# A simple variant of the q-binomial Theorem

Johann Cigler

Fakultät für Mathematik, Universität Wien

johann.cigler@univie.ac.at

**Abstract**
*We give a simple proof of a variant of the q-binomial theorem which generalizes a result by S. Nalci and O. Pashaev.*

We give a simple proof of a variant of the $q-$binomial Theorem which generalizes a result obtained by S. Nalci and O. Pashaev [2].

Let $\left[n\right]_q=\dfrac{q^n-1}{q-1}$, $[n]_q!=[1]_q[2]_q\ldots[n]_q$ and $\begin{bmatrix} n \\ k \end{bmatrix}_q=\dfrac{[n]_q!}{[k]_q![n-k]_q!}$ for $0\le k\le n.$

S. Nalci and O. Pashaev proved the following Theorem (in a different notation):

**Theorem 1**

*Let* $yx=Qxy.$ *Then*

$$(x+y)(x+qy)\cdots(x+q^{n-1}y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{\frac{Q}{q}} q^{kn-\binom{k+1}{2}}x^{n-k}y^k \tag{1.1}$$

*and*

$$(x+q^{n-1}y)(x+q^{n-2}y)\cdots(x+y)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{qQ} q^{\binom{k}{2}}x^{n-k}y^k. \tag{1.2}$$

For $q=1, p=1$ this reduces to the

**q-binomial Theorem (cf. e.g. [1])**

*Let* $yx=Qxy.$ *Then*

$$(x+y)^n=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}_{Q} x^{n-k}y^k. \tag{1.3}$$

We show more generally

**Theorem 2**

*Let* $yx = Qxy$ *and* $p, q$ *be arbitrary numbers. Then*

$$(x+y)(qx+py)\cdots(q^{n-1}x+p^{n-1}y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{qQ}{p}} q^{\binom{n-k}{2}} p^{kn-\binom{k+1}{2}} x^{n-k} y^{k}. \qquad (1.4)$$

Theorem 2 is equivalent with the $q-$binomial Theorem. The proof is almost trivial:

**Proof**

Let $\varepsilon$ be the linear operator on the polynomials in $x, y$ defined by $\varepsilon x = qx\varepsilon$ and $\varepsilon y = y\varepsilon$ and let $\eta$ satisfy $\eta x = x\eta$ and $\eta y = py\eta.$

Then $(y\varepsilon\eta)(x\varepsilon\eta) = qyx(\varepsilon\eta)^2 = qQxy(\varepsilon\eta)^2$ and $(x\varepsilon\eta)(y\varepsilon\eta) = pxy(\varepsilon\eta)^2$.
Therefore

$$(y\varepsilon\eta)(x\varepsilon\eta) = \frac{qQ}{p}(x\varepsilon\eta)(y\varepsilon\eta).$$

By the $q-$binomial theorem this gives

$$((x+y)\varepsilon\eta)^n = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{qQ}{p}} (x\varepsilon\eta)^{n-k} (y\varepsilon\eta)^{k} = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{qQ}{p}} q^{\binom{n-k}{2}} x^{n-k}\varepsilon^{n-k}\eta^{n-k} \left( p^{\binom{k}{2}} y^k \varepsilon^k \eta^k \right)$$

$$= \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{\frac{qQ}{p}} q^{\binom{n-k}{2}} p^{nk-\binom{k+1}{2}} x^{n-k} y^k \varepsilon^n \eta^n.$$

The left-hand side is

$$(x+y)\varepsilon\eta(x+y)\varepsilon\eta\cdots(x+y)\varepsilon\eta = (x+y)(qx+py)\cdots(q^{n-1}x+p^{n-1}y)\varepsilon^n\eta^n.$$

Applying these operators to the constant polynomial 1 gives (1.4).

For $q=1$ and $p \to q$ we get (1.1) and for $q=1,\ y \to q^{n-1}y$ and $p \to \frac{1}{q}$ we get (1.2).

If we let $p \to \frac{1}{p}$ and $y \to p^{n-1}y$ we get the symmetric identity

$$(x+p^{n-1}y)(qx+p^{n-2}y)\cdots(q^{n-1}x+y) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix}_{pqQ} q^{\binom{k}{2}} p^{\binom{n-k}{2}} x^{k} y^{n-k}. \qquad (1.5)$$

The generating function of these polynomials is given by

$$\sum_{n\ge 0}(x+p^{n-1}y)(qx+p^{n-2}y)\cdots(q^{n-1}x+y)\frac{z^n}{[n]_{pqQ}!}=\sum_{k\ge 0}q^{\binom{k}{2}}x^k\frac{z^k}{[k]_{pqQ}!}\sum_{j\ge 0}p^{\binom{j}{2}}y^j\frac{z^j}{[j]_{pqQ}!}.$$

For $Q=1$ and $p\to q$ identity (1.5) reduces to

$$(x+q^{n-1}y)(qx+q^{n-2}y)\cdots(q^{n-1}x+y)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}_{q^2}q^{\binom{k}{2}}q^{\binom{n-k}{2}}x^ky^{n-k} \qquad (1.6)$$

for commuting variables $x, y$ which (in a different notation) plays an important role in [3].

**References**


[1] J. Cigler, Operatormethoden für q-Identitäten, Monatsh. Math. 88 (1979), 87-105

[2] S. Nalci and O. Pashaev, Non-commutative q-binomial formula, arXiv:1202.2264

[3] S.K. Suslov, An introduction to basic Fourier series, Kluwer 2003


Wien, 20.2.2012