Some elementary results and conjectures about q-Newton binomials

Johann Cigler

Fakultät für Mathematik

Universität Wien

johann.cigler@univie.ac.at

Abstract

This paper gives an elementary presentation of some $q-$ analogues of the binomial theorem with unusual $q-$ powers and collects some results and conjectures about variants of Gauss's identity.

0. Introduction

This paper is concerned with elementary aspects of $q-$ Newton binomials of the form $\sum_{j=0}^{n} s^j q^{r(j)} \begin{bmatrix} n \\ j \end{bmatrix}_{q^k}$. It is self-contained and makes no use of the theory of basic hypergeometric series. After some background material we show that for each $m \in \mathbb{Z}$ the general $q-$ Newton polynomials $\sum_{k=0}^{n} q^{(m+1)\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q s^k x^{n-k}$ satisfy a recurrence of order $|m|+1$. Then motivated by Gauss's identity $\sum_{k=0}^{2n} (-1)^k \begin{bmatrix} 2n \\ k \end{bmatrix}_q = (q;q^2)_n$ and its counterpart $\sum_{k=0}^{n} q^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = (-q;q)_n$ we study some $q-$ Newton binomials which for $q \to \pm 1$ converge to 0. Computer experiments show that many of them have factorizations $a(q)b(q)$ into a non-trivial "nice" part $a(q)$ which has a closed expression and an "ugly" part $b(q)$ which is a polynomial with integer coefficients, but has nice values for $q = \pm 1$. We formulate some conjectures and provide proofs for some special cases. I want to thank Ole Warnaar for reference [1] and some useful comments.

1. Some background material

In [3] I tried to study polynomials of the form $\sum_{k=0}^{n} q^{m\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} x^k$ where m is an arbitrary integer. Similar efforts were made by Boris Kupershmidt in [6] and in my paper [4]. In the present essay I resume these topics with some new results and conjectures.

In the context of the general $q-$binomial theorem (1.3) I mostly consider polynomials $p(x,s)=\sum_{k=0}^{n} q^{m\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} s^k x^{n-k}$ in two variables x and s. But since $p(x,s)=x^n p\left(1,\frac{s}{x}\right)$ for real numbers x,s it suffices in this case to consider the univariate polynomials $p(1,x)$.

We let q be a real number and will mostly assume that $|q|<1$. As usual we set $[x]=[x]_q=\frac{1-q^x}{1-q}$ for real numbers x and let $[n]!=\prod_{j=1}^{n}[j]$ for $n\in\mathbb{N}$. The $q-$binomials $\begin{bmatrix} x \\ k \end{bmatrix}$ are defined as $\begin{bmatrix} x \\ k \end{bmatrix}=\begin{bmatrix} x \\ k \end{bmatrix}_q=\frac{\prod_{j=0}^{k-1}\left(1-q^{x-j}\right)}{\prod_{j=0}^{k-1}\left(1-q^{k-j}\right)}$. We shall also use the $q-$Pochhammer symbols

$(x;q)_n=\prod_{j=0}^{n-1}\left(1-q^j x\right)$ and $(x;q)_\infty=\prod_{j=0}^{\infty}\left(1-q^j x\right)$.

The $q-$binomial coefficients satisfy two recurrence relations

$$\begin{bmatrix} x+1 \\ k \end{bmatrix}=q^k\begin{bmatrix} x \\ k \end{bmatrix}+\begin{bmatrix} x \\ k-1 \end{bmatrix} \tag{1.1}$$

and

$$\begin{bmatrix} x+1 \\ k \end{bmatrix}=\begin{bmatrix} x \\ k \end{bmatrix}+q^{x+1-k}\begin{bmatrix} x \\ k-1 \end{bmatrix}. \tag{1.2}$$

My first attempts in this direction led to my paper [2], where I rediscovered Schützenberger's **general $q-$binomial theorem** [8] in the following form:

Let q be a real number and let A and B be linear operators on $\mathbb{C}[x,s]$ which $q-$commute, i.e. satisfy $BA=qAB$, then

$$(A+B)^n=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}A^k B^{n-k}, \tag{1.3}$$

where $\begin{bmatrix} n \\ k \end{bmatrix}=\begin{bmatrix} n \\ k \end{bmatrix}_q$ are $q-$binomial coefficients.

The proof is almost trivial since

$$(A+B)(A+B)^n=(A+B)\sum_k\begin{bmatrix} n \\ k \end{bmatrix}A^k B^{n-k}=\sum_k\left(\begin{bmatrix} n \\ k-1 \end{bmatrix}+q^k\begin{bmatrix} n \\ k \end{bmatrix}\right)A^k B^{n+1-k}=\sum_k\begin{bmatrix} n+1 \\ k \end{bmatrix}A^k B^{n+1-k}.$$

Let me first recall some well-known facts from this point of view. Let $\underline{x}$ and $\underline{s}$ be the multiplication operators with x and s respectively, and let ε be the augmenting operator defined by $\varepsilon f(x,s) = f(x,qs)$ for $f \in \mathbb{C}[x,s].$

The operators $A = \underline{s}\varepsilon$ and $B = \underline{x}\varepsilon$ are $q-$commuting because
$BAf(x,s) = \underline{x}\varepsilon\underline{s}\varepsilon f(x,s) = qsxf(x,q^2 s) = q\underline{s}\varepsilon\underline{x}\varepsilon f(x,s) = qBAf(x,s).$

This implies that $(\underline{s}\varepsilon + \underline{x}\varepsilon)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\underline{s}\varepsilon)^k (\underline{x}\varepsilon)^{n-k}$. By applying this identity to the constant polynomial $f(x,s) = 1$ we get the **Euler polynomials**

$$\sum_{k=0}^{n} q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} s^k x^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\underline{s}\varepsilon)^k (\underline{x}\varepsilon)^{n-k} 1 = (\underline{s}\varepsilon + \underline{x}\varepsilon)^n 1 = (x+s)(x+qs)\cdots(x+q^{n-1}s).$$
(1.4)

The $q-$commuting operators $A = \underline{s}$ and $B = \underline{x}\varepsilon$ in the same way give the **Rogers-Szegö polynomials**

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} s^k x^{n-k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (\underline{x}\varepsilon)^k (\underline{s})^{n-k} 1 = (\underline{s} + \underline{x}\varepsilon)^n 1. \qquad (1.5)$$

By comparing coefficients the general $q-$binomial theorem is equivalent with the $q-$ **exponential law**

$$\exp((A+B)z) = \exp(Az)\exp(Bz) \qquad (1.6)$$

for the **first** $q-$ **exponential series**

$$\exp(z) = \exp_q(z) = \sum_{n\geq 0} \frac{z^n}{[n]!}. \qquad (1.7)$$

All considered series will be **formal power series**.

There is also a **second** $q-$ **exponential series**

$$Exp(z) = Exp_q(z) = \sum q^{\binom{n}{2}} \frac{z^n}{[n]!}. \qquad (1.8)$$

Note that

$$\exp(z)Exp(-z) = 1. \qquad (1.9)$$

To show this let $A=\underline{s}$ and $B=-\underline{s}\varepsilon.$ These operators are again $q-$commuting. Therefore $\exp(\underline{s}z)\exp(-\underline{s}\varepsilon z)=\exp\left(\underline{s}(1-\varepsilon)z\right).$ Since $\underline{s}(1-\varepsilon)1=0$ we get

$$1=\exp\left(\underline{s}(1-\varepsilon)z\right)1=\exp(\underline{s}z)\exp(-\underline{s}\varepsilon z)1=\exp(\underline{s}z)\sum_{n\geq 0}(-1)^n q^{\binom{n}{2}}\frac{s^n z^n}{[n]!}=\exp(sz)Exp(-sz).$$

Instead of $\exp(z)$ we will also consider the variant $e(z)=e_q(z)=\exp\left(\frac{z}{1-q}\right)=\sum_{n\geq 0}\frac{z^n}{(q;q)_n}$ which is the most convenient one in the context of basic hypergeometric series.

From our point of view the $q-$analogue $\frac{\exp(z)-\exp(qz)}{(1-q)z}=\exp(z)$ of $\frac{d}{dx}e^x=e^x$ implies the well-known results

$$e_q(qz)=(1-z)e_q(z) \tag{1.10}$$

and by iteration

$$e_q(z)=\frac{1}{(z;q)_\infty}. \tag{1.11}$$

Therefore

$$e_{q^2}(z^2)=\frac{1}{\prod_{j\geq 0}\left(1-q^{2j}z^2\right)}=\frac{1}{\prod_{j\geq 0}\left(1+q^jz\right)\prod_{j\geq 0}\left(1-q^jz\right)}=e_q(-z)e_q(z). \tag{1.12}$$

Later we will need the following $q-$analogue of $e^{(x+y)^2}=e^{x^2}e^{2xy}e^{y^2}.$

Lemma 1.1 ([5], Corollary 3.3)

Let A *and* B *be linear operators on* $\mathbb{C}[x,s]$ *which satisfy* $BA=qAB.$

Then $\exp_{q^2}((A+B)^2)=\exp_{q^2}(A^2)\exp_q((1+q)AB)\exp_{q^2}(B^2)$ or equivalently

$$e_{q^2}\left((A+B)^2\right)=e_{q^2}(A^2)e_q(AB)e_{q^2}(B^2). \tag{1.13}$$

Proof

Since $B^nA=Aq^nB^n$ we get $e_q(B)A=Ae_q(qB)$ and therefore $e_q(B)Ae_q(B)^{-1}=Ae_q(qB)e_q(B)^{-1}=A(1-B)e_q(B)e_q(B)^{-1}=A(1-B).$

Thus $e_q(B)A^ne_q(B)^{-1}=\left(A(1-B)\right)^n$ and $e_q(B)e_q(A)e_q(B)^{-1}=e_q(A(1-B)).$

This implies

$$e_q(B)e_q(A) = e_q(A - AB)e_q(B) = e_q(A)e_q(-AB)e_q(B). \tag{1.14}$$

Therefore

$$e_{q^2}\left((A+B)^2\right) = e_q\left(-(A+B)\right)e_q\left((A+B)\right) = e_q\left(-A\right)e_q\left(-B\right)e_q\left(A\right)e_q\left(B\right)$$
$$= e_q\left(-A\right)e_q(A)e_q\left(AB\right)e_q\left(-B\right)e_q\left(B\right) = e_{q^2}\left(A^2\right)e_q\left(AB\right)e_{q^2}\left(B^2\right).$$

Let us denote by $r_n(x,s,m) = r_n(x,s,m,q)$ the **general q-Newton binomials**

$$r_n(x,s,m,q) = \sum_{k=0}^{n} q^{(m+1)\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q s^k x^{n-k} = \sum_{k=0}^{n} q^{(m+1)\binom{n-k}{2}} \begin{bmatrix} n \\ k \end{bmatrix}_q s^{n-k} x^k. \tag{1.15}$$

Note that for $m = 0$ we get the Euler polynomials and for $m = -1$ the Rogers-Szegö polynomials. Let us also note the well-known generating functions of these special polynomials.

The generating function of the Rogers-Szegö polynomials is

$$\sum_{n\ge 0} r_n(x,s,-1)\frac{z^n}{[n]!} = \exp(xz)\exp(sz). \tag{1.16}$$

This can be proved by comparing coefficients or by the exponential version of (1.5).

An equivalent version is

$$\sum_{n\ge 0} \frac{r_n(1,s,-1)}{(q;q)_n} z^n = \frac{1}{(z;q)_\infty (sz;q)_\infty}. \tag{1.17}$$

The generating function of the Euler polynomials is

$$\sum_{n\ge 0} r_n(x,s,0)\frac{z^n}{[n]!} = Exp(sz)\exp(xz) = \frac{\exp(xz)}{\exp(-sz)}. \tag{1.18}$$

It is equivalent with

$$\sum_{n\ge 0} \frac{r_n(1,-s,0)}{(q;q)_n} z^n = \sum_{n\ge 0} \frac{(s;q)_n}{(q;q)_n} z^n = \frac{(sz;q)_\infty}{(z;q)_\infty}, \tag{1.19}$$

which plays the role of the $q-$ **binomial theorem** within the theory of basic hypergeometric series.

The general $q-$ Newton binomials $r_n(x,s,m,q)$ satisfy the recurrence relation

$$r_n(x,s,m,q) = x r_{n-1}(x,qs,m,q) + s r_{n-1}(x,q^{m+1}s,m,q). \tag{1.20}$$

This can be seen by comparing coefficients of s^k which gives the identity

$$q^{(m+1)\binom{k}{2}}\begin{bmatrix} n \\ k \end{bmatrix} x^{n-k} = q^{(m+1)\binom{k}{2}+k}\begin{bmatrix} n-1 \\ k \end{bmatrix} x^{n-k} + q^{(m+1)\binom{k-1}{2}+(m+1)(k-1)}\begin{bmatrix} n-1 \\ k-1 \end{bmatrix} x^{n-k}$$

which is equivalent with $\begin{bmatrix} n \\ k \end{bmatrix} = q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} + \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$

From (1.20) we deduce the well known recurrence

$$r_n(x,s,-1) = \left(x+s\right) r_{n-1}(x,s,-1) + \left(q^{n-1}-1\right) sx r_{n-2}(x,s,-1). \tag{1.21}$$

For

$$r_n(x,s,-1) = x r_{n-1}(x,qs,-1) + s r_{n-1}(x,s,-1) = \left(x+s\right) r_{n-1}(x,s,-1) + x\left(r_{n-1}(x,qs,-1) - r_{n-1}(x,s,-1)\right)$$
$$= \left(x+s\right) r_{n-1}(x,s,-1) + \left(q^{n-1}-1\right) sx r_{n-2}(x,s,-1).$$

The last identity follows from

$$r_{n-1}(x,qs,-1) - r_{n-1}(x,s,-1) = \sum_k \begin{bmatrix} n-1 \\ k \end{bmatrix} \left(q^k-1\right) s^k x^{n-1-k} = \left(q^{n-1}-1\right) \sum_k \begin{bmatrix} n-2 \\ k-1 \end{bmatrix} s^k x^{n-1-k} = s r_{n-2}(x,s,-1)$$

by noting that $\begin{bmatrix} n \\ k \end{bmatrix} = \dfrac{1-q^n}{1-q^k} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}.$

For later use let us mention another recurrence which we state for the univariate polynomials

$$f(n,x,q) = r_n(1,x,-1,q) = \sum_{j=0}^{n} x^j \begin{bmatrix} n \\ j \end{bmatrix}_q . \tag{1.22}$$

They satisfy

$$f(n,x,q) = \left(1+(1+q)q^{n-2}x + x^2\right) f(n-2,x,q) - \left(1-q^{n-3}\right)\left(1-q^{n-2}\right) x^2 f(n-4,x,q). \tag{1.23}$$

Proof

By (1.21) we have

$$f(n,x,q) = (1+x) f(n-1,x,q) + \left(q^{n-1}-1\right) x f(n-2,x,q).$$

Therefore we get

$$f(n,x,q)=(1+x)f(n-1,x,q)+\left(q^{n-1}-1\right)xf(n-2,x,q),$$
$$f(n-1,x,q)=(1+x)f(n-2,x,q)+\left(q^{n-2}-1\right)xf(n-3,x,q),$$
$$(1+x)f(n-3,x,q)=f(n-2,x,q)+\left(1-q^{n-3}\right)xf(n-4,x,q).$$

This implies

$$f(n,x,q)=(1+x)\left((1+x)f(n-2,x,q)+\left(q^{n-2}-1\right)xf(n-3,x,q)\right)+\left(q^{n-1}-1\right)xf(n-2,x,q)$$
$$=\left((1+x)^2+\left(q^{n-1}-1\right)x\right)f(n-2,x,q)+\left(q^{n-2}-1\right)x\left(f(n-2,x,q)+\left(1-q^{n-3}\right)xf(n-4,x,q)\right)$$
$$=\left(1+(1+q)q^{n-2}x+x^2\right)f(n-2,x,q)-\left(1-q^{n-3}\right)\left(1-q^{n-2}\right)x^2f(n-4,x,q).$$

For $x=1$ and $s=-1$ (1.21) reduces to

$r_n(1,-1,-1)=\left(1-q^{n-1}\right)r_{n-2}(1,-1,-1).$ Since $r_0(1,-1,-1)=1$ and $r_1(1,-1,-1)=0$

and thus leads to

Gauss's identity

$$\sum_{k=0}^{2n+1}(-1)^k\begin{bmatrix}2n+1\\k\end{bmatrix}=0,$$
$$\sum_{k=0}^{2n}(-1)^k\begin{bmatrix}2n\\k\end{bmatrix}=\left(q;q^2\right)_n. \qquad (1.24)$$

This identity is equivalent with

$$\exp(z)\exp(-z)=\sum_{n\geq 0}\left(q;q^2\right)_n\frac{z^{2n}}{[2n]!} \qquad (1.25)$$

or equivalently with (1.12).

Let us also note the following q – analogue of $e^{\frac{x}{2}}e^{\frac{x}{2}}=e^x$:

$$\exp_{q^2}\left(\frac{z}{[2]}\right)\exp_{q^2}\left(\frac{qz}{[2]}\right)=\exp_q(z). \qquad (1.26)$$

or equivalently

$$e_{q^2}(qz)e_{q^2}(z)=e_q(z) \qquad (1.27)$$

which is obvious by (1.11).

By comparing coefficients this is equivalent with

$$\sum_{k=0}^{n} q^{k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = (1+q)(1+q^2)\cdots(1+q^n) = (-q;q)_n . \tag{1.28}$$

More generally we get

Theorem 1.1

For each $m \in \mathbb{N}$ *we get*

$$\sum_{k=0}^{n} q^{(2m+1)k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = (-q;q)_n \, a(m,n,q) \tag{1.29}$$

with

$$a(m,n,q) = \sum_{j=0}^{n} (-1)^j q^{j^2} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \frac{(q;q)_n}{(q;q)_{n-j}} \in \mathbb{Z}[q]. \tag{1.30}$$

The polynomial $a(m,n,q)$ *satisfies*

$$\begin{aligned} &a(m,n,1) = 1, \\ &a(m,2n,-1) = 1, \\ &a(m,2n+1,-1) = 2m+1. \end{aligned} \tag{1.31}$$

Proof

Comparing coefficients in

$$\sum_{n\geq 0} \sum_{k=0}^{n} q^{(2m+1)k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \frac{z^n}{(q^2;q^2)_n} = e_{q^2}(q^{2m+1}z) e_{q^2}(z) = (qz;q^2)_m \, e_{q^2}(qz) e_{q^2}(z) = (qz;q^2)_m \, e_q(z)$$

we get

$$\sum_{k=0}^{n} q^{(2m+1)k} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} = (q^2;q^2)_n \sum_{j=0}^{n} (-1)^j q^{j^2} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \frac{1}{(q;q)_{n-j}} = (-q;q)_n \sum_{j=0}^{n} (-1)^j q^{j^2} \begin{bmatrix} m \\ j \end{bmatrix}_{q^2} \frac{(q;q)_n}{(q;q)_{n-j}} .$$

Remark

Kupershmidt [6] has given another equivalent form of this theorem which has also been considered in [4]. Note that there is no such factorization for $\sum_{k=0}^{n} q^{2mk} \begin{bmatrix} n \\ k \end{bmatrix}_{q^2}$.

Boris A. Kupershmidt [6] has also shown that

$$r_n(x,s,-1) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \begin{bmatrix} n \\ 2k \end{bmatrix} (q;q^2)_k \, s^{2k} r_{n-2k}(x,s,0). \tag{1.32}$$

In fact he used another notation. To compare his version with the present one note that

$$\begin{bmatrix} 2n+1 \\ 2k \end{bmatrix} (q;q^2)_k = \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \left(q^{2n+1}; \frac{1}{q^2} \right)_k \quad \text{and} \quad \begin{bmatrix} 2n \\ 2k \end{bmatrix} (q;q^2)_k = \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \left(q^{2n-1}; \frac{1}{q^2} \right)_k .$$

He gave three proofs of this fact. Let us recall another proof which we have given in [4].

We use the generating functions.

$$\sum_{n\geq 0} r_n(x,s,-1)\frac{z^n}{[n]!} = \exp(xz)\exp(sz) = \left(\exp(sz)\exp(-sz)\right)\frac{\exp(xz)}{\exp(-sz)} = \sum_{j\geq 0} r_j(s,s,-1)\frac{z^j}{[j]!}\sum_{k\geq 0} r_k(x,s,0)\frac{z^k}{[k]!}$$
$$= \sum_{j\geq 0} \left(q;q^2\right)_j s^{2j}\frac{z^{2j}}{[2j]!}\sum_{k\geq 0} r_k(x,s,0)\frac{z^k}{[k]!}.$$

Comparing coefficients we get the above formula.

A. Berkovich and S.O. Warnaar [1] have given several other results for the Rogers-Szegö polynomials. I will recall them with a different proof.

Lemma 1.2 ([1], (8.9) and (8.11))

$$\sum_{n\geq 0}\frac{r_{2n}(1,s,-1,q)}{\left(q^2;q^2\right)_n} z^n = \frac{1}{e_q(-sz)} e_{q^2}(s^2 z) e_{q^2}(z) \tag{1.33}$$

$$\sum_{n\geq 0}\frac{r_{2n+1}(1,s,-1,q)}{\left(q^2;q^2\right)_n} z^n = (1+s)\frac{1}{e_q(-qsz)} e_{q^2}(s^2 z) e_{q^2}(z). \tag{1.34}$$

Proof

By (1.13) and (1.16) we get

$$\sum_{n\geq 0}\frac{r_{2n}(1,s,-1,q)}{\left(q^2;q^2\right)_n} z^n = e_{q^2}\left((\underline{s}+\varepsilon)^2 z\right)1 = e_{q^2}(\underline{s}^2 z) e_q(\underline{s}\varepsilon z) e_{q^2}(\varepsilon^2 z)1 = e_{q^2}(s^2 z)\frac{1}{e_q(-sz)} e_{q^2}(z).$$

For the second identity observe that by (1.10)

$$\sum_{n\geq 0}\frac{r_{2n+1}(1,s,-1,q)}{\left(q^2;q^2\right)_n} z^n = (\underline{s}+\varepsilon) e_{q^2}\left((\underline{s}+\varepsilon)^2 z\right)1 = (\underline{s}+\varepsilon) e_{q^2}(\underline{s}^2 z) e_q(\underline{s}\varepsilon z) e_{q^2}(\varepsilon^2 z)1$$
$$= (\underline{s}+\varepsilon) e_{q^2}(s^2 z)\frac{1}{e_q(-sz)} e_{q^2}(z) = s e_{q^2}(s^2 z)\frac{1}{e_q(-sz)} e_{q^2}(z) + e_{q^2}(q^2 s^2 z)\frac{1}{e_q(-qsz)} e_{q^2}(z)$$
$$= s(1+sz) e_{q^2}(s^2 z)\frac{1}{e_q(-qsz)} e_{q^2}(z) + \left(1-s^2 z\right) e_{q^2}(s^2 z)\frac{1}{e_q(-qsz)} e_{q^2}(z) = (1+s)\frac{e_{q^2}(s^2 z) e_{q^2}(z)}{e_q(-qsz)}.$$

Comparing coefficients and observing (1.16) we get

Corollary 1.1 ([1], (8.8))

$$r_{2n}(1,s,-1,q)=\sum_{k=0}^{n}\left(-q;q\right)_k q^{\binom{k}{2}}s^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} r_{n-k}(1,s^2,-1,q^2) \tag{1.35}$$

and

$$r_{2n+1}(1,s,-1,q)=(1+s)\sum_{k=0}^{n}\left(-q;q\right)_k q^{\binom{k+1}{2}}s^k \begin{bmatrix} n \\ k \end{bmatrix}_{q^2} r_{n-k}(1,s^2,-1,q^2). \tag{1.36}$$

Another result which is a $q-$analogue of $(1+s)^{2n}=\sum_{k=0}^{n}\binom{n}{k}s^{2k}\left(1+\frac{1}{s}\right)^k(1+s)^{n-k}$ is

Corollary 1.2 (A. Berkovich and S.O. Warnaar [1], Theorem 8.1)

$$r_{2n}(1,s,-1,q)=\sum_{k=0}^{n}s^{2k}r_k\left(1,\frac{q}{s},0,q^2\right)r_{n-k}\left(1,s,0,q^2\right)\begin{bmatrix} n \\ k \end{bmatrix}_{q^2}, \tag{1.37}$$

and

$$\begin{aligned} r_{2n+1}(1,s,-1,q)&=\sum_{k=0}^{n}s^{2k}r_k\left(1,\frac{q}{s},0,q^2\right)r_{n+1-k}\left(1,s,0,q^2\right)\begin{bmatrix} n \\ k \end{bmatrix}_{q^2} \\ &=(1+s)\sum_{k=0}^{n}s^{2k}r_k\left(1,\frac{q}{s},0,q^2\right)r_{n-k}\left(1,q^2s,0,q^2\right)\begin{bmatrix} n \\ k \end{bmatrix}_{q^2}. \end{aligned} \tag{1.38}$$

Proof

By comparing coefficients (1.37) is equivalent with

$$\sum_{n\geq 0}\frac{r_{2n}(1,s,-1,q)}{\left(q^2;q^2\right)_n}z^n=\sum_n s^{2n}r_n\left(1,\frac{q}{s},0,q^2\right)\frac{z^n}{\left(q^2;q^2\right)_n}\sum_n r_n(1,s,0,q^2)\frac{z^n}{\left(q^2;q^2\right)_n}$$

$$=\frac{e_{q^2}(s^2z)}{e_{q^2}\left(-qsz\right)}\frac{e_{q^2}(z)}{e_{q^2}\left(-sz\right)}=\frac{e_{q^2}(s^2z)e_{q^2}(z)}{e_q\left(-sz\right)}$$

and (1.38) with

$$\sum_{n\geq 0}\frac{r_{2n+1}(1,s,-1,q)}{\left(q^2;q^2\right)_n}z^n=(1+s)\sum_n s^{2n}r_n\left(1,\frac{q}{s},0,q^2\right)\frac{z^n}{\left(q^2;q^2\right)_n}\sum_n r_n(1,q^2s,0,q^2)\frac{z^n}{\left(q^2;q^2\right)_n}$$

$$=(1+s)\frac{e_{q^2}(s^2z)}{e_{q^2}\left(-qsz\right)}\frac{e_{q^2}(z)}{e_{q^2}\left(-q^2sz\right)}=(1+s)\frac{e_{q^2}(s^2z)e_{q^2}(z)}{e_q\left(-qsz\right)}.$$

2. Recurrence relations for the general $q-$Newton binomials

For each integer m the operators $A=\underline{s}\varepsilon^m$ and $B=\underline{x}\varepsilon$ are $q-$commuting:

$(\underline{x}\varepsilon)(\underline{s}\varepsilon^m)=q(\underline{s}\varepsilon^m)(\underline{x}\varepsilon).$

For

$$(\underline{x}\varepsilon)(\underline{s}\varepsilon^m)f(x,s)=x\varepsilon sf(x,q^m s)=xqsf(x,q^{m+1}s)$$
$$=qsxf(x,q^{m+1}s)=q(\underline{s}\varepsilon^m)xf(x,qs)=q(\underline{s}\varepsilon^m)(\underline{x}\varepsilon)f(x,s).$$

The identity (1.20) can be written as

$r_n(x,s,m)=(\underline{x}\varepsilon+\underline{s}\varepsilon^{m+1})r(n-1,x,s,m).$

By iteration we finally get

$$r_n(x,s,m)=(\underline{x}\varepsilon+\underline{s}\varepsilon^{m+1})^n 1, \tag{2.1}$$

i.e. $r_n(x,s,m)$ can be obtained by applying the operator $(\underline{x}\varepsilon+s\varepsilon^{m+1})^n$ to the constant function $f(x,s)=1.$

We now recall the recurrence relation of the sequence $(r_n(x,s,m))_{n\geq 0}$ for fixed x and $s,$ which has already been obtained in [3].

We show that for each $m\in\mathbb{Z}$ the sequence $(r_n(x,s,m))_{n\geq 0}$ satisfies a recurrence of order $|m|+1.$ This is the reason for choosing $(m+1)\binom{k}{2}$ instead of $m\binom{k}{2}$ in (1.15).

Theorem 2.1

For $m\geq 0$ the sequence $(r_n(x,s,m))_{n\geq 0}$ satisfies the recurrence

$$r_n(x,s,m)=q^{(m+1)(n-1)}sr_{n-1}(x,s,m)-\sum_{k=1}^{m+1}(-1)^k q^{\binom{k}{2}}\left(q^{n-k+1};q\right)_{k-1}\left(\begin{bmatrix}m+1\\k\end{bmatrix}-q^n\begin{bmatrix}m\\k\end{bmatrix}\right)x^k r_{n-k}(x,s,m) \tag{2.2}$$

of order $m+1.$

Proof

In order to formulate the following results in a simple way consider the algebra $\mathbb{C}(q)[N]$ of all polynomials in N whose coefficients are rational functions of $q.$

Consider the following polynomials $C_k(N) = \frac{\prod_{j=0}^{k-1}\left(1-q^{-j}N\right)}{(q;q)_k}$ which satisfy $C_k\left(q^n\right) = \begin{bmatrix} n \\ k \end{bmatrix}$. It is clear that each polynomial has a unique representation of the form $\sum a_k(q) C_k(N)$. Let E be the linear operator defined by $Ef(N) = f(qN)$ and Δ the operator defined by

$\Delta f(N) = \frac{f(qN) - f(N)}{N} = \frac{1}{N}\left(E-1\right) f(N)$. Δ operates on $\mathbb{C}(q)[N]$ since

$\Delta N^k = \left(q^k - 1\right) N^{k-1}$.

It can also be characterized by

$$\Delta q^{\binom{k}{2}} C_k(N) = q^{\binom{k-1}{2}} C_{k-1}(N).$$

By definition we have $E = 1 + N\Delta$.

Let Z be the linear operator defined by $ZC_k(N) = q^k C_k(N)$. Then $E = Z(1+\Delta)$.

For

$$Z\left(1+\Delta\right) C_k(N) = Z\left(C_k(N) + q^{1-k} C_{k-1}(N)\right) = q^k C_k(N) + C_{k-1}(N) = \frac{C_{k-1}(N)}{1-q^k}\left(q^k\left(1 - \frac{N}{q^{k-1}}\right) + 1 - q^k\right)$$

$$= \frac{C_{k-1}(N)}{1-q^k}(1-qN) = C_k(qN) = EC_k(N).$$

Therefore we can write

$$Z^{-1} = \left(1+\Delta\right) E^{-1} = \left(1 + \frac{1}{N}\left(E-1\right)\right) E^{-1} = \frac{1}{N} + \left(1 - \frac{1}{N}\right) E^{-1} = \frac{1}{N}\left(1 + \left(N-1\right) E^{-1}\right).$$

Note that Z and Z^{-1} are operators on $\mathbb{C}(q)[N]$, whereas the multiplication operators $\frac{1}{N}$ and the operator $\frac{N-1}{N}$ are only defined on the larger space $\mathbb{C}(q)[N, N^{-1}]$.

The operators $A = \frac{1}{N}$ and $B = \frac{N-1}{N} E^{-1}$ satisfy $BA = qAB$ and therefore by the general $q-$ binomial theorem we get for $\ell \geq 0$

$$Z^{-\ell} = \sum_{k=0}^{\ell} \begin{bmatrix} \ell \\ k \end{bmatrix} \frac{1}{N^{\ell-k}} \left(\frac{N-1}{N} E^{-1}\right)^k = \frac{1}{N^\ell} \sum_{k=0}^{\ell} \begin{bmatrix} \ell \\ k \end{bmatrix} \prod_{j=0}^{k-1}\left(N - q^j\right) E^{-k}. \qquad (2.3)$$

Note further that

$$Z^{-(\ell-1)} \Delta q^{\ell\binom{k}{2}} C_k(N) = Z^{-(\ell-1)} q^{\ell\binom{k}{2} - (k-1)} C_{k-1}(N) = q^{\ell\binom{k}{2} - (k-1) - (\ell-1)(k-1)} C_{k-1}(N) = q^{\ell\binom{k-1}{2}} C_{k-1}(N).$$

Consider now the polynomials in N

$$S_n(s,m,N)=\sum_{k=0}^{n} q^{(m+1)\binom{k}{2}} C_k(N)s^k. \tag{2.4}$$

Then for $\ell \geq n$ we get $S_\ell\left(s,m,q^n\right)=r_n(1,s,m)$.

Now observe that

$$\frac{S_{n+1}(s,m,qN)-S_{n+1}(s,m,N)}{N}=\Delta S_{n+1}(s,m,N)=\sum_{k=0}^{n+1} q^{m\binom{k}{2}} \Delta q^{\binom{k}{2}} C_k(N)s^k$$

$$=\sum_{k=1}^{n+1} q^{m\binom{k}{2}} q^{\binom{k-1}{2}} C_{k-1}(N)s^k=\sum_{k=0}^{n} q^{m\binom{k+1}{2}+\binom{k}{2}} C_k(N)s^{k+1}=sZ^m S_n\left(s,m,N\right)=sS_n\left(q^m s,m,N\right).$$

Thus

$$Z^{-m}\Delta S_{n+1}(s,m,N)=sS_n\left(s,m,N\right). \tag{2.5}$$

We have $NE^{-k}\Delta=q^k E^{-k}N\Delta=q^k\left(E^{1-k}-E^{-k}\right)$ and therefore for $m\geq 0$

$$N^{m+1}Z^{-m}\Delta=\sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}\prod_{j=0}^{k-1}\left(N-q^j\right)q^k E^{-k}(E-1)$$

$$=E+\sum_{k=0}^{m}\begin{bmatrix} m \\ k+1 \end{bmatrix}\prod_{j=0}^{k}\left(N-q^j\right)q^{k+1}E^{-k}-\sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}\prod_{j=0}^{k-1}\left(N-q^j\right)q^k E^{-k}$$

$$=E+\sum_{k=0}^{m}\begin{bmatrix} m \\ k+1 \end{bmatrix}\prod_{j=0}^{k-1}\left(N-q^j\right)\left(qN-q^{k+1}\right)q^k E^{-k}-\sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}\prod_{j=0}^{k-1}\left(N-q^j\right)q^k E^{-k}$$

$$=E+\sum_{k=0}^{m}\left(\begin{bmatrix} m \\ k+1 \end{bmatrix}qN-\begin{bmatrix} m \\ k+1 \end{bmatrix}q^{k+1}-\begin{bmatrix} m \\ k \end{bmatrix}\right)\prod_{j=0}^{k-1}\left(N-q^j\right)q^k E^{-k}$$

$$=E+\sum_{k=0}^{m}\left(\begin{bmatrix} m \\ k+1 \end{bmatrix}qN-\begin{bmatrix} m+1 \\ k+1 \end{bmatrix}\right)\prod_{j=0}^{k-1}\left(N-q^j\right)q^k E^{-k}.$$

Therefore

$$sN^{m+1}S_n\left(s,m,N\right)=N^{m+1}Z^{-m}\Delta S_{n+1}(s,m,N)$$

$$=S_{n+1}(s,m,qN)+\sum_{k=0}^{m}\left(\begin{bmatrix} m \\ k+1 \end{bmatrix}qN-\begin{bmatrix} m+1 \\ k+1 \end{bmatrix}\right)\prod_{j=0}^{k-1}\left(N-q^j\right)q^k S_{n+1}(s,m,q^{-k}N).$$

By choosing $N=q^{n-1}$ this reduces to

$$q^{(m+1)(n-1)}sr_{n-1}(1,s,m)=r_n(1,s,m)$$
$$+\sum_{k=0}^{m}\left(\begin{bmatrix} m \\ k+1 \end{bmatrix}q^n-\begin{bmatrix} m+1 \\ k+1 \end{bmatrix}\right)\prod_{j=0}^{k-1}\left(q^n-q^{j+1}\right)r_{n-k-1}(1,s,m)$$
$$=r_n(1,s,m)+\sum_{k=1}^{m+1}\left(\begin{bmatrix} m \\ k \end{bmatrix}q^n-\begin{bmatrix} m+1 \\ k \end{bmatrix}\right)\prod_{j=0}^{k-2}\left(q^n-q^{j+1}\right)r_{n-k}(1,s,m)$$
$$=r_n(1,s,m)+\sum_{k=1}^{m+1}\begin{bmatrix} m \\ k \end{bmatrix}q^n q^{\binom{k}{2}}\prod_{j=0}^{k-2}\left(q^{n-j-1}-1\right)r_{n-k}(1,s,m)-\sum_{k=1}^{m+1}\begin{bmatrix} m+1 \\ k \end{bmatrix}q^{\binom{k}{2}}\prod_{j=0}^{k-2}\left(q^{n-j-1}-1\right)r_{n-k}(1,s,m)$$

or

$$r_n(1,s,m)=q^{(m+1)(n-1)}sr_{n-1}(1,s,m)-\sum_{k=1}^{m+1}(-1)^k q^{\binom{k}{2}}\left(q^{n-k+1};q\right)_{k-1}\left(\begin{bmatrix} m+1 \\ k \end{bmatrix}-q^n\begin{bmatrix} m \\ k \end{bmatrix}\right)r_{n-k}(1,s,m).$$

Now observe that

$r_n(x,s,m)=x^n r_n\left(1,\frac{s}{x},m\right)$ and therefore

$$r_n(x,s,m)=q^{(m+1)(n-1)}sr_{n-1}(x,s,m)-\sum_{k=1}^{m+1}(-1)^k q^{\binom{k}{2}}\left(q^{n-k+1};q\right)_{k-1}\left(\begin{bmatrix} m+1 \\ k \end{bmatrix}-q^n\begin{bmatrix} m \\ k \end{bmatrix}\right)x^k r_{n-k}(x,s,m)$$

For $m=0$ we get

$$r_{n+1}(x,s,0)-xr_n(x,s,0)=q^n sr_n\left(x,s,0\right)$$

and therefore

$$r_n(x,s,0)=(x+s)(x+qs)\cdots(x+q^{n-1}s).$$

For $m=1$ we have

$$r_n(x,s,1)=q^{2(n-1)}sr_{n-1}(x,s,1)-\sum_{k=1}^{2}(-1)^k q^{\binom{k}{2}}\left(q^{n-k+1};q\right)_{k-1}\left(\begin{bmatrix} 2 \\ k \end{bmatrix}-q^n\begin{bmatrix} 1 \\ k \end{bmatrix}\right)x^k r_{n-k}(x,s,1)$$
$$=\left(q^{2(n-1)}s+\left(1+q-q^n\right)x\right)r_{n-1}(x,s,1)-q\left(1-q^{n-1}\right)x^2 r_{n-2}(x,s,1)$$

Theorem 2.2

The sequence $\left(r_n(x,s,-m)\right)_{m\geq 0}$ *with* $m>0$ *satisfies the recurrence relation*

$$r_n(x,s,-m)-xr_{n-1}(x,s,-m)=\frac{s}{q^{(m-1)(n-1)}}\sum_{k=0}^{m}\begin{bmatrix} m \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}}\left(q^{n-1},\frac{1}{q}\right)_k x^k r_{n-k-1}(x,s,-m). \quad (2.6)$$

Proof

We know already that

$$S_{n+1}(s,-m,qN)-S_{n+1}(s,-m,N)=sNZ^{-m}S_n\left(s,m,N\right)$$

and that

$$sNZ^{-m}S_n\left(s,m,N\right)=sN\,\frac{1}{N^m}\sum_{k=0}^{m}\begin{bmatrix} m\\ k\end{bmatrix}\prod_{j=0}^{k-1}\left(N-q^j\right)E^{-k}S_n\left(s,m,N\right)$$

Therefore we get

$$S_{n+1}(s,-m,qN)-S_{n+1}(s,-m,N)=s\,\frac{1}{N^{m-1}}\sum_{k=0}^{m}\begin{bmatrix} m\\ k\end{bmatrix}\prod_{j=0}^{k-1}\left(N-q^j\right)S_n\left(s,m,q^{-k}N\right).$$

For $N=q^{n-1}$ this reduces to

$$r_n(1,s,-m)-r_{n-1}(1,s,-m)=\frac{s}{q^{(m-1)(n-1)}}\sum_{k=0}^{m}\begin{bmatrix} m\\ k\end{bmatrix}(-1)^k q^{\binom{k}{2}}\left(q^{n-1},\frac{1}{q}\right)_k r_{n-k-1}(1,s,-m)$$

which implies (2.6).

As special case we get again

$$r_n(x,s,-1)-xr_{n-1}(x,s,-1)=sr_{n-1}(x,s,-1)-s(1-q^{n-1})xr_{n-2}(x,-1)$$

Let us also mention

Proposition 2.1

The polynomials $\sigma_n(m,s,q)=\sum_{j=0}^{n}q^{m^2\binom{j}{2}}\begin{bmatrix} n\\ mj\end{bmatrix}s^j$ *satisfy the recurrence*

$$\sum_{k=0}^{m}(-1)^k\begin{bmatrix} m\\ k\end{bmatrix}q^{\binom{k}{2}}\sigma_{n+m-k}(m,s,q)=q^{nm}s\sigma_n(m,s,q). \tag{2.7}$$

Proof

$\Delta^m q^{\binom{mk}{2}}C_{mk}(N)=q^{\binom{m(k-1)}{2}}C_{m(k-1)}(N)$ implies $q^{\binom{m}{2}}\Delta^m q^{m^2\binom{k}{2}}C_{mk}(N)=q^{m^2\binom{k-1}{2}}C_{m(k-1)}(N)$

and therefore

$$q^{\binom{m}{2}}\Delta^m\sum_{j=0}^{n}q^{m^2\binom{j}{2}}C_{mj}(N)s^j=\sum_{j=0}^{n}q^{m^2\binom{j-1}{2}}C_{m(j-1)}(N)s^j=s\sum_{j=0}^{n-1}q^{m^2\binom{j}{2}}C_{mj}(N)s^j.$$

Now observe that

$$\Delta^m = \frac{1}{q^{\binom{m}{2}} N^m}(E-1)(E-q)\cdots(E-q^{m-1}) = \frac{1}{q^{\binom{m}{2}} N^m}\sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} E^{m-k}.$$

Therefore we get

$$q^{\binom{m}{2}} N^m \Delta^m = \sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} E^{m-k}.$$

$$\sum_{k=0}^{m}(-1)^k \begin{bmatrix} m \\ k \end{bmatrix} q^{\binom{k}{2}} \sum_{j=0}^{n+m} q^{m^2\binom{j}{2}} C_{mj}(q^{m-k}N)s^j = N^m s \sum_{j=0}^{n+m-1} q^{m^2\binom{j}{2}} C_{mj}(N)s^j.$$

For $N = q^n$ this reduces to (2.7).

For $q = 1$ and a prime number p we get the recurrence

$$\sigma_n(p,-1,1) = \sum_{k=1}^{p-1}(-1)^{k-1}\binom{p}{k}\sigma_{n-k}(p,-1,1).$$

Since all $\binom{p}{k}$ are multiples of p we see that $p^{\left\lfloor \frac{n}{p} \right\rfloor}$ is a factor of $\sigma_n(p,-1,1)$.

In the general case we get the recurrence

$$\sigma_n(p,-1,q) = \left(q^{\binom{p}{2}} - q^{(n-p)p} \right)\sigma_{n-p}(p,-1,q) - \sum_{k=1}^{p-1}(-1)^k \begin{bmatrix} p \\ k \end{bmatrix}\sigma_{n-k}(p,-1,q).$$

All coefficients are multiples of $[p]$ since $q^{\binom{p}{2}} - q^{np} = [p](1-q)q^{\binom{p}{2}}\left[n - \frac{p-1}{2} \right]_{q^p}$.

This gives

Proposition 2.2

For a prime number p we have

$$\sigma_n(p,-1,q) = [p]^{\left\lfloor \frac{n}{p} \right\rfloor} c(n,p,q) \tag{2.8}$$

with $c(n,p,q) \in \mathbb{Z}[q]$.

3. Some generalizations of Gauss's identity and related facts

Theorem 3.1

For each integer $m \geq 0$

$$\sum_{j=0}^{n} (-1)^j q^{jm} \begin{bmatrix} n \\ j \end{bmatrix} = \left(q;q^2\right)_{\left\lfloor \frac{n+1}{2} \right\rfloor} c_n(m,q) \tag{3.1}$$

where $c_n(m,q) \in \mathbb{Z}[q]$ *is given by*

$$c_{2n}(m,q) = \sum_{\ell=0}^{n} q^{\binom{2n-2\ell}{2}} \begin{bmatrix} m \\ 2n-2\ell \end{bmatrix} \left(q^{2\ell+2};q^2\right)_{n-\ell} \tag{3.2}$$

and

$$c_{2n+1}(m,q) = \sum_{\ell=0}^{n} q^{\binom{2n+1-2\ell}{2}} \begin{bmatrix} m \\ 2n+1-2\ell \end{bmatrix} \left(q^{2\ell+2};q^2\right)_{n-\ell}. \tag{3.3}$$

For $m=0$ we get Gauss's identity since $c_{2n}(0,q)=1$ and $c_{2n+1}(0,q)=0.$ Other simple examples are

$$\sum_{j=0}^{n} (-1)^j q^j \begin{bmatrix} n \\ j \end{bmatrix} = \left(q;q^2\right)_{\left\lfloor \frac{n+1}{2} \right\rfloor},$$

$$\sum_{j=0}^{2n} (-1)^j q^{2j} \begin{bmatrix} 2n \\ j \end{bmatrix} = \left(q;q^2\right)_n \left(1+q-q^{2n+1}\right)$$

and

$$\sum_{j=0}^{2n+1} (-1)^j q^{2j} \begin{bmatrix} 2n+1 \\ j \end{bmatrix} = \left(q;q^2\right)_{n+1} \left(1+q\right).$$

Note that $c_n(m,0)=1$ and $c_{2n}(m,1)=1$ and $c_{2n+1}(m,1)=m.$

Proof

By (1.10) we have

$$\sum_n \sum_{j=0}^{n} (-1)^j q^{jm} \begin{bmatrix} n \\ j \end{bmatrix} \frac{z^n}{(q;q)_n} = e_q\left(-q^m z\right) e_q(z) = (-z;q)_m \, e_q(-z) e_q(z) = \sum_{j=0}^{m} q^{\binom{j}{2}} \begin{bmatrix} m \\ j \end{bmatrix} z^j \sum_{\ell \geq 0} \frac{z^{2\ell}}{\left(q^2;q^2\right)_\ell}.$$

Comparing coefficients gives

$$\sum_{j=0}^{n}(-1)^j q^{jm}\begin{bmatrix} n \\ j \end{bmatrix} = \left(q;q\right)_n \sum_{j+2\ell=n} q^{\binom{j}{2}}\begin{bmatrix} m \\ j \end{bmatrix}\frac{1}{\left(q^2;q^2\right)_\ell}.$$

This implies

$$\sum_{j=0}^{2n}(-1)^j q^{jm}\begin{bmatrix} 2n \\ j \end{bmatrix} = \left(q;q\right)_{2n} \sum_{j+2\ell=2n} q^{\binom{j}{2}}\begin{bmatrix} m \\ j \end{bmatrix}\frac{1}{\left(q^2;q^2\right)_\ell} = \left(q;q^2\right)_n \sum_{\ell=0}^{n} q^{\binom{2n-2\ell}{2}}\begin{bmatrix} m \\ 2n-2\ell \end{bmatrix}\frac{\left(q^2;q^2\right)_n}{\left(q^2;q^2\right)_\ell}$$

and

$$\sum_{j=0}^{2n+1}(-1)^j q^{jm}\begin{bmatrix} 2n+1 \\ j \end{bmatrix} = \left(q;q\right)_{2n+1} \sum_{j+2\ell=2n+1} q^{\binom{j}{2}}\begin{bmatrix} m \\ j \end{bmatrix}\frac{1}{\left(q^2;q^2\right)_\ell} = \left(q;q^2\right)_{n+1} \sum_{\ell=0}^{n} q^{\binom{2n+1-2\ell}{2}}\begin{bmatrix} m \\ 2n+1-2\ell \end{bmatrix}\frac{\left(q^2;q^2\right)_n}{\left(q^2;q^2\right)_\ell}.$$

Theorem 3.2

For each integer $m \geq 0,$

$$\sum_{j=0}^{n}(-1)^j q^{(2m+1)j}\begin{bmatrix} n \\ j \end{bmatrix}_{q^2} = \left(q;-q\right)_n c_n(m,q) \qquad (3.4)$$

with $c_n(m,q) \in \mathbb{Z}[q].$

For $m = 0$ *we get*

$$\sum_{j=0}^{n}(-1)^j q^j\begin{bmatrix} n \\ j \end{bmatrix}_{q^2} = (q;-q)_n = \left(q;q^2\right)_{\left\lfloor \frac{n+1}{2} \right\rfloor}\left(-q^2;q^2\right)_{\left\lfloor \frac{n}{2} \right\rfloor}. \qquad (3.5)$$

Proof

By replacing $q \to -q$ we can use the same argument as in Theorem 1.1.

Computer experiments lead to

Conjecture 3.1

For integers $r \geq 0, m \geq 0, k > 0$

$$\sum_{j=0}^{n}(-1)^j q^{rj^2+mj}\begin{bmatrix} n \\ j \end{bmatrix}_{q^{2^k}} = \left(q;q^2\right)_{\left\lfloor \frac{n+1}{2} \right\rfloor} c_n(r,m,k,q) \qquad (3.6)$$

with $c_n(r,m,k,q) \in \mathbb{Z}[q].$ *For* $r = 0$ *and* $m = 1$ *all coefficients of* $c_n(0,1,k,q)$ *are non-negative.*

Let us now consider how fast $\sum_{j=0}^{n}(-1)^j q^{mj}\begin{bmatrix} n \\ j \end{bmatrix}_{q^k}$ converges to 0 for $q\to 1$.

Theorem 3.3

$$\lim_{q\to 1}\frac{\sum_{j=0}^{2n}(-1)^j q^{mj}\begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k}}{\left(q;q^2\right)_n}=k^n \tag{3.7}$$

and

$$\lim_{q\to 1}\frac{\sum_{j=0}^{2n+1}(-1)^j q^{mj}\begin{bmatrix} 2n+1 \\ j \end{bmatrix}_{q^k}}{\left(q;q^2\right)_{n+1}}=mk^n. \tag{3.8}$$

Proof

Let $f(n,x,q)=\sum_{j=0}^{n}x^j\begin{bmatrix} n \\ j \end{bmatrix}_q$.

Then $f(2n,-q^m,q^k)=\sum_{j=0}^{2n}(-1)^j q^{mj}\begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k}$

by (1.23) satisfies the recurrence

$$f(2n,-q^m,q^k)=\left(1-(1+q^k)q^{2k(n-1)+m}+q^{2m}\right)f(2n-2,-q^m,q^k)$$
$$-\left(1-q^{k(2n-3)}\right)\left(1-q^{2k(n-1)}\right)q^{2m}f(2n-4,-q^m,q^k).$$

Therefore

$$h(n,q)=\frac{f(2n,-q^m,q^k)}{\left(q;q^2\right)_n}$$

satisfies the recurrence

$$h(n,q)=\frac{1-(1+q^k)q^{2k(n-1)+m}+q^{2m}}{1-q^{2n-1}}h(n-1,q)-\frac{\left(1-q^{k(2n-3)}\right)\left(1-q^{2k(n-1)}\right)q^{2m}}{\left(1-q^{2n-1}\right)\left(1-q^{2n-3}\right)}h(n-2,q)$$

with initial values

$h(0,q)=1$ and $h(1,q)=[m]-q^{m+k}[m-k]$.

For $q\to 1$ we get

$$h(n,1)=\frac{k(4n-3)}{2n-1}h(n-1,1)-\frac{2k^2(n-1)}{2n-1}h(n-2,1)$$

with initial values $h(0,1)=1$ and $h(1,1)=k.$

This implies $h(n,1)=k^n$ and thus (3.7).

In the same way $f(2n+1,-q^m,q^k)=\sum_{j=0}^{2n+1}(-1)^j q^{mj}\begin{bmatrix}2n+1\\ j\end{bmatrix}_{q^k}$

satisfies the recurrence

$$f(2n+1,-q^m,q^k)=\left(1-(1+q^k)q^{k(2n-1)+m}+q^{2m}\right)f(2n-1,-q^m,q^k)$$
$$-\left(1-q^{k(2n-2)}\right)\left(1-q^{k(2n-1)}\right)q^{2m}f(2n-3,-q^m,q^k).$$

Therefore

$$H(n,q)=\frac{f(2n+1,-q^m,q^k)}{\left(q;q^2\right)_{n+1}}$$

satisfies the recurrence

$$H(n,q)=\frac{1-(1+q^k)q^{k(2n-1)+m}+q^{2m}}{1-q^{2n+1}}H(n-1,q)-\frac{\left(1-q^{k(2n-1)}\right)\left(1-q^{2k(n-1)}\right)q^{2m}}{\left(1-q^{2n-1}\right)\left(1-q^{2n+1}\right)}H(n-2,q)$$

with initial values

$$H(0,q)=[m] \text{ and } H(1,q)=[m]\frac{[m+k]+q^{2m}[2k-m]}{[3]}.$$

For $q\to 1$ this gives

$$H(n,1)=\frac{k(4n-1)}{2n+1}H(n-1,1)-\frac{2k^2\left(n-1\right)}{2n+1}H(n-2,1)$$

with initial values $H(0,1)=m$ and $H(1,1)=km.$ This gives (3.8).

The same argument gives

Corollary 3.1

$$\lim_{q\to -1}\frac{\sum_{j=0}^{n}q^{(2m+1)j}\begin{bmatrix}2n\\ j\end{bmatrix}_{q^{2k}}}{\left(-q;q^2\right)_n}=(2k)^n \tag{3.9}$$

and

$$\lim_{q\to -1}\frac{\sum_{j=0}^{n} q^{(2m+1)j}\begin{bmatrix}2n+1\\ j\end{bmatrix}_{q^{2k}}}{\left(-q;q^2\right)_{n+1}}=(2k)^n(2m+1). \qquad (3.10)$$

More generally let us consider the same question for $\sum_{j=0}^{n}(-1)^j q^{rj^2+mj}\begin{bmatrix}n\\ j\end{bmatrix}_{q^k}$.

In a first draft of this paper I stated two conjectures, one of which I posted as question in MathOverflow. This has been proved by Will Sawin [7].

Lemma 3.1 (W. Sawin [7])

For all $r,m\in\mathbb{Z}$

$$\lim_{q\to 1}\frac{\sum_{j=0}^{2n}(-1)^j q^{rj^2+mj}\begin{bmatrix}2n\\ j\end{bmatrix}_{q^k}}{\left(q;q^2\right)_n}=\left(k-2r\right)^n, \qquad (3.11)$$

Proof

For $k=2r$ we get from (1.4)

$$\sum_{j=0}^{n}(-1)^j q^{\frac{k}{2}j^2+mj}\begin{bmatrix}n\\ j\end{bmatrix}_{q^k}=\left(1-q^{\frac{k}{2}+m}\right)\left(1-q^{3\frac{k}{2}+m}\right)\cdots\left(1-q^{(2n-1)\frac{k}{2}+m}\right)=\left(q^{\frac{k}{2}+m};q^k\right)_n. \qquad (3.12)$$

Therefore

$$\lim_{q\to 1}\frac{\sum_{j=0}^{2n}(-1)^j q^{\frac{k}{2}j^2+m}\begin{bmatrix}2n\\ j\end{bmatrix}_{q^k}}{\left(q;q^2\right)_n}=\lim_{q\to 1}\frac{\left(q^{\frac{k}{2}+m};q^k\right)_{2n}}{\left(q;q^2\right)_n}=[n=0]$$

and

$$\lim_{q\to 1}\frac{\sum_{j=0}^{2n+1}(-1)^j q^{\frac{k}{2}j^2+mj}\begin{bmatrix}2n+1\\ j\end{bmatrix}_{q^k}}{\left(q;q^2\right)_{n+1}}=\lim_{q\to 1}\frac{\left(q^{\frac{k}{2}+m};q^k\right)_{2n+1}}{\left(q;q^2\right)_{n+1}}=\left(\frac{k}{2}+m\right)[n=0].$$

Thus in this case the Lemma is true.

Let $f(n,r,m,k)=\sum_{j=0}^{n}(-1)^j q^{rj^2+mj}\begin{bmatrix}n\\ j\end{bmatrix}_{q^k}$.

To obtain the limit (3.11) we observe that the denominator has the root $q=1$ with multiplicity n. If the limit exists then the nominator also has the root $q=1$ with multiplicity at least n.

Sawin's trick was to use de L'Hospital's rule combined with Leibniz's rule by writing $f(n,r,m,k)$ in the form

$$f(n,r,m,k)=\sum_{j=0}^{n}(-1)^j q^{rj^2+mj}\begin{bmatrix} n \\ j \end{bmatrix}_{q^k}=\sum_{j=0}^{n}(-1)^j q^{\left(r-\frac{k}{2}\right)j^2+mj}\cdot q^{\frac{k}{2}j^2}\begin{bmatrix} n \\ j \end{bmatrix}_{q^k}.$$

This gives

$$\frac{\partial^i f(2n,r,m,k)}{\partial q^i}(1)=\sum_{a=0}^{i}\binom{i}{a}\sum_{j=0}^{2n}(-1)^j \frac{\partial^a q^{\left(r-\frac{k}{2}\right)j^2+mj}}{\partial q^a}(1)\frac{\partial^{i-a} q^{\frac{k}{2}j^2}\begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k}}{\partial q^{i-a}}(1).$$

Now $\frac{\partial^a q^{\left(r-\frac{k}{2}\right)j^2+mj}}{\partial q^a}(1)$ is a polynomial of degree $2a$ in $j.$ Therefore the right-hand side can be written as a linear combination of the terms

$$F(b,c,2n)=\sum_{j=0}^{2n}(-1)^j j^b \frac{\partial^c q^{\frac{k}{2}j^2}\begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k}}{\partial q^c}(1) \tag{3.13}$$

where $b+2c\le 2a+2(i-a)=2i.$

Let us now consider the known case $f\left(2n,\frac{k}{2},m,k\right).$

$$\frac{\partial^i f\left(2n,\frac{k}{2},m,k\right)}{\partial q^i}(1)=\frac{\partial^i \sum_{j=0}^{2n}(-1)^j q^{\frac{k}{2}j^2+mj}\begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k}}{\partial q^i}(1)$$

$$=\sum_{a=0}^{i}\binom{i}{a}\sum_{j=0}^{2n}(-1)^j \frac{\partial^a q^{mj}}{\partial q^a}(1)\frac{\partial^{i-a} q^{\frac{k}{2}j^2}\begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k}}{\partial q^{i-a}}(1)$$

$$=\sum_{a=0}^{i}\binom{i}{a}\sum_{j=0}^{2n}(-1)^j (mj)(mj-1)\cdots(mj-a+1)\frac{\partial^{i-a} q^{\frac{k}{2}j^2}\begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k}}{\partial q^{i-a}}(1)$$

$$=\sum_{a=0}^{i}\binom{i}{a}\sum_{b=0}^{a}[j^b](mj)(mj-1)\cdots(mj-a+1)F(b,i-a,2n).$$

It is clear that $F(0,0,2n)=[n=0].$

Now let us assume by induction that

$F(b,c,2n)=0$ for $b+c<i$ for some fixed $i\le 2n.$

Then we get

$$\frac{\partial^i \sum_{j=0}^{n} (-1)^j q^{\frac{k}{2}j^2+mj} \begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k}}{\partial q^i}(1) = \sum_{a=0}^{i} \binom{i}{a} [j^a](mj)(mj-1)\cdots(mj-a+1)F(a,i-a,2n)$$

$$= \sum_{a=0}^{i} \binom{i}{a} m^a F(a,i-a,2n).$$

By (3.12) this polynomial is identically 0 for $i < 2n.$

Therefore all coefficients $F(a,i-a,2n)$ vanish. Thus $F(b,c,2n) = 0$ for $b+c = i.$

This implies that $F(b,c,2n) = 0$ for $b+2c < 2n$ and therefore the first $n-1$ partial derivatives vanish.

The only $F(b,c,2n)$ with $b+2c \le 2n$ which we did not yet compute is

$$F(2n,0,2n) = \sum_{j=0}^{2n} (-1)^j j^{2n} q^{\frac{k}{2}j^2} \begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k} (1) = \sum_{j=0}^{2n} (-1)^j j^{2n} \binom{2n}{j} = (2n)!.$$

Since j^{2n} only occurs for $a = n$ we get

$$\frac{\partial^n f(2n,r,m,k)}{\partial q^n}(1) = \sum_{a=0}^{n} \binom{n}{a} \sum_{j=0}^{2n} (-1)^j \frac{\partial^a q^{\left(r-\frac{k}{2}\right)j^2+mj}}{\partial q^a}(1) \frac{\partial^{i-a} q^{\frac{k}{2}j^2} \begin{bmatrix} 2n \\ j \end{bmatrix}_{q^k}}{\partial q^{i-a}}(1)$$

$$= F(2n,0,2n)[j^{2n}] \frac{\partial^n q^{\left(r-\frac{k}{2}\right)j^2+mj}}{\partial q^n}(1).$$

Now

$$\left[j^{2n}\right] \frac{\partial^n q^{\left(r-\frac{k}{2}\right)j^2+mj}}{\partial q^n}(1)$$

$$= \left[j^{2n}\right] \left(\left(r-\frac{k}{2}\right)j^2+mj \right) \left(\left(r-\frac{k}{2}\right)j^2+mj-1 \right) \cdots \left(\left(r-\frac{k}{2}\right)j^2+mj-n+1 \right) = \left(r-\frac{k}{2}\right)^n.$$

Therefore we finally get

$$\frac{\partial^n f(2n,r,m,k)}{\partial q^n}(1) = (2n)! \left(r-\frac{k}{2}\right)^n.$$

By Gauss's theorem we have $\left(q;q^2\right)_n = f(2n,0,0,1).$

Thus

$$\frac{\partial^n f(2n,0,0,1)}{\partial q^n}(1) = (2n)!\left(-\frac{1}{2}\right)^n.$$

This implies

$$\lim_{q\to 1}\frac{\sum_{j=0}^{2n}(-1)^j q^{rj^2+mj}\begin{bmatrix}2n\\ j\end{bmatrix}_{q^k}}{\left(q;q^2\right)_n} = \lim_{q\to 1}\frac{\sum_{j=0}^{2n}(-1)^j q^{rj^2+mj}\begin{bmatrix}2n\\ j\end{bmatrix}_{q^k}}{\sum_{j=0}^{2n}(-1)^j \begin{bmatrix}2n\\ j\end{bmatrix}_{q^k}} = \frac{\left(r-\frac{k}{2}\right)^n}{\left(-\frac{1}{2}\right)^n} = \left(k-2r\right)^n.$$

Let us now consider binomials of the form $\sum_{j=0}^{n} q^{rj^2+mj}\begin{bmatrix}n\\ j\end{bmatrix}_{q^k}$.

It is easy to check that $\lim_{q\to -1}\sum_{j=0}^{n} q^{rj^2+mj}\begin{bmatrix}n\\ j\end{bmatrix}_{q^k} = 0$ for $n>0$ if and only if $r+m\equiv 1 \bmod 2$ and $k\equiv 0 \bmod 2.$

Therefore we get

Corollary 3.2

If $r+m\equiv 1\bmod 2$ *then*

$$\lim_{q\to -1}\frac{\sum_{j=0}^{2n} q^{rj^2+mj}\begin{bmatrix}2n\\ j\end{bmatrix}_{q^{2k}}}{\left(-q;q\right)_{2n}} = \left(k-r\right)^n. \tag{3.14}$$

This follows from $(-q)^{rj^2+mj} = (-1)^j q^{rj^2+mj}$ and $\lim_{q\to -1}\frac{(-q;q)_{2n}}{\left(-q;q^2\right)_n} = 2^n.$

The analogous results for $2n+1$ are stated as

Conjecture 3.2

$$\lim_{q\to 1}\frac{\sum_{j=0}^{2n+1}(-1)^j q^{rj^2+mj}\begin{bmatrix}2n+1\\ j\end{bmatrix}_{q^k}}{\left(q;q^2\right)_{n+1}} = \left((2n+1)r+m\right)\left(k-2r\right)^n \tag{3.15}$$

and for $r+m\equiv 1\bmod 2$

$$\lim_{q\to -1}\frac{\sum_{j=0}^{2n+1} q^{rj^2+mj}\begin{bmatrix}2n+1\\ j\end{bmatrix}_{q^{2k}}}{(-q;q)_{2n+1}}=\big((2n+1)r+m\big)\big(k-r\big)^n. \tag{3.16}$$

Conjecture 3.3

For a prime number p let $v_p(x)$ denote the $p-$adic valuation of x, which is the largest non-negative integer m such that p^m divides x and let $V_p(x)=p^{v_p(x)}$. For integers $m,r\geq 0$ we get

$$\sum_{j=0}^{n}(-1)^j q^{rj^2+mj}\begin{bmatrix}n\\ j\end{bmatrix}_{q^p}=c_n(r,m,p,q)\prod_{j=1}^{\left\lfloor\frac{n+1}{2}\right\rfloor}\left(1-q^{\frac{2j-1}{V_p(2j-1)}}\right), \tag{3.17}$$

with $c_n(r,m,p,q)\in\mathbb{Z}[q]$.

Finally let us state the following

Conjecture 3.4

For any odd prime p and any $m\in\mathbb{N}$ there exists a factorization

$$A_p(m,n,q)=\sum_{j=0}^{n} q^{(2m+1)j}\begin{bmatrix}n\\ j\end{bmatrix}_{q^{2p}}=\prod_{k=1}^{n}\left(1+q^{\frac{k}{V_p(k)}}\right)c_p(m,n,q) \tag{3.18}$$

with $c_p(m,n,q)\in\mathbb{Z}[q]$.

This polynomial satisfies $c_p(m,n,1)=1$ and

$$c_p(0,n,-1)=p^{b(n,p)} \tag{3.19}$$

with

$$b(n,p)=\left\lfloor\frac{n}{2}\right\rfloor+v_p(n!)-v_p\left(\left\lfloor\frac{n}{2}\right\rfloor!\right). \tag{3.20}$$

Example

Let us consider for example $A_3(2,n,q)=\sum_{j=0}^{n} q^{5j}\begin{bmatrix}n\\ j\end{bmatrix}_{q^6}$.

From $\left(\frac{k}{V_3(k)}\right)_{k\geq 1}=(1,2,1,4,5,2,7,8,1,\cdots)$

we find

$A_3(2,0,q)=1,\ \ A_3(2,1,q)=(1+q)c_3(2,1,q),\ \ A_3(2,2,q)=(1+q)\left(1+q^2\right)c_3(2,2,q),$

$A_3(2,3,q)=(1+q)^2\left(1+q^2\right)c_3(2,3,q),\ \ A_3(2,4,q)=(1+q)^2\left(1+q^2\right)\left(1+q^4\right)c_3(2,4,q),$

$A_3(2,5,q)=(1+q)^2\left(1+q^2\right)\left(1+q^4\right)\left(1+q^5\right)c_3(2,5,q),\cdots.$

The first terms of the sequence $\left(c_3(2,n,q)\right)_{n\geq 0}$ are

$1,\ 1-q+q^2-q^3+q^4,\ 1-q+q^4-q^6+q^8,\ \left(1-q+q^2-q^3+q^4\right)\left(1-q+q^4-q^5+q^8-q^9+q^{10}-q^{13}+q^{14}\right),\cdots$

It is clear that $c_3(2,n,1)=1.$

For $q=-1$ we get $\left(c_3(2,n,-1)\right)_{n\geq 0}=\left(1,\ 5,\ 3,\ 5\cdot 3^2,\ 3^3,\ 5\cdot 3^3,\ 3^4,\ 5\cdot 3^4,\ 3^5,\ 5\cdot 3^7,\cdots\right).$

In order to compute $c_p(m,n,-1)$ we observe that by Corollary 3.1 we have

$$\lim_{q\to -1}\frac{\sum_{j=0}^{n} q^{(2m+1)j}\begin{bmatrix}2n\\ j\end{bmatrix}_{q^{2k}}}{\left(-q;q\right)_{2n}}=k^n \tag{3.21}$$

and

$$\lim_{q\to -1}\frac{\sum_{j=0}^{n} q^{(2m+1)j}\begin{bmatrix}2n+1\\ j\end{bmatrix}_{q^{2k}}}{\left(-q;q\right)_{2n+1}}=k^n(2m+1). \tag{3.22}$$

It remains to compute $\lim_{q\to -1}\frac{\left(-q;q\right)_n}{\prod_{j=1}^{n}\left(1+q^{\frac{j}{V_p(j)}}\right)}.$ To this end observe that for $j\not\equiv 0 \bmod p$ we have

$V_p(j)=1$ and therefore $\frac{\left(-q;q\right)_n}{\prod_{j=1}^{n}\left(1+q^{\frac{j}{V_p(j)}}\right)}=\prod_{2jp\leq n}\frac{1+q^{2jp}}{1+q^{\frac{2jp}{V_p(2jp)}}}\prod_{(2j+1)p\leq n}\frac{1+q^{(2j+1)p}}{1+q^{\frac{(2j+1)p}{V_p((2j+1)p)}}}.$

For $q\to -1$ the first product converges to 1.

Since $\lim_{q\to -1}\frac{1+q^{(2j+1)p}}{1+q^{\frac{(2j+1)p}{V_p((2j+1)p)}}}=V_p((2j+1)p)$ and $V_p\left(2\cdot 4\cdots 2\left\lfloor\frac{n}{2}\right\rfloor\right)=V_p\left(\left\lfloor\frac{n}{2}\right\rfloor !\right)$

we get $\lim_{q\to -1}\frac{\left(-q;q\right)_n}{\prod_{j=1}^{n}\left(1+q^{a(j,p)}\right)}=p^{v_p(n!)-v_p\left(\left\lfloor\frac{n}{2}\right\rfloor !\right)}.$

4. Conclusion

In the main part of this paper we have obtained some evidence that for non-negative integers r,m,k the polynomials $\sum_{j=0}^{n}(-1)^j q^{rj^2+mj}\begin{bmatrix} n \\ j \end{bmatrix}_{q^{2^k}}$ are divisible by $\left(q;q^2\right)_{\left\lfloor \frac{n+1}{2} \right\rfloor}$ in $\mathbb{Z}[q]$ and that for odd primes p the polynomials $\sum_{j=0}^{n}(-1)^j q^{rj^2+mj}\begin{bmatrix} n \\ j \end{bmatrix}_{q^p}$ have the factor $\prod_{j=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor}\left(1-q^{\frac{2j-1}{V_p(2j-1)}}\right)$ in $\mathbb{Z}[q]$.

Further we were led to the conjecture that for odd prime numbers p and non-negative integers m the polynomials $\sum_{j=0}^{n} q^{(2m+1)j}\begin{bmatrix} n \\ j \end{bmatrix}_{q^{2p}}$ are not only divisible by $\prod_{j=1}^{\left\lfloor \frac{n+1}{2} \right\rfloor}\left(1+q^{\frac{2j-1}{V_p(2j-1)}}\right)$ but even by $\prod_{k=1}^{n}\left(1+q^{\frac{k}{V_p(k)}}\right)$. Till now we could only provide proofs for some special cases.

References

[1] A. Berkovich and S.O. Warnaar, Positivity preserving transformations for $q-$binomial coefficients, TAMS 357 (2005), 2291-2351

[2] J. Cigler, Operatormethoden für q-Identitäten, Monatshefte Math. 88 (1979), 87-105

[3] J. Cigler, Ein paar einfache $q-$Analoga des binomischen Lehrsatzes, Preprint 2005

[4] J. Cigler, Finite $q-$identities related to well-known theorems of Euler and Gauss, arXiv: math/0610546

[5] T. H. Koornwinder, Special functions and $q-$commuting variables, arXiv: q-alg/9608008

[6] B. A. Kupershmidt, $q-$Newton binomial: from Euler to Gauss, J. Nonlinear Math. Phys. 7 (2000), 244-262

[7] W. Sawin, http://mathoverflow.net/questions/212926/

[8] M.P. Schützenberger, Une interprétation de certaines solutions de l'équation fonctionnelle: $F(x+y)=F(x)F(y)$, C. R. Acad. Sci. Paris 236 (1953), 352-353