

## BEST RESPONSE DYNAMICS FOR CONTINUOUS ZERO-SUM GAMES

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**ABSTRACT.** We study best response dynamics in continuous time for continuous concave-convex zero-sum games and prove convergence of its trajectories to the set of saddle points, thus providing a dynamical proof of the minmax theorem. Consequences for the corresponding discrete time process with small or diminishing step-sizes are established, including convergence of the fictitious play procedure.

**1. Introduction.** This paper contributes to the literature studying game theoretical problems through dynamical tools. A first example was a proof of the minmax theorem for finite games through a differential equation by Brown and von Neumann [7]. Another one is fictitious play, a discrete time process also introduced by Brown [6] and shown to converge to the set of optimal strategies for zero-sum games by Robinson [17]. Recently, continuous time dynamics have been used to give alternative proofs of this convergence result ([11], [12]). This approach is in the spirit of numerical dynamics [20] or more generally, stochastic approximation theory [2]: fictitious play is nothing but an Euler discretization procedure with diminishing stepsizes of a certain continuous time process. However, one technicality arises from the fact that the approximating continuous time dynamics, the best response dynamics studied below, is not given by a smooth differential equation, but by a discontinuous and even multivalued one. Thus we extend the basic result on numerical approximation (convergence to a global attractor) to the more general multivalued setting of a differential inclusion (section 5). More refined results involving chain recurrent components are established in [3].

In this paper we generalize the above mentioned convergence result of the best reply dynamics on the strategy space, from the bilinear case to the setup of concave-convex functions. We thus obtain a purely dynamic proof of the minmax theorem

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in the saddle case. Another consequence is a simple proof of Brown–Robinson’s convergence result in this framework.

**2. The main result.** Let  $X, Y$  be compact, convex subsets of some finite dimensional Euclidean spaces and  $U : X \times Y \rightarrow \mathbb{R}$  be a continuous saddle function, i.e., concave in  $x$  and convex in  $y$ . This defines a two-person zero-sum game where  $U$  is the payoff and  $X$  (resp.  $Y$ ) is the strategy set of the maximizer (resp. minimizer). Denote

$$A(y) = \max_{x \in X} U(x, y), \quad B(x) = \min_{y \in Y} U(x, y), \quad (1)$$

$A$  is convex and continuous on  $Y$ ,  $B$  is concave and continuous on  $X$ , using the Maximum theorem (Berge [4], p. 123). One has  $B(x) \leq U(x, y) \leq A(y)$ , for all  $x, y$  in  $X \times Y$  hence

$$\underline{w} = \max_{x \in X} B(x) = \max_{x \in X} \min_{y \in Y} U(x, y) \leq \min_{y \in Y} \max_{x \in X} U(x, y) = \min_{y \in Y} A(y) = \bar{w}$$

Let

$$V(x, y) = A(y) - B(x). \quad (2)$$

Then  $V : X \times Y \rightarrow \mathbb{R}$  is convex, nonnegative and continuous and its minimum,  $\bar{w} - \underline{w} \geq 0$ , is reached on a product of compact convex sets,  $X(U) \times Y(U)$ . If this minimum is 0, then the game is said to have a value  $w = \bar{w} = \underline{w}$ . More precisely,  $V(x, y) = 0$  iff  $A(y) = \bar{w} = \underline{w} = B(x)$  or equivalently  $U(x', y) \leq U(x, y) \leq U(x, y')$  for all  $x', y'$  in  $X \times Y$ :  $(x, y)$  belongs to  $X(U) \times Y(U)$ , the set of saddle points of  $U$  on  $X \times Y$ .

Under the above assumptions on  $U$ , the minmax theorem ([21], [9]) applies and the game is known to have a value. In fact weaker assumptions suffice (quasiconcave and u.s.c. in  $x$ , quasiconvex and l.s.c. in  $y$ ,  $X$  and  $Y$  convex compact subsets of a locally convex Hausdorff topological vector space [19]).

The following theorem provides an alternative proof for the existence of a value. Introduce the best response correspondences

$$\text{BR}_1(y) = \text{Argmax}_{x \in X} U(x, y), \quad \text{BR}_2(x) = \text{Argmin}_{y \in Y} U(x, y). \quad (3)$$

Since  $U$  is continuous, the Maximum theorem [4, p. 123] implies that  $\text{BR}_1$  (resp.  $\text{BR}_2$ ) is an upper semi-continuous correspondence from  $Y$  to  $X$  (resp. from  $X$  to  $Y$ ) with nonempty closed convex values. Consider the *best response dynamics* ([10], [12], [13], [14]) on  $X \times Y$

$$\dot{x} \in \text{BR}_1(y) - x, \quad \dot{y} \in \text{BR}_2(x) - y. \quad (4)$$

A *solution* of this differential inclusion is an absolutely continuous function  $t \mapsto (x(t), y(t))$  from  $[0, +\infty)$  to  $X \times Y$  satisfying (4) for almost all  $t \geq 0$ . Given a solution  $(x(t), y(t))$  of (4), denote  $v(t) = V(x(t), y(t))$ ,  $\alpha(t) = x(t) + \dot{x}(t) \in \text{BR}_1(y(t))$  and  $\beta(t) = y(t) + \dot{y}(t) \in \text{BR}_2(x(t))$ .

**Theorem.** (i) (4) has a solution for every initial condition  $(x(0), y(0)) \in X \times Y$ .  
(ii) Along every solution of (4),  $v(t)$  is absolutely continuous and satisfies

$$\dot{v}(t) \leq -v(t) \quad \text{for almost all } t \quad (5)$$

hence

$$v(t) \leq e^{-t} v(0). \quad (6)$$

Thus the game has a value and every solution of (4) converges to the nonempty set of saddle points  $X(U) \times Y(U)$ , which is a uniform global attractor for (4).

Versions of this result have been shown by Brown [6], Harris [11] and Hofbauer [12] for finite zero-sum games:  $X$  and  $Y$  being simplices and  $U$  bilinear. In this case equality holds in (5):  $\dot{v}(t) = -v(t)$ .

The Theorem will be proven in Section 4.

**3. Properties of the trajectories.** The best reply correspondence  $\text{BR} = \text{BR}_1 \times \text{BR}_2$  is u.s.c. from the compact space  $X \times Y$  to itself and has nonempty convex compact values. In addition the set  $X$  is convex and closed and its tangent cone  $T_X(x)$  at  $x$  contains all vectors  $x' - x$  with  $x' \in X$ . In particular  $\text{BR}_1(y) - x \subset T_X(x)$  and similarly for  $\text{BR}_2$ . The existence of solutions to (4) follows from general existence results (Aubin and Cellina [1, Chapter 4, Section 2, Theorem 1], or Clarke et al. [8, Section 4.2]). In addition one has a Lipschitz property: since  $X$  and  $Y$  are compact, the derivatives in (4) are uniformly bounded by the diameters of these sets.

**3.1. Boundary behavior.** The dimension of a convex set  $C$  is the dimension of the affine space it generates. A facet of a convex set  $C$  of dimension  $n$  is defined as  $C \cap H$  where  $H$  is a closed half space of dimension  $n$  which intersects  $C$  and such that the corresponding open half space does not.  $C \cap H$  is a convex set of dimension at most  $n - 1$ . The set of faces of  $C$  consists of  $C$  itself, all facets of  $C$ , facets of facets and so on. Let  $F_C(x)$  be the intersection of the faces of  $C$  containing  $x$ , and hence the minimal face of  $C$  containing  $x$ . If  $x$  is in the relative interior of  $C$  then  $F_C(x) = C$ . The relative boundary of a convex set equals the union of all its facets. Hence  $x$  is in the relative interior of the minimal face  $F_C(x)$ . This is also true if  $F_C(x) = \{x\}$  is a point.

**Lemma 1.** *Any trajectory  $(x(t), y(t))$  that belongs to a face of  $X \times Y$  at time  $T > 0$  is included in this face on  $[0, T]$ .*

*Proof.* First consider a facet of  $X \times Y$  with corresponding closed half space  $H$ . By applying a linear transformation one can assume that  $H$  corresponds to either  $\{x_1 \leq 0\}$  or  $\{y_1 \leq 0\}$  or  $\{x_1 \leq 0 \text{ and } y_1 \leq 0\}$ . Consider the first of these cases (the others are analogous). Suppose  $x_1(T) = 0$ , for some  $T > 0$ . Equation (4) and  $\text{BR}_1(y) \subseteq X \subseteq \{x_1 \geq 0\}$  implies that  $\dot{x}_1(t) \geq -x_1(t)$ , so that  $x_1(t) \geq e^{-(t-s)}x_1(s)$ , for all  $t \geq s \geq 0$ . Hence  $x_1(t) = 0$  for  $t \in [0, T]$ . It follows that  $\dot{x}_1(t) = 0$  as well. Let  $X' = X \cap \{x_1 = 0\}$  and  $\text{BR}'_1 = \text{BR}_1 \cap X'$ . Then  $(x(t), y(t))$  is a solution of the differential inclusion

$$\dot{x} \in \text{BR}'_1(y) - x, \quad \dot{y} \in \text{BR}_2(x) - y$$

on  $X' \times Y$  for  $t \in [0, T]$ . The proof then proceeds by induction on the dimension of  $X \times Y$ .  $\square$

**Corollary 2.** *On any trajectory  $(x(t), y(t))$ , for all but finitely many  $t$ ,  $(x(s), y(s))$  belongs to the relative interior of the minimal face  $F_X(x(t)) \times F_Y(y(t)) \subset X \times Y$  that contains  $(x(t), y(t))$ , for  $s$  close to  $t$ .*

*Proof.* Given any trajectory  $(x(t), y(t))$ , there are strictly increasing sequences of times  $t_i$  and of natural numbers  $m_i \geq 0$  such that on  $]t_i, t_{i+1}[$ ,  $x(t)$  belongs to the relative interior of a face of dimension  $m_i$ , and similarly for  $y(t)$ .  $\square$

**3.2. Directional derivatives on trajectories.** In the following lemma on convex functions, the one sided right and left derivatives of a function  $g : \mathbb{R} \rightarrow \mathbb{R}$  are

$$\begin{aligned}\frac{d^+}{dt} g(t) &= \dot{g}(t^+) = \lim_{h \downarrow 0} \frac{g(t+h) - g(t)}{h} \\ \frac{d^-}{dt} g(t) &= \dot{g}(t^-) = -\lim_{h \downarrow 0} \frac{g(t-h) - g(t)}{h}\end{aligned}$$

and the directional derivative of a function  $G : \mathbb{R}^n \rightarrow \mathbb{R}$  at  $x$  in direction  $v$  is

$$DG(x)(v) = \lim_{h \downarrow 0} \frac{G(x+hv) - G(x)}{h}.$$

**Lemma 3.** *Let  $t \mapsto y(t) \in \mathbb{R}^n$  be Lipschitz and  $f : \mathcal{U} \subset \mathbb{R}^n \rightarrow \mathbb{R}$  be a convex function where  $\mathcal{U}$  is an open convex set containing  $y(t)$ . Then  $t \mapsto f(y(t))$  is locally Lipschitz and its one-sided derivatives are given by*

$$\frac{d^\pm}{dt} f(y(t)) = \pm Df(y(t))(\pm \dot{y}(t^\pm)) \quad (7)$$

whenever  $\dot{y}(t^\pm)$  exists.

*Proof.* First  $t \mapsto f(y(t))$  is locally Lipschitz since  $f$  is locally Lipschitz (Rockafellar [18, Thm. 10.4]). If  $\dot{y}(t^+)$  exists then  $y(t+h) = y(t) + h\dot{y}(t^+) + o(h)$ . This implies

$$\begin{aligned}\frac{d^+}{dt} f(y(t)) &= \lim_{h \downarrow 0} \frac{f(y(t+h)) - f(y(t))}{h} \\ &= \lim_{h \downarrow 0} \frac{f(y(t) + h\dot{y}(t^+) + o(h)) - f(y(t))}{h} \\ &= \lim_{h \downarrow 0} \frac{f(y(t) + h\dot{y}(t^+)) - f(y(t))}{h}.\end{aligned}$$

Hence

$$\frac{d^+}{dt} f(y(t)) = Df(y(t))(\dot{y}(t^+))$$

which exists by convexity [18, Thm. 23.1]. Similarly

$$\frac{d^-}{dt} f(y(t)) = -\lim_{h \downarrow 0} \frac{f(y(t-h)) - f(y(t))}{h} = -Df(y(t))(-\dot{y}(t^-)).$$

□

The following is a version of the envelope theorem, compare [5, Section 4.3.1].

**Lemma 4.** *Let  $(x(t), y(t))$  be a solution of (4). Then the function  $t \mapsto A(y(t))$  is locally Lipschitz except for finitely many  $t$  and satisfies for almost all  $t > 0$*

$$\frac{d}{dt} A(y(t)) = D_y U(\alpha(t), y(t))(\dot{y}(t)) \quad (8)$$

where the right hand side means the directional partial derivative with respect to  $y$  in direction  $\dot{y}(t)$ .

*Proof.* Start with

$$\begin{aligned}A(y(s)) - A(y(t)) &= U(\alpha(s), y(s)) - U(\alpha(t), y(t)) \\ &\geq U(\alpha(t), y(s)) - U(\alpha(t), y(t))\end{aligned} \quad (9)$$

By Corollary 2 and Lemma 3, with  $\mathcal{U}$  being the relative interior of the minimal face  $F_Y(y(t))$ , the function  $t \mapsto A(y(t))$  is locally Lipschitz, except for finitely many  $t$ , and hence the limits

$$\mathcal{A}^+ = \lim_{s \downarrow t} \frac{A(y(s)) - A(y(t))}{s - t}, \quad \mathcal{A}^- = \lim_{s \uparrow t} \frac{A(y(s)) - A(y(t))}{s - t} \tag{10}$$

exist and coincide for almost all  $t > 0$ . Again by Lemma 3, (7), for each  $t > 0$  for which  $\dot{y}(t)$  exists,

$$\mathcal{B}^+ = \lim_{s \downarrow t} \frac{U(\alpha(t), y(s)) - U(\alpha(t), y(t))}{s - t} = D_y U(\alpha(t), y(t))(\dot{y}(t)), \tag{11}$$

and

$$\mathcal{B}^- = \lim_{s \uparrow t} \frac{U(\alpha(t), y(s)) - U(\alpha(t), y(t))}{s - t} = -D_y U(\alpha(t), y(t))(-\dot{y}(t)). \tag{12}$$

The convexity of  $U$  in  $y$  implies  $\mathcal{B}^- \leq \mathcal{B}^+$ , while [18, Thm. 23.1] and (9) show  $\mathcal{B}^+ \leq \mathcal{A}^+ = \mathcal{A}^- \leq \mathcal{B}^-$ . Hence  $\mathcal{B}^- = \mathcal{B}^+ = \mathcal{A}^+ = \mathcal{A}^-$ .  $\square$

Similar to (8), one obtains

$$\frac{d}{dt} B(x(t)) = D_x U(x(t), \beta(t))(\dot{x}(t)). \tag{13}$$

#### 4. Convergence theorems.

##### 4.1. Trajectories.

*Proof of the Theorem.* Along any solution of (4), (8) and (13) give, for almost all  $t$

$$\begin{aligned} \frac{d}{dt} v(t) &= D_y U(\alpha(t), y(t))(\dot{y}(t)) - D_x U(x(t), \beta(t))(\dot{x}(t)) \\ &= D_y U(\alpha(t), y(t))(\dot{y}(t)) - U(\alpha(t), \beta(t)) \\ &\quad + U(\alpha(t), \beta(t)) - D_x U(x(t), \beta(t))(\dot{x}(t)) \end{aligned} \tag{14}$$

$$\leq -U(\alpha(t), y(t)) + U(x(t), \beta(t)) \tag{15}$$

$$= -A(y(t)) + B(x(t))$$

$$= -V(x(t), y(t))$$

$$= -v(t) \tag{16}$$

where (15) follows from (14) by convexity (resp. concavity) of  $U$  in  $y$  (resp.  $x$ ). (Note that equality holds if  $U$  is bilinear, compare the remark after the Theorem.) By Lemma 4,  $v$  is locally Lipschitz except for finitely many  $t$ . Since  $v$  is continuous, it is absolutely continuous on  $[0, +\infty)$ . Thus (16) implies (6) and  $v(t)$  converges to 0 on each trajectory. This shows the existence of a value:  $\bar{w} = \underline{w} = w$ .

Since  $v(0)$  is uniformly bounded, (6) implies that for any  $\epsilon > 0$ , there exists  $T$  such that along every solution,  $t \geq T$  implies

$$B(x(t)) \geq w - \epsilon. \tag{17}$$

$B$  being u.s.c. and  $X$  compact there exists  $T'$  such that for  $t \geq T'$

$$d(x(t), X(U)) \leq \epsilon \tag{18}$$

where  $d$  denotes the usual Euclidean distance. A dual result holds for  $y(t)$ .  $\square$

We define  $M$  to be an *invariant set* of (4) if for each point  $m$  in  $M$ , there exists a solution  $m(t)$  of (4), defined for all positive and negative  $t \in \mathbb{R}$  with  $m(t) \in M$  and  $m(0) = m$ . The maximal invariant set is the union of all such complete trajectories.

**Corollary 5.** *The maximal invariant set of (4) is the set  $X(U) \times Y(U)$  of saddle points.*

*Proof.* (6) and boundedness of  $v$  imply that  $v(t) \equiv 0$  along any complete solution.  $\square$

**4.2. Payoffs.** In this section we prove convergence of a certain average payoff along the trajectories.

**Proposition 6.** *Define*

$$C(t_0, T) = \frac{1}{T} \int_{\ln(t_0)}^{\ln(t_0+T)} U(\alpha(t), \beta(t)) e^t dt.$$

*Then  $C(t_0, T) \rightarrow w$  as  $T \rightarrow \infty$ .*

*Proof.* Concavity implies

$$U(\alpha(t), \beta(t)) \leq U(x(t), \beta(t)) + D_x U(x(t), \beta(t))(\dot{x}(t)).$$

Using (13) this gives, for almost all  $t$

$$U(\alpha(t), \beta(t)) \leq B(x(t)) + \frac{d}{dt} B(x(t)).$$

This implies

$$\begin{aligned} C(t_0, T) &\leq \frac{1}{T} \int_{\ln(t_0)}^{\ln(t_0+T)} (B(x(t)) + \frac{d}{dt} B(x(t))) e^t dt \\ &= \frac{1}{T} \int_{\ln(t_0)}^{\ln(t_0+T)} \frac{d}{dt} (B(x(t)) e^t) dt \\ &= \frac{1}{T} (B(x(\ln(t_0+T))) (t_0+T) - B(x(\ln(t_0))) t_0). \end{aligned} \quad (19)$$

Hence  $\limsup_{T \rightarrow \infty} C(t_0, T) \leq w$ . A dual inequality gives the result.  $\square$

## 5. Discrete counterpart.

**5.1. Vanishing stepsizes.** Recall that a fictitious play process (Brown [6]) associated to the game  $U$  satisfies

$$p_{n+1} \in \text{BR}_1(Q_n), \quad q_{n+1} \in \text{BR}_2(P_n) \quad (20)$$

with initial values  $p_1 = P_1 \in X$ ,  $q_1 = Q_1 \in Y$  and for  $n \in \mathbb{N}$ ,

$$P_n = \frac{1}{n} \sum_{k=1}^n p_k, \quad Q_n = \frac{1}{n} \sum_{k=1}^n q_k.$$

Hence (20) gives first

$$\begin{aligned} (n+1)P_{n+1} - nP_n &\in \text{BR}_1(Q_n) \\ (n+1)Q_{n+1} - nQ_n &\in \text{BR}_2(P_n) \end{aligned} \quad (21)$$

and finally the difference inclusion

$$\begin{aligned} P_{n+1} - P_n &\in \frac{1}{n+1} [\text{BR}_1(Q_n) - P_n] \\ Q_{n+1} - Q_n &\in \frac{1}{n+1} [\text{BR}_2(P_n) - Q_n]. \end{aligned} \quad (22)$$

The corresponding equation in continuous time writes

$$\begin{aligned} \dot{P}(t) &\in \frac{1}{t}[\text{BR}_1(Q(t)) - P(t)] \\ \dot{Q}(t) &\in \frac{1}{t}[\text{BR}_2(P(t)) - Q(t)]. \end{aligned} \tag{23}$$

Changing time scale

$$x(t) = P(e^t), \quad y(t) = Q(e^t)$$

leads to the best response dynamics (4)

$$\dot{x} \in \text{BR}_1(y) - x, \quad \dot{y} \in \text{BR}_2(x) - y.$$

From (6), one obtains, as in [11], that any solution of (23) satisfies

$$v(P(t), Q(t)) \leq \frac{1}{t}v(P(1), Q(1))$$

hence convergence to 0 at a rate  $1/t$ . Similarly  $d(P(t), X(U))$  and  $d(Q(t), Y(U))$  go to 0 as  $t \rightarrow \infty$ . In addition, in the finite case, the same rate of convergence holds [11].

To study the asymptotic properties of a solution of (22) it is enough to show that they are analogous to those of (23) or of (4). In fact the analysis applies to a much more general framework. Consider a differential inclusion of the form

$$\dot{z} \in \Phi(z) - z \tag{24}$$

where  $Z$  is a compact convex subset of an Euclidean space,  $\Phi$  is an u.s.c. compact convex valued correspondence from  $Z$  to itself and  $z(0) \in Z$ .

A discrete counterpart can be written as

$$P_{n+1} = \alpha_{n+1}p_{n+1} + (1 - \alpha_{n+1})P_n \tag{25}$$

with  $P_1 \in Z$ ,  $p_{n+1} \in \Phi(P_n)$ ,  $\alpha_n \in [0, 1]$  decreasing to 0 as  $n \rightarrow \infty$  and  $\sum_n \alpha_n = +\infty$ .

Then the following comparison result holds:

**Proposition 7.** *Assume that  $Z_0 \subset Z$  is a global uniform attractor of (24) in the sense that for any  $\varepsilon > 0$ , there exists  $T$  such that for any solution  $z$  of (24) with  $z(0) \in Z$  and any  $t \geq T$*

$$d(z(t), Z_0) \leq \varepsilon.$$

*Then for any  $\varepsilon > 0$ , there exists  $N$  such that any solution of (25) with  $P_1 \in Z$  satisfies, for all  $n \geq N$ ,*

$$d(P_n, Z_0) \leq \varepsilon.$$

*Proof.* The proof relies on the following approximation result [1, Ch. 2, sect. 2]: Given  $T_1 > 0$ , let  $\mathcal{A}(\Phi, T_1, u)$  be the set of solutions  $z$  of (24) on  $[0, T_1]$  with  $z(0) = u$ , endowed with the topology of uniform convergence. Let  $D(x, y) = \max_{0 \leq t \leq T_1} \|x(t) - y(t)\|$ . For any  $\varepsilon > 0$ , there exists  $\delta > 0$  such that, given a correspondence  $\Psi$  with graph included in a  $\delta$ -neighborhood of the graph of  $\Phi$

$$\min\{D(y, z) : z \in \mathcal{A}(\Phi, T_1, u)\} \leq \varepsilon \tag{26}$$

for any  $u$  in  $Z$  and for any solution  $y$  of

$$\dot{y} \in \Psi(y) - y \tag{27}$$

with  $y(0) = u$  (or even  $\|y(0) - u\| \leq \delta$ ).

Consider a linear interpolation of the points  $P_n$  thus defining a path  $y(t)$  with  $t_1 = 0, t_n = \sum_{m=2}^n \alpha_m$  and

$$y(t_n) = P_n, \quad \frac{y(t) - y(t_n)}{t - t_n} = \frac{y(t_{n+1}) - y(t_n)}{t_{n+1} - t_n} \in \Phi(y(t_n)) - y(t_n), \quad t \in [t_n, t_{n+1}].$$

The divergence of the sum of the sequence  $\alpha_m$  implies that  $y$  is defined for all  $t \geq 0$ . Moreover, since  $\alpha_m$  decreases to 0, for any  $\delta > 0$ , there exists  $T_2$  such that for  $t \geq T_2$ ,  $y$  satisfies the inclusion (27) with  $\Psi(y) = N^\delta(\Phi(N^\delta(y))) \cap Z$  where  $N^\delta$  stands for  $\delta$ -neighborhood. Alternatively the graph of  $\Psi$  is the intersection of a  $\delta$ -neighborhood of the graph of  $\Phi$  with  $Z \times Z$ .

Given  $\varepsilon > 0$ , let  $T$  be as in the statement of Proposition 7. Put  $T_1 = T$  thus defining  $\delta$  and finally choose  $T_2$  as above adapted to this  $\delta$ . Let  $N$  such that  $t_N \geq T_1 + T_2$ . Then  $n \geq N$  implies that  $d(P_n, Z_0) \leq 2\varepsilon$ . In fact, on the trajectory of  $y$  after time  $t_n - T_1 \geq T_2$  the approximation property applies. Along this time interval of length  $T_1$ , any  $z$  solution of (24) starting from  $y(T_1)$  reaches  $Z_0$  within  $\varepsilon$  at time  $t_n$  and  $y$  remains  $\varepsilon$  close to the set of such  $z$  during this period.  $\square$

While the above analysis is similar to that of Harris [11], the following alternative approach from Hofbauer [12] uses the explicit construction of a solution of the differential inclusion ([1, Ch. 2, sect. 1] or [8, Section 4.1]).

**Proposition 8.** *The set of limit points of a solution of (25) is an invariant set for the dynamics (24) and hence is contained in its maximal invariant set.*

*Proof.* Consider a solution  $P_n$  of (25) and its linear interpolation  $y$  as above. Let  $L$  be the set of limit points of  $y(t)$  as  $t \rightarrow \infty$ , which equals the set of limit points of  $P_n$  as  $n \rightarrow \infty$ , since  $\alpha_n$  goes to 0 and its sum diverges. Let  $z \in L$ . Then there exists a sequence  $T_n \rightarrow \infty$  such that  $y(T_n) \rightarrow z$ . Given any  $T > 0$ , the sequence of trajectories  $y(t+T_n)$  on  $[-T, T]$  is equicontinuous and hence contains a subsequence that converges uniformly to a function  $z(t)$  from  $[-T, T]$  to  $L$  with  $z(0) = z$ . In addition, this subsequence can be chosen so that  $\dot{y}(t+T_n)$  converges weakly to  $\dot{z}(t)$  and since  $\Phi$  is convex valued,  $\dot{z}(t) \in \Phi(z(t)) - z(t)$ . This being true for any  $T$ , one obtains a complete solution of (25) through  $z$ .  $\square$

Proposition 7 together with the Theorem, or alternatively Proposition 8 and Corollary 5, show the convergence of fictitious play (22) to the set of saddle points. The same approximation implies the convergence of average payoff (Rivière [16], Monderer et al. [15]).

**Proposition 9.**

$$\lim_{N \rightarrow \infty} \frac{1}{N} \sum_{k=n}^{n+N} U(p_k, q_k) = w$$

*Proof.* Similar computations as above show that  $p_n = P_n + (n + 1)(P_{n+1} - P_n)$  satisfying (22) will have trajectories uniformly close to  $P(s) + s\dot{P}(s)$  satisfying (23), hence  $\frac{1}{N} \sum_{k=n}^{n+N} U(p_k, q_k)$  will be near

$$\frac{1}{N} \int_n^{n+N} U(s\dot{P}(s) + P(s), s\dot{Q}(s) + Q(s)) ds$$

which is, with  $x(t) = P(e^t)$

$$\begin{aligned} \frac{1}{N} \int_{\ln(n)}^{\ln(n+N)} U(x(t) + \dot{x}(t), y(t) + \dot{y}(t))e^t dt &= \frac{1}{N} \int_{\ln(n)}^{\ln(n+N)} U(\alpha(t), \beta(t))e^t dt \\ &= C(n, N) \end{aligned} \tag{28}$$

and Proposition 6 applies. □

**5.2. Small stepsizes.** We consider here alternative discrete procedures satisfying

$$P_{n+1} = \alpha_{n+1}p_{n+1} + (1 - \alpha_{n+1})P_n \tag{29}$$

with  $P_1 \in Z$ ,  $p_{n+1} \in \Phi(P_n)$ , but where the step size  $\alpha_n \in [0, 1]$  does not necessarily goes to 0.

**Proposition 10.** *Assume that  $Z_0 \subset Z$  is a global uniform attractor of (24) as in Proposition 7. Then for any  $\varepsilon > 0$ , there exists  $\alpha$  and  $N$  such that any solution of (29) with  $P_1 \in Z$  and  $\alpha_n \leq \alpha$  satisfies, for all  $n \geq N$*

$$d(P_n, Z_0) \leq \varepsilon.$$

*Proof.* The proof is similar to the proof of Proposition 7, using the same approximation argument. □

As an example we consider the following version of geometric fictitious play where the past is discounted at a rate  $\rho < 1$ . Explicitly

$$p_{n+1} \in \text{BR}_1(Q_n), \quad q_{n+1} \in \text{BR}_2(P_n) \tag{30}$$

with initial values  $p_1 = P_1 \in X$ ,  $q_1 = Q_1 \in Y$  and

$$P_n = \frac{\sum_{k=0}^{n-1} \rho^k p_{n-k}}{\sum_{k=0}^{n-1} \rho^k}, \quad Q_n = \frac{\sum_{k=0}^{n-1} \rho^k q_{n-k}}{\sum_{k=0}^{n-1} \rho^k}.$$

This gives the difference inclusion

$$\begin{aligned} P_{n+1} - P_n &\in \frac{1 - \rho}{1 - \rho^{n+1}} [\text{BR}_1(Q_n) - P_n] \\ Q_{n+1} - Q_n &\in \frac{1 - \rho}{1 - \rho^{n+1}} [\text{BR}_2(P_n) - Q_n]. \end{aligned} \tag{31}$$

Hence for any  $\varepsilon > 0$ , there exists  $\bar{\rho} < 1$  and  $N$  such that  $\bar{\rho} \leq \rho < 1$  and  $n > N$  imply

$$d((P_n, Q_n), X(U) \times Y(U)) \leq \varepsilon.$$

So for discount rates close to 1, geometric fictitious play will converge to a small neighborhood of the set of saddle points. However, in general, the set of saddle points itself is unstable for (31).

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