Simple proofs of Bressoud's and Schur's polynomial versions of the Rogers-Ramanujan identities.

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Abstract
We give simple elementary proofs of Bressoud's and Schur's polynomial versions of the Rogers-Ramanujan identities.

1. Bressoud's identity

In [1] George E. Andrews and Kimmo Eriksson gave a simple proof of David Bressoud's ([2]) polynomial version of the Rogers-Ramanujan identities. I want to show that their proof can be further simplified by starting with the identity

$$\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix} n \\ k-j\end{bmatrix}\begin{bmatrix} n \\ k+j\end{bmatrix}=\begin{bmatrix} n \\ k\end{bmatrix}. \tag{1}$$

Ole Warnaar has informed me that this identity has been obtained in [6], Lemma 3.1 as limit case of Rogers' q-Dougall sum. In [6] he already used (1) to prove (a generalization of) Bressoud's identity (11). Christian Krattenthaler has told me that (1) can be considered as limit case of Jackson's q-Dixon summation. It is also a special case of Paule's transformation T1 of [4].
A simple computer proof can be given if we write the left hand side of (1) in the equivalent form

$$f(n,k)=\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\frac{1+q^j}{2}\begin{bmatrix} n \\ k-j\end{bmatrix}\begin{bmatrix} n \\ k+j\end{bmatrix}.$$

Then the implementation qZeil of the $q-$Zeilberger algorithm gives

$$f(n,k)=\frac{1-q^n}{1-q^{n-k}}f(n-1,k),$$

from which (1) is obvious if we observe that $f(k,k)=1.$

If you don't trust the computer set $a(n,k,j)=(-1)^j q^{\frac{j(3j-1)}{2}}(1+q^j)\begin{bmatrix} n \\ k-j\end{bmatrix}\begin{bmatrix} n \\ k+j\end{bmatrix}$ and

$b(n,k,j)=\frac{q^{n-k+2j}(1-q^{k-j})(1-q^{n-j-k})}{(1+q^j)(1-q^{n-k})(1-q^n)}a(n,k,j)$ and verify that for $n>k$

$$a(n,k,j)-\frac{1-q^n}{1-q^{n-k}}a(n-1,k,j)=b(n,k,j)-b(n,k,j-1).$$

Here I give an elementary proof of (1) which uses only the recurrence relations for the $q-$binomial coefficients:

To this end let

$$f(n,k)=\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix}=\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j+1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix}. \qquad (2)$$

From the recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}=\begin{bmatrix} n \\ k \end{bmatrix}+q^{n-k+1}\begin{bmatrix} n \\ k-1 \end{bmatrix} \qquad (3)$$

for the $q-$binomial coefficients we also get

$$\sum_j(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n+1 \\ k+j \end{bmatrix}=f(n,k). \qquad (4)$$

This follows from

$$\sum_j(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n+1 \\ k+j \end{bmatrix}=\sum_j(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix}$$
$$+\sum_j(-1)^j q^{\frac{j(3j-1)}{2}+n-k-j+1}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j-1 \end{bmatrix}=f(n,k)+q^{n-k+1}\sum_j(-1)^j q^{\frac{3j(j-1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j-1 \end{bmatrix}.$$

The last sum vanishes, because $j\to -j+1$ defines a sign reversing involution.

The other recurrence relation

$$\begin{bmatrix} n+1 \\ k \end{bmatrix}=q^k\begin{bmatrix} n \\ k \end{bmatrix}+\begin{bmatrix} n \\ k-1 \end{bmatrix} \qquad (5)$$

gives

$$\sum_j(-1)^j q^{\frac{j(3j+1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}=q^{k+1}\sum_j(-1)^j q^{\frac{3j(j+1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j+1 \end{bmatrix}+f(n,k)=f(n,k).$$

Therefore we get

$$f(n,k)=\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j+1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n \\ k+j \end{bmatrix}=\sum_j(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n+1 \\ k+j \end{bmatrix}$$
$$=\sum_j(-1)^j q^{\frac{j(3j+1)}{2}}\begin{bmatrix} n \\ k-j \end{bmatrix}\begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix}. \qquad (6)$$

This implies

$$f(n+1,k)=\sum_j (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n+1 \\ k-j \end{bmatrix} \begin{bmatrix} n+1 \\ k+j \end{bmatrix} = \sum_j (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n+1 \\ k+j \end{bmatrix}$$
$$+q^{n-k+1}\sum_j (-1)^j q^{\frac{j(3j+1)}{2}} \begin{bmatrix} n \\ k-j-1 \end{bmatrix} \begin{bmatrix} n+1 \\ k+j \end{bmatrix}$$
$$= f(n,k)+q^{n-k+1}\sum_j (-1)^j q^{\frac{j(3j+1)}{2}} \begin{bmatrix} n \\ (k-1)-j \end{bmatrix} \begin{bmatrix} n+1 \\ (k-1)+j+1 \end{bmatrix} = f(n,k)+q^{n-k+1} f(n,k-1).$$

Therefore the sequence $\left(f(n,k)\right)$ satisfies the recurrence relation (3) for the $q-$binomial coefficients and the corresponding boundary values.

This proves

Theorem 1
The following identities hold:

$$\sum_{j=-k}^{k} (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n \\ k+j \end{bmatrix} = \sum_j (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n+1 \\ k+j \end{bmatrix}$$
$$= \sum_j (-1)^j q^{\frac{j(3j+1)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n+1 \\ k+j+1 \end{bmatrix} = \begin{bmatrix} n \\ k \end{bmatrix}. \tag{7}$$

From (7) we obtain

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2} = \sum_j (-1)^j q^{\frac{j(5j-1)}{2}} \sum_{k\geq|j|} q^{(k-j)(k+j)} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n \\ k+j \end{bmatrix}. \tag{8}$$

The $q-$Vandermonde formula

$$\begin{bmatrix} m+n \\ k \end{bmatrix} = \sum_j \begin{bmatrix} m \\ j \end{bmatrix} \begin{bmatrix} n \\ k-j \end{bmatrix} q^{(m-j)(k-j)} \tag{9}$$

implies

$$\sum_{k\geq|j|} q^{(k-j)(k+j)} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n \\ k+j \end{bmatrix} = \begin{bmatrix} 2n \\ n-2j \end{bmatrix}. \tag{10}$$

Therefore (8) reduces to Bressoud's identity

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2} = \sum_j (-1)^j q^{\frac{j(5j-1)}{2}} \begin{bmatrix} 2n \\ n-2j \end{bmatrix}. \tag{11}$$

In the same way we get

$$\sum_j (-1)^j q^{\frac{j(3j-3)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n+1 \\ k+j \end{bmatrix} = q^k \sum_j (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n \\ k+j \end{bmatrix}$$
$$+ \sum_j (-1)^j q^{\frac{j(3j-3)}{2}} \begin{bmatrix} n \\ k-j \end{bmatrix} \begin{bmatrix} n \\ k+j-1 \end{bmatrix} = q^k f(n,k) = q^k \begin{bmatrix} n \\ k \end{bmatrix}.$$

This implies as above

$$\sum_k \begin{bmatrix} n \\ k \end{bmatrix} q^{k^2+k} = \sum_j (-1)^j q^{\frac{j(5j-3)}{2}} \begin{bmatrix} 2n+1 \\ n+1-2j \end{bmatrix}. \tag{12}$$

As is well known (cf. e.g. [1]) by letting $n \to \infty$ in (11) we get the first Rogers-Ramanujan identity

$$\sum_{k \ge 0} \frac{q^{k^2}}{(1-q)(1-q^2)\cdots(1-q^k)} = \frac{1}{\prod_{k \ge 1}(1-q^k)} \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{j(5j-1)}{2}}. \tag{13}$$

In the same way from (12) we get the second Rogers-Ramanujan identity

$$\sum_{k \ge 0} \frac{q^{k^2+k}}{(1-q)(1-q^2)\cdots(1-q^k)} = \frac{1}{\prod_{k \ge 1}(1-q^k)} \sum_{j \in \mathbb{Z}} (-1)^j q^{\frac{j(5j-3)}{2}}. \tag{14}$$

2. Schur's identity

The identity which corresponds to (1) for Schur's polynomial version is

Theorem 2

$$g(n,k) = \sum_{j=-k}^{k} (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} \left\lfloor \frac{n+j}{2} \right\rfloor \\ k-j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n-j+1}{2} \right\rfloor \\ k+j \end{bmatrix} = \begin{bmatrix} n-k \\ k \end{bmatrix}. \tag{15}$$

This identity has been obtained in [3] by other means.

By using (5) we get

$$g(n+2,k) = q^k \sum_{j=-k}^{k} (-1)^j q^{\frac{j(3j-3)}{2}} \begin{bmatrix} \left\lfloor \frac{n+j}{2} \right\rfloor \\ k-j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n-j+3}{2} \right\rfloor \\ k+j \end{bmatrix} + \sum_{j=-k}^{k} (-1)^j q^{\frac{j(3j-1)}{2}} \begin{bmatrix} \left\lfloor \frac{n+j}{2} \right\rfloor \\ k-j-1 \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n-j+3}{2} \right\rfloor \\ k+j \end{bmatrix}.$$

For the first sum we get again by using (5)

$$\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-3)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+3}{2}\right\rfloor\\ k+j\end{bmatrix}=q^k\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+1}{2}\right\rfloor\\ k+j\end{bmatrix}$$
$$+\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-3)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+1}{2}\right\rfloor\\ k+j-1\end{bmatrix}=q^k g(n,k)+\ell(n,k),$$

where

$$\ell(n,k)=\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-3)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+1}{2}\right\rfloor\\ k+j-1\end{bmatrix}=0,$$

because $j\to -j+1$ induces a sign reversing involution.

Therefore we have

$$\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-3)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+3}{2}\right\rfloor\\ k+j\end{bmatrix}=q^k g(n,k). \qquad (16)$$

The second term in the above formula gives

$$\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j-1\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+3}{2}\right\rfloor\\ k+j\end{bmatrix}=q^k\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j+1)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j-1\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+1}{2}\right\rfloor\\ k+j\end{bmatrix}$$
$$+\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j-1\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+1}{2}\right\rfloor\\ k+j-1\end{bmatrix}=q^k\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j+1}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j}{2}\right\rfloor\\ k+j-1\end{bmatrix}+g(n,k-1).$$

Let now

$$h(n,k)=\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j+1}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j}{2}\right\rfloor\\ k+j-1\end{bmatrix}.$$

Then

$$h(n,k)=\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j+1}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j}{2}\right\rfloor\\ k+j-1\end{bmatrix}$$

$$=q^k\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-3)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j-1}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j}{2}\right\rfloor\\ k+j-1\end{bmatrix}+\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j-1}{2}\right\rfloor\\ k-j-1\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j}{2}\right\rfloor\\ k+j-1\end{bmatrix}$$

$$=q^k\ell(n-1,k)+g(n-1,k-1)=g(n-1,k-1).$$

Therefore we get the recursion

$$g(n+2,k)=q^{2k}g(n,k)+q^k g(n-1,k-1)+g(n,k-1). \tag{17}$$

It is easy to verify that $g(n,0)=1=\begin{bmatrix}n-0\\ 0\end{bmatrix}$, $g(k,k)=0=\begin{bmatrix}k-k\\ k\end{bmatrix}$ for $k\geq 1$ and $g(k+1,k)=\begin{bmatrix}k+1-k\\ k\end{bmatrix}=0$ for $k\geq 2.$

By this recurrence and the initial values $g(n,k)$ is uniquely determined for all $n\geq k$.

Since

$$\begin{bmatrix}n+2-k\\ k\end{bmatrix}=q^k\begin{bmatrix}n+1-k\\ k\end{bmatrix}+\begin{bmatrix}n+1-k\\ k-1\end{bmatrix}=q^{2k}\begin{bmatrix}n-k\\ k\end{bmatrix}+q^k\begin{bmatrix}n-1-(k-1)\\ k-1\end{bmatrix}+\begin{bmatrix}n-(k-1)\\ k-1\end{bmatrix}$$

we see that $g(n,k)=\begin{bmatrix}n-k\\ k\end{bmatrix}$ for all $n\geq k$.

By summing over all k and using the $q-$Vandermonde formula we get

$$\sum_{k=0}^{n}q^{k^2}\begin{bmatrix}n-k\\ k\end{bmatrix}=\sum_{k=0}^{n}q^{k^2}\sum_{j=-k}^{k}(-1)^j q^{\frac{j(3j-1)}{2}}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+1}{2}\right\rfloor\\ k+j\end{bmatrix}$$

$$=\sum_{j=-n}^{n}(-1)^j q^{\frac{j(5j-1)}{2}}\sum_{k\geq|j|}q^{(k-j)(k+j)}\begin{bmatrix}\left\lfloor\frac{n+j}{2}\right\rfloor\\ k-j\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-j+1}{2}\right\rfloor\\ k+j\end{bmatrix}=\sum_{j=-n}^{n}(-1)^j q^{\frac{j(5j-1)}{2}}\begin{bmatrix}n\\ \left\lfloor\frac{n+5j}{2}\right\rfloor\end{bmatrix}.$$

This gives Schur's ([5]) polynomial version of the first Rogers-Ramanujan identity

$$\sum_{k=0}^{n}q^{k^2}\begin{bmatrix}n-k\\ k\end{bmatrix}=\sum_{j=-n}^{n}(-1)^j q^{\frac{j(5j-1)}{2}}\begin{bmatrix}n\\ \left\lfloor\frac{n+5j}{2}\right\rfloor\end{bmatrix}. \tag{18}$$

In the same way from (16) we get

$$\sum_{k=0}^{n} q^{k^2+k} \begin{bmatrix} n-k \\ k \end{bmatrix} = \sum_{k=0}^{n} q^{k^2} \sum_{j=-k}^{k} (-1)^j q^{\frac{j(3j-3)}{2}} \begin{bmatrix} \left\lfloor \frac{n+j}{2} \right\rfloor \\ k-j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n-j+3}{2} \right\rfloor \\ k+j \end{bmatrix}$$

$$= \sum_{j=-n}^{n} (-1)^j q^{\frac{j(5j-3)}{2}} \sum_{k \geq |j|} q^{(k-j)(k+j)} \begin{bmatrix} \left\lfloor \frac{n+j}{2} \right\rfloor \\ k-j \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n-j+3}{2} \right\rfloor \\ k+j \end{bmatrix} = \sum_{j=-n}^{n} (-1)^j q^{\frac{j(5j-3)}{2}} \begin{bmatrix} n+1 \\ \left\lfloor \frac{n-5j+3}{2} \right\rfloor \end{bmatrix}.$$

This is Schur's polynomial version of the second Rogers-Ramanujan identity, which is usually written in the form

$$\sum_{k=0}^{n-1} q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} = \sum_{j=-n}^{n} (-1)^j q^{\frac{j(5j-3)}{2}} \begin{bmatrix} n \\ \left\lfloor \frac{n-5j+2}{2} \right\rfloor \end{bmatrix}. \tag{19}$$

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