A New q-Analog of Stirling Numbers

By

J. Cigler

(Vorgelegt in der Sitzung der math.-nat. Klasse am 15. Oktober 1992)

Abstract

The number of spanning subsets of a finite vector space is closely related to a q-analog of the Stirling numbers (cf. [3]). The purpose of this note is to study these numbers in more detail.

1. The q-Stirling Numbers $\left\{ {n \atop k} \right\}_q$

The Stirling number $\left\{ {n \atop k} \right\}$ of the second kind counts the number of partitions π of $\{0, 1, \ldots, n-1\}$ into k nonempty subsets $B_0, B_1, \ldots, B_{k-1}$. We use the notation proposed by D. Knuth ([5], [6]). We also use his version of Iverson's convention, setting $[P(n)] = 1$ if the statement $P(n)$ is true and $[P(n)] = 0$ if it is false.

We now associate a weight $w(\pi)$ with each partition π. To this end we distinguish that part of π which contains the number 0 and call it B_0.

Then $w(\pi) := q^{\sum_{i \in B_0} i}$.

For each set A of partitions let $w(A) := \sum_{\pi \in A} w(\pi)$.

Let $A_{n,k}$ be the set of all partitions of $\{0, 1, \ldots, n-1\}$ into k nonempty parts.

Then we define

$$\left\{ {n \atop k} \right\}_q := w(A_{n,k}), \quad n, k \geqslant 1,$$

and

$$\left\{ {0 \atop k} \right\}_q := [k=0], \quad \left\{ {n \atop 0} \right\}_q := [n=0], \quad n,k \geqslant 0.$$

E.g. $\left\{ {4 \atop 2} \right\}_q = 1+q+q^2+2q^3+q^4+q^5$, since $w(A_{4,2}) = w(0/123) + w(01/23) + w(02/13) + w(03/12) + w(012/3) + w(013/2) + w(023/1) = 1 + q + q^2 + q^3 + q^3 + q^4 + q^5$.

The numbers $\left\{ {n \atop k} \right\}_q$ satisfy the following recurrence

$$\left\{ {n+1 \atop k} \right\}_q = \left\{ {n \atop k-1} \right\}_q + (k-1+q^n) \left\{ {n \atop k} \right\}_q, \quad n \geqslant 0, k \geqslant 1.$$

To see this write $A_{n+1,k} = C_1 \cup C_2 \cup C_3$.

C_1 is the set of all $\pi \in A_{n+1,k}$ such that $\{n\}$ is one of the nonempty parts of π.

C_2 is the set of all π such that $n \in B_i, i \neq 0$, and $B_i \backslash \{n\} \neq \varnothing$.

C_3 is the set of all π such that $n \in B_0$.

Then obviously

$$w(C_1) = \left\{ {n \atop k-1} \right\}_q, \quad w(C_2) = (k-1) \left\{ {n \atop k} \right\}_q,$$

and $w(C_3) = q^n \left\{ {n \atop k} \right\}_q$.

We get the following table:

n	$\left\{ {n \atop 0} \right\}_q$	$\left\{ {n \atop 1} \right\}_q$	$\left\{ {n \atop 2} \right\}_q$	$\left\{ {n \atop 3} \right\}_q$
0	1	0	0	0
1	0	1	0	0
2	0	q	1	0
3	0	q^3	$1+q+q^2$	1
4	0	q^6	$1+q+q^2+2q^3+q^4+q^5$	$3+q+q^2+q^3$

In order to give an explicit formula for $\left\{ {n \atop k} \right\}_q$ we need the q-binomial coefficients

$$\binom{n}{k}_q = \frac{[n]!}{[k]![n-k]!}$$

with $[n]! = [n][n-1]\cdots[2][1]$ and $[n] = \dfrac{q^n-1}{q-1}$.

It is well known (cf. e.g. [2]), that

$$(a+x)(a+qx)\cdots(a+q^{n-1}x) = \sum \binom{n}{i}_q q^{\binom{i}{2}} x^i a^{n-i}.$$

Replacing x by qx and comparing coefficients of x^i gives

$$\sum_{1 \leqslant j_1 < j_2 < \cdots j_i \leqslant n} q^{j_1 + j_2 + \cdots + j_i} = \binom{n}{i}_q q^{\binom{i+1}{2}}.$$

Writing $\pi \in A_{n+1,k+1}$ in the form

$$\pi = \{0, j_1, \ldots j_i\}/B_1/\cdots/B_k$$

we get therefore

$$\left\{ {n+1 \atop k+1} \right\}_q = w(A_{n+1,k+1}) = \sum_i \sum_{j_1 < \cdots < j_i} q^{j_1 + \cdots + j_i} \left\{ {n-i \atop k} \right\}.$$

We thus get

$$\left\{ {n+1 \atop m+1} \right\}_q = \sum_i \binom{n}{i}_q \left\{ {i \atop m} \right\} q^{\binom{n-i+1}{2}}. \tag{1}$$

As special case we note

$$\left\{ {n+1+k \atop n+1} \right\}_q = \sum_{i=0}^{k} f_i(n) \binom{n+k}{k-i}_q q^{\binom{k-i+1}{2}}, \tag{2}$$

where $f_i(n) = \left\{ {n+i \atop n} \right\}$ are the usual Stirling polynomials (cf. [4]) of degree $2i$.

E.g. we have

$$\left\{ {n+1 \atop n} \right\}_q = [n]q + \binom{n}{2},$$

$$\left\{ {n+2 \atop n} \right\}_q = \binom{n+1}{2} q^3 + [n+1]q\binom{n}{2} + \left\{ {n+1 \atop n-1} \right\}.$$

From the well known formula

$$\frac{\Delta^k}{k!} x^n|_{x=0} = \begin{Bmatrix} n \\ k \end{Bmatrix}.$$

(cf. e.g. [7], [8]) we get

$$\begin{aligned}\begin{Bmatrix} n \\ k+1 \end{Bmatrix}_q &= \sum \binom{n-1}{i}_q \begin{Bmatrix} n-1-i \\ k \end{Bmatrix} q^{\binom{i+1}{2}} \\ &= \frac{\Delta^k}{k!} \sum \binom{n-1}{i}_q q^{\binom{i+1}{2}} x^{n-i-1}|_{x=0} \\ &= \frac{\Delta^k}{k!}(q+x)(q^2+x)\cdots(q^{n-1}+x)|_{x=0}.\end{aligned}$$

Observing that $\Delta x^{\underline{n}} = \Delta x(x-1)\cdots(x-n+1) = n x^{\underline{n-1}}$ we get immediately

$$\sum \begin{Bmatrix} n \\ k+1 \end{Bmatrix}_q x^{\underline{k}} = (q+x)(q^2+x)\cdots(q^{n-1}+x)$$

or by multiplying both sides with $(1+x)$ the slightly more symmetric formula

$$(x+1)(x+q)\cdots(x+q^{n-1}) = \sum_k \begin{Bmatrix} n \\ k \end{Bmatrix}_q (x+1)^{\underline{k}} \tag{3}$$

By choosing successively $x = 0, 1, 2, \ldots$ we get

$$\begin{Bmatrix} n \\ 1 \end{Bmatrix}_q = q^{\binom{n}{2}}, \quad 2(q+1)(q^2+1)\cdots(q^{n-1}+1) = 2\begin{Bmatrix} n \\ 1 \end{Bmatrix} + 2\begin{Bmatrix} n \\ 2 \end{Bmatrix},$$

i.e.

$$\begin{Bmatrix} n \\ 2 \end{Bmatrix}_q = (1+q)(1+q^2)\cdots(1+q^{n-1}) - q^{\binom{n}{2}},$$

etc.

It is well known that the Stirling numbers $\begin{Bmatrix} n \\ k \end{Bmatrix}$ can be uniquely extended to all $n, k \in \mathbb{Z}$ satisfying the same recurrence. The same is of course also true for $\begin{Bmatrix} n \\ k \end{Bmatrix}_q$.

We have therefore a uniquely determined set of numbers $\begin{Bmatrix} n \\ k \end{Bmatrix}_q$ satisfying

$$\begin{Bmatrix} n \\ 0 \end{Bmatrix}_q = [n=0], \quad \begin{Bmatrix} 0 \\ k \end{Bmatrix}_q = [k=0], \quad n, k \in \mathbb{Z} \tag{4}$$

$$\begin{Bmatrix} n+1 \\ k \end{Bmatrix}_q = \begin{Bmatrix} n \\ k-1 \end{Bmatrix}_q + (k-1+q^n)\begin{Bmatrix} n \\ k \end{Bmatrix}_q, \quad n, k \in \mathbb{Z}. \tag{5}$$

Setting $\begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}} = \begin{Bmatrix} -k \\ -n \end{Bmatrix}_q$ we get

$$\begin{aligned}\begin{bmatrix} n+1 \\ k \end{bmatrix}_{\frac{1}{q}} = \begin{Bmatrix} -k \\ -n-1 \end{Bmatrix}_q &= \begin{Bmatrix} -k+1 \\ -n \end{Bmatrix}_q + (n+1-q^{-k})\begin{Bmatrix} -k \\ -n \end{Bmatrix}_q \\ &= \begin{bmatrix} n \\ k-1 \end{bmatrix}_{\frac{1}{q}} + \left(n+1-\left(\frac{1}{q}\right)^k\right)\begin{bmatrix} n \\ k \end{bmatrix}_{\frac{1}{q}}.\end{aligned}$$

2. The q-Stirling Numbers $\begin{bmatrix} n \\ k \end{bmatrix}_q$

We now define the q-Stirling numbers $\begin{bmatrix} n \\ k \end{bmatrix}_q$ of the first kind as $\begin{bmatrix} n \\ k \end{bmatrix}_q = \begin{Bmatrix} -k \\ -n \end{Bmatrix}_{\frac{1}{q}}$ or equivalently as the uniquely determined numbers satisfying.

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q = [n=0], \quad \begin{bmatrix} k \\ 0 \end{bmatrix}_q = [k=0], \quad n, k \in \mathbb{Z} \tag{6}$$

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q + (n+1-q^k)\begin{bmatrix} n \\ k \end{bmatrix}_q, \quad n, k \in \mathbb{Z} \tag{7}$$

We get the following table:

n	$\begin{bmatrix} n \\ 0 \end{bmatrix}_q$	$\begin{bmatrix} n \\ 1 \end{bmatrix}_q$	$\begin{bmatrix} n \\ 2 \end{bmatrix}_q$	$\begin{bmatrix} n \\ 3 \end{bmatrix}_q$
0	1	0	0	0
1	0	1	0	0
2	0	$2-q$	1	0
3	0	$(2-q)(3-q)$	$5-q-q^2$	1
4	0	$(2-q)(3-q)(4-q)$	$26-9q-8q^2+q^3+q^4$	$9-q-q^2-q^3$

It is easy to verify that the generating function is given by

$$(x+1)(x+2)\cdots(x+n)=\sum\begin{bmatrix}n\\k\end{bmatrix}_q (x+1)(x+q)\cdots(x+q^{k-1}). \qquad (8)$$

This follows at once from

$$(x+n+1)\begin{bmatrix}n\\k\end{bmatrix}_q=\begin{bmatrix}n\\k\end{bmatrix}_q((x+q^k)+(n+1-q^k)).$$

Let now D_q be the q-derivation $(D_qf)(x)=\dfrac{f(qx)-f(x)}{(q-1)x}$.
Then $D_q(x+1)\cdots(x+q^{n-1})=[n](x+1)\cdots(x+q^{n-2})$.
Therefore we get

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{D_q^k}{[k]!}(x+1)(x+2)\cdots(x+n)|_{x=-1}.$$

Observing that $(x+1)\cdots(x+n)=\sum\begin{bmatrix}n+1\\k\end{bmatrix}x^{k-1}$ we get

$$\begin{bmatrix}n\\k\end{bmatrix}_q=\frac{1}{[k]!}D_q^k\sum\begin{bmatrix}n+1\\l\end{bmatrix}x^{l-1}|_{x=-1}$$

or

$$\begin{bmatrix}n\\m\end{bmatrix}_q=\sum_k\begin{bmatrix}n+1\\k+1\end{bmatrix}\binom{k}{m}_q(-1)^{k-m}. \qquad (9)$$

As a special case we get

$$\begin{bmatrix}n-1\\n-k-1\end{bmatrix}_q=\sum_{i=0}^{k}\begin{bmatrix}n\\n-i\end{bmatrix}\binom{n-i-1}{k-i}_q(-1)^{i-k}. \qquad (10)$$

Remark. This is of course the same formula as (2).

We have only to observe that $\begin{bmatrix}-n\\-n-i\end{bmatrix}_{\frac{1}{q}}=\left\{\begin{matrix}n+i\\n\end{matrix}\right\}_q$

and

$$\binom{-m}{p}_{\frac{1}{q}}=(-1)^pq^{\binom{p+1}{2}}\binom{m+p-1}{p}_q.$$

Then

$$\left\{\begin{matrix}n+k+1\\n+1\end{matrix}\right\}_q=\begin{bmatrix}-n-1\\-n-k-1\end{bmatrix}_{\frac{1}{q}}=\sum\begin{bmatrix}-n\\-n-i\end{bmatrix}\binom{-n-i-1}{k-i}_{\frac{1}{q}}(-1)^{i-k}$$

$$=\sum\left\{\begin{matrix}n+i\\n\end{matrix}\right\}_q\binom{n+k}{k-i}_q q^{\binom{k-i+1}{2}}.$$

As special cases we get

$$\begin{bmatrix}n\\n-1\end{bmatrix}_q=\binom{n+1}{2}-[n]$$

$$\begin{bmatrix}n\\n-2\end{bmatrix}_q=\binom{n}{2}_q-[n-1]\binom{n+1}{2}+\frac{n(n^2-1)(3n+2)}{4!}.$$

From the generating function it is immediate that

$$\begin{bmatrix}n\\1\end{bmatrix}_q=(2-q)(3-q)\cdots(n-q).$$

Remark. The Stirling numbers $s(n,k)$ introduced in [3] are given by

$$s(n,k)=(-1)^{n-k}\begin{bmatrix}n\\k\end{bmatrix}_q.$$

3. The q-Stirling Numbers $\begin{bmatrix}n\\k\end{bmatrix}_q^*$

The classical Stirling numbers satisfy the inversion formulas

$$\sum\begin{bmatrix}n\\k\end{bmatrix}\left\{\begin{matrix}k\\m\end{matrix}\right\}(-1)^{n-k}=[m=n]$$

and

$$\sum\left\{\begin{matrix}n\\k\end{matrix}\right\}\begin{bmatrix}k\\m\end{bmatrix}(-1)^{n-k}=[m=n].$$

In order to get a q-analog of these formulas we introduce another class of q-Stirling numbers $\begin{bmatrix}n\\k\end{bmatrix}_q^*$.

We define them via the generating function

$$\sum\begin{bmatrix}n\\k\end{bmatrix}_q^*(-1)^{n-k}(x+1)\cdots(x+q^{k-1})=(x+1)^n. \qquad (11)$$

Then clearly

$$\sum \begin{bmatrix} n \\ k \end{bmatrix}_q^* \begin{Bmatrix} k \\ m \end{Bmatrix}_q (-1)^{n-k} = [n=m] \tag{12}$$

and

$$\sum \begin{Bmatrix} n \\ k \end{Bmatrix}_q \begin{bmatrix} k \\ m \end{bmatrix}_q^* (-1)^{n-k} = [n=m]. \tag{13}$$

From $(x+1)^{\underline{n+1}} = (x+1-n)(x+1)^{\underline{n}}$ we get

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^* = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^* + (n-1+q^k)\begin{bmatrix} n \\ k \end{bmatrix}_q^*, \quad n,k \in \mathbb{Z}. \tag{14}$$

Of course we also have

$$\begin{bmatrix} n \\ 0 \end{bmatrix}_q^* = [n=0] \quad \text{and} \quad \begin{bmatrix} 0 \\ k \end{bmatrix}_q^* = [k=0]. \tag{15}$$

We get the following table:

n	$\begin{bmatrix} n \\ 0 \end{bmatrix}_q^*$	$\begin{bmatrix} n \\ 1 \end{bmatrix}_q^*$	$\begin{bmatrix} n \\ 2 \end{bmatrix}_q^*$	$\begin{bmatrix} n \\ 3 \end{bmatrix}_q^*$
0	1	0	0	0
1	0	1	0	0
2	0	q	1	0
3	0	$q(q+1)$	q^2+q+1	1
4	0	$q(q+1)(q+2)$	$q^4+q^3+q^2+3q+2$	q^3+q^2+q+3

From the generating function we find setting $x = -q, -q^2, \ldots$ explicit formulas for $\begin{bmatrix} n \\ k \end{bmatrix}_q^*$.

E.g.

$$\begin{bmatrix} n \\ 1 \end{bmatrix}_q^* = q(q+1)\cdots(q+n-2).$$

$$\begin{bmatrix} n \\ 2 \end{bmatrix}_q^* = \frac{q^2(q^2+1)\cdots(q^2+n-2) - q(q+1)\cdots(q+n-2)}{q^2-q}.$$

Remark. It turns out that $\begin{bmatrix} n \\ k \end{bmatrix}_q^* = t(n,k)$, the q-Stirling numbers connected with multisets introduced in [3].

It is easy to obtain analytically the formula

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q^* = \sum_i \begin{bmatrix} n \\ k+i \end{bmatrix} \binom{k+i}{k}_q \cdot q^i. \tag{16}$$

But it may be more instructive to give a purely combinatorial proof.

It is well known that $\begin{bmatrix} n \\ k \end{bmatrix}$ is the number of permutations π of $\{1,2,\ldots,n\}$ with k cycles $C_0, C_1, \ldots, C_{k-1}$.

We order the cycles with respect to decreasing largest elements: Let $\max(C_i)$ be the largest element of C_i. Write this element as last element of the cycle and order the cycles so that $\max(C_i) > \max(C_{i+1})$. E.g. $\pi = [476][31][82][5]$ becomes $\pi = [2\underline{8}][64\underline{7}][\underline{5}][1\underline{3}]$ with $C_0 = [28]$.

In this form we may forget the brackets, since the last elements of the cycles are the successive absolute maxima. Given a permutation $a_1 a_2 \ldots a_n$ we call the corresponding decomposition into nonempty parts C_0, $C_1, \ldots, C_{k-1}$ the natural decomposition of π and the ordering according to decreasing largest elements the natural ordering of the parts.

Since $\max(C_0) = n$ we have a natural decomposition $C_0 = \{n\}$, $C_{01}, \ldots, C_{0i}$, but we shall prefer to write the one element cycle $\{n\}$ at the end after C_{0i} in order to indicate the special role of C_0.

A permutation π may thus be uniquely described a set of $k+i$ cycles of some permutation $\pi' \in \mathfrak{S}_{n-1}$ and a subset of i cycles (those belonging to C_0).

E.g. $\pi = [562149][38][7]$ is uniquely determined by the set of cycles

$$[38], \quad [7], \quad [56], \quad [214]$$

together with the subset

$$\{[56], \quad [214]\}.$$

Every choice of a set of $k+i$ cycles of some permutation of $\{1,2,\ldots,n-1\}$ together with a specified subset of i cycles determines a unique permutation π of $\{1,2,\ldots,n\}$ with $z(\pi) = k+1$ cycles.

This is the combinatorial content of the well known formula (cf. [5])

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix} = \sum_{i=0}^{n-k} \begin{bmatrix} n \\ k+i \end{bmatrix} \binom{k+i}{i}.$$

We now introduce a weight $w(\pi)$ on the permutations such that $\begin{bmatrix} n \\ k \end{bmatrix}_q^*$ becomes the weight of the set of all permutations π, such that the natural decomposition has exactly k parts.

For $\pi = [C_{01}|C_{02}|\cdots C_{0i}|n]C_1|C_2|\cdots|C_{k-1}$ let

$$w(\pi) := q^{j_1 + j_2 + \cdots + j_i},$$

where $j_l = m$ if C_{0l} lies between C_{m-1} and C_m in the natural ordering of parts.

If $\max(C_{0l}) < \max(C_{k-1})$, then $j_l = k$.

Example

$\pi = [123] = [[12][3]]$

[12] comes after $C_0 = [123]$. Thus $w([123]) = q^1$.

$\pi = [213] = [[2][1][3]]$

We have $[123] < [2] < [1]$, thus $w([213]) = q^{1+1} = q^2$.

$\pi = [23][1] = [[2][3]][1], [23] < [2] < [1]$, thus $w(\pi) = q^1$.

$\pi = [13][2] = [[1][3]][2], [3] < [2] < [1]$, thus $w(\pi) = q^2$.

$\pi = [3][12], [3] < [12]$, thus $w(\pi) = 1$.

$\pi = [3][2][1], w(\pi) = 1$.

From this we get

$$\begin{bmatrix} 3 \\ 1 \end{bmatrix}_q^* = w([123]) + w([213]) = q + q^2,$$

$$\begin{bmatrix} 3 \\ 2 \end{bmatrix}_q^* = w([23][1]) + w([13][2]) + w([3][12]) = q + q^2 + 1,$$

$$\begin{bmatrix} 3 \\ 3 \end{bmatrix}_q^* = w([3][2][1]) = 1.$$

We show first that $\begin{bmatrix} n \\ k \end{bmatrix}_q^*$ defined in this way satisfies the recurrence relation together with the (trivial) boundary conditions.

To this end let for $\pi \in \mathfrak{S}_{n+1}$ denote π' the permutation obtained from π by eliminating 1 and reducing each element by 1.

E.g. if $\pi = 2\ 4\ 6\ 1\ 3\ 5$ then $\pi' = 1\ 3\ 5\ 2\ 4$.

Then

$$\begin{aligned} w(\pi) &= w(\pi') && \text{if } C_{0i} \neq \{1\} \\ &= q^{z(\pi)} w(\pi') && \text{if } C_{0i} = \{1\}. \end{aligned}$$

Note that in the first case each $\max(C_{0i})$ remains unchanged. In the second case C_{0i} contributes q^k to the weight of π and 1 to the weight of π'.

Let B_1 be the set of permutations $\pi \in \mathfrak{S}_{n+1}$ such that $C_{k-1} = \{1\}$. Then $w(B_1) = \begin{bmatrix} n \\ k-1 \end{bmatrix}_q^*$.

Let B_2 be the set of permutations $\pi \in \mathfrak{S}_{n+1}$ such that $C_{k-1} \neq \{1\}$ and $C_{0i} \neq \{1\}$. Then $w(B_2) = (n-1)\begin{bmatrix} n \\ k \end{bmatrix}_q^*$, because there are $n-1$ possibilities for 1 (before each element). Let B_3 be the set of permutations with $C_{0i} = \{1\}$. Then

$$w(B_3) = q^k \begin{bmatrix} n \\ k \end{bmatrix}_q^*.$$

Obviously

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}_q^* = w(B_1) + w(B_2) + w(B_3),$$

which proves our first assertion.

Consider now the set B_i of all permutations $\pi \in \mathfrak{S}_{n+1}$ with $k+1$ parts, such that C_0 contains i parts $C_{01}, \ldots C_{0i}$ apart from $[n]$.

Then

$$w(B_i) = \sum_{1+k \geqslant j_1 \geqslant j_2, \ldots \geqslant j_i \geqslant 1} q^{j_1 + j_2 + \cdots + j_i},$$

where the sum runs over all $j_1, \ldots, j_i$ with the stated properties.

This is obvious because for each given set of $k+i$ parts $D_1, \ldots, D_{k+i}$ with $D_1 < \cdots < D_{k+i}$ and given $j_1 \geqslant j_2 \geqslant \cdots \geqslant j_i$ those D_l belonging to the distinguished subset $C_{01}, \ldots, C_{0i}$ are uniquely determined.

From the well known formula

$$\frac{1}{(1-x)(1-qx)\cdots(1-q^k x)} = \sum_i \binom{i+k}{i}_q x^i$$

it follows that

$$\binom{i+k}{i}_q = \sum_{0 \leqslant r_1 \leqslant \cdots \leqslant r_i \leqslant k} q^{r_1 + r_2 + \cdots + r_i}.$$

Thus

$$w(B_i) = \binom{i+k}{i}_q q^i.$$

From this we get immediately the desired formula (16),

$$\begin{bmatrix} n+1 \\ k+1 \end{bmatrix}_q^* = \sum_i \begin{bmatrix} n \\ k+i \end{bmatrix} \binom{k+i}{k}_q q^i.$$

A special case is

$$\begin{bmatrix} n+k+1 \\ n+1 \end{bmatrix}_q^* = \sum_{i=0}^{k} \begin{bmatrix} n+k \\ n+i \end{bmatrix} \binom{n+i}{n}_q q^i. \tag{17}$$

which gives e.g.

$$\begin{bmatrix} n \\ n-1 \end{bmatrix}_q^* = \binom{n-1}{2} + q[n-1].$$

4. The q-Stirling Numbers $\left\{ {n \atop k} \right\}_q^*$

These numbers are given by

$$\left\{ {n \atop k} \right\}_q^* = \begin{bmatrix} -k \\ -n \end{bmatrix}_{\frac{1}{q}}^*.$$

Therefore we have

$$\left\{ {n+1 \atop k} \right\}_q^* = \begin{bmatrix} -k \\ -n-1 \end{bmatrix}_{\frac{1}{q}}^* = \begin{bmatrix} -k+1 \\ -n \end{bmatrix}_{\frac{1}{q}}^* - (q^{-n} - k - 1) \begin{bmatrix} -k \\ -n \end{bmatrix}_{\frac{1}{q}}^*.$$

This gives

$$\left\{ {n+1 \atop k} \right\}_q^* = \left\{ {n \atop k-1} \right\}_q^* + (k+1-q^n) \left\{ {n \atop k} \right\}_q^*, \quad n, k \in \mathbb{Z} \tag{18}$$

$$\left\{ {n \atop 0} \right\}_q^* = [n=0], \quad \left\{ {0 \atop k} \right\}_q^* = [k=0], \quad n, k \in \mathbb{Z}. \tag{19}$$

This leads to the following table

n	$\left\{ {n \atop 0} \right\}_q^*$	$\left\{ {n \atop 1} \right\}_q^*$	$\left\{ {n \atop 2} \right\}_q^*$	$\left\{ {n \atop 3} \right\}_q^*$
0	1	0	0	0
1	0	1	0	0
2	0	$2-q$	1	0
3	0	$(2-q^2)(2-q)$	$5-q-q^2$	1
4	0	$(2-q^3)(2-q^2)(2-q)$	$q^5+q^4-4q^3-5q^2-5q+19$	$9-q^3-q^2-q$

The generating function is given by

$$\sum \left\{ {n \atop k} \right\}_q^* (x-1)^{\underline{k}} = (x-1)(x-q)\cdots(x-q^{n-1}), \tag{20}$$

which gives

$$\left\{ {n \atop m} \right\}_q^* = \sum_k \binom{n}{k}_q \left\{ {k+1 \atop m+1} \right\} (-1)^{m-k} q^{\binom{m-k}{2}}. \tag{21}$$

Finally we remark that it is also possible to generalize the generating functions to negative indices. Writing (3) in the form

$$(x+1)(x+q)\cdots(x+q^{n-1}) = \sum_{k=0}^{\infty} \left\{ {n \atop n-k} \right\}_q (x+1)^{\underline{n-k}},$$

this formula holds true for negative n.

The simplest interpretation is via formal power series in $\frac{1}{x}$ as in [1]. We shall not go into details but state only the formal result:

$$\frac{1}{\left(x+\frac{1}{q}\right)\left(x+\frac{1}{q^2}\right)\cdots\left(x+\frac{1}{q^n}\right)} = \sum \left\{ {-n \atop -n-k} \right\}_q \frac{1}{(x+2)\cdots(x+n+k+1)}$$

or equivalently

$$\frac{1}{(x+q)(x+q^2)\cdots(x+q^n)} = \sum \begin{bmatrix} n+k \\ n \end{bmatrix}_q \frac{1}{(x+2)\cdots(x+n+k+1)}.$$

Analogous formulas hold in the other cases.

References

[1] Cigler, J.: Some remarks on Rota's umbral calculus. Indag. Math. **40** (1978), 27–42.

[2] Cigler, J.: Operatormethoden für q-Identitäten. Monatshefte für Mathematik **88** (1979), 87–105.

[3] Cigler, J.: Über die Anzahl erzeugender Mengen in endlichen Vektorräumen. Anzeiger d. Österr. Akad. Wiss. **5** (1992), 27–33.

[4] Gessel, I., and R. P. Stanley: Stirling polynomials. J. Combin. Theory **A 24** (1978), 24–33.

[5] Graham, R. L., D. E. Knuth, and O. Patashnik: Concrete Mathematics. Addison-Wesley, 1989.

[6] Knuth, D. E.: Two notes on notation. Amer. Math. Monthly **99** (1992), 403–422.

[7] Rota, G.-C.: Finite Operator Calculus. Academic Press, 1975.

[8] Stanley, R. P.: Enumerative Combinatorics. Wadsworth, 1986.

Author's address: J. Cigler, Institut für Mathematik, Universität Wien, 1090 Wien, Strudlhofgasse 4.