## A curious class of Hankel determinants

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#### Abstract

We consider Hankel determinants of the sequence of Catalan numbers modulo 2 (interpreted as integers 0 and 1) and more generally Hankel determinants where the sum over all permutations reduces to a single signed permutation.

### **0. Introduction**

Let  $C_n = \frac{1}{n+1} {2n \choose n}$  be a Catalan number. It is well known that  $\det(C_{i+j})_{i,j=0}^{n-1} = 1$  for all

 $n \in \mathbb{N}$ . Of course this remains true if we consider all terms modulo 2. It is also well known that  $C_n \equiv 1 \mod 2$  if and only if  $n = 2^k - 1$  for some k.

But what happens if we consider the sequence  $C_n \mod 2$  as a sequence of integers from  $\{0,1\}$ ? The attempt to answer this question gave rise to the present paper. After completion of a first version I discovered the paper [1] by Roland Bacher, where similar questions are considered from a different point of view. There is some overlap between these approaches which I will be mention at the appropriate places.

Let  $(a_n)_{n\geq 0}$  satisfy  $a_n = 1$  if n+1 is a power of 2 and  $a_n = 0$  else.

Computer experiments led me to guess that

$$d(n) = \det\left(a_{i+j}\right)_{i,j=0}^{n-1} = (-1)^{\binom{n}{2}}$$
(0.1)

and that the determinant

$$\det\left(a_{i+j}\right)_{i,j=0}^{n-1} = \sum_{\pi} \operatorname{sgn}(\pi) a_{0+\pi(0)} a_{1+\pi(1)} \cdots a_{n-1+\pi(n-1)}$$
(0.2)

which in general is a sum over all permutations  $\pi$  of  $\{0, 1, \dots, n-1\}$  is reduced to a single term

$$\operatorname{sgn}(\pi_n) a_{0+\pi_n(0)} a_{1+\pi_n(1)} \cdots a_{n-1+\pi_n(n-1)} \neq 0$$
 (0.3)

for a uniquely determined permutation  $\pi_n$ .

For example

$$d(5) = \det \left(a_{i+j}\right)_{i,j=0}^{4} = \det \begin{pmatrix} 1 & 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \end{pmatrix} = 1$$
(0.4)

reduces to the term  $a_1 a_3^2 a_7^2$  corresponding to  $\pi_5 = 02143$ .

Thus 
$$\delta([0]_2) = 0$$
,  $\delta([1]_2) = 0$ ,  $\delta([10]_2) = 0$ ,  $\delta([11]_2) = 1$ ,  $\delta([100]_2) = 1$ ,  $\delta([101]_2) = 1$ ,  
 $\delta([110]_2) = 0$ ,  $\delta([111]_2) = 1, \cdots$ .

Consider for example

$$D(5) = \det \left( a_{i+j+1} \right)_{i,j=0}^{4} = \det \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 1 & 0 & 0 \end{pmatrix} = -1^{\delta(5)} = -1.$$
(0.5)

The determinant reduces to  $-a_1a_3a_7^3$ .

More generally we study Hankel determinants for sequences  $(a_n)_{n\geq 0}$  such that  $a_n = x_n$  if n+1 is a power of 2 and  $a_n = 0$  else, where  $x_n$  are arbitrary numbers. For some choices of  $x_n$  we get curious results.

For example for  $x_{2^{k}-1} = x^{k}$  we get  $d(n) = (-1)^{\binom{n}{2}} x^{2a(n)}$ , where a(n) is the total number of 1's in the binary expansions of the numbers  $\leq n-1$ . For example

$$\det \begin{pmatrix} 1 & x & 0 & 1 & 0 \\ x & 0 & x^2 & 0 & 0 \\ 0 & x^2 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & x^3 \\ 0 & 0 & 0 & x^3 & 0 \end{pmatrix} = x^{2a(5)} = x^{10}.$$
 (0.6)

The total number of 1's in the binary expansions of  $0 = [0]_2, 1 = [1]_2, 2 = [10]_2, 3 = [11]_2, 4 = [100]_2$  is a(5) = 5.

If we choose  $x_1 = 1$  and  $x_{2^{k-1}} = (-1)^k$  for k > 1 we get the Golay-Rudin-Shapiro sequence D(n) = r(n) which satisfies  $r(n) = (-1)^{\rho(n)}$  where  $\rho(n)$  denotes the number of pairs 11 in the binary expansion of *n*.

For example

$$\det \begin{pmatrix} 1 & 0 & 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & -1 \\ 1 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 & 0 & 0 \end{pmatrix} = (-1)^{\rho(6)} = -1.$$
 (0.7)

Here  $6 = [110]_2$  gives  $\rho(6) = 1$ .

This choice of  $x_n$  also leads to the continued fraction

$$\sum_{k\geq 0} (-1)^k z^{2^{k-1}} = \frac{1}{1 + \frac{r(0)r(2)z}{1 + \frac{r(1)r(3)z}{1 + \frac{r(2)r(4)z}{1 + \ddots}}}} = \frac{1}{1 + \frac{z}{1 - \frac{z}{1 + \frac{z}{1 - \frac{$$

### 1. Hankel determinants of Catalan numbers modulo 2

Let  $a_n \in \{0,1\}$  satisfy  $a_n \equiv C_n \mod 2$  or with other words let  $a_n = 1$  if n+1 is a power of 2 and  $a_n = 0$  else.

Then

$$d(n) = \det\left(a_{i+j}\right)_{i,j=0}^{n-1} = (-1)^{\binom{n}{2}}.$$
(1.1)

The following proof uses an idea due to Darij Grinberg [5] who called a permutation  $\pi$  nimble if for each *i* in its domain the number  $i + \pi(i) + 1$  is a power of 2. Thus a permutation  $\pi$  is nimble if and only if  $a_{0+\pi(0)}a_{1+\pi(1)}\cdots a_{n-1+\pi(n-1)} \neq 0$ .

## Theorem 1.1

For each  $n \in \mathbb{N}$  there exists a unique nimble permutation  $\pi_n$  of  $\{0,1,\dots,n-1\}$  such that

$$d(n) = \det\left(a_{i+j}\right)_{i,j=0}^{n-1} = sgn(\pi_n)a_{0+\pi_n(0)}a_{1+\pi_n(1)}\cdots a_{n-1+\pi_n(n-1)} = (-1)^{\binom{n}{2}}.$$
 (1.2)

#### Proof

For n = 0 we set d(0) = 1 by convention.

Let  $k \ge 1$  and  $2^{k-1} < n \le 2^k$ . Let us try to construct a nimble permutation  $\pi$ .

By definition we must have  $n-1+\pi(n-1) = 2^{\ell}-1$  for some  $\ell$ . Since  $n-1 \ge 2^{k-1}$  we get  $2^{\ell}-1 \ge 2^{k-1}$  and therefore  $\ell = k$  which implies  $\pi(n-1) = 2^k - 1 - (n-1) = 2^k - n$ . (Since  $2(n-1) < 2^{k+1} - 1$  we have  $\ell \le k$ ).

If we define  $\pi(n-1-j) = 2^k - 1 - (n-1-j)$  for all j for which  $2^k + j - n \le n-1$ , i.e. for  $n-1-j \in [2^k - n, n-1]$ , we get a nimble order reversing permutation of the interval  $[2^k - n, n-1]$ .

Let us show that each nimble permutation  $\sigma$  on [0, n-1] reduces to this permutation on  $[2^k - n, n-1]$ .

If  $n-1-j \ge 2^{k-1}$  the same argument as above gives that  $\sigma(n-1-j)$  must be  $\pi(n-1-j) = 2^k - 1 - (n-1-j) = 2^k + j - n.$ 

If  $2^{k} - n \le n - 1 - j \le 2^{k-1} - 1$  then  $2^{k} - 1 - (n - 1 - j) \ge 2^{k} - 1 - 2^{k-1} + 1 = 2^{k-1}$ .

Choose *i* such that  $\sigma(i) = 2^k - 1 - (n - 1 - j)$ . Then  $i + \sigma(i) = i + 2^k - 1 - (n - 1 - j) = 2^\ell - 1$  for some  $\ell$  and  $2^\ell - 1 \ge 2^{k-1}$ .

This implies that  $\ell = k$  and thus i = n-1-j and  $\sigma(n-1-j) = \pi(n-1-j)$ .

Since  $\pi$  is order reversing on  $[2^k - n, n-1]$  its sign is

$$\operatorname{sgn}(\pi) = (-1)^{\binom{n-1-\binom{2^k-n}{j+1}}{2}} = (-1)^{\binom{2n-2^k}{2}} = (-1)^n.$$

Thus we have seen that for  $1 \le 2^{k-1} < n \le 2^k$  there exists a uniquely determined nimble permutation  $\pi$  on the interval  $[2^k - n, n - 1]$ .

Since  $2^k - n - 1 \le 2^{k-1}$  we can suppose by induction that there is a unique nimble permutation of the interval  $[0, 2^k - n - 1]$ . This gives us the desired nimble permutation  $\pi_n$ .

It remains to show that  $\operatorname{sgn}(\pi_n) = (-1)^{\binom{n}{2}}$ .

This follows by induction because 
$$\binom{2n-2^k}{2} + \binom{2^k-n}{2} - \binom{n}{2} = 2(2^{k-1}-n)(2^k-n)$$
 is even.

If we write a permutation  $\pi$  in the notation  $\pi = \pi(0)\pi(1)\cdots\pi(n-1)$  the first nimble permutations are  $\pi_1 = 0$ ,  $\pi_2 = 10$ ,  $\pi_3 = 021$ ,  $\pi_4 = 3210$ ,  $\pi_5 = 02143$ .

For example choose n = 3. Since  $2^1 < 3 \le 2^2$  we have k = 2.

$$\left(a_{i+j}\right)_{i,j=0}^{2} = \begin{pmatrix} 1 & 1 & 0 \\ 1 & 0 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

The above construction gives the permutation  $\pi = 21$  on  $\{1, 2\}$  with  $i + \pi(i) = 3 = 2^2 - 1$ .

There remains  $(a_{i+j})_{i,j=0}^{0} = (1)$  with  $\pi(0) = 0$  and  $i + \pi(i) = 2^{0} - 1 = 0$ . Thus  $\pi_{3} = 021$  with  $sgn(\pi_{3}) = -1 = (-1)^{\binom{3}{2}}$ .

# **2.** Hankel determinants of the sequence $(C_{n+1})$ modulo **2**

Let as above  $a_n = 1$  if  $n = 2^k - 1$  for some k and  $a_n = 0$  else.

Note that  $a_{2n} = 0$  and  $a_{2n+1} = 1$  if and only if  $2n + 1 = 2^{k+1} - 1$  for some k or equivalently  $n = 2^k - 1$ . Thus  $a_{2n+1} = a_n$ . Therefore we get  $(a_1, a_2, a_{3, \dots}) = (a_0, 0, a_1, 0, a_2, 0, 0)$ . This means that in this case the shifted sequence  $(a_{n+1})_{n\geq 0}$  coincides with the aerated sequence  $(A_n)_{n\geq 0} = (a_0, 0, a_1, 0, a_2, 0, \dots)$ .

Here we have  $A_n = 1$  if  $n = 2(2^k - 1) = 2^{k+1} - 2$  and  $A_n = 0$  else.

If  $f(x) = \sum_{n} a_n x^n = \sum_{k} x^{2^{k-1}}$  is the generating function of the sequence  $(a_n)_{n\geq 0}$  then the generating function of the aerated sequence  $(A_n)_{n\geq 0}$  is  $f(x^2)$ . Since  $A_n = a_{n+1}$  we even have  $f(x) = 1 + xf(x^2)$ .

All Hankel determinants of the sequence  $(C_{n+1})_{n\geq 0}$  are 1. Therefore we know in advance that no Hankel determinant  $D(n) = \det(A_{i+j})_{i=0}^{n-1}$  vanishes.

The first Hankel determinants of the sequence  $(A_n)_{n\geq 0}$  are

#### Theorem 2.1

Let  $A_{2^{k+1}-2} = 1$  for each k and  $A_n = 0$  else and let  $D(n) = \det(A_{i+j})_{i,j=0}^{n-1}$ .

If  $2^{k-1} \le n < 2^k$  then

$$D(n) = (-1)^n D(2^k - n - 1).$$
(2.1)

#### Proof

Let us call a permutation  $\pi$  *m*-nimble if  $i + \pi(i) + m = 2^{\ell} - 1$  for some  $\ell$  for each *i* in its domain. Then  $A_{0+\pi(0)}A_{1+\pi(1)}\cdots A_{n-1+\pi(n-1)} \neq 0$  if and only if  $\pi$  is 1-nimble.

Since  $2^{k-1} \le n < 2^k$  we have  $2^{k-1} \le n-1 + \pi(n-1) + 1 = 2^{\ell} - 1 = n + \pi(n-1) < 2^k + 2^k - 1 = 2^{k+1} - 1$  and therefore  $\ell = k$  which implies  $n + \pi(n-1) = 2^k - 1$  or  $\pi(n-1) = 2^k - 1 - n$ .

We can now define a 1-nimble permutation  $\pi$  of the interval  $[2^k - 1 - n, n - 1]$  by  $\pi(n-1-j) = 2^k - 1 - n + j$  for  $0 \le j \le 2n - 2^k$ .

Then  $\pi$  is an order reversing permutation of the interval  $[2^k - 1 - n, n - 1]$ .

Let  $\sigma$  be any 1-nimble permutation on [0, n-1]. Then  $\sigma = \pi$  on  $[2^k - 1 - n, n-1]$ .

If 
$$n-1-j \ge 2^{k-1}-1$$
 we get  $2^{\ell} = n-1-j + \sigma(n-1-j) + 2 \ge 2^{k-1}+1$  which implies  $\ell = k$ .

If  $n-1-j < 2^{k-1}-1$  then  $2^k - 1 - n + j > 2^{k-1} - 1$ . Let  $\sigma$  be a 1-nimble permutation. Then  $\sigma(i) = 2^k - 1 - n + j$  for some *i* and  $i + 2^k - 1 - n + j + 2 = 2^\ell$  for some  $\ell$ . This implies  $\ell = k$  and i = n - 1 - j.

Thus we get a uniquely determined 1–nimble permutation of the interval  $[2^k - 1 - n, n - 1]$ . The sign of this permutation is  $(-1)^{\binom{2n+1-2^k}{2}} = (-1)^n$ .

By induction we get (2.1).

For example for n = 4 we have k = 3 and

$$\left(A_{i+j}\right)_{i,j=0}^{3} = \left(a_{i+j+1}\right)_{i,j=0}^{3} = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

The corresponding permutation is  $\pi = 2103$  with  $3 + \pi(3) = 6 = 2^3 - 2$  and  $i + \pi(i) = 2 = 2^2 - 2$  on  $\{0, 1, 2\}$ .

Let us now try to find some regularities of the sequence of determinants D(n).

#### **Corollary 2.2**

For k > 0 the sequence D(n) satisfies

$$D(2^{k}+n) = (-1)^{n} D(2^{k}-1-n)$$
(2.2)

for  $0 \le n < 2^k$  with initial values D(0) = D(1) = 1.

For example for k = 3 we get

D(7-n)	-1	1	-1	-1	-1	1	1	1
D(8+n)	-1	-1	-1	1	-1	-1	1	-1

#### **Corollary 2.3**

Let k > 0.

For  $0 \le n < 2^k$  we get

$$D(2^{k+1}+n) = -D(n).$$
 (2.3)

For  $2^k \leq n < 2^{k+1}$  we get

$$D(2^{k+1}+n) = D(n).$$
 (2.4)

### Proof

By (2.2) with k+1 instead of k we get

 $D(2^{k+1}+n) = (-1)^n D(2^{k+1}-1-n) = (-1)^n D(2^k+2^k-1-n) = (-1)^n (-1)^{2^k-1-n} D(2^k-1-(2^k-1-n))$ which gives (2.3).

Again by (2.2) we have

$$D(2^{k+1}+2^{k}+i) = (-1)^{2^{k}+i}D(2^{k+1}-1-(2^{k}+i)) = (-1)^{i}D(2^{k}-1-i) = (-1)^{i}(-1)^{i}D(2^{k}+i)$$

which gives (2.4).

For example for k = 2 we get for  $0 \le n < 8$ 

D(n)	1	1	1	-1	-1	-1	1	-1
$\overline{D(8+n)}$	-1	-1	-1	1	-1	-1	1	-1

### **Corollary 2.4**

Let  $\delta(n)$  be the number of pairs  $\varepsilon_{i+1}\varepsilon_i$  in the binary expansion of n such that  $\varepsilon_{i+1}\varepsilon_i = 10$  for  $i \ge 1$  and  $\varepsilon_1\varepsilon_0 = 11$ . Then

$$D(n) = (-1)^{\delta(n)}.$$
 (2.5)

#### Proof

This is true for n < 4.

If it is true for  $0 \le n < 2^{k+1}$  then by (2.3) it is true for  $2^{k+1} + n$  with  $n < 2^k$  because for  $n = [v]_2$  we get  $2^{k+1} + n = [10v]_2$  and  $\delta(2^{k+1} + n) = \delta(n) + 1$ .

By (2.4) it is also true for  $2^{k} + n = [1v]_{2}$  because  $D(2^{k+1} + 2^{k} + n) = D(n)$  and  $\delta([11v]_{2}) = \delta([1v]_{2})$ .

### Examples

For n = 9 we have  $9 = [1001]_2$  and thus  $\delta(9) = 1$ .

For n = 15 we get  $\delta(15) = 1$  because  $15 = [1111]_2$ . There is no pair 10 for  $i \ge 1$  but 1 pair 11 for i = 0.

### Theorem 2.5

The Hankel determinants  $D(n) = \det \left(A_{i+j}\right)_{i,j=0}^{n-1}$  satisfy

$$D(2n) = (-1)^{\binom{n}{2}} D(n),$$
  

$$D(2n+1) = (-1)^{\binom{n+1}{2}} D(n),$$
  

$$D(0) = 1$$
(2.6)

#### Proof

Let us give two different proofs.

1) Let  $(M_{i,j})_{i,j=0}^{n-1}$  be a matrix for which  $M_{i,j} = 0$  whenever i + j is odd. Then (cf. e.g.[4])

$$\det\left(M_{i,j}\right)_{i,j=0}^{n-1} = \det\left(M_{2i,2j}\right)_{i,j=0}^{\left\lfloor\frac{n-1}{2}\right\rfloor} \det\left(M_{2i+1,2j+1}\right)_{i,j=0}^{\left\lfloor\frac{n-2}{2}\right\rfloor}.$$
(2.7)

Choose  $M_{i,j} = A_{i+j}$ . Then

$$\begin{pmatrix} M_{2i,2j} \end{pmatrix}_{i,j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} = \begin{pmatrix} A_{2i+2j} \end{pmatrix}_{i,j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} = \begin{pmatrix} a_{i+j} \end{pmatrix}_{i,j=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor}, \begin{pmatrix} M_{2i+1,2j+1} \end{pmatrix}_{i,j=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} = \begin{pmatrix} A_{2i+2j+2} \end{pmatrix}_{i,j=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor} = \begin{pmatrix} A_{i+j} \end{pmatrix}_{i,j=0}^{\left\lfloor \frac{n-2}{2} \right\rfloor}$$

because  $2i + 2j = 2^{k+1} - 2$  implies  $i + j = 2^k - 1$  and  $2i + 2j + 2 = 2^{k+1} - 2$  implies  $i + j = 2^k - 2$ .

Thus (2.7) gives 
$$D(n) = d\left(\left\lfloor \frac{n+1}{2} \right\rfloor\right) D\left(\left\lfloor \frac{n}{2} \right\rfloor\right)$$
 which gives (2.6), because  $d(n) = (-1)^{\binom{n}{2}}$ .

2) Another proof uses Corollary 2.4.

Let us first give another formulation of (2.6):

$$D(2n) = D(n) \text{ if } n \equiv 0,1 \mod 4, D(2n) = -D(n) \text{ if } n \equiv 2,3 \mod 4,$$
  

$$D(2n+1) = D(n) \text{ if } n \equiv 0,3 \mod 4, D(2n+1) = -D(n) \text{ if } n \equiv 1,2 \mod 4,$$
  

$$D(0) = 1$$
(2.8)

Let now  $n = [v\varepsilon_1\varepsilon_0]_2$ . Then  $2n = ([v\varepsilon_1\varepsilon_00]_2)$  and  $2n + 1 = ([v\varepsilon_1\varepsilon_01]_2)$ .

The assertion for 2n follows from

$$\delta([v0\varepsilon_0]_2) = \delta([v0\varepsilon_0]_2), \ \delta([v10]_2) + 1 = \delta([v100]_2) \text{ and } \delta([v11]_2) - 1 = \delta([v110]_2).$$

The assertion for 2n+1 follows from

 $\delta([v00]_2) = \delta([v001]_2), \ \delta([v01]_2) + 1 = \delta([v011]_2), \ \delta([v10]_2) + 1 = \delta([v101]_2), \text{ and}$  $\delta([v11]_2) = \delta([v111]_2).$ 

## Remark

As already mentioned a result by R. Bacher [1] implies that  $D(n) = \prod_{j=0}^{n-1} S(j)$  where  $(S(n))_{n\geq 0} = (1,1,-1,1,1,-1,-1,1,1,1,\cdots)$  is the paperfolding sequence defined by

$$S(2n) = (-1)^{n},$$
  

$$S(2n+1) = S(n),$$
  

$$S(0) = 1.$$
  
(2.9)

This can easily be verified since |D(n)| = 1 implies S(n) = D(n)D(n+1). By (2.6) we get

$$S(2n) = D(2n)D(2n+1) = (-1)^n D(n)^2 = (-1)^n,$$
  

$$S(2n+1) = D(2n+1)D(2n+2) = D(n)D(n+1) = S(n)$$

To obtain further information let us compare the above approach to Hankel determinants with the approach via orthogonal polynomials and continued fractions (cf. e.g. [3], [7]).

Let me sketch the relevant results: Let  $(u_n)_{n\geq 0}$  be a given sequence. Define a linear functional L on the polynomials by  $L(x^n) = \frac{u_n}{u_0}$ . If  $H_n = \det(u_{i+j})_{i,j=0}^{n-1} \neq 0$  for each n, then there exists a (uniquely determined) sequence of monic polynomials  $(p_n(x))_{n\geq 0}$  with deg  $p_n = n$  such that  $L(p_n p_m) = 0$  for  $m \neq n$  and  $L(p_n^2) \neq 0$ . We call these polynomials  $p_n$  orthogonal with respect to L. By Favard's theorem there exist (uniquely determined) numbers  $s_n$  and  $t_n$  such that  $p_n(x) = (x - s_{n-1}) p_{n-1}(x) - t_{n-2} p_{n-2}(x)$  for all n. The numbers  $t_n$  are given by  $t_n = \frac{H_n H_{n+2}}{H_{n+1}^2}$ . These give rise to the continued fraction

$$\sum_{n\geq 0} u_n z^n = \frac{u_0}{1 - s_0 z - \frac{t_0 z^2}{1 - s_1 z - \frac{t_1 z^2}{1 - \ddots}}}.$$
(2.10)

Let us suppose that  $u_0 = 1$ . Then the matrix  $H_n = (u_{i+j})_{i,j=0}^{n-1}$  has a unique canonical decomposition

$$H_n = A_n D_n(t) A_n^t \tag{2.11}$$

where  $A_n = (a(i, j))_{i,j=0}^{n-1}$  is a lower triangular matrix with diagonal a(i,i) = 1 and  $D_n(t)$  is the diagonal matrix with entries  $d_{i,i}(t) = \prod_{k=0}^{i-1} t_k$ .

The entries a(i, j) satisfy

$$a(i, j) = a(i-1, j-1) + s_j a(i-1, j) + t_j a(i-1, j+1)$$
(2.12)

with a(0, j) = [j = 0] and a(n, -1) = 0. See e.g. [7], (2.30). Let us consider the decomposition (2.11) of the matrix  $H_n = (a_{i+j})_{i,j=0}^{n-1}$ .

Let  $(s(n))_{n\geq 0} = (1,1,-1,1,-1,-1,-1,1,\cdots)$  satisfy  $s(2n) = (-1)^n s(n)$  and s(2n+1) = s(n) with s(0) = 1 and let  $D_n(s)$  be the diagonal matrix  $D_n(s) = (s(i)[i=j])_{i,j=0}^{n-1}$ . Let

 $D_n(t) = \left((-1)^i [i=j]\right)_{i,j=0}^{n-1}$ . Let  $x^* \in \{0,1\}$  be the residue modulo 2 of the number x and define  $B_n = (h(i-i))^{n-1} = \left(s(i) \left(2i+1\right)^*\right)^{n-1}$ 

define  $B_n = (b(i, j))_{i,j=0}^{n-1} = \left(s(i) {\binom{2i+1}{i-j}}^*\right)_{i,j=0}^{n-1}$ .

In [1], Theorem 1.2 it is shown that  $H_n = D_n(s)B_nD_n(t)B_n^tD_n(s)$ .

Note that b(i,i) = s(i). Therefore we get the canonical decomposition

$$H_n = A_n D_n(t) A_n^t \tag{2.13}$$

with

$$A_n = \left(a(i,j)\right)_{i,j=0}^{n-1} = D_n(s)B_n D_n(s), \qquad (2.14)$$

i.e.  $a(i, j) = s(i)s(j) {\binom{2i+1}{i-j}}^*$ .

For example we have for  $H_4$ 

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 0 & -1 & 1 & 0 \\ 1 & 1 & -1 & 1 \end{pmatrix} \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix} \begin{pmatrix} 1 & 1 & 0 & 1 \\ 0 & 1 & -1 & 1 \\ 0 & 0 & 1 & -1 \\ 0 & 0 & 0 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 & 0 & 1 \\ 1 & 0 & 1 & 0 \\ 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \end{pmatrix}$$

It is clear that (2.13) implies (0.1).

For the aerated sequence  $(u_0, 0, u_1, 0, u_2, 0, \cdots)$  we get  $s_n = 0$  for all *n*. In this case we write  $T_n$  instead of  $t_n$ .

Since  $D(n) \neq 0$  for all *n* and moreover  $D(n) = \pm 1$  we write in this case

$$T_n = (-1)^{\tau_n} = D(n)D(n+2) = (-1)^{\delta(n)+\delta(n+2)}$$
(2.15)

for  $\tau_n \in \{0,1\}$ . We also have  $T_n = D(n)D(n+2) = D(n)D(n+1)D(n+2) = S(n)S(n+1).$ 

[1] Theorem 10.1 implies that if we define a sequence  $(v(n)) = (1,1,1,1,-1,1,1,1,-1,-1,-1,\cdots)$ satisfying v(2n+1) = v(n),  $v(4n) = (-1)^n v(2n)$ , v(4n+2) = v(2n) and v(0) = 1 then

$$\left(A_{i+j}\right)_{i,j=0}^{n-1} = \left(a_{i+j+1}\right)_{i,j=0}^{n-1} = C_n D_n(t) C_n^t$$
(2.16)

Here  $D_n(t)$  is the diagonal matrix with entries S(i) and  $C_n = (c(i, j))_{i,j=0}^{n-1}$  with

$$c(i, j) = \left(\frac{2i+2}{i-j}\right)^* v(i)v(j).$$

By (2.10) and z in place of  $z^2$  we get the continued fraction (cf. [1])

$$\sum_{k} z^{2^{k}-1} = \frac{1}{1 - \frac{T_0 z}{1 - \frac{T_1 z}{1 -$$

### Remark

The corresponding orthogonal polynomials  $p_n(x)$  are 1, x,  $x^2 - 1$ ,  $x^3$ ,  $x^4 + x^2 - 1$ ,  $x^5 - x$ ,  $x^6 + x^4 - 1$ ,  $x^7, \cdots$ . They satisfy  $p_n(x) = xp_{n-1}(x) - T_{n-2}p_{n-2}(x)$  and  $L(x^n) = A_n$ .

### Theorem 2.6

The numbers  $T_n$  satisfy

$$T_{2n} = T_{2n-1}T_{n-1},$$
  

$$T_{2n+1} = -T_{2n},$$
  

$$T_0 = 1, T_1 = -1.$$
  
(2.18)

#### Proof

By (2.6) we get

$$T_{2n} = D(2n)D(2n+2) = (-1)^{\binom{n}{2} + \binom{n+1}{2}} D(n)D(n+1)$$
$$T_{2n-1} = D(2n-1)D(2n+1) = (-1)^{\binom{n}{2} + \binom{n+1}{2}} D(n-1)D(n)$$

implies  $T_{2n}T_{2n-1} = D(n-1)D(n+1) = T_{n-1}$ .

The second assertion follows from

$$T_{2n}T_{2n+1} = (-1)^{\binom{n}{2} + \binom{n+1}{2}} D(n)D(n+1)(-1)^{\binom{n+1}{2} + \binom{n+2}{2}} D(n)D(n+1) = (-1)^{\binom{n}{2} + \binom{n+2}{2}} = -1.$$

#### Remark

OEIS A104977 states that the numbers  $T_n$  which occur in the continued fraction (2.17) satisfy  $T_n = (-1)^{b(n+2)+1}$ , if b(n) denotes the number of "non-squashing partitions of n into distinct

parts". As has been shown in [9] the numbers b(n) of non-squashing partitions of n into distinct parts satisfy

$$b(2m) = b(2m-1) + b(m) - 1,$$
  
 $b(2m+1) = b(2m) + 1.$ 

Since b(2) = 1 and b(3) = 2 we get  $T_n = (-1)^{b(n+2)+1}$  by comparing with (2.18).

Let us now obtain some further properties of the sequence  $(T_n)_{n>0}$ .

For  $2^k \le n < 2^{k+1} - 2$  we have by (2.1)

.

$$D(n) = (-1)^{n} D(2^{k+1} - n - 1),$$
  

$$D(n+1) = (-1)^{n+1} D(2^{k+1} - n - 2),$$
  

$$D(n+2) = (-1)^{n} D(2^{k+1} - n - 3).$$

This implies

$$\frac{D(n)D(n+2)}{D(n+1)^2} = \frac{D(2^{k+1}-n-1)D(2^{k+1}-n-3)}{D(2^{k+1}-n-2)^2}.$$

Therefore we have

$$T_n = T_{2^{k+1}-3-n} \tag{2.19}$$

for  $2^k \le n \le 2^{k+1} - 3$ .

By  $T_{2n+1} = -T_{2n}$  we only need to consider  $n \equiv 0 \mod 2$  or  $n \equiv 0, 2 \mod 4$ .

For  $n \equiv 0 \mod 4$  we get

$$T_{4n} = (-1)^n, (2.20)$$

because

$$T_{4n} = D(4n)D(4n+2) = (-1)^n D(2n)(-1)^{\binom{2n+1}{2}} D(2n+1) = (-1)^n D(2n)(-1)^{\binom{2n+1}{2}} (-1)^{\binom{n+1}{2}} D(n)$$
$$= (-1)^{n+n+\binom{n+1}{2}+\binom{n}{2}} = (-1)^n.$$

Then we get

$$T_{4n+2} = (-1)^{n+1} T_{2n}.$$
 (2.21)

for  $T_{4n+2} = T_{4n+1}T_{2n} = (-1)^{n+1}T_{2n}$ .

To compute  $T_{4n+2}$  we look at  $T_{8n+2}$  and  $T_{8n+6}$ .

$$T_{8n+2} = (-1)^{n+1}.$$
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(2.22)

because by (2.20)  $T_{8n+2} = T_{8n+1}T_{4n} = -T_{8n}T_{4n} = (-1)^{n+1}.$ 

Now we claim that for  $k \ge 2$ 

$$T_{2^{k+1}n+2^{k}-2} = (-1)^{n+1}.$$
(2.23)

By induction, (2.18) and (2.20) we get

$$T_{2^{k+1}n+2^{k}-2} = T_{2^{k+1}n+2^{k}-3} T_{2^{k}n+2^{k-1}-2} = -T_{2^{k+1}n+2^{k}-4} T_{2^{k}n+2^{k-1}-2} = (-1)^{n} T_{4(2^{k-1}n+2^{k-2}-1)} = (-1)^{n-1}$$

As special case we get  $T_{2^{k}-2} = -1$  for  $k \ge 2$ .

This gives

#### Theorem 2.7

The numbers  $T_n$  satisfy

$$T_n = T_{2^{k+1}-3-n}$$
 for  $2^k \le n \le 2^{k+1} - 3, k \ge 2,$  (2.24)

and

$$T_{4n} = (-1)^n,$$
  

$$T_{2^{k+1}n+2^k-2} = (-1)^{n+1} \text{ for } k \ge 2.$$
(2.25)

Together with  $T_0 = 1$  and  $T_1 = -1$  this gives another view on the structure of the sequence  $(T_n)$ .

The sequence begins with  $T_0, T_1, -1, 1, T_1, T_0, -1, 1, T_0, T_1, 1, -1, T_1, T_0, -1, 1, \cdots$ 

 $T_{4n} = (-1)^n$  and  $T_{4n+1} = -T_{4n}$  gives a part  $T_0, T_1, \cdot, \cdot, T_1, T_0, \cdot, \cdot, T_0, T_1, \cdot, \cdot, T_1, T_0, \cdot, \cdot, \cdots$  with period 8.  $T_{8n+2} = (-1)^{n+1}$  gives  $\cdot, \cdot, -1, 1, \cdot, \cdot, \cdot, \cdot, \cdot, 1, -1, \cdots$  with period 16,  $T_{8n+2} = (-1)^{n+1}$  gives a periodic part with period 32 of

 $T_{16n+8-2} = (-1)^{n+1}$  gives a periodic part with period 32, etc.

Let more generally  $M_k = T_0, \dots, T_{2^k-3}$  be a beginning block and  $\overline{M}_k = T_{2^k-3}, \dots, T_0$  this block in reverse order then we get

$$M_k, -1, 1, M_k, -1, 1, M_k, 1, -1, M_k, -1, 1, \cdots$$

R. Bacher [1] gives a simpler formulation of (2.23) which (in our notation) can be summarized as

$$T_{2n+1} = -T_{2n},$$

$$T_{4n} = (-1)^{n},$$

$$T_{8n+2} = (-1)^{n+1},$$

$$T_{8n+6} = T_{4n+2}.$$
(2.26)

To show that this is equivalent it suffices to show that this implies  $T_{2^{k+1}n+2^k-2} = (-1)^{n+1}$  for k > 3.

This follows by induction from

 $T_{2^{k+1}n+2^{k}-2} = T_{2^{3}(2^{k-2}n+2^{k-3}-1)+6} = T_{2^{2}(2^{k-2}n+2^{k-3}-1)+2} = T_{2^{k}n+2^{k-1}-2}.$ 

Let us recall (cf. [3]) that there is a simple relation between  $t_n$  and  $T_n$ .

$$t_{n} = T_{2n}T_{2n+1},$$
  

$$s_{0} = T_{0},$$
  

$$s_{n} = T_{2n-1} + T_{2n}.$$
  
(2.27)

This gives  $t_n = -1$  and  $s_0 = 1$ .

The first terms of the sequence  $(s_n)$  are  $(s_n)_{n\geq 0} = (1, -2, 0, 0, 2, 0, -2, 0, 2, -2, 0, \cdots)$ .

In terms of the paperfolding sequence S(n) we get for n > 0  $s_n = D(2n-1)D(2n+1) + D(2n)D(2n+2) = S(2n-1)S(2n) + S(2n)S(2n+1)$ = S(2n)(S(2n-1) + S(2n+1)).

By (2.10) this gives another continued fraction for  $\sum_{k\geq 0} z^{2^{k}-1}$  (cf. [1], Theorem1.4):

$$\sum_{k\geq 0} z^{2^{k}-1} = \frac{1}{1-z+\frac{z^{2}}{1+2z+\frac{z^{2}}{1+\frac{z^{2}}{\cdot}}}}.$$
(2.28)

### 3. Hankel determinants of shifted Catalan numbers modulo 2

Let  $m \ge 2$ . Consider the Hankel determinants

$$d(n,m) = \det\left(a_{i+j+m}\right)_{i,j=0}^{n-1}$$
(3.1)

of the sequence  $(a_{n+m})_{n\geq 0}$ .

Let us give some examples:

$$(d(n,2))_{n\geq 0} = (1,0,-1,0,1,0,-1,0,\cdots), (d(n,3))_{n\geq 0} = (1,1,0,0,-1,1,0,0,-1,-1,0,0)$$

## Theorem 3.1.

Let  $1 \le 2^{K} < m \le 2^{K+1}$ . Then  $d(n,m) = \det(a_{i+j+m})_{i,j=0}^{n-1} = \pm 1$  if  $n \equiv 0 \mod 2^{K+1}$  or  $n \equiv -m \mod 2^{K+1}$  and d(n,m) = 0 else.

## Remark

It is well known that  $\det (C_{i+j+m})_{i,j=0}^{n-1} = H_{n,m} = \prod_{j=1}^{m-1} \prod_{i=1}^{j} \frac{2n+i+j}{i+j}.$ 

Therefore we get

### **Corollary 3.2**

Let 
$$2^{K} < m \le 2^{K+1}$$
. Then  $\prod_{j=1}^{m-1} \prod_{i=1}^{j} \frac{2n+i+j}{i+j} \equiv 1 \mod 2$  if and only if  $n \equiv 0 \mod 2^{K+1}$  or  $n \equiv -m \mod 2^{K+1}$ .

It would be nice to find a direct proof of this Corollary.

For the proof of Theorem 3.1 we need some more information.

### Lemma 3.3.

Let  $0 \le 2^k - m < n < 2^k$  for some k. Then

$$d(n,m) = 0.$$
 (3.2)

### Proof.

The matrix  $(a_{i+j+m})_{i,j=0}^{n-1}$  contains the vanishing row  $(a_{2^k}, a_{2^{k+1}}, \dots, a_{2^{k+n-1}})$ because  $m \le 2^k$  and  $2^k + n - 1 < 2^{k+1} - 1$ .

Let for example m = 3 and n = 7. Then

Recall that a permutation  $\pi$  is m-nimble if for all i in its domain  $i + \pi(i) = 2^k - m - 1$  for some k. An m-nimble permutation can only exist if the last row of  $(a_{i+j+m})$  contains an element of the form  $2^k - 1 - m$ . For fixed m, n there can be at most one of the numbers m+n-1+i with  $0 \le i \le n-1$  such that  $m+n-1+i = 2^k - 1$  because the extreme case would be  $m+n-1 = 2^{k-1}$  and  $m+2n-2 = 2^{k+1} - 1$  which is impossible. But it is possible that all elements of the last row are 0. For example in  $(a_{i+j+3})_{i,j=0}^5$  the last row is  $(a_5, a_6, a_7, a_8, a_9) = (0, 0, 0, 0, 0)$  because none of the numbers i+4 for  $5 \le i \le 9$  is a power of 2.

Let us first consider the case  $m \equiv 1 \mod 2$ .

**3.1.**  $m \equiv 1 \mod 2$ . Let  $m = 2r + 1 \ge 3$ .

For  $n = 2^{k-1} - r$  an *m*-nimble permutation gives  $\pi(n-1) = n-1 = 2^{k-1} - r - 1$  and

for  $n = 2^{k-1} - r + j$  it implies  $\pi(n-1) = 2^{k-1} - r - 1 - j = n - 1 - 2j$ .

Thus we get an *m*-nimble permutation  $\pi$  on the interval [n-1-2j, n-1] which satisfies  $\pi(n-1-i) = \pi(n-1)+i$  for  $0 \le i \le 2j$ .

As above  $\pi$  is uniquely determined and therefore we get

## Lemma 3.4.

Let m = 2r+1 and k be given. Then for  $0 \le j \le 2^{k-1} - r - 1$ 

$$d\left(2^{k-1}-r+j,m\right) = \left(-1\right)^{\binom{2\,j+1}{2}} d\left(2^{k-1}-r-j-1,m\right). \tag{3.4}$$

In example (3.3) we have m = 3 and  $n = 7 = 2^3 - 1 = 2^3 - r$ . Thus  $\pi$  is the identity on the element  $\{6\}$ . In this case d(7,3) = 0 because d(6,3) = 0 since this matrix has a row of zeroes.

#### Lemma 3.5.

Let m = 2r + 1. If  $a \le 2^k - m$  for some k we have

$$d(2^{k}+a,m) = (-1)^{\binom{2a+2r+1}{2}} d(2^{k}-a-m,m)$$

Proof

$$d\left(2^{k}+a,m\right) = d\left(2^{k}-r+a+r,m\right) = (-1)^{\binom{2a+2r+1}{2}}d\left(2^{k}-r-a-r-1,m\right)$$
$$= (-1)^{\binom{2a+2r+1}{2}}d\left(2^{k}-a-m,m\right).$$

#### **Corollary 3.6.**

Let m = 2r + 1. Then  $d(n,m) \neq 0$  if and only if  $n \equiv 0, -m \mod 2^{K+1}$ , where  $1 \le 2^{K} < m \le 2^{K+1}$ .

#### Proof

By Lemma 3.3 and Lemma 3.4 the determinants d(n,m) for  $2^k - r \le n \le 2^{k+1} - r - 1$  can be reduced those for  $2^{k-1} - r \le n \le 2^k - r - 1$ .

By induction we need only consider the case k = K + 1.

If  $1 \le n < 2^{K+1} - m$  then the first row of  $(a_{i+j+m})_{i,j=0}^{n-1}$  is  $(a_m, a_{m+1}, \dots, a_{m+n-1})$  and since  $m+n-1 < 2^{K+1}-1$  all terms vanish.

If  $m = 2^{K+1} - 1$  then this is trivially true because there is no such *n*.

If  $2^{K+1} - m < n < 2^{K+1}$  then the row  $(a_{2^{K+1}}, a_{2^{K+1}+1}, \cdots, a_{2^{K+1}+n-1})$  vanishes because  $2^{K+1} + n - 1 < 2^{K+1} + 2^{K+1} - 1 = 2^{K+2} - 1.$ 

For  $n = 2^{K+1} - m$  we have by Lemma 3.4

$$d\left(2^{K+1}-m,m\right) = d\left(2^{K}-r+2^{K}-r-1,m\right) = (-1)^{\binom{2^{K+1}-m}{2}}.$$

For  $n = 2^{K+1}$  we get

$$d(2^{K+1},m) = d(2^{K}-r+r,m) = (-1)^{\binom{m}{2}} d(2^{K}-r-r-1,m) = (-1)^{\binom{m}{2}} d(2^{K}-m,m).$$

**3.2.**  $m \equiv 0 \mod 2$ .

Consider now the case  $m = 2r \ge 2$ .

If  $n = 2^{k-1} - r + j$  for some k with  $1 \le j \le 2^{k-1} - r$  we get  $\pi(n-1) = 2^{k-1} - r - j = n - 2j$ because  $i + \pi(i) = 2^k - (2r) - 1$ . Now define  $\pi$  on the interval [n-2j, n-1] by  $\pi(n-1-i) = \pi(n-1)+i$  for  $0 \le i \le 2j-1$ . This implies that

$$d\left(2^{k-1}-r+j,m\right) = \left(-1\right)^{\binom{2j}{2}} d\left(2^{k-1}-r-j,m\right)$$
(3.5)

for  $1 \le j \le 2^{k-1} - r$ .

Therefore the determinants d(n,m) for  $2^k - r + 1 \le n \le 2^{k+1} - r$  can be reduced to those for  $2^{k-1} - r + 1 \le n \le 2^k - r$ .

Therefore it suffices to consider the case k = K.

For m = 2 we have K = 1 and d(0, 2) = 1 and d(1, 2) = 0.

If  $n \equiv 0 \mod 2$  then d(n, 2) can be reduced by (3.5) to d(0, 2) = 1 and if  $n \equiv 1 \mod 2$  to d(1, 2) = 0.

For m = 2r with r > 1 we get

$$d\left(2^{K+1}-m,m\right) = d\left(2^{K}-r+2^{K}-r,m\right) = (-1)^{\binom{2^{K+1}-m}{2}} = (-1)^{r}.$$

For  $n = 2^{K+1}$  we get

$$d(2^{K+1},m) = d(2^{K+1}-r+r,m) = (-1)^{\binom{m}{2}} d(2^{K+1}-r-r,m) = (-1)^{\binom{m}{2}} d(2^{K+1}-m,m) = 1.$$

This gives

### Lemma 3.7.

Let 
$$1 \le 2^{K} < m = 2r \le 2^{K+1}$$
. Then  $d(n,m) = \det \left(a_{i+j+m}\right)_{i,j=0}^{n-1} = \pm 1$  if  $n \equiv 0 \mod 2^{K+1}$  or  $n \equiv -m \mod 2^{K+1}$ , and  $d(n,m) = 0$  else.

Theorem 3.7 and Corollary 3.6 imply Theorem 3.1.

### Remark

With the condensation method (cf. [7], (2.16)) we get more precisely

$$(d(n,2)) = (1,0,-1,0,1,0,-1,\cdots).$$
 (3.6)

This method gives

$$d(n,0)d(n-2,2) = d(n-1,2)d(n-1,0) - d(n-1,1)^{2}.$$

Since  $d(n,0) = (-1)^{\binom{n}{2}}$  and  $d(n-1,1)^2 = 1$  we get

$$d(n,2) = (-1)^n d(n-1,2) + (-1)^{\binom{n}{2}}$$
 with initial value  $d(0,2) = 1$ .

This gives  $d(2n, 2) = (-1)^n$  and d(2n+1, 2) = 0.

In the general case computer experiments lead to

## **Conjecture 3.8**

Let  $1 \le 2^{K} < m \le 2^{K+1}$ . For m = 2r > 2 we have

$$d(2^{K+1}n,m) = 1,$$
  

$$d(2^{K+1}n-m,m) = (-1)^{r}.$$
(3.7)

For  $m = 2r + 1 \ge 3$  we have

$$d(2^{K+1}n,m) = d(2^{K+1}n,1),$$
  

$$d(2^{K+1}n-m,m) = (-1)^{n+\varepsilon(m)}d(2^{K+1}n-m,1),$$
(3.8)

where  $\varepsilon(m) \in \{0,1\}$ .

### 4. A slightly more general case

Let  $(x_k)_{k\geq 0}$  be an arbitrary sequence of numbers or indeterminates and define a sequence  $(a_n)_{n\geq 0}$  by  $a_n = x_n$  if  $n = 2^k - 1$  and  $a_n = 0$  else.

### Theorem 4.1.

Let  $a_n = x_n$  if  $n = 2^k - 1$  and  $a_n = 0$  else.

Let 
$$d(n) = \det\left(a_{i+j}\right)_{i,j=0}^{n-1}$$
,  $\alpha(n) = 2^{\lceil \log_2(n) \rceil} - 1$  and  $\beta(n) = 2n - 1 - \alpha(n)$ .

Then

$$d(n) = (-1)^{\binom{\beta(n)}{2}} x_{\alpha(n)}^{\beta(n)} d(n - \beta(n)).$$
(4.1)

#### Proof

By convention d(0) = 1. For n = 1 we have  $\alpha(1) = 0$ ,  $\beta(1) = 1$  and

 $d(1) = x_0 = (-1)^{\binom{1}{2}} x_{\alpha(1)}^{\beta(1)} d(1 - \beta(1)).$  Therefore (4.1) is true.

For n > 1 choose k such that  $2^{k-1} < n \le 2^k$ . Then  $k-1 < \log_2(n) \le k$  and  $\alpha(n) = 2^k - 1$ . As in the proof of Theorem 1 we find a permutation  $\pi$  of the interval  $\left[2^k - n, n-1\right] = \left[\alpha(n) + 1 - n, n-1\right]$  such that  $i + \pi(i) = \alpha(n)$ , which implies that  $a_{i+\pi(i)} = x_{\alpha(n)}$ . Since there are  $\beta(n) = 2n - 1 - \alpha(n)$  elements in the interval  $\left[\alpha(n) + 1 - n, n-1\right]$  we get (4.1).

Let us consider an example. The Hankel matrix  $(a_{i+j})_{i,j=0}^4$  is

$\int x_0$	$x_1$	0	$x_3$	0 )
$x_1$	0	<i>x</i> <sub>3</sub>	0	0
0	<i>x</i> <sub>3</sub>	0	0	0
$x_3$	0	0	0	<i>x</i> <sub>7</sub>
$\left( 0 \right)$	0	0	<i>x</i> <sub>7</sub>	0)

We have  $\alpha(5) = 7$  because  $2^2 < 5 \le 2^3$  and  $\beta(5) = 10 - 1 - 7 = 2$ . We get the permutation  $\pi = 43$  with sgn $(\pi) = -1$  and  $d(4) = -x_7^2 d(3)$ .

For n=3 we get  $\alpha(3) = 3$  and  $\beta(3) = 6 - 1 - 3 = 2$ . This gives  $d(3) = -x_3^2 d(1)$ .

Thus we finally get  $d(5) = x_0 x_3^2 x_7^2$ .

The sign is  $(-1)^{\binom{5}{2}} = 1$ . This can also be obtained from  $x_0 x_3^2 x_7^2$  as  $(-1)^{\binom{1}{2} + \binom{2}{2} + \binom{2}{2}} = (-1)^2 = 1$ .

The first terms of the sequence  $(d(n))_{n\geq 0}$  are

1,  $x_0$ ,  $-x_1^2$ ,  $-x_0x_3^2$ ,  $x_3^4$ ,  $x_0x_3^2x_7^2$ ,  $-x_1^2x_7^4$ ,  $-x_0x_7^6$ ,  $x_7^8$ ,...

By (4.1) see that  $d(1) = x_0$ ,  $d(2) = -x_1^2$  and  $d(2^k) = x_{2^{k-1}}^{2^k}$  for k > 1.

### Lemma 4.2

For  $k \ge 1$  we get

$$d(2^{k} + n) = (-1)^{n} x_{2^{k+1} - 1}^{2n} d(2^{k} - n)$$
(4.2)

for  $0 < n \le 2^k$ .

### Proof

By assumption we have  $2^{k} < 2^{k} + n \le 2^{k+1}$ . Therefore  $\alpha (2^{k} + n) = 2^{k+1} - 1$  and  $\beta (2^{k} + n) = 2n$ .

By (4.1) we get (4.2).

#### Lemma 4.3

For k > 1 we have

$$d(2^{k} - n) = (-1)^{n} x_{2^{k} - 1}^{2^{k} - 2^{n}} d(n)$$
(4.3)

for  $n \in \{0, 1, 2, \cdots, 2^k\}$ .

#### Proof

If  $n < 2^{k-1}$  then  $2^{k-1} < 2^k - n \le 2^k$  and we see that  $\beta(2^k - n) = 2^k - 2n$  and

$$d(2^{k}-n) = (-1)^{\binom{2^{k}-2n}{2}} x_{2^{k}-1}^{2^{k}-2n} d(n).$$

If  $n = 2^{k-1}$  then (4.3) is trivially true.

If  $n > 2^{k-1}$  then  $i = 2^k - n < 2^{k-1}$  and therefore  $d(2^k - i) = (-1)^{\binom{2^k - 2i}{2}} x_{2^{k-1}}^{2^k - 2i} d(i)$  or equivalently

$$d(n) = (-1)^{\binom{2n-2}{2}} x_{2^{k}-1}^{2n-2^{k}} d(2^{k}-n) \text{ which equals (4.3).}$$

Let e.g. k = 3.

d(8-n)	$x_{7}^{8}$	$-x_0 x_7^6$	$-x_1^2 x_7^4$	$x_0 x_3^2 x_7^2$	$x_{3}^{4}$	$-x_0 x_3^2$	$-x_{1}^{2}$	$x_0$	1
<i>d</i> ( <i>n</i> )	1	$x_0$	$-x_{1}^{2}$	$-x_0 x_3^2$	$x_{3}^{4}$	$x_0 x_3^2 x_7^2$	$-x_1^2 x_7^4$	$-x_0 x_7^6$	$x_{7}^{8}$
$\frac{d(8-n)}{d(n)}$	$x_{7}^{8}$	$-x_{7}^{6}$	$x_{7}^{4}$	$-x_{7}^{2}$	1	$-x_{7}^{-2}$	$x_{7}^{-4}$	$-x_{7}^{-6}$	$x_{7}^{-8}$

Let us write d(n) in the form  $d(n) = (-1)^{\sum_{i\geq 0}^{l} \binom{\lambda_i(n)}{2}} \prod_{i\geq 0} x_{2^i-1}^{\lambda_i(n)}$  for some integers  $\lambda_i(n)$ .

Then we get

# Theorem 4.4

$$\begin{aligned} \lambda_{k}(n) &= 0 \quad \text{for } 0 \leq n \leq 2^{k-1}, \\ \lambda_{k}(2^{k-1}+i) &= 2i \quad \text{for } 0 \leq i \leq 2^{k-1}, \\ \lambda_{k}(2^{k}+i) &= 2^{k} - 2i \quad \text{for } 0 \leq i \leq 2^{k-1}, \\ \lambda_{k}(n) &= 0 \quad \text{for } 2^{k} + 2^{k-1} \leq i \leq 2^{k+1}. \end{aligned}$$

$$(4.4)$$

and for  $n > 2^{k+1}$ 

$$\lambda_k(n) = \lambda_k(n \mod 2^{k+1}). \tag{4.5}$$

For example we get

$$\begin{aligned} \left(\lambda_0(n)\right)_{n\geq 0} &= (0,1,\cdots), \\ \left(\lambda_1(n)\right)_{n\geq 0} &= (0,0,2,0\cdots), \\ \left(\lambda_2(n)\right)_{n\geq 0} &= (0,0,0,2,4,2,0,0,\cdots). \end{aligned}$$

### Proof

Formula (4.4) is obvious from the above considerations. For example  $\lambda_k(2^k + i) = 2^k - 2i$  follows from Lemma 4.2 and Lemma 4.3 because  $\lambda_k(2^k + i) = \lambda_k(2^k - i) = 2^k - 2i$  and  $\lambda_k(i) = 0$  for  $i \leq 2^{k-1}$ .

Again from Lemma 4.2 and Lemma 4.3 we get  $\lambda_k (2^R + n) = \lambda_k (2^R - n) = \lambda_k (n)$  for  $n \le 2^R$ . By applying this several times we get (4.5).

Consider for example n = 11.

 $11 \equiv 1 = 2^{0} \mod 2 \text{ implies } \lambda_{0}(11) = 1,$   $11 \equiv 3 = 2^{1} + 1 \mod 2^{2} \text{ implies } \lambda_{1}(11) = \max(2^{1} - 2, 0) = 0,$   $11 \equiv 3 = 2^{2} - 1 \mod 2^{3} \text{ implies } \lambda_{2}(11) = \max(2^{2} - 2, 0) = 2,$   $11 \equiv 11 = 2^{3} + 3 \mod 2^{4} \text{ implies } \lambda_{3}(11) = \max(2^{3} - 6, 0) = 2,$   $11 \equiv 11 = 2^{4} - 5 \mod 2^{5} \text{ implies } \lambda_{4}(11) = \max(2^{4} - 10, 0) = 6.$ For  $k \ge 5$  we have  $11 = 2^{k} - (2^{k} - 11)$  and thus  $\lambda_{k}(11) = 0.$ 

Therefore we get  $d(11) = -x_0 x_3^2 x_7^2 x_{15}^6$ .

We know already that the sign is  $(-1)^{\binom{11}{2}} = -1$ , but now we can also derive this from the  $\lambda_k$  because

$$(-1)^{\binom{1}{2} + \binom{2}{2} + \binom{2}{2} + \binom{6}{2}} = (-1)^{0+1+1+15} = -1.$$

An immediate Corollary of Lemma 4.2 is

### Theorem 4.5

The sequence (d(n)) satisfies the recurrence

$$d(n) = (-1)^n x_{2^{k}-1}^{2n-2^k} d\left(2^k - n\right)$$
(4.6)

for  $1 < 2^{k-1} < n \le 2^k$  with initial values d(0) = 1,  $d(1) = x_0$  and  $d(2) = -x_1^2$ .

# **Corollary 4.6**

Let  $t_n = \frac{d(n)d(n+2)}{d(n+1)^2}$ . Then we have

$$t_{2n} = -\frac{x_1^2}{x_0^2},$$

$$t_{2^k n + 2^{k-1} - 1} = -\frac{x_0^2 x_{2^k - 1}^2}{x_{2^{k-1} - 1}^4} \quad \text{for } k > 1.$$
(4.7)

This can be proved in an analogous way as Theorem 5.7

As special case of Theorem 4.5 let us choose  $x_{\gamma_{k-1}} = x^k$ . Then we get

### **Corollary 4.7**

Let  $b_{2^{k}-1} = x^{k}$  and  $b_{n} = 0$  else. Then

$$d_{n} = \det\left(b_{i+j}\right)_{i,j=0}^{n-1} = (-1)^{\binom{n}{2}} x^{2a(n)}, \qquad (4.8)$$

where a(n) is the total number of 1's in the binary expansions of the numbers  $\leq n-1$ .

#### Proof

A search in OEIS led to the conjecture that  $d_n = (-1)^{\binom{n}{2}} x^{2a(n)}$ , where a(n) is the total number of 1's in the binary expansions of the numbers 0, 1, ..., n-1. (OEIS A000788). The following proof follows Darij Grinberg [6].

By (4.6) we have  $d_n = (-1)^n x^{2k(n-2^{k-1})} d_{2^k - n}$  for  $1 < 2^{k-1} < n \le 2^k$ .

Therefore it suffices to show that  $k(n-2^{k-1}) = a(n) - a(2^k - n)$ , the total number of 1's in the binary expansions of the numbers  $2^k - n$ ,  $2^k - n + 1$ ,  $\cdots$ , n-1.

Let the binary expansion of *n* be  $n = [\varepsilon_{k-1} \cdots \varepsilon_0]$ . Let us write all binary expansions with *k* digits  $\varepsilon_i = 0, 1$ .

The total number of 1's in the binary expansions of  $\{n, \dots, 2^k - 1\}$  is the total number of 0's in the binary expansions of  $\{2^k - 1 - n, \dots, 1, 0\}$  of length k which is  $(2^k - n)k - a(2^k - n)$ .

Thus  $a(2^k) - a(n) = (2^k - n)k - a(2^k - n)$ . Now  $a(2^k) = k2^{k-1}$  since each  $\varepsilon_i$  occurs  $2^{k-1}$  times.

Therefore we have  $a(n) - a(2^{k} - n) = k2^{k-1} - k(2^{k} - n) = k(n - 2^{k-1}).$ 

This proves (4.8) by induction since the initial values a(0) = a(1) = 0 and a(2) = 1 give  $d_0 = 1$ ,  $d_1 = 1$ , and  $d_2 = -x^2$ .

For example a(3) is the number of 1's in 1,10, i.e. a(3) = 2. Thus

$$d_{3} = \det \begin{pmatrix} 1 & x & 0 \\ x & 0 & x^{2} \\ 0 & x^{2} & 0 \end{pmatrix} = -x^{4+0} = x^{4}.$$

## Example 4.8

Let  $c(2^{k}-1) = x^{2^{k}-1}$  and c(n) = 0 else. Then

$$\mathbf{d}_{n} = \det\left(c(i+j)\right)_{i,j=0}^{n-1} = (-1)^{\binom{n}{2}} x^{2\binom{n}{2}}.$$
(4.9)

For example

$$\mathbf{d}_{4} = \det \begin{pmatrix} 1 & x & 0 & x^{3} \\ x & 0 & x^{3} & 0 \\ 0 & x^{3} & 0 & 0 \\ x^{3} & 0 & 0 & 0 \end{pmatrix} = x^{12} = x^{2\binom{4}{2}}$$

For the proof observe that  $\mathbf{d}_n = (-1)^n x^{\binom{2^k - 1}{2n - 2^k}} \mathbf{d}_{2^k - n}$  for  $1 < 2^{k - 1} < n \le 2^k$ .

It suffices to verify that  $2\binom{2^k-n}{2}+(2^k-1)(2n-2^k)=2\binom{n}{2}$ .

**5. The matrices**  $(a_{i+j+1})_{i,j=0}^{n-1}$ .

### Theorem 5.1

Let 
$$D(n) = \det(a_{i+j+1})_{i,j=0}^{n-1}$$
,  $\gamma(n) = 2^{\lfloor \log_2(n) \rfloor} - 1$  and  $\delta(n) = 2n - \gamma(n)$ .

Then

$$D(n) = x_{\gamma(n)}^{\delta(n)} (-1)^n D(\gamma(n) - n).$$
(5.1)

### Proof

For given n > 0 choose k such that  $2^{k-1} < n+1 \le 2^k$ . Then  $k-1 < \log_2(n+1) \le k$  and  $\gamma(n) = 2^k - 1$ .

Let  $\pi$  be a 1-nimble permutation. Then as above we see that  $\pi$  induces an order reversing permutation on the interval  $[2^k - 1 - n, n - 1]$ . Here we have  $i + \pi(i) = 2^k - 2$ .

Since there are  $\delta(n) = 2n - \gamma(n)$  elements in the interval  $[2^k - 1 - n, n - 1]$  we get (5.1) by induction.

### **Corollary 5.2**

The sequence D(n) satisfies

$$D(2^{k} + n) = (-1)^{n} x_{2^{k+1} - 1}^{2n+1} D(2^{k} - 1 - n)$$
(5.2)

for  $0 \le n < 2^k$ .

Let us compute the first values with this recursion:

$$D(0) = 1 D(1) = x_1$$
  

$$D(3) = -x_3^3 D(2) = x_1 x_3$$
  

$$D(0) = 1 D(1) = x_1 D(2) = x_1 x_3 D(3) = -x_3^3$$
  

$$D(7) = -x_7^7 D(6) = x_1 x_7^5 D(5) = -x_1 x_3 x_7^3 D(4) = -x_3^3 x_7$$

For example for  $n = 2^{k+1} - 1$  we have  $\gamma(n) = n$  and  $\delta(n) = n$ .

This implies for k > 1

$$D(2^{k}-1) = -x_{2^{k}-1}^{2^{k}-1}.$$
(5.3)

# Lemma 5.3

For  $0 \le n < 2^{k-1}$  we get

$$D(2^{k}+n) = -x_{2^{k+1}-1}^{2^{n+1}} x_{2^{k}-1}^{2^{k}-1-2^{n}} D(n).$$
(5.4)

#### Proof

By (5.2) we get

$$D(2^{k}+n) = (-1)^{n} x_{2^{k+1}-1}^{2n+1} D(2^{k}-1-n) = (-1)^{n} x_{2^{k+1}-1}^{2n+1} D(2^{k-1}+2^{k-1}-1-n)$$

Again by (5.2) we have  $D(2^{k-1}+2^{k-1}-1-n) = (-1)^{n+1} x_{2^{k}-1}^{2^{k}-1-2n} D(n).$ 

Thus 
$$D(2^{k}+n) = -x_{2^{k+1}-1}^{2n+1}x_{2^{k}-1}^{2^{k}-1-2n}D(n).$$

# Lemma 5.4

For  $2^{k-1} \le n < 2^k$  we get

$$D(2^{k}+n) = x_{2^{k+1}-1}^{2n+1} x_{2^{k}-1}^{2^{k}-1-2n} D(n).$$
(5.5)

#### Proof

$$D\left(2^{k}+2^{k-1}+i\right) = (-1)^{i} x_{2^{k+1}-1}^{2^{k}+2i+1} D\left(2^{k}-1-2^{k-1}-i\right) = (-1)^{i} x_{2^{k+1}-1}^{2^{k}+2i+1} D\left(2^{k-1}-1-i\right)$$
$$= (-1)^{i} x_{2^{k+1}-1}^{2^{k}+2i+1} (-1)^{i} x_{2^{k}-1}^{-2i-1} D\left(2^{k-1}+i\right)$$

which is equivalent with (5.5).

Let us write

$$D(n) = (-1)^{\sum_{i=1}^{\binom{\mu_{i}(n)}{2}}} \prod_{i\geq 1} x_{2^{i}-1}^{\mu_{i}(n)}.$$
(5.6)

Then  $\mu_k(n)$  only depends on the residue class modulo  $2^{k+2}$ .

## Theorem 5.5

Let  $0 \le i \le 2^{k-1} - 1$ . Then

$$\mu_{k}(i) = 0,$$
  

$$\mu_{k}(2^{k-1} + i) = 2i + 1,$$
  

$$\mu_{k}(2^{k} + i) = 2^{k} - 2i - 1,$$
  

$$\mu_{k}(2^{k} + 2^{k-1} + i) = 0.$$
  
(5.7)

For  $n \ge 2^{k+1}$  we have  $\mu_k(n) = \mu_k(n \mod 2^{k+1})$ .

# Proof

Formula (5.7) follows from (5.1).

By (5.4) we have 
$$\mu_k(2^R + n) = \mu_k(n)$$
 for  $R \ge k+1$  which implies  $\mu_k(n) = \mu_k(n \mod 2^{k+1})$ .

(I) 0

Thus for  $n < 2^k$  we have  $\mu_k(i) =$ 

Let us for example compute D(11).

 $11 \equiv 3 = 2 + 1 \mod 2^2$  implies  $\mu_1(11) = \max(2 - 1 - 2, 0) = 0$ ,

 $11 \equiv 3 = 2 + 1 \mod 2^3$  implies  $\mu_2(11) = \max(2+1,0) = 3$ ,

 $11 \equiv 11 = 2^3 + 3 \mod 2^4$  implies  $\mu_3(11) = \max(2^3 - 1 - 6, 0) = 1$ ,

 $11 \equiv 11 = 8 + 3 \mod 2^5$  implies  $\mu_4(11) = \max(6+1,0) = 7$ .

Therefore we get  $D(11) = (-1)^{\binom{3}{2} + \binom{1}{2} + \binom{7}{2}} x_3^3 x_7 x_{15}^7 = x_3^3 x_7 x_{15}^7.$ 

Let us now determine the numbers

$$T_n = \frac{D(n)D(n+2)}{D(n+1)^2}.$$
 (5.8)

From

$$D(2^{k} + 2^{k-1} - 2) = -x_{2^{k+1}-1}^{2^{k}-3} x_{2^{k}-1}^{3} D(2^{k-1} - 2),$$
  

$$D(2^{k} + 2^{k-1} - 1) = -x_{2^{k+1}-1}^{2^{k}-1} x_{2^{k}-1}^{1} D(2^{k-1} - 1),$$
  

$$D(2^{k} + 2^{k-1}) = x_{2^{k+1}-1}^{2^{k}+1} x_{2^{k}-1}^{-1} D(2^{k-1}),$$
  

$$D(2^{k} + 2^{k-1} + 1) = x_{2^{k+1}-1}^{2^{k}+3} x_{2^{k}-1}^{-3} D(2^{k} + 1)$$

we get

$$T_{2^{k}+2^{k-1}-2} = -T_{2^{k-1}-2}$$
 and  $T_{2^{k}+2^{k-1}-1} = -T_{2^{k-1}-1}$ .

For m > 0 and  $0 \le j \le 3$  we get by (5.4)

$$D(2^{k+m}+2^{k-1}-2+j) = -x_{2^{k+m+1}-1}^{2^{k}-3+2j}x_{2^{k+m}-1}^{2^{k+m}+3-2^{k}-2j}D(2^{k-1}-2+j)$$

which implies  $T_{2^{k+m}+2^{k-1}-2} = T_{2^{k-1}-2}$  and  $T_{2^{k+m}+2^{k-1}-1} = T_{2^{k-1}-1}$ .

The same argument using (5.4) gives  $T_{2^{k+m}+2^k+2^{k-1}-2+j} = T_{2^k+2^{k-1}-2+j} = -T_{2^{k-1}-2+j}$  for m > 1 and j = 0, 1.

There remains  $T_{2^{k+2}+2^{k+1}+2^k-2+j}$ . By (5.5) we get  $T_{2^{k+2}+2^{k+1}+2^k-2+j} = T_{2^{k+1}+2^k-2+j} = -T_{2^k-2+j}$ .

This gives

#### Theorem 5.6

The numbers  $T_{2^{k+1}n+2^k-2+j}$  satisfy

$$T_{2^{k+1}n+2^{k}-2+j} = (-1)^{n} T_{2^{k}-2+j}$$
(5.9)

for j = 0, 1.

The first terms of the sequence  $(T_n)_{n\geq 0}$  are

$$\frac{x_3}{x_1}, \quad -\frac{x_3}{x_1}, \quad -\frac{x_1x_7}{x_3^2}, \quad \frac{x_1x_7}{x_3^2}, \quad -\frac{x_3}{x_1}, \quad \frac{x_3}{x_1}, \quad -\frac{x_1x_{15}}{x_7^2}, \quad \frac{x_1x_{15}}{x_7^2}, \cdots$$

### Theorem 5.7

The numbers  $T_n$ ,  $n \ge 0$ , satisfy

$$T_{2n+1} = -T_{2n},$$

$$T_{4n} = (-1)^n \frac{x_3}{x_1}$$

$$T_{2^{k+1}n+2^k-1} = (-1)^n \frac{x_1 x_{2^{k+1}-1}}{x_{2^k-1}^2} \text{ for } k \ge 2.$$
(5.10)

## Proof

For  $2^{k-1} \le n < 2^k - 2$  we have

$$D(n) = (-1)^{n} x_{2^{k}-1}^{2n+1-2^{k}} D(2^{k}-n-1),$$
  

$$D(n+1) = (-1)^{n+1} x_{2^{k}-1}^{2n+3-2^{k}} D(2^{k}-n-2),$$
  

$$D(n+2) = (-1)^{n} x_{2^{k}-1}^{2n+5-2^{k}} D(2^{k}-n-3).$$

This implies

$$\frac{D(n)D(n+2)}{D(n+1)^2} = \frac{x_{2^{k}-1}^{2^{n+1}-2^k}D(2^k-n-1)x_{2^{k}-1}^{2^{n+5}-2^k}D(2^k-n-3)}{x_{2^{k}-1}^{2^{n+3}-2^k}D(2^k-n-2)x_{2^{k}-1}^{2^{n+3}-2^k}D(2^k-n-2)} = \frac{D(2^k-n-1)D(2^k-n-3)}{D(2^k-n-2)^2}.$$

Therefore we have

$$T_n = T_{2^k - 3 - n} \tag{5.11}$$

for  $2^{k-1} \le n \le 2^k - 3$ .

It remains to prove

$$T_{2^{k}-2} = -\frac{x_{1}x_{2^{k+1}-1}}{x_{2^{k}-1}^{2}},$$

$$T_{2^{k}-1} = \frac{x_{1}x_{2^{k+1}-1}}{x_{2^{k}-1}^{2}}$$
(5.12)

for  $k \ge 2$ .

By (5.4) we have

$$D(2^{k}+1) = -x_{1}x_{2^{k+1}-1}^{3}x_{2^{k}-1}^{2^{k}-3},$$
  

$$D(2^{k}) = -x_{2^{k+1}-1}x_{2^{k}-1}^{2^{k}-1}.$$
  
By (5.3)  $D(2^{k}-1) = -x_{2^{k}-1}^{2^{k}-1}$  and by (5.2)  $D(2^{k}-2) = x_{1}x_{2^{k}-1}^{2^{k}-3}.$ 

This gives (5.12).

Let us prove that  $T_{4n} = (-1)^n \frac{x_3}{x_1}$  and  $T_{4n+1} = (-1)^{n+1} \frac{x_3}{x_1}$ .

By induction we get using (5.11)

$$T_{4(2^{k}+j)} = T_{2^{k+3}-3-4(2^{k}+j)} = T_{4(2^{k}-j-1)+1} = (-1)^{2^{k}+j} \frac{x_{3}}{x_{1}}$$

$$T_{4(2^{k}+j)+1} = T_{2^{k+3}-3-4(2^{k}+j)-1} = T_{4(2^{k}-j-1)} = (-1)^{2^{k}+j+1} \frac{x_{3}}{x_{1}}.$$

## Remarks

Let us derive some connections between D(n) and d(n).

# Lemma 5.8

$$\frac{d(n)d(n+1)}{D(n)^2} = (-1)^n x_0.$$
(5.13)

## Proof

For  $0 < n < 2^{k}$  we have by (4.2)

$$d(2^{k}+n) = (-1)^{n} x_{2^{k+1}-1}^{2n} d(2^{k}-n),$$
  
$$d(2^{k}+n+1) = (-1)^{n+1} x_{2^{k+1}-1}^{2n+2} d(2^{k}-n-1)$$

and therefore

$$d(2^{k}+n)d(2^{k}+n+1) = -x_{2^{k+1}-1}^{4n+2}d(2^{k}-n)d(2^{k}-n-1).$$

By (5.2) we have

$$D(2^{k}+n)^{2} = x_{2^{k+1}-1}^{4n+2} D(2^{k}-1-n)^{2}.$$

This implies

$$h(2^{k}+n) = \frac{d(2^{k}+n)d(2^{k}+n+1)}{D(2^{k}+n)^{2}} = \frac{-d(2^{k}-n)d(2^{k}-n-1)}{D(2^{k}-1-n)^{2}} = -h(2^{k}-n-1)$$

for  $0 < n < 2^k$ . Further we have

$$h(2^{k}) = \frac{d(2^{k})d(2^{k}+1)}{D(2^{k})^{2}} = \frac{d(2^{k})(-1)x_{2^{k+1}-1}^{2}d(2^{k}-1)}{x_{2^{k+1}-1}^{2}D(2^{k}-1)^{2}} = -h(2^{k}-1).$$

Therefore we get

$$h\left(2^{k}+n\right)=-h\left(2^{k}-n-1\right)$$

for  $0 \le n < 2^k$  and  $k \ge 1$ . This gives by induction  $h(2^k + n) = -h(2^k - 1 - n) = (-1)^{2^k - n} x_0 = (-1)^{2^k + n} x_0.$ 

## Example 5.9

Let  $b_{2^{k}-1} = x^{k}$  and  $b_{n} = 0$  else. Then

$$D_{n} = \det\left(b_{i+j+1}\right)_{i,j=0}^{n-1} = \left(-1\right)^{\delta(n)} x^{a(n)+a(n+1)},$$
(5.14)

where a(n) denotes the total number of 1's in the binary expansions of the first n-1 positive integers.

This follow immediately from (4.8) and (5.13).

For the numbers  $T_n$  we get

$$T_n = (-1)^{\tau_n} x^{s_2(n+2)-s_2(n)}, (5.15)$$

where  $s_2(n)$  denotes the sum of digits of the binary expansion of n.

For

$$T_n = \frac{D(n)D(n+2)}{D(n+1)^2} = (-1)^{\tau_n} \frac{x^{a(n)+a(n+1)}x^{a(n+2)+a(n+3)}}{x^{2a(n+1)+2a(n+2)}} = (-1)^{\tau_n} x^{a(n+3)-a(n+2)-(a(n+1)-a(n))}$$
$$= (-1)^{\tau_n} x^{s_2(n+2)-s_2(n)}.$$

The first terms of  $(T_n)_{n\geq 0}$  are  $x, -x, -1, 1, -x, x, -\frac{1}{x}, \frac{1}{x}, x, -x, 1, -1, -x, x, -\frac{1}{x^2}, \frac{1}{x^2}, x, -x, \cdots$ .

By (5.10) we have

$$\begin{split} T_{2n+1} &= -T_{2n}, \\ T_{4n} &= (-1)^n x \\ T_{2^{k+1}n+2^k-1} &= (-1)^n \frac{x^{k+2}}{x^{2k}} = (-1)^n x^{2-k} & \text{for } k \ge 2 \end{split}$$

This is in accord with

$$s_2(2^{k+1}n+2^k+1)-s_2(2^{k+1}n+2^{k-1}+2^{k-2}+\cdots+1)=2-k.$$

# Example 5.10

Let  $c(2^{k}-1) = x^{2^{k}-1}$  and c(n) = 0 else. Then

$$\mathbf{D}_{n} = \det\left(c(i+j+1)\right)_{i,\,j=0}^{n-1} = \left(-1\right)^{\tau_{n}} x^{n^{2}}.$$
(5.16)

## Proof

We know that  $\mathbf{d}_n = (-1)^{\binom{n}{2}} x^{\binom{n}{2}}$ . Since  $\binom{n}{2} + \binom{n+1}{2} = n^2$  we get (5.16).

In this case we get  $T_n = (-1)^{\tau_n} x^2$ .

#### Theorem 5.11

If we choose g(1) = 1, and  $g(2^k - 1) = (-1)^k$  for k > 1 and g(n) = 0 else then

$$\det\left(g(i+j+1)\right)_{i,j=0}^{n-1} = r(n) \tag{5.17}$$

where r(n) denotes the Golay-Rudin-Shapiro sequence, which is defined by

$$r(2n) = r(n),$$
  

$$r(2n+1) = (-1)^{n} r(n),$$
 (5.18)  

$$r(0) = 1.$$

#### Proof

Some information about the Golay-Rudin-Shapiro sequence can be found in OEIS [8] A020985. As has been observed in [2] the Golay-Rudin-Shapiro sequence counts the number of pairs 11 in the binary expansion of n modulo 2:

$$r(n) = (-1)^{\varepsilon_0 \varepsilon_1 + \dots + \varepsilon_{k-1} \varepsilon_k} \quad \text{if} \quad n = \left[\varepsilon_k \varepsilon_{k-1} \cdots \varepsilon_0\right]_2. \tag{5.19}$$

Thus  $r(0) = (-1)^0 = 1$ .

If  $n = [\varepsilon_k \varepsilon_{k-1} \cdots \varepsilon_0]_2$  then  $2n = [\varepsilon_k \varepsilon_{k-1} \cdots \varepsilon_0 0]_2$  and  $2n+1 = [\varepsilon_k \varepsilon_{k-1} \cdots \varepsilon_0 1]_2$ 

which implies r(2n) = r(n) and  $r(2n+1) = r(n)(-1)^n$  because  $\varepsilon_0 1 = 11$  if *n* is odd.

The Golay-Rudin-Shapiro sequence can also be characterized by the recursion

$$r(2^{k} + n) = r(n) \text{ for } 0 \le n < 2^{k-1},$$
  

$$r(2^{k} + n) = -r(n) \text{ for } 2^{k-1} \le n < 2^{k}$$
(5.20)  
for  $k \ge 2$  and  $r(0) = r(1) = 1.$ 

If  $n < 2^{k-1}$  then  $2^k + n = [10 \cdots]_2$  and thus  $r(2^k + n) = r(n)$ .

If  $2^{k-1} \le n < 2^k$  then  $2^k + n = [11\cdots]_2$  and thus  $r(2^k + n) = -r(n)$ .

The proof of (5.17) now follows from (5.4) and (5.5).

From (2.10) we get

**Corollary 5.12** 

$$\sum_{k\geq 0} (-1)^{k} z^{2^{k}-1} = \frac{1}{1 + \frac{r(0)r(2)z}{1 + \frac{r(1)r(3)z}{1 + \frac{r(2)r(4)z}{1 + \ddots}}}}.$$
(5.21)

# **6.** Hankel determinants of shifted sequences $(a_{n+m})_{n>0}$ .

All results are very similar to the case  $x_n = 1$ . Therefore we only need to make slight alterations.

Let us state the first terms of  $d(n,m) = \det(a_{i+j+m})_{i,j=0}^{n-1}$  for m=3 and m=5:

$$(d(n,3)) = (1, x_3, 0, 0, -x_3 x_7^3, x_7^5, 0, 0, -x_7^5 x_{15}^3, -x_3 x_7^3 x_{15}^5, 0, 0, -x_3 x_{15}^{11}, \cdots),$$
  
$$(d(n,5)) = (1, 0, 0, -x_7^3, 0, 0, 0, 0, -x_7^3 x_{15}^5, 0, 0, -x_{15}^{11}, \cdots).$$

As in Lemma 3.3 we see that

$$d(n,m) = \det\left(a_{i+j+m}\right)_{i,j=0}^{n-1} = 0$$
(6.1)

if  $2^k - m < n < 2^k$  for some k.

# Lemma 6.1

Let m = 2r+1 and k be given. Then for  $0 \le j \le 2^{k-1} - r - 1$ 

$$d\left(2^{k-1}-r+j,m\right) = (-1)^{\binom{2j+1}{2}} x_{2^{k-1}}^{2j+1} d\left(2^{k-1}-r-j-1,m\right).$$
(6.2)

For example  $d(9,3) = d(8-1+2,3) = (-1)^{10} x_{15}^5 d(8-1-2-1,3) = x_{15}^5 d(4,3).$ 

## Lemma 6.2

Let m = 2r + 1. Then for  $a \le 2^R - m$  for some R we have

$$d\left(2^{R}+a,m\right) = (-1)^{\binom{2a+2r+1}{2}} x_{2^{R+1}-1}^{2a+m} d\left(2^{R}-a-m,m\right).$$
(6.3)

if  $a \leq 2^R - m$ .

# Lemma 6.3

Let m = 2r and k be given. Then for  $0 \le j \le 2^{k-1} - r$ 

$$d\left(2^{k-1}-r+j,m\right) = (-1)^{\binom{2j}{2}} x_{2^{k}-1}^{2j} d\left(2^{k-1}-r-j,m\right).$$
(6.4)

#### Lemma 6.4

Let m = 2r. Then for  $a \le 2^R - m$  for some R we have

$$d\left(2^{R}+a,m\right) = (-1)^{\binom{2a+m}{2}} x_{2^{R+1}-1}^{2a+m} d\left(2^{R}-a-m,m\right).$$
(6.5)

#### Theorem 6.5

 $d(n,m) \neq 0$  if and only if  $n \equiv 0, -m \mod 2^{K+1}$  if  $2^{K} < m \le 2^{K+1}$ .

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