Some Hankel determinants with nice evaluations

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Talk at the occasion of Peter Paule's 60^{th} birthday

Introduction

For each n we consider the Hankel determinant

$$H_n = \det\left(a_{i+j}\right)_{i,j=0}^{n-1}.$$

We are interested in the sequence $\left(H_n\right)_{n\geq 0}$ with $H_0 = 1$.

It is well known that the sequence of Catalan numbers $C_n = \frac{1}{n+1}\binom{2n}{n}$

can be characterized by the fact that all Hankel determinants

of the sequences $\left(C_n\right)_{n\geq 0}$ and $\left(C_{n+1}\right)_{n\geq 0}$ are 1.

The generating function of the Catalan numbers $C(x) = \sum_{n\geq 0} C_n x^n = \frac{1-\sqrt{1-4x}}{2x}$

satisfies $C(x) = 1 + xC(x)^2$.

Let $C(x)^r = \sum_{n\geq 0} C_n^{(r)} x^n$. Then we get $C_n^{(2)} = C_{n+1}$ and $C_n^{(r)} = \frac{r}{2n+r}\binom{2n+r}{n}$.

In the first part of this talk I want to give some overview about the Hankel determinants

$$d_r(n)=\det\left(\binom{2i+2j+r}{i+j}\right)_{i,j=0}^{n-1} \quad \text{and} \quad D_r(n)=\det\left(C_{i+j}^{(r)}\right)_{i,j=0}^{n-1} \quad \text{for } r\geq 0.$$

Many of these determinants are easy to guess and show an interesting modular pattern, but strangely enough I found almost nothing about them in the literature except for $r=0$ and $r=1$. Only after I posted a question in MathOverflow I learned that at least Egecioglu, Redmond and Ryavec (arXiv:0804.0440) had considered $d_r(n)$. Proofs seem only to be known for $r\leq 3$.

$$\left(d_0(n)\right)_{n\geq 0}=\left(1,1,2,2^2,2^3,\cdots\right),$$
$$\left(d_1(n)\right)_{n\geq 0}=\left(1,1,1,1,1,\cdots\right),$$
$$\left(d_2(n)\right)_{n\geq 0}=\left(1,1,-1,-1,1,1,-1,-1,\cdots\right),$$
$$\left(d_3(n)\right)_{n\geq 0}=\left(1,1,-4,3,3,-8,5,5,-12,7,7,-16,\cdots\right),$$
$$\left(d_4(n)\right)_{n\geq 0}=\left(1,1,-8,8,1,1,-16,-16,1,1,-24,-24,\cdots\right),$$
$$\left(d_5(n)\right)_{n\geq 0}=\left(1,1,-13,-16,61,9,9,-178,-64,370,25,25,-695,-144,1127,\cdots\right)$$

It seems that

$$d_{2k+1}\big((2k+1)n\big)=d_{2k+1}\big((2k+1)n+1\big)=(2n+1)^k,$$

$$d_{2k+1}\big((2k+1)n+k+1\big)=(-1)^{\binom{k+1}{2}}4^k\,(n+1)^k,$$

$$d_{2k}\left(2kn\right)=d_{2k}\left(2kn+1\right)=(-1)^{kn},$$

$$d_{2k}\left(2kn+k\right)=-d_{2k}\left(2kn+k+1\right)=(-1)^{kn+\binom{k}{2}}4^{k-1}(n+1)^{k-1}.$$

The other values are not so nice.

For example

$$d_5(5n+2)=-\frac{(n+1)(2n+1)\big(50n+39\big)}{3},\quad d_5(5n+4)=-\frac{(n+1)(2n+3)\big(50n+61\big)}{3}.$$

But

$$d_{2k}(2kn-1)+d_{2k}(2kn+2)=(-1)^{kn}\left(\binom{2k+1}{2}-2\right),$$

$$d_{2k+1}\big((2k+1)n-1\big)+d_{2k+1}\big((2k+1)n+2\big)=\left(2-\binom{2k+2}{2}\right)(2n+1)^k.$$

Some background material

If $d_n = \det\left(a_{i+j}\right)_{i,j=0}^{n-1} \neq 0$ for each n we can define the polynomials

$$p_n(x) = \frac{1}{d_n}\begin{vmatrix} a_0 & a_1 & \cdots & a_{n-1} & 1 \\ a_1 & a_2 & \cdots & a_n & x \\ a_2 & a_3 & \cdots & a_{n+1} & x^2 \\ \vdots & & & & \vdots \\ a_n & a_{n+1} & \cdots & a_{2n-1} & x^n \end{vmatrix}.$$

If we define a linear functional L on the polynomials by $L\left(x^n\right) = a_n$ then

$L\left(p_n p_m\right) = 0$ for $n \neq m$ and $L\left(p_n^2\right) \neq 0$ (Orthogonality).

There exist s_n and t_n such that

$$p_n(x) = (x - s_{n-1})p_{n-1}(x) - t_{n-2}p_{n-2}(x).$$

The numbers t_n are given by $t_n = \dfrac{d_n d_{n+2}}{d_{n+1}^2}$.

For arbitrary s_n and t_n define numbers $a_n(j)$ by

$$
\begin{aligned}
&a_0(j) = [j = 0],\\
&a_n(0) = s_0 a_{n-1}(0) + t_0 a_{n-1}(1),\\
&a_n(j) = a_{n-1}(j-1) + s_j a_{n-1}(j) + t_j a_{n-1}(j+1).
\end{aligned}
$$

Then we get

$$
\det\left(a_{i+j}(0)\right)_{i,j=0}^{n-1} = \prod_{i=1}^{n-1}\prod_{j=0}^{i-1} t_j.
$$

If we start with the sequence $\left(a_n\right)_{n\geq 0}$ and guess s_n and t_n and if we also can guess $a_n(j)$ and show that $a_n(0) = a_n$ then all our guesses are correct and the Hankel determinant is given by the above formula.

For the aerated sequence $\left(1,0,1,0,2,0,5,0,14,0,\cdots\right)$ of Catalan numbers it is easy to guess that $s_n = 0$ and $t_n = 1$ and that $a_{2n+k}(k) = C_n^{(k+1)}$ and all other $a_n(j) = 0.$ Thus $a_{2n}(0) = C_n$ and $a_{2n+1}(0) = 0.$ Therefore all Hankel determinants are 1.

There is a well-known equivalence with continued fractions, so-called J-fractions:

$$\sum_{n\geq 0} a_n x^n = \cfrac{1}{1-s_0x-\cfrac{t_0x^2}{1-s_1x-\cfrac{t_1x^2}{1-\ddots}}}.$$

For some sequences this gives a simpler approach to Hankel determinants.

The generating function of the Catalan numbers satisfies $C(x)=1+xC(x)^2$.

Therefore

$$C(x)=\frac{1}{1-xC(x)} \text{ and } C(x^2)=\frac{1}{1-x^2C(x^2)}=\cfrac{1}{1-\cfrac{x^2}{1-\cfrac{x^2}{1-\ddots}}}.$$

This again implies that the Hankel determinants of the aerated sequence of Catalan numbers are 1 and also that $D_1(n)=1$.

Some other examples of J-fractions

$$C(x)^2 = \frac{1}{1-2x-x^2C(x)^2}$$

implies $D_2(n)=1$.

$$\frac{1}{\sqrt{1-4x}} = \frac{1}{1-2xC(x)} = \frac{1}{1-2x-2x^2C(x)^2}$$

implies $d_0(n)=2^{n-1}$.

$$\sum_{n\geq 0}\binom{2n+1}{n}x^n = \frac{1}{2}\sum_{n\geq 0}\binom{2n+2}{n+1}x^n = \frac{1}{2x}\left(\frac{1}{\sqrt{1-4x}}-1\right) = \frac{C(x)}{\sqrt{1-4x}}$$

and $C(x)\left(1-3x-x^2C(x)^2\right)=\sqrt{1-4x}$ give

$$\frac{C(x)}{\sqrt{1-4x}} = \frac{1}{1-3x-x^2C(x)^2}$$

and thus $d_1(n)=1$.

$$d_2(n) = \det\left(\binom{2i+2j+2}{i+j}\right)_{i,j=0}^{n-1}$$

It is easy to guess that $s_{2k} = 4$, $s_{2k+1} = 0$ and $t_k = 1$.

We also guess that $a_n(2k) = \binom{2n+2}{n-2k}$.

This implies $a_n(2k+1) = \binom{2n}{n-2k-1} - \binom{2n}{n-2k-3}$.

It remains to verify the trivial identity

$$\binom{2n+2}{n-2k} = \binom{2n-2}{n-2k} - \binom{2n}{n-2k-2} + 4\binom{2n}{n-1-2k} - \binom{2n-2}{n-2k-2} + \binom{2n-2}{n-2k-4}.$$

Therefore we get $(d_2(n))_{n\geq 0} = (1,1,-1,-1,1,1,-1,-1,\cdots)$.

A proof with J-fractions

By induction we get

$$B_r(x)=\sum_{n\geq 0}\binom{2n+r}{n}x^n=\frac{C(x)^r}{\sqrt{1-4x}}.$$

This implies

$$B_2(x)+x^2B_2(x)^2=\frac{1}{1-4x}.$$

For $C(x)\sqrt{1-4x}=2-C(x)$ and $xC(x)^2=C(x)-1$ and therefore

$$(1-4x)\left(\frac{C(x)^2}{\sqrt{1-4x}}+x^2\frac{C(x)^4}{1-4x}\right)=C(x)\left(C(x)\sqrt{1-4x}\right)+\left(xC(x)^2\right)^2$$

$$=C(x)\left(2-C(x)\right)+\left(C(x)-1\right)^2=1.$$

This implies

$$B_2(x)=\frac{1}{1-4x}\frac{1}{1+x^2B_2(x)}=\frac{1}{1-4x+x^2\left(1-4x\right)B_2(x)}=\frac{1}{1-4x+\cfrac{x^2}{1+x^2B_2(x)}}.$$

For $r \geq 3$ the situation becomes more complicated. Since no Hankel determinant vanishes the above method should in principle be applicable. It seems that it is possible for each fixed r to guess s_n and t_n. But for $r \geq 5$ I could not guess $a_n(j)$.

Let me sketch the case $r = 3$: Here we get $d_3(3n) = d_3(3n+1) = 2n+1$ and $d_3(3n+2) = -4(n+1)$.

$$s_{3n} = 5,\ s_{3n+1} = \frac{2n+1}{4(n+1)},\ s_{3n+2} = \frac{2n+3}{4(n+1)},$$

$$t_{3n} = -\frac{4(n+1)}{2n+1},\ t_{3n+1} = \frac{(2n+1)(2n+3)}{4^2(n+1)^2},\ t_{3n+2} = -\frac{4(n+1)}{2n+3}.$$

$$a_n(3k) = \binom{2n+3}{n-3k},$$

$$a_n(3k+1) = \binom{2n+1}{n-3k-1} + \frac{2k+1}{4(k+1)}\binom{2n+1}{n-3k-2} - \frac{2k+1}{4(k+1)}\binom{2n+1}{n-3k-3},$$

$$a_n(3k+2) = \binom{2n+1}{n-3k-2} + \binom{2n+1}{n-3k-3} - \frac{4(k+1)}{2k+3}\binom{2n+1}{n-3k-4}.$$

I have only found the following curious regularities:

Let $r \geq 2.$

Then

$$s_{rn} = r + 2,$$
$$s_{rn} + s_{rn+1} + \cdots + s_{rn+r-1} = 2r,$$
$$t_{rn} t_{rn+1} \cdots t_{rn+r-1} = 1.$$

Furthermore it seems that

$$a_n(rk) = \binom{2n+r}{n-rk}.$$

$$D_r(n) = \det\left(C_{i+j}^{(r)}\right)_{i,j=0}^{n-1}.$$

These determinants show a similar pattern. But some of them vanish. For example for $r = 3$ it is known (C. Krattenthaler and J.C. 2011) that $D_3(n) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^k \binom{n-k}{k}$ or

$\left(D_3(n)\right)_{n\geq 0} = (1, 1, 0, -1 -1, 0, \cdots)$, which is periodic with period 6.

For $r > 3$ apparently no results appear in the literature. But we will show that for odd r there are always vanishing determinants. Therefore the method of orthogonal polynomials is not directly applicable. I have studied the case $r = 3$ in more detail and looked for other tricks to compute these determinants.

Guo-Niu Han, arXiv:1406.1593, has shown that each formal power series has a unique expansion as a so-called H-fraction

$$\sum_{n\geq 0} a_n x^n = \cfrac{x^{k_0}}{1 - s_0(x)x - \cfrac{t_0 x^{2+k_0+k_1}}{1 - s_1(x)x - \cfrac{t_1 x^{2+k_1+k_2}}{1 - \ddots}}}$$

and proved a formula for the non-vanishing Hankel determinants.

The case r=3 as H-fraction

The powers of the generating function of the Catalan numbers satisfy

$$C(x)^r L_r(-x) = 1 + x^r C(x)^{2r},$$

where

$$\left(L_r(x)\right)_{r\geq 0} = \left(2,\ 1,\ 1+2x,\ 1+3x,\ 1+4x+2x^2,\ 1+5x+5x^2, \cdots\right)$$

are Lucas polynomials. This gives rise to continued fractions.

For $r=3$ we get the H-fraction

$$C(x)^3 = \cfrac{1}{1-3x-\cfrac{x^3}{1-3x-\cfrac{x^3}{1-3x-\ddots}}}$$

from which we get again $\left(D_3(n)\right)_{n\geq 0} = \left(1,1,0,-1,-1,0,\cdots\right).$

Analogously $x^{k-1}C(x)^{2k}$ and $x^{k-1}C(x)^{2k+1}$ give H-fractions.

A valuable Lemma

Another helpful trick is the following Lemma (Szegö 1939):

Let $p_n(x)$ be monic polynomials which are orthogonal

with respect to the linear functional L with moment $L\left(x^n\right) = a_n$

and let $r_n(x) = a_n x - a_{n+1}$. Then

$$\det\left(r_{i+j}(x)\right)_{i,j=0}^{n-1} = \det\left(a_{i+j}\right)_{i,j=0}^{n-1} p_n(x).$$

For the proof let $p_n(x) = b_{n,0} + b_{n,1}x + \cdots + b_{n,n-1}x^{n-1} + x^n$ and

$$B_n = \begin{pmatrix} x & -1 & 0 & \cdots & 0 \\ 0 & x & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -1 \\ b_{n,0} & b_{n,1} & b_{n,2} & \cdots & x + b_{n,n-1} \end{pmatrix}.$$

Then we get

$$\left(r_{i+j}(x)\right)_{i,j=0}^{n-1} = B_n \left(a_{i+j}\right)_{i,j=0}^{n-1}.$$

For $a_n = C_{n+1}$ we get $s_n = 2,\ t_n = 1$ and

$$p_n(x) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k} (x-2)^{n-2k}.$$

Since $C_n^{(3)} = C_{n+2} - C_{n+1}$ the Lemma implies

$$D_3(n) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} (-1)^k \binom{n-k}{k}.$$

The Lemma also gives another proof of the Theorem

(Cvetkovic, Rajkovic and Ivkovic)

$$\det\left(C_{i+j} + C_{i+j+1}\right)_{i,j=0}^{n-1} = F_{2n+1}.$$

Narayana polynomials

Another trick is to introduce another parameter such that no determinant vanishes.

The Narayana polynomials

$$C_n(t)=\sum_{k=0}^{n}\binom{n}{k}\binom{n-1}{k}\frac{1}{k+1}t^k$$

for $n>0$ and $C_0(t)=1$ satisfy $C_n(1)=C_n$. The first terms are

$1,\ 1,\ 1+t,\ 1+3t+t^2,\ 1+6t+6t^2+t^3,\cdots.$

For the sequence $\left(C_{n+1}(t)\right)_{n\geq 0}$ we get $s_n=1+t$ and $t_n=t$.

The orthogonal polynomials are

$$p_n(x,t)=\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\binom{n-k}{k}t^k(x-1-t)^{n-2k}.$$

By the Lemma we get

$$\det\left(C_{i+j+1}(t)+C_{i+j+2}(t)\right)_{i,j=0}^{n-1}=t^{\binom{n}{2}}\sum_{k=0}^{\left\lfloor\frac{n}{2}\right\rfloor}(-1)^k\binom{n-k}{k}(t+2)^{n-2k}.$$

Another proof by the Lindström-Gessel-Viennot theorem has been given by C. Krattenthaler.

For $t=1$ we can again get $D_3(n)$.

More interesting is the case $t=-1$. Here we get

$$\left(C_{n+1}(-1)+C_{n+2}(-1)\right)_{n\geq 0}=\left(1,-1,-1,2,2,-5,-5,14,14,-42,-42,\cdots\right).$$

The corresponding Hankel determinants are Fibonacci numbers

$$\left(d_n\right)_{n\geq 0}=\left(1,1,-2,-3,5,8,-13,-21,\cdots\right).$$

For $(1,1,1,2,2,5,5,14,14,\cdots)$ we get the Hankel determinants

$$\left(d_n\right)_{n\geq 0}=\left(1,1,0,-1,-1,0,1,1,0,-1,-1,0,\cdots\right).$$

These results can also be obtained directly with the method of orthogonal polynomials.

For odd r some Hankel determinants vanish

We can prove that

$$D_{2k+1}(k+1)=0.$$

A search for a linear relation led to

$$R(k,n)=\sum_{j=0}^{k}(-1)^{k-j}\left(\binom{k+j}{2j+1}+\binom{k+j+1}{2j+1}\right)C_{n+j}^{(2k+1)}=0$$

for $0\leq n\leq k$ if $k>0.$

More generally we get

$$\sum_{n\geq 0}R(k,n)x^n=x^{k+1}C(x)^{4k+2}.$$

Christian Krattenthaler has provided a proof using hypergeometric identities.

It can also be proved with Peter Paule's implementation of Zeilberger's algorithm.

I want to congratulate Peter Paule und his team for the very valuable Mathematica packages which were indispensible for my work since my interest turned to experimental mathematics.

$D_{2k+1}(n)$

For $r>3$ I have only conjectures:

$$\left(D_5(n)\right)_{n\geq 0}=\left(1,1,-5,0,5,1,1,-10,0,10,1,1,-15,0,15,\cdots\right)$$

$$\left(D_7(n)\right)_{n\geq 0}=\left(1,1,-14,-7^2,0,7^2,329,-1,-1,-315,(2\cdot 7)^2,0,-(2\cdot 7)^2,-1687,\cdots\right).$$

More generally

$$D_{2k+1}((2k+1)n)=D_{2k+1}((2k+1)n+1)=(-1)^{kn},$$

$$D_{2k+1}\left((2k+1)n+k+1\right)=0,$$

$$D_{2k+1}\left((2k+1)n+k+2\right)=-D_{2k+1}\left((2k+1)n+k\right)=(-1)^{nk+\binom{k}{2}+1}\left(\left(2k+1\right)(n+1)\right)^{k-1},$$

$$D_{2k+1}((2k+1)n-1)+D_{2k+1}((2k+1)n+2)=(-1)^{kn+1}(k-1)(2k+1).$$

$D_4(n)$

For $\left(C_n^{(4)}\right)_{n\geq 0} = (1, 4, 14, 48, 165, 572, 2002, \cdots)$ we get

$$\left(D_4(n)\right)_{n\geq 0} = \left(1, 1, -2, -2, 3, 3, -4, -4, \cdots\right).$$

Here we have $s_{2k} = 4,\ s_{2k+1} = 0,\ t_{2k} = -\frac{k+2}{k+1}$ and $t_{2k+1} = -\frac{k+1}{k+2}.$

The corresponding $a_n(j)$ satisfy

$$\sum_{n\geq 0} a_n(2k)x^n = x^{2k}C(x)^{4k+4},$$

$$\sum_{n\geq 0} a_n(2k+1)x^n = x^{2k+1}C(x)^{4k+4} - \frac{k+1}{k+2}x^{2k+3}C(x)^{4k+8}.$$

$D_{2k}(n,t)$

Define $C_n^{(2k)}(t)$ by

$$\sum_{n\geq 0} C_n^{(2k)}(t)x^n = \left(\sum_{n\geq 0} C_{n+1}(t)x^n\right)^k :$$

This implies that $C_n^{(2k)}(1) = C_n^{(2k)}$.

Let

$$D_{2k}(n,t) = \det\left(C_{i+j}^{(2k)}(t)\right)_{i,j=0}^{n-1}.$$

If we use the $q-$notation $[n]_q = 1+q+\cdots+q^{n-1}$ then we get

$$D_4(2n,t) = (-1)^n t^{2(n^2-n)} \left[n+1\right]_{t^2},$$
$$D_4(2n+1,t) = (-1)^n t^{2n^2} \left[n+1\right]_{t^2}.$$

$D_6(n,t)$

The first terms of $D_6(n)$ are

$$1^2, 1^2, -3^2, -2^2, -2^2, 3^2\left(1^2+2^2\right), 3^2, 3^2, -3^2\left(1^2+2^2+3^2\right), \cdots.$$

Conjecture:

$$D_6(3n) = D_6(3n+1) = (-1)^n (n+1)^2,$$

$$D_6(3n+2) = 3^2(-1)^{n+1}\sum_{j=1}^{n+1} j^2.$$

$$D_6(3n,t) = (-1)^n t^{9\binom{n}{2}} [n+1]_{t^3}^2,$$

$$D_6(3n+1,t) = (-1)^n t^{3\frac{n(3n-1)}{2}} [n+1]_{t^3}^2,$$

$$D_6(3n+2,t) = (-1)^{n+1} 3[3]_t\, t^{3\frac{n(3n+1)}{2}} r_n(t)$$

with

$$\left(r_n(t)\right)_{n\ge 0} = \left(1,\ 1+3t^3+t^6,\ 1+3t^3+6t^6+3t^9+t^{12}, \cdots\right).$$

Some more conjectures

$$D_{2k}(kn) = D_{2k}(kn+1) = (-1)^{n\binom{k}{2}}(n+1)^{k-1},$$

$$D_{2k}(2kn-1) + D_{2k}(2kn+2) = -k(2k-3)(2n+1)^{k-1}.$$

$$D_{2k}(kn,t) = (-1)^{\binom{k}{2}n} t^{k^2\binom{n}{2}} [n+1]_{t^k}^{k-1},$$

$$D_{2k}(kn+1,t) = (-1)^{\binom{k}{2}n} t^{k^2\binom{n}{2}+kn} [n+1]_{t^k}^{k-1}.$$

Catalan numbers modulo 2

It is well known that $C_n \equiv 1 \bmod 2$ iff $n = 2^k - 1$ for some k: Let $f(x) = C(x) \bmod 2.$

Then $f(x) = 1 + xf\left(x^2\right)$ which implies $f(x) = \sum_{k \geq 0} x^{2^k - 1}.$

Let now $a_{2^k - 1} = 1$ and $a_n = 0$ else. Then

$$d_n = \det\left(a_{i+j}\right)_{i,j=0}^{n-1} = (-1)^{\binom{n}{2}}.$$

In this case the determinant is reduced to a single term

$$d_n = \operatorname{sgn} \pi_n a_{0+\pi_n(0)} \cdots a_{n-1+\pi_n(n-1)} \neq 0$$

for a uniquely determined permutation π_n.

For example $\pi_5 = 02143$ and

$$d_5 = \det\begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = 1.$$

Similar determinants have previously been considered by R. Bacher (2004) from another point of view. I have posted some questions about such determinants on MO and received some proofs from Darij Grinberg.

More generally let $b_{2^k-1} = x^k$ and $b_n = 0$ else. For example

$$B_5 = \begin{pmatrix} 1 & x & 0 & x^2 & 0 \\ x & 0 & x^2 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 \\ x^2 & 0 & 0 & 0 & x^3 \\ 0 & 0 & 0 & x^3 & 0 \end{pmatrix}.$$

The corresponding determinants are

$$\det B_n = (-1)^{\binom{n}{2}} x^{2a(n)},$$

where $a(n)$ is the total number of 1's in the binary expansions of the numbers $1, 2, \cdots, n-1$.

In the above example we get $a(5) = 5$ because the number of 1's in 1, 10, 11, 100 is 5.

The aerated sequence $(a_0, 0, a_1, 0, a_2, 0, \cdots)$.

Let $a_{2^k-1} = 1$ and $a_n = 0$ else and let $A_{2n} = a_n$ and $A_{2n+1} = 0$ be the aerated sequence.

It is easy to see that $A_n = a_{n+1}$.

For example

$$\left(A_{i+j}\right)_{i,j=0}^{3} = \left(a_{i+j+1}\right)_{i,j=0}^{3} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

In this case too the determinant is reduced to a single permutation.

We get

$$D_n = \det\left(A_{i+j}\right)_{i,j=0}^{n-1} = (-1)^{\delta_n},$$

where δ_n is the number of pairs $\varepsilon_{i+1}\varepsilon_i = 10$ for $i \geq 1$ or $\varepsilon_1\varepsilon_0 = 11$ in the binary expansion of n.

For example $\delta_4 = 1$ because $4 = 100$ or $\delta_{75} = 3$ because $75 = 1001011$.

The determinants satisfy $D_{2n} = (-1)^{\binom{n}{2}} D_n$ and $D_{2n+1} = (-1)^{\binom{n+1}{2}} D_n$.

An approach via orthogonal polynomials

These determinants have also been studied by R.Bacher who found the interesting formula

$$D_n = \prod_{j=0}^{n-1} S(j),$$

where $\left(S(n)\right)_{n\geq 0} = \left(1,1,-1,1,1,-1,-1,1,1,1,\cdots\right)$ is the so-called paperfolding sequence

which satisfies

$S(2n) = (-1)^n, \quad S(2n+1) = S(2n)$ and $S(0) = 1.$

The method of orthogonal polynomials gives $s_n = 0$ and $T_n = S(n)S(n+1).$

The numbers T_n are uniquely determined by the recursion

$$T_{2n} = T_{2n-1}T_{n-1},$$
$$T_{2n+1} = -T_{2n},$$
$$T_0 = 1, T_1 = -1.$$

Golay-Rudin-Shapiro sequence

Let $g(1)=1$ and $g\left(2^k-1\right)=(-1)^k$ for $k>1$ and $g(n)=0$ else.

Then

$$\det\left(g(i+j+1)\right)_{i,j=0}^{n-1}=r(n),$$

where $r(n)$ is the Golay-Rudin-Shapiro sequence defined by

$$r(2n)=r(n),$$
$$r(2n+1)=(-1)^n r(n),$$
$$r(0)=1.$$

Equivalently $r(n)=(-1)^{R(n)}$, where $R(n)$ denotes the number of pairs 11 in the binary expansion of n.

Associated continued fractions

Let me finally state two associated continued fractions:

$$\sum_{k\geq 0} x^{2^k-1} = \cfrac{1}{1-\cfrac{S(0)S(1)x}{1-\cfrac{S(1)S(2)x}{1-\ddots}}}$$

and

$$\sum_{k\geq 0} (-1)^k x^{2^k-1} = \cfrac{1}{1+\cfrac{r(0)r(2)x}{1+\cfrac{r(1)r(3)x}{1+\ddots}}}.$$