constant one at the NE) converge to the set where $V(x) = \max_i (Ax)_i = 0$ which is a closed orbit under the BR dynamics. It is called the *Shapley triangle* of the game, as a tribute to [**52**], see figure 2 (right). In this case the equilibrium payoff $\frac{a-b}{3}$ is smaller than 0, the payoff for a tie. This is the intuitive reason why the population tries to get away from the NE and closer to the pure states.

Interestingly, the times averages of the solutions of the replicator dynamics approach for $b > a$ the very same Shapley triangle, see [**13**]. The general reason for this is explained in Sorin's chapter [**56**, ch. 3].

For similar cyclic games with $n = 4$ strategies several Shapley polygons can coexist, see [**16**]. For $n \geq 5$ chaotic dynamics is likely to occur.

Smoothed best replies. The BR dynamics can be approximated by smooth dynamics such as the *logit dynamics* [**5, 12, 31**]

$$\dot x = L(\frac{Ax}{\varepsilon}) - x \tag{2.4}$$

with

$$L : \mathbb{R}^n \to \Delta, \quad L_k(u) = \frac{e^{u_k}}{\sum_j e^{u_j}}.$$

with $\varepsilon > 0$. As $\varepsilon \to 0$, this approaches the best reply dynamics, and every family of rest points[5] $\hat x_\varepsilon$ accumulates in the set of Nash equilibria.

There are (at least) two ways to motivate and generalize this 'smoothing'.

Whereas $BR(x)$ is the set of maximizers of the linear function $z \mapsto \sum_i z_i a_i(x)$ on Δ, consider $b_{\varepsilon v}(x)$, the unique maximizer of the function $z \mapsto \sum_i z_i a_i(x) + \varepsilon v(z)$ on $\operatorname{int}\Delta$, where $v : \operatorname{int}\Delta \to \mathbb{R}$ is a strictly concave function such that $|v'(z)| \to \infty$ as z approaches the boundary of Δ. If v is the entropy $-\sum z_i \log z_i$, the corresponding smoothed best reply dynamics

$$\dot x = b_{\varepsilon v}(x) - x \tag{2.5}$$

reduces to the logit dynamics (2.4) above [**12**]. Another choice[6] is $v(x) = \sum_i \log x_i$ used by Harsányi [**17**] in his logarithmic games.

Another way to perturb best replies are stochastic perturbations. Let ε be a random vector in $\mathbb{R}^n$ distributed according to some positive density function. For $z \in \mathbb{R}^n$, let

$$C_i(z) = \operatorname{Prob}(z_i + \varepsilon_i \geq z_j + \varepsilon_j \quad \forall j), \tag{2.6}$$

and $b(x) = C(a(x))$ the resulting stochastically perturbed best reply function. It can be shown [**23**] that each such stochastic perturbation can be represented by a deterministic perturbation as described before. The main idea is that there is a

[5] These are the quantal response equilibria of McKelvey and Palfrey [**37**].

[6] A completely different approximate best reply function appears already in Nash's Ph.D. thesis [**40**], in his first proof of the existence of equilibria by Brouwer's fixed point theorem.

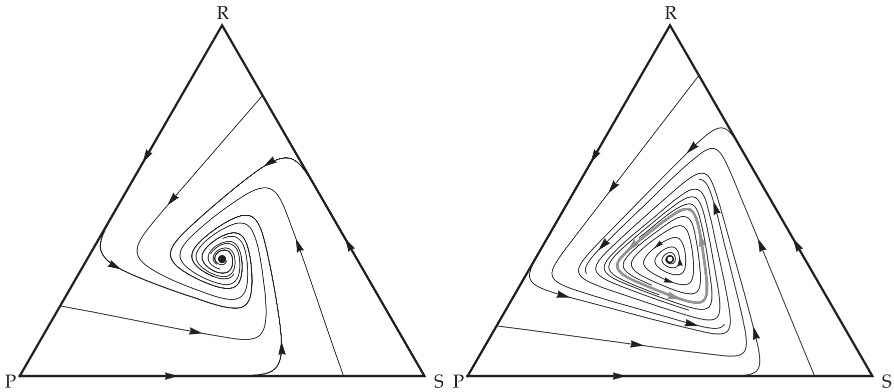

FIGURE 3. Logit dynamics for Rock-Paper-Scissors games: $a \geq b$ versus $a < b$

potential function $W : \mathbb{R}^n \to \mathbb{R}$, with $\frac{\partial W}{\partial a_i} = C_i(a)$ which is convex, and has $-v$ as its Legendre transform. If the (ε_i) are i.i.d. with the extreme value distribution $F(x) = \exp(-\exp(-x))$ then $C(a) = L(a)$ is the logit choice function and we obtain (2.4).

THEOREM 2.4. [**22**] *In a negative semidefinite game* (1.3)*, the smoothed BR dynamics* (2.5) *(including the logit dynamics) has a unique equilibrium $\hat{x}_\varepsilon$. It is globally asymptotically stable.*

The proof uses the Lyapunov function

$$V(x) = \pi_{\varepsilon v}(b_{\varepsilon v}(x), x) - \pi_{\varepsilon v}(x, x) \geq 0 \quad \text{with} \quad \pi_{\varepsilon v}(z, x) = z\cdot a(x) + \varepsilon v(z).$$

I will return to these perturbed dynamics in section 5. For more information on the logit dynamics see Sorin's chapter [**56**] and references therein, and [**43**].

The Brown–von Neumann–Nash dynamics. The *Brown–von Neumann–Nash dynamics* (BNN) is defined as

$$\dot{x}_i = \hat{a}_i(x) - x_i \sum_{j=1}^{n} \hat{a}_j(x), \tag{2.7}$$

where

$$\hat{a}_i(x) = [a_i(x) - x\cdot a(x)]_+ \tag{2.8}$$

(with $u_+ = \max(u, 0)$) denotes the positive part of the excess payoff for strategy i. This dynamics is closely related to the continuous map $f : \Delta \to \Delta$ defined by

$$f_i(x) = \frac{x_i + h\hat{a}_i(x)}{1 + h\sum_{j=1}^{n} \hat{a}_j(x)} \tag{2.9}$$

which Nash [**41**] used (for $h = 1$) to prove the existence of equilibria, by applying Brouwer's fixed point theorem: It is easy to see that $\hat{x}$ is a fixed point of f iff it is a rest point of (2.7) iff $\hat{a}_i(\hat{x}) = 0$ for all i, i.e. iff $\hat{x}$ is a Nash equilibrium of the game.

Rewriting the Nash map (2.9) as a difference equation, and taking the limit $\lim_{h\to 0} \frac{f(x)-x}{h}$ yields (2.7). This differential equation was considered earlier by

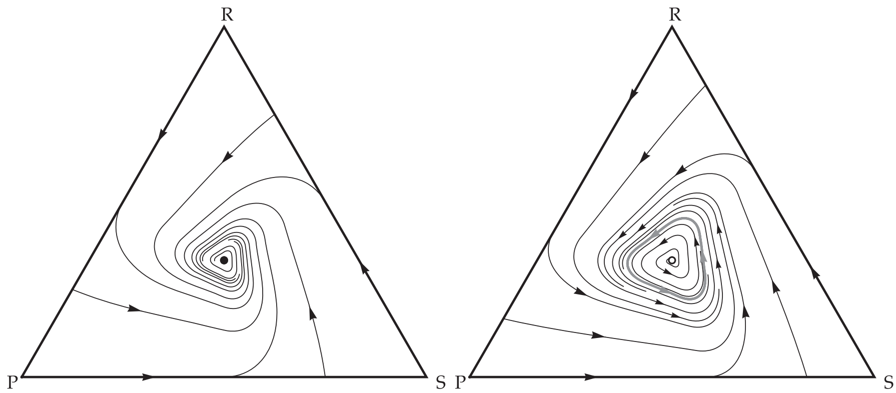

FIGURE 4. BNN dynamics for Rock-Paper-Scissors games: $a \geq b$ versus $a < b$

Brown and von Neumann [**7**] in the special case of zero–sum games, for which they proved global convergence to the set of equilibria.

In contrast to the best reply dynamics, the BNN dynamics (2.7) is Lipschitz (if payoffs are Lipschitz) and hence has unique solutions.

Equation (2.7) defines an 'innovative better reply' dynamics. A strategy not present that is a best (or at least a better) reply against the current population will enter the population.

THEOREM 2.5. [**7, 22, 24, 42**] *For a negative semidefinite game* (1.3)*, the convex set of its equilibria is globally asymptotically stable for the BNN dynamics* (2.7)*.*

The proof uses the Lyapunov function $V = \frac{1}{2}\sum_i \hat{a}_i(x)^2$, since $V(x) \geq 0$ with equality at NE, and

$$\dot{V} = \dot{x}{\cdot}A\dot{x} - \dot{x}{\cdot}Ax \sum_i \hat{a}_i(x) \leq 0,$$

with equality ony at NE.

Dynamics based on pairwise comparison. The BNN dynamics is a prototype of an innovative dynamics. A more natural way to derive innovative dynamics is the following,

$$\dot{x}_i = \sum_j x_j \rho_{ji} - x_i \sum_j \rho_{ij}, \tag{2.10}$$

in the form of an input–output dynamics. Here $x_i \rho_{ij}$ is the flux from strategy i to strategy j, and $\rho_{ij} = \rho_{ij}(x) \geq 0$ is the rate at which an i player switches to the j strategy.

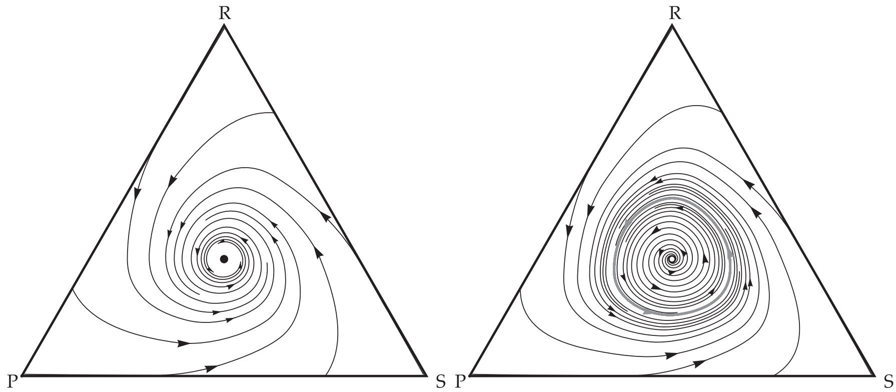

FIGURE 5. Smith's pairwise difference dynamics for Rock-Paper-Scissors games: $a \geq b$ versus $a < b$

A natural assumption on the *revision protocol*[7] ρ is

$$\rho_{ij} > 0 \Leftrightarrow a_j > a_i, \qquad \text{and} \qquad \rho_{ij} \geq 0.$$

Here switching to *any* better reply is possible, as opposed to the BR dynamics where switching is only to the optimal strategies (usually there is only one of them), or the BNN dynamics where switching occurs only to strategies better than the population average.

An important special case is when the switching rate depends on the payoff difference only, i.e.,

$$\rho_{ij} = \phi(a_j - a_i) \tag{2.11}$$

where ϕ is a function with $\phi(u) > 0$ for $u > 0$ and $\phi(u) = 0$ for $u \leq 0$. The resulting dynamics (2.10) is called *pairwise comparison dynamics*. The natural choice seems $\phi(u) = u_+$, given by the proportional rule

$$\rho_{ij} = [a_j - a_i]_+. \tag{2.12}$$

The resulting *pairwise difference dynamics* (PD)

$$\dot{x}_i = \sum_j x_j [a_i - a_j]_+ - x_i \sum_j [a_j - a_i]_+ \tag{2.13}$$

was introduced by Michael J. Smith [**55**] in the transportation literature as a dynamic model for congestion games. He also proved the following global stability result.

THEOREM 2.6. [**55**] *For a negative semidefinite game* (1.3)*, the convex set of its equilibria is globally asymptotically stable for the PD dynamics* (2.13)*.*

[7] All the basic dynamics considered so far can be written in the form (2.10) with a suitable revision protocol ρ (with some obvious modification in the case of the multi–valued BR dynamics). Given the revision protocol ρ, the payoff function a, and a finite population size N, there is a natural finite population model in terms of a Markov process on the grid $\{x \in \Delta : Nx \in \mathbb{Z}^n\}$. The differential equation (2.10) provides a very good approximation of the behavior of this stochastic process, at least over finite time horizons and for large population sizes. For all this see Sandholm's chapter [**49**].

The proof uses the Lyapunov function $V(x) = \sum_{i,j} x_j[a_i(x) - a_j(x)]_+^2$, by showing $V(x) \geq 0$ and $V(x) = 0$ iff x is a NE, and

$$2\dot{V} = \dot{x}\cdot A\dot{x} + \sum_{k,j} x_k \rho_{kj} \sum_i (\rho_{ji}^2 - \rho_{ki}^2) < 0$$

except at NE. This result extends to pairwise comparison dynamics (2.10,2.11), see [**24**].

The payoff projection dynamics. A more recent proof of the existence of Nash equilibria, due to *Gül–Pearce–Stacchetti* [**15**] uses the payoff projection map

$$P_h x = \Pi_\Delta(x + ha(x))$$

Here $h > 0$ is fixed and $\Pi_\Delta : \mathbb{R}^n \to \Delta$ is the projection onto the simplex Δ, assigning to each vector $u \in \mathbb{R}^n$ the point in the compact convex set Δ which is closest to u. Now $\Pi_\Delta(z) = y$ iff for all $x \in \Delta$, the angle between $x - y$ and $z - y$ is obtuse, i.e., iff $(x-y)\cdot(z-y) \leq 0$ for all $x \in \Delta$. Hence, $P_h\hat{x} = \hat{x}$ iff for all $x \in \Delta$, $(x - \hat{x})\cdot a(\hat{x}) \leq 0$, i.e., iff $\hat{x}$ is a Nash equilibrium. Since the map $P_h : \Delta \to \Delta$ is continuous Brouwer's fixed point theorem implies the existence of a Nash equilibrium.

Writing this map as a difference equation, we obtain in the limit $h \to 0$

$$\dot{x} = \lim_{h\to 0} \frac{\Pi_\Delta(x + ha(x)) - x}{h} = \Pi_{T(x)} a(x) \tag{2.14}$$

with

$$T(x) = \{\xi \in \mathbb{R}^n : \sum_i \xi_i = 0, \xi_i \geq 0 \text{ if } x_i = 0\}$$

being the cone of feasible directions at x into Δ.

This is the *payoff projection dynamics* (PP) of Lahkar and Sandholm [**34**]. The latter equality in (2.14) and its dynamic analysis use some amount of convex analysis, in particular the Moreau decomposition, see [**1, 34**].

For $x \in \operatorname{int}\Delta$ we obtain

$$\dot{x}_i = a_i(x) - \frac{1}{n}\sum_k a_k(x)$$

which, for a linear game, is simply a linear dynamics. It appeared in many places as a suggestion for a simple game dynamics, but how to treat this on the boundary has been rarely dealt with. Indeed, the vector field (2.14) is discontinuous on $\operatorname{bd}\Delta$. However, essentially because P_h is Lipschitz, solutions exist for all $t \geq 0$ and are unique (in forward time). This can be shown by rewriting (2.14) as a viability problem in terms of the normal cone ([**1, 34**])

$$\dot{x} \in a(x) - N_\Delta(x), \quad x(t) \in \Delta.$$

THEOREM 2.7. [**34**] *In a negative definite game* (1.2)*, the unique NE is globally asymptotically stable for the payoff projection dynamics* (2.14)*.*

The proof uses as Lyapunov function the Euclidean distance to the equilibrium $V(x) = \sum_i (x_i - \hat{x}_i)^2$.

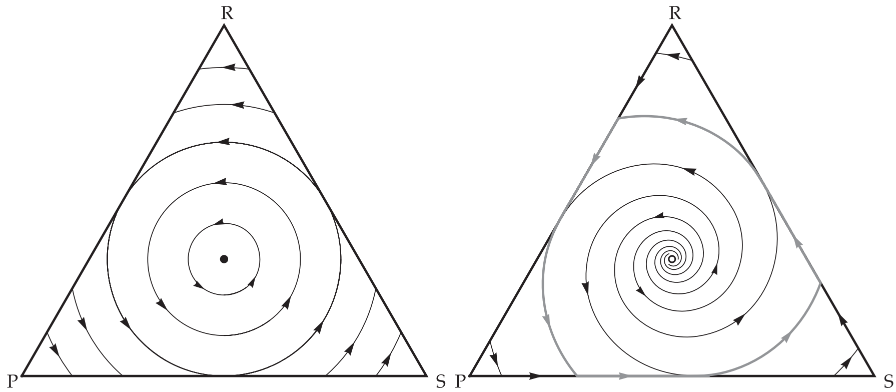

FIGURE 6. Payoff projection dynamics for Rock-Paper-Scissors games: $a = b$ versus $a < b$

Summary. As we have seen many of the special dynamics are related to maps that have been used to prove existence of Nash equilibria. The best response dynamics, the perturbed best response dynamics, and the BNN dynamics correspond to the three proofs given by Nash himself: [**39, 40, 41**]. The payoff projection dynamics is related to [**15**]. Even the replicator dynamics can be used to provide such a proof, if only after adding a mutation term, see [**26, 27**], or Sigmund's chapter [**53**, (11.3)]:

$$\dot{x}_i = x_i\,(a_i(x) - x{\cdot}a(x)) + \varepsilon_i - x_i \sum_j \varepsilon_j, \quad i = 1, \dots, n \tag{2.15}$$

with $\varepsilon_i > 0$ describing mutation rates.

Moreover, there is a result analogous to Theorem 2.4.

THEOREM 2.8. *For a negative semidefinite game* (1.3), *and any* $\varepsilon_i > 0$, (2.15) *has a unique rest point* $\hat{x}(\varepsilon) \in \Delta$. *It is globally asymptotically stable, and for* $\varepsilon \to 0$ *it approaches the set of NE of the game.*

I show a slightly more general result. With the notation $\phi_i(x_i) = \frac{\varepsilon_i}{x_i}$, let us rewrite (2.15) as

$$\dot{x}_i = x_i\left((Ax)_i + \phi_i(x_i) - x{\cdot}Ax - \bar{\phi}\right) \tag{2.16}$$

where $\bar{\phi} = \sum_i x_i \phi_i(x_i)$. In the following, I require only that each ϕ_i is a strictly decreasing function.

THEOREM 2.9. *If* A *is a negative semidefinite game* (1.3) *and the functions* ϕ_i *are strictly decreasing, then there is a unique rest point* $\hat{x}(\phi)$ *for* (2.16) *which is globally asymptotically stable for* (2.16).

PROOF. By Brouwer's fixed point theorem, (2.16) has a rest point $\hat{x} \in \Delta$. Consider now the function $L(x) = \sum_i \hat{x}_i \log x_i$ defined on int Δ. Then

$$\begin{aligned} \dot{L} &= \sum_i \hat{x}_i \frac{\dot{x}_i}{x_i} = \sum_i \hat{x}_i \left((Ax)_i + \phi_i(x_i) - x\cdot Ax - \bar{\phi}\right) \\ &= \sum_i (\hat{x}_i - x_i)\left((Ax)_i - \phi_i(x_i)\right) \\ &= -(\hat{x} - x)\cdot A(\hat{x} - x) - \sum_i (x_i - \hat{x}_i)\left(\phi_i(\hat{x}_i) - \phi_i(x_i)\right) \geq 0 \end{aligned}$$

with equality only at $x = \hat{x}$. Hence L is a Lyapunov function for $\hat{x}$, and hence $\hat{x}$ is globally asymptotically stable (w.r.t. int Δ). □

The six basic dynamics described so far enjoy the following common properties.

1. The unique NE of a negative definite game (in particular, any interior ESS) is globally asymptotically stable.
2. Interior NE of a positive definite game ('anti-ESS') are repellors.

Because of the nice behaviour of negative (semi-)definite games with respect to these basic dynamics, Sandholm christened them **stable games**.

For nonlinear games in a single population these are games whose payoff function $a : \Delta \to \mathbb{R}^n$ satisfies

$$(a(x) - a(y))(x - y) \leq 0 \qquad \forall x, y \in \Delta \tag{2.17}$$

or equivalently, if a is smooth,

$$z \cdot a'(x)z \leq 0 \qquad \forall x \in \Delta, z \in \mathbb{R}^n_0$$

Examples are congestion games [**48**], the war of attrition [**36**], the sex–ratio game [**36**], the habitat selection game [**10**], or simply the nonlinear payoff function $a(x) = Ax + \phi(x)$ in (2.16).

The global stability theorems 2.1, 2.3, 2.4, 2.5, 2.6, 2.14 hold for general stable N population games, see [**24, 48**].

3. Bimatrix games

The replicator dynamics for an $n \times m$ bimatrix game (A, B) reads

$$\begin{aligned} \dot{x}_i &= x_i\big((Ay)_i - x\cdot Ay\big), \quad i = 1, \dots, n \\ \dot{y}_j &= y_j\big((B^T x)_j - x\cdot By\big) \quad j = 1, \dots, m \end{aligned}$$

For its properties see [**26, 27**] and especially [**21**]. N person games are treated in [**61**] and [**44**].

The best reply dynamics for bimatrix games reads

$$\dot{x} \in BR^1(y) - x \qquad \dot{y} \in BR^2(x) - y \tag{3.1}$$

See Sorin [**56**, section 1] for more information.

For 2×2 games the state space $[0, 1]^2$ is two-dimensional and one can completely classify the dynamic behaviour. There are four robust cases for the replicator dynamics, see [**26, 27**], and additionally 11 degenerate cases. Some of these degenerate cases arise naturally as extensive form games, such as the Entry Deterrence

Game, see Cressman's chapter [**10**]. A complete analysis including all phase portraits are presented in [**9**] for (BR) and (REP), and in [**46**] for the BNN and the Smith dynamics.

For bimatrix games, stable games include zero-sum games, but not much more. We call an $n \times m$ bimatrix game (A, B) a *rescaled zero-sum game* [**26, 27**] if

$$\exists c > 0 : \quad u{\cdot}Av = -cu{\cdot}Bv \qquad \forall u \in \mathbb{R}^n_0, v \in \mathbb{R}^m_0 \tag{3.2}$$

or equivalently, there exists an $n \times m$ matrix C, $\alpha_i, \beta_j \in \mathbb{R}$ and $\gamma > 0$ s.t.

$$a_{ij} = c_{ij} + \alpha_j, \quad b_{ij} = -\gamma c_{ij} + \beta_i, \quad \forall i = 1, \dots, n, j = 1, \dots, m$$

For 2×2 games, this includes an open set of payoff matrices, corresponding to games with a cyclic best reply structure, or equivalently, those with a unique and interior Nash equilibrium. Simple examples are the *Odd or Even* game [**53**, (1.1)], or the Buyers and Sellers game [**10**]. However, for larger n, m this is a thin set of games, e.g. for 3×3 games, this set has codimension 3.

For such rescaled zero-sum games, the set of Nash equilibria is stable for (REP), (BR) and the other basic dynamics.

One of the main open problems in evolutionary game dynamics concerns the converse.

CONJECTURE 3.1. *Let (p, q) be an isolated interior equilibrium of a bimatrix game (A, B), which is stable for the BR dynamics or for the replicator dynamics. Then $n = m$ and (A, B) is a rescaled zero sum game.*

4. Dominated Strategies

A pure strategy i (in a single population game with payoff function $a : \Delta \to \mathbb{R}^n$) is said to be *strictly dominated* if there exists some $y \in \Delta$ such that

$$a_i(x) < y{\cdot}a(x) \tag{4.1}$$

for all $x \in \Delta$. A rational player will not use such a strategy.

In the best response dynamics, $\dot{x}_i = -x_i$ and hence $x_i(t) \to 0$ as $t \to \infty$.

Similarly, for the replicator dynamics, $L(x) = \log x_i - \sum_k y_k \log x_k$ satisfies $\dot{L}(x) < 0$ for $x \in \operatorname{int} \Delta$ and hence $x_i(t) \to 0$ along all interior orbits of (REP).

A similar result holds for extensions of (REP), given by differential equations of the form

$$\dot{x}_i = x_i g_i(x) \tag{4.2}$$

where the functions g_i satisfy $\sum x_i g_i(x) = 0$ on Δ. The simplex Δ and its faces are invariant. Such an equation is said to be *payoff monotonic* [**61**] if for any i, j, and $x \in \Delta$

$$g_i(x) > g_j(x) \Leftrightarrow a_i(x) > a_j(x). \tag{4.3}$$

All dynamics arising from an imitative revision protocol have this property. For such payoff monotonic dynamics, if the pure strategy i is strictly dominated by another pure strategy j, i.e., $a_i(x) < a_j(x)$ for all $x \in \Delta$ then $\frac{x_i}{x_j}$ goes monotonically to zero, and hence $x_i(t) \to 0$. However, if the dominating strategy is mixed, this need no longer be true, see [**20, 30**].

The situation is even worse for all other basic dynamics from section 2, in particular, (BNN), (PD) and (PP). As shown in [**4, 25, 34**] there are games with a pure strategy i being strictly dominated by another pure strategy j such that

i survives in the long run, i.e., $\liminf_{t\to+\infty} x_i(t) > 0$ for an open set of initial conditions.

5. Supermodular Games and Monotone Flows

An interesting class of games are the *supermodular games* (also known as games with strict strategic complementarities [**59**]). They make use of the natural order among pure strategies and are defined by

$$a_{i+1,j+1} - a_{i+1,j} - a_{i,j+1} + a_{i,j} > 0 \qquad \forall i,j \tag{5.1}$$

where $a_{i,j} = a_{ij}$ are the entries of the payoff matrix A. This means that for any $i < n$, $a_{i+1,k} - a_{ik}$ increases strictly with k.

In the case of $n = 2$ strategies this reduces to $a_{22} - a_{21} - a_{12} + a_{11} > 0$, which means that the game is positive definite (1.5). In particular, every bistable 2×2 game is supermodular.

For $n \geq 3$ there is no simple relation between supermodular games and positive definite games, although they share some properties, such as the instability of interior NE. For example, the RPS game with $b > a$ is positive definite but not supermodular. Indeed, a supermodular game cannot have a best reply cycle among the pure strategies, see below. On the other hand, an $n\times n$ pure coordination game (where the payoff matrix is a positive diagonal matrix) is positive definite, but supermodular only if $n = 2$.

Stochastic dominance defines a partial order on the simplex Δ:

$$p \succeq p' \quad \Leftrightarrow \quad \sum_{k=1}^{m} p_k \leq \sum_{k=1}^{m} p'_k \quad \forall m = 1,\dots,n-1. \tag{5.2}$$

If all inequalities in (5.2) are strict, we write $p \succ p'$. The intuition is that p has more mass to the right than p'. This partial order extends the natural order on the pure strategies:

$1 \prec 2 \prec \cdots \prec n$. Here k is identified with the kth unit vector, i.e., a corner of Δ.

LEMMA 5.1. *Let (u_k) be an increasing sequence, and $x \preceq y$. Then $\sum u_k x_k \leq \sum u_k y_k$. If (u_k) is strictly increasing and $x \preceq y$, $x \neq y$ then $\sum u_k x_k < \sum u_k y_k$. If (u_k) is increasing but not constant and $x \prec y$ then $\sum u_k x_k < \sum u_k y_k$.*

The proof follows easily from Abel summation (the discrete analog of integration by parts): set $x_k - y_k = c_k$ and $u_{n-1} = u_n - v_{n-1}$, $u_{n-2} = u_n - v_{n-1} - v_{n-2}$, etc.

LEMMA 5.2. *For $i < j$ and $x \preceq y$, $x \neq y$:*

$$(Ax)_j - (Ax)_i < (Ay)_j - (Ay)_i. \tag{5.3}$$

PROOF. Take $u_k = a_{jk} - a_{ik}$ as strictly increasing sequence in the previous lemma. □

The crucial property of supermodular games is the monotonicity of the best reply correspondence.

THEOREM 5.3. [**59**] *If $x \preceq y$, $x \neq y$ then $\max BR(x) \leq \min BR(y)$, i.e., no pure best reply to y is smaller than a pure best reply to x.*

PROOF. Let $j = \max BR(x)$. Then for any $i < j$ (5.3) implies that $(Ay)_j > (Ay)_i$, hence $i \notin BR(y)$. Hence every element in $BR(y)$ is $\geq j$. $\square$

Some further consequences of Lemma 5.2 and Theorem 5.3 are:

The extreme strategies 1 and n are either strictly dominated strategies or pure Nash equilibria.

There are no best reply cycles: Every sequence of pure strategies which is sequential best replies is finally constant and ends in a pure NE.

For results on the convergence of fictitious play and the best response dynamics in supermodular games see [**3, 32**].

THEOREM 5.4. *Mixed (=nonpure) equilibria of supermodular games are unstable under the replicator dynamics.*

PROOF. W.l.o.g., we can assume that the equilibrium $\hat{x}$ is interior (otherwise restrict to a face). A supermodular game satisfies $a_{ij} + a_{ji} < a_{ii} + a_{jj}$ for all $i \neq j$ (set $x = i, y = j$ in (5.3)). Hence, if we normalize the game by $a_{ii} = 0$, $\hat{x}{\cdot}A\hat{x} = \sum_{i,j}(a_{ij} + a_{ji})\hat{x}_i\hat{x}_j < 0$. Now it is shown in [**27**, p.164] that $-\hat{x}{\cdot}A\hat{x}$ equals the trace of the Jacobian of (REP) at $\hat{x}$, i.e., the sum of all its eigenvalues. Hence at least one of the eigenvalues has positive real part, and $\hat{x}$ is unstable. $\square$

For different instability results of mixed equilibria see [**11**].

The following is a generalization of Theorem 5.3 to perturbed best replies, due to [**23**]. I present here a different proof.

THEOREM 5.5. *For every supermodular game*

$$x \preceq y, x \neq y \quad \Rightarrow \quad C(a(x)) \prec C(a(y))$$

holds if the choice function $C : \mathbb{R}^n \to \Delta_n$ *is* C^1 *and the partial derivatives* $C_{i,j} = \frac{\partial C_i}{\partial x_j}$ *satisfy for all* $1 \leq k, l < n$

$$\sum_{i=1}^{k}\sum_{j=1}^{l} C_{i,j} > 0, \tag{5.4}$$

and for all $1 \leq i \leq n$,

$$\sum_{j=1}^{n} C_{i,j} = 0. \tag{5.5}$$

PROOF. It is sufficient to show that the perturbed best reponse map is strongly monotone:

$$x \preceq y, x \neq y \quad \Rightarrow \quad C(Ax) \prec C(Ay)$$

From Lemma 5.2 we know: If $x \preceq y$, $x \neq y$ then $(Ay - Ax)_i$ increases strictly in i. Hence, with $a = Ax$ and $b = Ay$, it remains to show:

LEMMA 5.6. *Let* $a, b \in \mathbb{R}^n$ *with* $b_1 - a_1 < b_2 - a_2 < \cdots < b_n - a_n$. *Then* $C(a) \prec C(b)$.

This means that for each k: $C_1(a) + \cdots + C_k(a) \geq C_1(b) + \cdots + C_k(b)$.

Taking derivative in direction $u = b - a$, this follows from $\sum_{i=1}^{k}\sum_{j=1}^{n} C_{i,j}u_j < 0$ which by Lemma 5.1 holds whenever $(x_j - y_j =)c_j = \sum_{i=1}^{k} C_{i,j}$ satisfies $\sum_{j=1}^{l} c_j > 0$ for $l = 1, \ldots, n-1$ and $\sum_{j=1}^{n} c_j = 0$. $\square$

The conditions (5.4, 5.5) on C hold for every stochastic choice model (2.6), since there $C_{i,j} < 0$ for $i \neq j$. As a consequence, the perturbed best reply dynamics

$$\dot{x} = C(a(x)) - x \tag{5.6}$$

generates a strongly monotone flow: If $x(0) \preceq y(0), x(0) \neq y(0)$ then $x(t) \prec y(t)$ for all $t > 0$. The theory of monotone flows developped by Hirsch and others (see [**54**]) implies that almost all solutions of (5.6) converge to a rest point of (5.6).

It seems that the other basic dynamics do not respect the stochastic dominance order (5.2). They do not generate a monotone flow for every supermodular game. Still there is the open problem

PROBLEM 5.7. *In a supermodular game, do almost all orbits of (BR), (REP), (BNN), (PD), (PP) converge to a NE?*

For the best reponse dynamics this entails to extend the theory of monotone flows to cover discontinuous differential equations or differential inclusions.

6. Partnership games and general adjustment dynamics

We consider now games with a symmetric payoff matrix $A = A^T$ ($a_{ij} = a_{ji}$ for all i, j). Such games are known as *partnership games* [**26, 27**] and *potential games* [**38**]. The basic population genetic model of Fisher and Haldane is equivalent to the replicator dynamics for such games, which is then a gradient system with respect to the Shahshahani metric and the mean payoff $x{\cdot}Ax$ as potential, see e.g. [**26, 27**]. The resulting increase of mean fitness or mean payoff $x{\cdot}Ax$ in time is often referred to as the *Fundamental Theorem of Natural Selection*. This statement about the replicator dynamics generalizes to the other dynamics considered here.

The generalization is based on the concept, defined by Swinkels [**58**], of a (myopic) adjustment dynamics which satisfies $\dot{x}{\cdot}Ax \geq 0$ for all $x \in \Delta$, with equality only at equilibria. If $A = A^T$ then the mean payoff $x \cdot Ax$ is increasing for every adjustment dynamics since $(x \cdot Ax)^{\cdot} = 2\dot{x} \cdot Ax \geq 0$. It is obvious that the best response dynamics (2.2) is an adjustment dynamics and it is easy to see that the other special dynamics from section 2 are as well.

As a consequence, we obtain the following result.

THEOREM 6.1. [**20, 22**] *For every partnership game $A = A^T$, the potential function $x \cdot Ax$ increases along trajectories. Hence every trajectory of every adjustment dynamics (in particular* (2.1), (2.2), (2.7), *and* (2.13)*) converges to (a connected set of) equilibria. A strict local maximizer of $x \cdot Ax$ is asymptotically stable for every adjustment dynamics.*

Generically, equilibria are isolated. Then the above result implies convergence for each trajectory. Still, continua of equilibria occur in many interesting applications, see e.g. [**45**]. Even in this case, it is known that every trajectory of the replicator dynamics converges to a rest point, and hence each interior trajectory converges to a Nash equilibrium, see e.g. [**26**, ch. 23.4] or [**27**, ch. 19.2]. It is an open problem whether the same holds for the other basic dynamics.

For the perturbed dynamics (2.5) (for a concave function v on int Δ) and (2.15) there is an analog of Theorem 6.1: A suitably perturbed potential function serves as a Lyapunov function.

THEOREM 6.2. [**22, 26**] *For every partnership game $A = A^T$:*
the function $P(x) = \frac{1}{2}x{\cdot}Ax + \varepsilon v(x)$ increases monotonically along solutions of (2.5),
the function $P(x) = \frac{1}{2}x \cdot Ax + \sum_i \varepsilon_i \log x_i$ is a Lyapunov function for (2.15).
Hence every solution converges to a connected set of rest points.

For bimatrix games the adjustment property is defined as

$$\dot{x}{\cdot}Ay \geq 0, \quad x{\cdot}B\dot{y} \geq 0.$$

A bimatrix game is a partnership/potential game if $A = B$, i.e., if both players obtain the same payoff [**26**, ch. 27.2]. Then the potential $x{\cdot}Ay$ increases monotonically along every solution of every adjustment dynamics.

For the general situation of potential games between N populations with nonlinear payoff functions see [**48**].

7. A universal Shapley example

The simplest example of persistent cycling in a game dynamics is probably the RSP game (1.4) with $b > a$ for the BR dynamics (2.2) which leads to a triangular shaped limit cycle, see figure 1 (right). Historically, Shapley [**52**] gave the first such example in the context of 3×3 bimatrix games (but it is less easy to visualize because of the 4d state space). Our six basic dynamics show a similar cycling behavior for positive definite RSP games.

But given the huge pool of adjustment dynamics, we now ask: Is there an evolutionary dynamics, which converges for each game from each initial condition to an equilibrium?

Such a dynamics is assumed to be given by a differential equation

$$\dot{x} = f(x, a(x)) \tag{7.1}$$

such that f depends continuously on the population state x and the payoff function a.

For N player binary games (each player chooses between two strategies only) general evolutionary dynamics are easy to describe:
The better of the two strategies increases, the other one decreases, i.e.,

$$\dot{x}_{i1} = -\dot{x}_{i2} > 0 \Leftrightarrow a^i(1, x^{-i}) > a^i(2, x^{-i}) \tag{7.2}$$

holds for all i at all (interior) states. Here x_{ij} denotes the frequency of strategy j used by player i, and $a^i(j, x^{-i})$ his payoff. In a common interest game where each player has the same payoff function $P(x)$, along solutions $x(t)$, $P(x(t))$ increases monotonically:

$$\dot{P} = \sum_{i=1}^{N}\sum_{k=1}^{2} a^i(k, x^{-i})\dot{x}_{ik} = \sum_{i=1}^{N} \left[a^i(1, x^{-i}) - a^i(2, x^{-i})\right] \dot{x}_{i1} \geq 0. \tag{7.3}$$

A familiy of $2 \times 2 \times 2$ games. Following [**29**], we consider 3 players, each with 2 pure strategies. The payoffs are summarized in the usual way as follows.

-1,-1,-1	$0, 0, \varepsilon$		$\varepsilon, 0, 0$	$0, \varepsilon, 0$
$0, \varepsilon, 0$	$\varepsilon, 0, 0$		$0, 0, \varepsilon$	-1,-1,-1

The first player (left payoff) chooses the row, the second chooses the column, the third (right payoff) chooses one of the matrices. For $\varepsilon \neq 0$, this game has a unique equilibrium $E = \left(\frac{1}{2}, \frac{1}{2}, \frac{1}{2}\right)$ at the centroid of the state space, the cube $[0, 1]^3$. This equilibrium is regular for all ε. For $\varepsilon > 0$, this game has a best response cycle among the six pure strategy combinations $122 \to 121 \to 221 \to 211 \to 212 \to 112 \to 122$.

For $\varepsilon = 0$, this game is a potential game: Every player gets the same payoff

$$P(x) = -x_{11}x_{21}x_{31} - x_{21}x_{22}x_{32}.$$

The minimum value of P is -1 which is attained at the two pure profiles 111 and 222. At the interior equilibrium E, its value is $P(E) = -\frac{1}{4}$. P attains its maximum value 0 at the set Γ of all profiles, where two players use opposite pure strategies, whereas the remaining player may use any mixture. All points in Γ are Nash equilibria. Small perturbations in the payoffs ($\varepsilon \neq 0$) can destroy this component of equilibria.

For every natural dynamics, $P(x(t))$ increases. If $P(x(0)) > P(E) = -\frac{1}{4}$ then $P(x(t)) \to 0$ and $x(t) \to \Gamma$. Hence Γ is an attractor (an asymptotically stable invariant set) for the dynamics, for $\varepsilon = 0$.

For small $\varepsilon > 0$, there is an attractor Γ_ε near Γ whose basin contains the set $\{x : P(x) > -\frac{1}{4} + \gamma(\varepsilon)\}$, with $\gamma(\varepsilon) \to 0$, as $\varepsilon \to 0$. This follows from the fact that attractors are upper–semicontinuous against small perturbations of the dynamics (for proofs of this fact see, e.g., [**25, 2**]). But for $\varepsilon > 0$, the only equilibrium is E.

Hence we have shown

THEOREM 7.1. [**29**] *For each dynamics satisfying the assumptions* (7.2) *and continuity in payoffs, there is an open set of games and an open set of initial conditions* $x(0)$ *such that* $x(t)$ *stays away from the set of NE, for large* $t > 0$.

Similar examples can be given as 4×4 symmetric one population games, see [**27**], and 3×3 bimatrix games, see [**29**]. The proofs follow the same lines: For $\varepsilon = 0$ these are potential games, the potential maximizer is a quadrangle or a hexagon, and this component of NE disappears for $\varepsilon \neq 0$ but continues to a nearby attractor for the dynamics.

A different general nonconvergence result is due to Hart and Mas-Colell [**18**].

For specific dynamics there are many examples with cycling and even chaotic behavior: Starting with Shapley [**52**] there are [**8, 13, 16, 47, 57**] for the best response dynamics. For other examples and a more complete list of references see [**48**, ch. 9].

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