Hankel determinants of Schröder-like numbers

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Abstract

After a short survey about Schröder numbers and some generalizations which I call Schröder-like numbers I study some $q-$analogues which have simple Hankel determinants. Some special cases have already been considered in [2] and [3].

1. Schröder numbers

In the first paragraphs I state some results about Schröder numbers and Hankel determinants which are either well known or simple modifications of well-known results.
The (large) Schröder numbers r_n can be defined by the generating function

$$F(z)=\sum_{n\geq 0} r_n z^n = \frac{1-z-\sqrt{1-6z+z^2}}{2z}. \tag{1.1}$$

The first terms of this sequence are (cf. [4] A006318)

$$\left(r_n\right)_{n\geq 0}=\left(1,2,6,22,90,\cdots\right). \tag{1.2}$$

The generating function satisfies

$$F(z)=1+zF(z)+zF(z)^2. \tag{1.3}$$

A closely related sequence (cf. [4], A001003) is the sequence $(1,1,3,11,45,197,\cdots)$ of little Schröder numbers s_n, defined by $s_0=1$ and $s_n=\frac{r_n}{2}$ for $n>0.$
Their generating function

$$f(z)=\frac{1+F(z)}{2} \tag{1.4}$$

satisfies

$$f(z)=1-zf(z)+2zf(z)^2. \tag{1.5}$$

Historical remarks about these numbers can be found in [7].

It is well known that the Hankel determinants of these numbers are (cf. [4] A006318, A001003)

$$\det\left(r_{i+j}\right)_{i,j=0}^{n-1}=2^{\binom{n}{2}}, \tag{1.6}$$

$$\det\left(r_{i+j+1}\right)_{i,j=0}^{n-1}=2^{\binom{n+1}{2}}, \tag{1.7}$$

$$\det\left(s_{i+j}\right)_{i,j=0}^{n-1} = \det\left(s_{i+j+1}\right)_{i,j=0}^{n-1} = 2^{\binom{n}{2}} \tag{1.8}$$

and

$$\det\left(s_{i+j+2}\right)_{i,j=0}^{n-1} = 2^{\binom{n}{2}}\left(2^{n+1}-1\right). \tag{1.9}$$

There are many ways to prove such results. My favorite method uses orthogonal polynomials. In the sequel I consider only sequences of numbers $(a(n))$ with the property that $a(0)=1$ and $\det(a(i+j))_{i,j=0}^{n-1} \neq 0$ for all $n \geq 1$.
In this situation the polynomials

$$p(n,x) = \frac{1}{\det(a(i+j))_{i,j=0}^{n-1}} \det\begin{pmatrix} a(0) & a(1) & \cdots & a(n-1) & 1 \\ a(1) & a(2) & \cdots & a(n) & x \\ a(2) & a(3) & \cdots & a(n+1) & x^2 \\ \vdots & & & & \vdots \\ a(n) & a(n+1) & \cdots & a(2n-1) & x^n \end{pmatrix} \tag{1.10}$$

are well defined and orthogonal with respect to the linear functional F defined by $F(x^n) = a(n)$.

By Favard's theorem they satisfy a recurrence of the form

$$p(n,x) = (x - s(n-1))\,p(n-1,x) - t(n-2)\,p(n-2,x). \tag{1.11}$$

Let $a(n,k)$ be the uniquely determined coefficients in the expansion

$$x^n = \sum_{k=0}^{n} a(n,k)\,p(k,x). \tag{1.12}$$

They satisfy

$$\begin{aligned} &a(0,k) = [k=0] \\ &a(n,0) = s(0)a(n-1,0) + t(0)a(n-1,1) \\ &a(n,k) = a(n-1,k-1) + s(k)a(n-1,k) + t(k)a(n-1,k+1). \end{aligned} \tag{1.13}$$

Obviously

$$a(n,0) = F(x^n) = a(n). \tag{1.14}$$

We say that the sequences $\left(s(n)\right)$ and $\left(t(n)\right)$ are associated with the sequence $(a(n))$. They contain the same information as the sequence $(a(n))$.

For the large Schröder numbers the associated sequences are (see (2.10))

$$\begin{aligned} &s(0) = 2, s(n) = 3 \\ &t(n) = 2. \end{aligned} \tag{1.15}$$

This gives the Schröder triangle ([4] A133367)

1				
2	1			
6	5	1		
22	23	8	1	
90	107	49	11	1

For the little Schroeder numbers the generating function (1.5) implies (see (2.11))

$$s(0)=1, s(n)=3 \tag{1.16}$$

and

$$t(n)=2. \tag{1.17}$$

This gives the triangle

1				
1	1			
3	4	1		
11	17	7	1	
45	76	40	10	1

The numbers $a(n,k)$ have a well known combinatorial interpretation.
Consider lattice paths, so-called Motzkin paths, with upward steps $(n,k)\to(n+1,k+1)$, horizontal steps $(n,k)\to(n+1,k)$ and downward steps $(n,k+1)\to(n+1,k)$ which start in $(0,0)$ and never fall under the $x-$axis. To each path we associate a weight in the following way: Each upward step has weight 1, each horizontal step at height k has weight $s(k)$ and each downward step which ends on height k has weight $t(k)$. The weight of a path is the product of the weights of its steps. Then $a(n,k)$ is the sum of the weights of all paths from $(0,0)$ to (n,k).
Let now $f_j(z)$ be the generating function of all paths which start and end at height j and never fall under this height. Then $f_j(z)$ is the generating function $f_j(z)=\sum_{n\geq 0} a_j(n,0)z^n$, where $a_j(n,k)$ is given by (1.13) with $s_j(n)=s(n+j)$ and $t_j(n)=t(n+j)$.

Then a simple combinatorial argument gives

$$f(z)=f_0(z)=1+s(0)zf(z)+t(0)z^2 f(z) f_1(z). \tag{1.18}$$

This identity is equivalent with

$$f(z)=\frac{1}{1-s(0)z-t(0)z^2 f_1(z)}$$

and therefore with the continued fraction

$$f(z)=\cfrac{1}{1-s(0)z-\cfrac{t(0)z^2}{1-s(1)z-\cfrac{t(1)z^2}{1-s(2)z-\ddots}}}. \qquad (1.19)$$

Let

$$d(n,k)=\det\left(a(i+j+k)\right)_{i,j=0}^{n-1}. \qquad (1.20)$$

be the Hankel determinant of order k of $\left(a(n)\right)$.

Then (cf. e.g. [6])

$$d(n,0)=\prod_{i=1}^{n-1}\prod_{k=0}^{i-1}t(k)=t(0)^{n-1}t(1)^{n-2}\cdots t(n-3)^2t(n-2) \qquad (1.21)$$

and

$$d(n,1)=(-1)^n p(n,0)d(n,0). \qquad (1.22)$$

Sometimes the sequence $\left(d(n,0)\right)$ is called the Hankel transform of the sequence $\left(a(n)\right)$. Under the stated requirements it would be better to call $\left(d(n,0)\right)_{n\geq 1}\times\left(d(n,1)\right)_{n\geq 1}$ the Hankel transform of $\left(a(n)\right)$ since the sequence $\left(a(n)\right)$ is uniquely determined by the sequences of Hankel determinants $\left(d(n,0)\right)$ and $\left(d(n,1)\right)$.

It is often convenient to consider a sequence $(a(n))$ which satisfies $a(2n)=c(n)$ and $a(2n+1)=0$. In this case $s(n)=0$ for all $n.$ The corresponding lattice paths have no horizontal steps, i.e. they are so called Dyck paths.
Let

$$\left(a_0(n)\right)=\left(a(2n,0)\right)=\left(c(n)\right) \qquad (1.23)$$

Then the associated sequences are

$$s_0(0)=t(0), s_0(n)=t(2n-1)+t(2n), t_0(n)=t(2n)t(2n+1) \qquad (1.24)$$

For

$$\left(a_1(n)\right)=\left(a(2n+1,1)\right) \qquad (1.25)$$

we get

$$s_1(0)=t(0)+t(1), s_1(n)=t(2n)+t(2n+1), t_1(n)=t(2n+1)t(2n+2). \qquad (1.26)$$

As an example consider the sequence $(a(n))$ with $a(2n)=r_n$ and $a(2n+1)=0.$ In this case we get $s(n)=0, t(2n)=2, t(2n+1)=1$ (cf. (2.9)).

The corresponding triangle is

1						
0	1					
2	0	1				
0	3	0	1			
6	0	5	0	1		
0	11	0	6	0	1	
22	0	23	0	8	0	1

Since $a_1(n) = s_{n+1}$ are again little Schröder numbers, (1.26) gives a triangle for the little Schroeder numbers s_{n+1}. This is [4] A110440.

1				
3	1			
11	6	1		
45	31	9	1	
197	156	60	12	1

The generating function for the sequence (s_{n+1}) is $g(z) = \frac{f(z)-1}{z}$ and satisfies $g(z) = 1 + 3zg(z) + 2z^2 g(z)^2$.

2. Schröder-like numbers

We first want to study Schröder-like numbers $A(n,x,y)$ defined by the generating function $F(z) = \sum_{n\geq 0} A(n,x,y)z^n$ which satisfies

$$F(z) = 1 + xzF(z) + yzF(z)^2. \qquad (2.1)$$

A useful combinatorial interpretation of $A(n,x,y)$ can be obtained in the following way: Consider lattice paths with upward and downward steps of length 1 and horizontal steps of length $2.$ If each upward step has weight 1, each downward step has weight xz and each horizontal step has weight $yz,$ then the weight of the set of all non-negative paths from $(0,0)$ to $(2n,0)$ is $A(n,x,y)z^n$.

This implies that

$$A(n,x,y) = \sum_{k=0}^{n} \binom{n+k}{2k} C_k x^{n-k} y^k = \sum_{k=0}^{n} \binom{2n-k}{k} C_{n-k} x^k y^{n-k}, \qquad (2.2)$$

where $C_k = \frac{1}{k+1}\binom{2k}{k}$ is a Catalan number.

We first choose a $k-$subset $1 \le a_1 < a_2 < \cdots a_k < 2n$ of $\{1, \cdots, 2n\}$ which contains no successive elements. This is equivalent with $1 \le a_1 < a_2 - 1 < \cdots a_k - k + 1 \le 2n - k.$ Therefore there are $\binom{2n-k}{k}$ possibilities to choose such a subset. Each a_j is the starting point of a horizontal step. On the $2n-2k$ remaining points there are C_{n-k} non-negative Dyck paths.

By solving a quadratic equation we get the explicit formula

$$F(z) = \frac{1 - xz - \sqrt{1 - 2(x+2y)z + x^2z^2}}{2yz}. \tag{2.3}$$

The numbers $A(n,x,y)$ satisfy the simple recurrence relation

$$A(n,x,y) = \frac{(2n-1)(x+2y)A(n-1,x,y) - (n-2)x^2 A(n-2,x,y)}{n+1} \tag{2.4}$$

with initial values

$$A(0,x,y) = 1 \text{ and } A(1,x,y) = x + y.$$

In order to show this we differentiate (2.1) and get
$F'(z) = xF(z) + xzF'(z) + yF(z)^2 + 2yzF(z)F'(z)$
Substituting (2.1) gives
$zF'(z)(1 - xz - 2yzF(z)) = yzF(z)^2 + xzF(z) = F(z) - 1$

Therefore

$$zF'(z) = \frac{F(z)-1}{(1 - xz - 2yzF(z))} = \frac{F(z)-1}{\sqrt{1-2(x+2y)z+x^2z^2}} = \frac{(F(z)-1)\sqrt{1-2(x+2y)z+x^2z^2}}{1-2(x+2y)z+x^2z^2}$$

$$= \frac{(F(z)-1)\left(1 - xz - 2yzF(z)\right)}{1-2(x+2y)z+x^2z^2} = \frac{\left(xz + 1 + (x+2y)zF(z) - F(z)\right)}{1-2(x+2y)z+x^2z^2}$$

i.e.

$$zF'(z)\left(1-2(x+2y)z+x^2z^2\right) = \left(xz + 1 + (x+2y)zF(z) - F(z)\right)$$

Comparing coefficients we get
$nA(n,x,y) + 2(x+2y)(n-1)A(n-1,x,y) + x^2(n-2)A(n-2,x,y)$
$= -A(n,x,y) + (x+2y)A(n-1,x,y)$
and thus (2.4).

The simplest cases occur for $x = 0$ where $A(n,0,y) = C_n y^n$ and for $x + 2y = 0,$ where we get $A(2n+2,2,-1) = 0$ and $A(2n+1,2,-1) = (-1)^n C_n.$

The equation $F(z)=1+xzF(z)+yzF(z)^2$ implies

$F(z)-yzF(z)^2=1+xzF(z)=1-yzF(z)+(x+y)zF(z)$
or
$F(z)=1+(x+y)zF(z)f(z)$ with

$$f(z)=\frac{1}{1-yzF(z)}. \tag{2.5}$$

Therefore (2.1) is equivalent with

$$\begin{aligned} &F(z)=1+(x+y)zF(z)f(z), \\ &f(z)=1+yzF(z)f(z). \end{aligned} \tag{2.6}$$

This also gives

$$f(z)=\frac{x+yF(z)}{x+y}. \tag{2.7}$$

It is easily verified that

$$f(z)=1-xzf(z)+(x+y)zf(z)^2. \tag{2.8}$$

If we write $f(z)=\sum_{n\geq 0}a(n,x,y)z^n$ then the numbers $a(n,x,y)$ are a generalization of the little Schröder numbers.

Comparing with (1.18) we see that $F(z^2)$ corresponds to

$$\begin{aligned} &s(n)=0, \\ &t(2n)=x+y, t(2n+1)=y. \end{aligned} \tag{2.9}$$

Denote the corresponding $a(n,k)$ by $\alpha(n,k)$.

Since $\alpha(2n,0)=A(n,x,y)$ we get for the original sequence $\left(A(0,x,y),A(1,x,y),A(2,x,y),\cdots\right)$

$$\begin{aligned} &s(0)=x+y, s(n)=x+2y \\ &t(n)=y(x+y). \end{aligned} \tag{2.10}$$

Observing that $a(n,x,y)=A(n,-x,x+y)$ we get for the sequence $\left(a(n,x,y)\right)$

$$\begin{aligned} &s(0)=y, s(n)=x+2y \\ &t(n)=y(x+y). \end{aligned} \tag{2.11}$$

Let

$$a_1(n)=\alpha(2n+1,1)=\frac{\alpha(2n+2,0)}{t(0)}=\frac{A(n+1,x,y)}{x+y}=\frac{a(n+1,x,y)}{y}. \qquad (2.12)$$

Its generating function is $g(z)=\frac{f(z)-1}{yz}$ and satisfies

$$g(z)=1+(x+2y)zg(z)+y(x+y)z^2g(z)^2. \qquad (2.13)$$

Therefore the corresponding values are

$$s(n)=x+2y, t(n)=y(x+y). \qquad (2.14)$$

From (2.1) we get the continued fraction

$$F(z)=\cfrac{1}{1-xz-\cfrac{yz}{1-xz-\cfrac{yz}{1-xz-\ddots}}} \qquad (2.15)$$

By (2.6) and (1.18) we see that this can also be written as a so- called J-fraction

$$F(z)=\cfrac{1}{1-(x+y)z-\cfrac{y(x+y)z^2}{1-(x+2y)z-\cfrac{y(x+y)z^2}{1-(x+2y)z-\cfrac{y(x+y)z^2}{1-(x+2y)z-\ddots}}}}. \qquad (2.16)$$

It would be interesting if there is also a combinatorial proof, i.e. a bijection between the lattice paths defining these continued fractions.

The Hankel determinants are

$$D(n,0)=\det\left(A(i+j,x,y)\right)_{i,j=0}^{n-1}=\left(y(x+y)\right)^{\binom{n}{2}} \qquad (2.17)$$

and

$$D(n,1)=\det\left(A(i+j+1,x,y)\right)_{i,j=0}^{n-1}=y^{\binom{n}{2}}\left((x+y)\right)^{\binom{n+1}{2}}. \qquad (2.18)$$

The first result is obvious. The second follows from (1.22). Let $r(n)=(-1)^n p(n,0)$. Then

$$r(n)=s(n-1)r(n-1)-t(n-2)r(n-2)$$
$$=(x+2y)r(n-1)-y(x+y)r(n-2)$$

with initial values $r(0)=1$ and $r(1)=(x+y)$.
This gives by induction $r(n)=(x+y)^n$.

By changing $x \to -x, y \to x+y$ we get

$$d(n,0) = \det\left(a(i+j,x,y)\right)_{i,j=0}^{n-1} = \left(y(x+y)\right)^{\binom{n}{2}} \tag{2.19}$$

and

$$d(n,1) = \det\left(a(i+j+1,x,y)\right)_{i,j=0}^{n-1} = y^{\binom{n+1}{2}}\left(x+y\right)^{\binom{n}{2}}. \tag{2.20}$$

The last identity also follows from (2.14). This can also be used to compute

$d(n,2) = \det\left(a(i+j+2,x,y)\right)_{i,j=0}^{n-1} = y^{\binom{n}{2}}\det\left(a_1(i+j+1)\right)_{i,j=0}^{n-1}$.

Using (1.22) we see that $d(n,2) = r(n)d(n,1)$, where

$r(n) = (x+2y)r(n-1) - y(x+y)r(n-2)$ with initial values $r(0)=1$ and

$r(1) = x+2y = \dfrac{(x+y)^2 - y^2}{x}$.

This gives $r(n) = \dfrac{(x+y)^{n+1} - y^{n+1}}{x}$.

Therefore

$$d(n,2) = y^{\binom{n+1}{2}}\left(x+y\right)^{\binom{n}{2}}\frac{(x+y)^{n+1} - y^{n+1}}{x}. \tag{2.21}$$

Remark

If $F(z) = 1 + (x-y)zF(z) + yzF(z)^2$ then $G(z) = \dfrac{F(z)-1}{xz}$ satisfies

$$G(z) = 1 + (x+y)zG(z) + xyz^2G(z)^2. \tag{2.22}$$

Let $F(z) = \sum_{n\geq 0} A(n)z^n$ and $G(z) = \sum_{n\geq 0} B(n)z^n$.

Then $A(0)=1$ and $A(n) = xB(n-1)$ for $n \geq 1$.

There are many interesting sequences whose generating function satisfies (2.22).

E.g. for $(x,y)=(2,1)$ we get $A(n) = r_n$ and $B(n) = s_{n+1}$,

for $(x,y)=(1,1)$ we get the Catalan numbers and for $(x,y) = \left(\dfrac{1+i\sqrt{3}}{2}, \dfrac{1-i\sqrt{3}}{2}\right)$ we get the Motzkin numbers $B(n) = M_n$.

3. q-analogues of Schröder-like numbers

Barcucci et al. [1] have introduced (large) $q-$Schröder numbers by the generating function

$$F(z)=1+zF(z)+qzF(z)F(qz). \tag{2.23}$$

The first terms are
$(1,1+q,(1+q)(1+q+q^2),(1+q)(1+2q+3q^2+3q^3+q^4+q^5),\cdots)$

We want to consider more generally the $q-$Schröder-like numbers $A(n,x,y)$ with generating function

$$F(z)=1+xzF(z)+yzF(z)F(qz). \tag{2.24}$$

For $(x,y)=(0,1)$ this reduces to $F(z)=1+zF(z)F(qz).$ Therefore $A(n,0,1)=C_n(q)$ are the $q-$Catalan numbers of Carlitz.

Theorem 1

Let $F(z)=\sum_{n\geq 0}A(n,x,y)z^n$ *satisfy the identity* $F(z)=1+xzF(z)+yzF(z)F(qz).$

Then

$$D(n,0)=\det\left(A(i+j,x,y)\right)_{i,j=0}^{n-1}=q^{\frac{n(n-1)^2}{2}}y^{\binom{n}{2}}(x+y)^{n-1}(x+qy)^{n-2}\cdots(x+q^{n-2}y), \tag{2.25}$$

$$\begin{aligned}D(n,1)&=\det\left(A(i+j+1,x,y)\right)_{i,j=0}^{n-1}=q^{\binom{n}{2}}(x+y)(x+qy)\cdots(x+q^{n-1}y)D(n,0)\\&=q^{\frac{n^2(n-1)}{2}}y^{\binom{n}{2}}(x+y)^n(x+qy)^{n-1}\cdots(x+q^{n-1}y)\end{aligned} \tag{2.26}$$

and

$$D(n,2)=q^{\frac{(n-1)n(n+1)}{2}}y^{\binom{n}{2}}\prod_{j=0}^{n-1}(x+q^jy)^{n-j}\left(\prod_{j=1}^{n+1}(x+q^{j-1}y)-q^{\binom{n+1}{2}}y^{n+1}\right). \tag{2.27}$$

An analogue of the little Schröder numbers is given by the generating function

$$f(z)=\sum_{n\geq 0}a(n,x,y)z^n=\frac{x+yF(z)}{x+y}. \tag{2.28}$$

It is easily verified that it satisfies the equation

$$f(z)=1-xzf(qz)+(x+y)zf(z)f(qz) \tag{2.29}$$

and that

$$F(z)=1+(x+y)zF(z)f(qz). \tag{2.30}$$

Theorem 2

Let $f(z)=\sum_{n\geq 0}a(n,x,y)z^n$ *satisfy the identity* $f(z)=1-xzf(qz)+(x+y)zf(z)f(qz).$

Then

$$d(n,0)=\det\left(a(i+j,x,y)\right)_{i,j=0}^{n-1}=q^{3\binom{n}{3}}y^{\binom{n}{2}}\prod_{j=1}^{n-1}\left(x+q^jy\right)^{n-j} \tag{2.31}$$

and

$$d(n,1)=\det\left(a(i+j+1,x,y)\right)_{i,j=0}^{n-1}=q^{n\binom{n}{2}}y^{\binom{n+1}{2}}(x+qy)^{n-1}\cdots(x+q^{n-1}y). \tag{2.32}$$

Whereas for $q=1$ the formulae for the Hankel determinants for $a(n,x,y)$ could be reduced to those of $A(n,x,y)$ this is not true for the general case.

An analogue of (2.6) is

$$\begin{aligned}F(z)&=1+(x+y)zF(z)f(qz),\\ f(z)&=1+yzF(z)f(qz).\end{aligned} \tag{2.33}$$

But in this form it seems to be of no use to find the continued fraction.
Therefore I use an idea of the proof I have given in [2].
There is a uniquely determined series $h(z)=h(z,y)=1+\sum_{n\geq 1}h_nz^n$ such that

$$F(z)=F(z,y)=\frac{h(qz,y)}{h(z,y)}. \tag{2.34}$$

From the defining equation for $F(z)$ we get $\frac{h(qz)}{h(z)}=1+xz\frac{h(qz)}{h(z)}+yz\frac{h(qz)}{h(z)}\frac{h(q^2z)}{h(qz)}$ and therefore

$h(qz)=h(z)+xzh(qz)+yzh(q^2z).$
Comparing coefficients we get
$(q^n-1)h_n=q^{n-1}\left(x+q^{n-1}y\right)h_{n-1}.$
This implies

$$h(z)=\sum_{k\geq 0}q^{\binom{k}{2}}\frac{(x+y)(x+qy)\cdots(x+q^{k-1}y)}{(q-1)(q^2-1)\cdots(q^k-1)}z^k. \tag{2.35}$$

On the other hand we have

$(x+y)h(z,qy)=xh(z,y)+yh(qz,y),$
i.e.

$$\frac{h(z,qy)}{h(z,y)}=\frac{x+yF(z)}{x+y}=f(z) \tag{2.36}$$

and

$$h(qz,y)-h(z,y)=\sum_{k\geq 0}q^{\binom{k}{2}}\frac{(x+y)(x+qy)\cdots(x+q^{k-1}y)}{(q-1)(q^2-1)\cdots(q^k-1)}(q^k-1)z^k$$
$$=(x+y)z\sum_{k\geq 0}q^{\binom{k-1}{2}}\frac{(x+qy)\cdots(x+q^{k-1}y)}{(q-1)(q^2-1)\cdots(q^{k-1}-1)}(qz)^{k-1}=(x+y)zh(qz,qy)$$

i.e.

$$F(z)=1+(x+y)z\frac{h(qz,qy)}{h(z,y)}. \tag{2.37}$$

From

$$F(z)f(qz)=\frac{h(qz,y)}{h(z,y)}\frac{h(qz,qy)}{h(qz,y)}=\frac{h(qz,qy)}{h(z,y)}=\frac{h(z,qy)}{h(z,y)}\frac{h(qz,qy)}{h(z,qy)}=f(z)F(z,qy)$$

we see that (2.33) can be written in the form

$$\begin{aligned}F(z,y)&=1+(x+y)zF(z,y)f(qz,y)\\ f(z,y)&=1+yzf(z,y)F(z,qy).\end{aligned} \tag{2.38}$$

This gives the continued fraction

$$F(z)=\cfrac{1}{1-\cfrac{(x+y)z}{1-\cfrac{qyz}{1-\cfrac{q(x+qy)z}{1-\cfrac{q^3yz}{1-\ddots}}}}} \tag{2.39}$$

From this we can deduce the associated sequences $s(n)$ and $t(n)$ for $F(z^2)$.
We get

$$s(n)=0,t(2n)=q^n(x+q^ny),t(2n+1)=q^{2n+1}y. \tag{2.40}$$

This again implies the associated sequences for $F(z)$ by (1.24).
They are

$$\begin{aligned}&s(0)=x+y,s(n)=q^n(x+q^{n-1}y(1+q))\\ &t(n)=q^{3n+1}y(x+q^ny).\end{aligned} \tag{2.41}$$

From this (2.25) follows immediately from (1.21).

In order to show (2.26) let $r(n)=(-1)^n p(n,0).$ This gives
$r(n)=q^{n-1}(x+q^{n-2}(1+q)y)r(n-1)-q^{3n-5}y(x+q^{n-2}y)r(n-2)$
with initial values $r(0)=1$ and $r(1)=x+y.$

It has to be shown that $r(n)=q^{\binom{n}{2}}(x+y)\cdots(x+q^{n-1}y),$

i.e.

$$q^{\binom{n}{2}}(x+y)\cdots(x+q^{n-1}y)=q^{n-1}(x+q^{n-2}(1+q)y)q^{\binom{n-1}{2}}(x+y)\cdots(x+q^{n-2}y)$$
$$-q^{3n-5}y(x+q^{n-2}y)q^{\binom{n-2}{2}}(x+y)\cdots(x+q^{n-3}y).$$

But this is easily verified.

From (2.38) we get for $f(z^2)$

$$s(n)=0,t(2n)=q^{2n}y,t(2n+1)=q^n(x+q^{n+1}y). \tag{2.42}$$

This implies that the associated sequences for $f(z)$ are

$$s(0)=y,s(n)=q^{n-1}(x+q^{n-1}(1+q)y),t(n)=q^{3n}y(x+q^{n+1}y). \tag{2.43}$$

This implies formulas (2.31) and (2.32).

Observing that $a(0,x,y)=\frac{x+y}{x+y},a(n,x,y)=\frac{yA(n,x,y)}{x+y}$ we get by expanding with respect to the first line

$$d(n,0)=\frac{y^{n-1}}{(x+y)^n}\left(yD(n,0)+xD(n-1,2)\right) \tag{2.44}$$

or

$$(x+y)^n q^{3\binom{n}{3}}y^{\binom{n}{2}}\prod_{j=1}^{n-1}\left(x+q^jy\right)^{n-j}-y^nq^{\frac{n(n-1)^2}{2}}y^{\binom{n}{2}}(x+y)^{n-1}(x+qy)^{n-2}\cdots(x+q^{n-2}y)$$
$$=q^{3\binom{n}{3}}\prod_{j=0}^{n-2}\left(x+q^jy\right)^{n-j-1}y^{\binom{n}{2}}\left((x+y)(x+qy)\cdots(x+q^{n-1}y)-y^nq^{\frac{n(n-1)}{2}}\right)=xy^{n-1}D(n-1,2)$$

This implies

$$D(n,2)=q^{\frac{(n-1)n(n+1)}{2}}y^{\binom{n}{2}}\prod_{j=0}^{n-1}(x+q^jy)^{n-j}\left(\prod_{j=1}^{n+1}(x+q^{j-1}y)-q^{\binom{n+1}{2}}y^{n+1}\right) \tag{2.45}$$

Let now $a_1(n)=\dfrac{A(n+1,x,y)}{x+y}$.

Its generating function is $g(z,y)=F(z)f(qz)=\dfrac{h(qz,y)h(qz,qy)}{h(z,y)h(qz,y)}=\dfrac{h(qz,qy)}{h(z,y)}$ by (2.38).

Substituting $F(z,y)=1+(x+y)zg(z,y)$ into (2.24) we find that

$$g(z,y)=1+(x+y)zg(z,y)+qyzg(qz,y)+qy(x+y)z^2g(z,y)g(qz,y). \qquad (2.46)$$

Since

$$\frac{g(qz,y)}{g(z,y)}=\frac{h(q^2z,qy)}{h(qz,y)}\frac{h(z,y)}{h(qz,qy)}=\frac{h(q^2z,qy)}{h(qz,qy)}\frac{h(z,y)}{h(qz,y)}=\frac{F(qz,qy)}{F(z,y)}$$

we get

$$\begin{aligned}&g(z,y)=1+(x+y)zg(z,y)+qyzg(qz,y)\left(1+(x+y)zg(z,y)\right)\\&=1+(x+y)zg(z,y)+qyzg(qz,y)F(z)=1+(x+y)zg(z,y)+qyzg(z,y)F(qz,qy)\\&=1+(x+y)zg(z,y)+qyzg(z,y)+q^2yz^2(x+qy)g(z,y)g(qz,qy),\end{aligned}$$

i.e.

$$g(z,y)=1+(x+(1+q)y)zg(z,y)+q^2y(x+qy)z^2g(z,y)g(qz,qy). \qquad (2.47)$$

This implies that the associated sequences are
$s(n)=q^n(x+q^n(1+q)y)$ and $t(n)=q^{3n+2}y(x+q^{n+1}y)$.

We can now give another proof of (2.45).
By (1.22) we get $D(n,2)=D(n,1)r(n)$,
where $r(n)=s(n-1)r(n-1)-t(n-2)r(n-2)$
with initial values $r(0)=1$ and $r(1)=s(0)=x+(1+q)y=\dfrac{(x+y)(x+qy)-qy^2}{x}$.

We have to show that $r(n)=q^{-\binom{n}{2}}\left(\prod_{j=1}^{n}(x+q^{j-1}y)-q^{\binom{n}{2}}y^n\right)$ or

$$\begin{aligned}&\prod_{j=1}^{n+1}(x+q^{j-1}y)-q^{\binom{n+1}{2}}y^{n+1}=(x+q^{n-1}(1+q)y)\left(\prod_{j=1}^{n}(x+q^{j-1}y)-q^{\binom{n}{2}}y^n\right)\\&-q^{n-1}y(x+q^{n-1}y)\left(\prod_{j=1}^{n-1}(x+q^{j-1}y)-q^{\binom{n-1}{2}}y^{n-1}\right).\end{aligned}$$

This is easily verified.

Remark

Let $g(z) = g(z, x, y, q)$ satisfy

$$g(z) = 1 + (x+y)zg(z) + qxyz^2 g(z)g(qz). \tag{2.48}$$

This gives $q-$analogues of several classical sequences.

The associated sequences are $s(n) = q^n(x+y)$ and $t(n) = q^{2n+1}xy.$

Define

$$f(z) = f(z, x, y, q) = 1 + xzg(z). \tag{2.49}$$

Then $f(z) = \sum_{n\geq 0} b(n, x, y, q)z^n$ satisfies the identity

$$f(z) = 1 + xzf(z) - yzf(qz) + yzf(z)f(qz). \tag{2.50}$$

This implies

$$f(z,x,y,q) = \frac{1 - yzf(qz,x,y,q)}{1 - yzf(qz,x,y,q) - xz} = \cfrac{1}{1 - \cfrac{xz}{1 - yzf(qz,x,y,q)}}$$

$$= \cfrac{1}{1 - \cfrac{xz}{1 - \cfrac{yz}{1 - \cfrac{qxz}{1 - qyzf(q^2z,x,y,q)}}}} = \cdots = \cfrac{1}{1 - \cfrac{xz}{1 - \cfrac{yz}{1 - \cfrac{qxz}{1 - \cfrac{qyz}{1 - \cfrac{q^2xz}{1 - \ddots}}}}}}$$

Therefore

$$f(z,x,y,q) = \frac{1}{1 - xzf(z,y,qx,q)}.$$

We thus get

$$f(z,x,y,q) = 1 + xzf(z,x,y,q)f(z,y,qx,q) \tag{2.51}$$

or

$$f(z,x,y,q) = 1 + xzf(z,x,y,q) + xyz^2 f(z,x,y,q)g(z,y,qx,q). \tag{2.52}$$

Therefore the associated sequences are

$$s(0) = x, s(n) = q^{n-1}(qx+y), t(n) = q^{2n}y. \tag{2.53}$$

It is easily verified that

$$d(n,0) = q^{2\binom{n}{3}}(xy)^{\binom{n}{2}}, \tag{2.54}$$

$$d(n,1) = x^n(xy)^{\binom{n}{2}} q^{\sum_{k=0}^{n-1} k^2} \tag{2.55}$$

and

$$d(n,2)=x^n(qxy)^{\binom{n}{2}}\frac{x^{n+1}-y^{n+1}}{x-y}q^{\sum_{k=0}^{n-1}k^2}=q^{2\binom{n+1}{3}}x^n(xy)^{\binom{n}{2}}\frac{x^{n+1}-y^{n+1}}{x-y}. \qquad (2.56)$$

From (2.50) we get the recurrence relation

$$b(n,x,y,q)=xb(n-1,x,y,q)+y\sum_{k=0}^{n-2}q^k b(k,x,y,q)b(n-1-k,x,y,q) \qquad (2.57)$$

with initial value $b(0,x,y,q)=1.$

For $x=1$ this reduces to a variant of the Pólya-Gessel $q-$Catalan numbers $C_n(y;q,q^{-1})$ (cf. [5],(5.5)). The well-known fact that $C_n(q;q^2,q^{-2})=C_n(q)$ can easily be seen by comparing the associated sequences (2.41) of $C_n(q)$ and (2.53) of $b(n,1,q,q^2)$ which turn out to be $s(0)=1, s(n)=q^{2n-1}(1+q)$ and $t(n)=q^{4n+1}.$

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