

# q-Fibonacci polynomials and q-Genocchi numbers

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## Abstract

We show that Genocchi and Bernoulli numbers are closely related to Fibonacci polynomials and derive some  $q$ -analogues.

## 1. Fibonacci polynomials and Genocchi numbers

Define the sequence  $(G_{2n})_{n \geq 1} = (1, 1, 3, 17, 155, 2073, 38227, 929569, \dots)$  of Genocchi numbers  $G_{2n}$  by their exponential generating function

$$\frac{2z}{1+e^z} = z + z \frac{1-e^z}{1+e^z} = \sum_{n \geq 0} g_n \frac{z^n}{n!} = z + \sum_{n \geq 1} (-1)^n G_{2n} \frac{z^{2n}}{(2n)!}. \quad (1.1)$$

It is well-known that  $G_{2n} = (-1)^n 2(1-2^{2n})B_{2n}$ , where  $(B_n)$  is the sequence of Bernoulli numbers defined by

$$B_n = \sum_{k=0}^n \binom{n}{k} B_k \quad (1.2)$$

for  $n > 1$  with  $B_0 = 1$ .

Let

$$F_n(s) = \sum_{k=0}^{n-1} \binom{n-k-1}{k} s^k \quad (1.3)$$

denote the Fibonacci polynomials. They are characterized by the recursion

$$F_n(s) = F_{n-1}(s) + sF_{n-2}(s) \quad (1.4)$$

with initial values  $F_0(s) = 0$  and  $F_1(s) = 1$  and are explicitly given by the Binet formula

$$F_n(s) = \frac{\alpha^n - \beta^n}{\alpha - \beta} \quad (1.5)$$

with  $\alpha = \frac{1 + \sqrt{1+4s}}{2}$  and  $\beta = \frac{1 - \sqrt{1+4s}}{2} = 1 - \alpha$ .

They satisfy the identity

$$\sum_{n \geq 0} \frac{F_n(s)}{n!} z^n = -e^z \sum_{n \geq 0} \frac{F_n(s)}{n!} (-z)^n. \quad (1.6)$$

For (1.6) is equivalent with  $e^{\alpha z} - e^{\beta z} = -e^z (e^{-\alpha z} - e^{-\beta z})$ , which is trivially true.

An easy consequence is

$$(1 + e^z) \sum_{n \geq 0} \frac{F_{2n}(s)}{(2n)!} z^{2n} = (e^z - 1) \sum_{n \geq 0} \frac{F_{2n+1}(s)}{(2n+1)!} z^{2n+1}. \quad (1.7)$$

If we define the linear functional  $L$  on  $\mathbb{C}[s]$  by

$$L(F_{2k+1}(s)) = [k = 0], \quad (1.8)$$

we get

**Theorem 1.1 ( D. Dumont and J. Zeng [4], Corollary 1)**

$$L(F_{2n}(s)) = (-1)^{n-1} G_{2n}. \quad (1.9)$$

In order to show this apply  $L$  to (1.7). This gives

$$\sum_{n \geq 0} \frac{L(F_{2n}(s))}{(2n)!} z^{2n} = -z \frac{1 - e^z}{1 + e^z} = z - \frac{2z}{1 + e^z} = \sum_{n \geq 1} (-1)^{n-1} G_{2n} \frac{z^{2n}}{(2n)!}.$$

**Corollary 1.2**

$$g_n = -L(F_n(s)) \quad (1.10)$$

for  $n \neq 1$ .

As another consequence we get

**Corollary 1.3**

$$F_{2n}(s) = \sum_{k=0}^{n-1} a(n, k) F_{2k+1} \quad (1.11)$$

with

$$a(n, k) = (-1)^{n-k-1} \frac{1}{2k+1} \binom{2n}{2k} G_{2n-2k}. \quad (1.12)$$

E.g. we have

$$(a(n, k))_{1 \leq n \leq 5, 0 \leq k \leq 4} = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 2 & 0 & 0 & 0 \\ 3 & -5 & 3 & 0 & 0 \\ -17 & 28 & -14 & 4 & 0 \\ 155 & -255 & 126 & -30 & 5 \end{pmatrix}.$$

The proof follows by writing (1.6) in the form

$$\sum_{n \geq 0} \frac{F_{2n}(s)}{(2n)!} z^{2n} = -z \frac{1 - e^z}{1 + e^z} \sum_{n \geq 0} \frac{F_{2n+1}(s)}{(2n+1)!} z^{2n} \quad (1.13)$$

and comparing coefficients.

## 2. Fibonacci polynomials and Bernoulli numbers

In an analogous way we define a linear functional  $M$  by

$$M(F_{2n}) = [n = 1]. \quad (2.1)$$

Then we get

$$M(F_{2n+1}(s)) = (2n+1)B_{2n}. \quad (2.2)$$

This follows from (1.7) and the well-known identity

$$\frac{z^2 e^z + 1}{2 e^z - 1} = \sum_{n \geq 0} \frac{B_{2n}}{(2n)!} z^{2n+1}.$$

This implies as above

$$F_{2n+1}(s) = \sum_{j=0}^n \binom{2n+1}{2j+1} \frac{B_{2n-2j}}{j+1} F_{2j+2}(s). \quad (2.3)$$

The first terms of the sequence  $((2n+1)B_{2n})_{n \geq 0}$  are

$$1, \frac{1}{2}, -\frac{1}{6}, \frac{1}{6}, -\frac{3}{10}, \dots$$

We can now give a simple proof of

**Theorem 2.1 (A. v. Ettingshausen [5], L. Seidel [9], M. Kaneko [7])**

$$\sum_{i=0}^{n+1} \binom{n+1}{i} (n+i+1) B_{n+i} = 0. \quad (2.4)$$

**Proof**

We need the identity

$$\sum_{k=0}^n (-1)^{n-k} \binom{n}{k} F_{n+k}(s) = \sum_{k=0}^n (-1)^k \binom{n}{k} F_{2n-k}(s) = 0 \quad (2.5)$$

which is immediate from  $\sum_{k=0}^n (-1)^k \binom{n}{k} (\alpha^{2n-k} - \beta^{2n-k}) = (\alpha^2 - \alpha)^n - (\beta^2 - \beta)^n = s^n - s^n = 0$ .

This implies

$$\sum_{i=0}^{n+1} \binom{n+1}{2i} F_{2n+2-2i}(s) = \sum_{i=0}^{n+1} \binom{n+1}{2i+1} F_{2n+1-2i}(s).$$

If we apply the linear functional  $M$  to (2.5) we get

$$\sum_{i=0}^{n+1} \binom{n+1}{2i+1} (2n-2i+1) B_{2n-2i} = 0 \text{ for } n > 1. \text{ Since } B_{2i+1} = 0 \text{ for } i > 0 \text{ we have also}$$

$$\sum_{i=0}^{n+1} \binom{n+1}{i} (n+i+1) B_{n+i} = 0 \text{ for } n > 1. \text{ It is easy to verify that (2.4) holds for } n = 0 \text{ and } n = 1 \text{ too.}$$

**Remark 2.2**

This theorem has first been proved by A. v. Ettingshausen [5] and has been rediscovered by L. Seidel [9], VIII, and by M. Kaneko [7]. Therefore I will call it **v. Ettingshausen-Seidel-Kaneko identity**.

The proof by v. Ettingshausen starts with the definition of the Bernoulli numbers

$$\sum_{k \geq 0} \binom{r}{k} B_k = B_r \text{ for } r \geq 2 \text{ and } B_0 = 1. \text{ He computes the differences } \Delta^w B_r \text{ for both sides using}$$

the fact that  $\Delta^w \binom{r}{k} = \binom{r}{k-w}$ . This gives

$$\sum_{i=0}^w (-1)^{w-i} \binom{w}{i} B_{r+i} = \Delta^w B_r = \sum_{k=0}^r B_k \Delta^w \binom{r}{k} = \sum_{k=0}^r B_k \binom{r}{k-w} = \sum_{j=0}^r \binom{r}{j} B_{w+j}.$$

Choosing  $r = w = n$  gives  $\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} B_{n+i} = \sum_{i=0}^n \binom{n}{i} B_{n+i}$  or equivalently

$$\sum_i \binom{n}{2i+1} B_{2n-2i-1} = 0, \quad (2.6)$$

which implies the well-known result  $B_{2i+1} = 0$  for  $i \geq 1$ .

For  $w = n, r = n + 1$  he gets

$$\sum_{i=0}^n (-1)^{n-i} \binom{n}{i} B_{n+1+i} = \sum_{j=0}^{n+1} \binom{n+1}{j} B_{n+j} = B_n + \sum_{i=0}^n \binom{n+1}{i+1} B_{n+i+1}. \quad (2.7)$$

Since  $B_{n+1+i} = 0$  for  $n+i \equiv 0 \pmod{2}$  this identity is the same as

$$-\sum_{i=0}^n \binom{n}{i} B_{n+1+i} = B_n + \sum_{i=0}^n \binom{n+1}{i+1} B_{n+i+1}. \text{ Because of } \binom{n+1}{i+1} + \binom{n}{i} = \frac{n+i+2}{n+1} \binom{n+1}{i+1}$$

this gives  $(n+1)B_n + \sum_{i=0}^n \binom{n+1}{i+1} (n+i+2)B_{n+i+1} = 0$  or equivalently

$$\sum_{i=0}^n \binom{n+1}{i} (n+i+1)B_{n+i} = 0.$$

Seidel's and Gessel's proofs are along similar lines although Seidel used a somewhat clumsy terminology.

These proofs of (2.4) show first that  $B_{2n+1} = 0$  for  $n \geq 1$ . This is usually done by observing

that (1.2) is equivalent with  $e^z \sum_n B_n \frac{z^n}{n!} = z + \sum_n B_n \frac{z^n}{n!}$  or  $B(z) = \sum_n B_n \frac{z^n}{n!} = \frac{z}{e^z - 1}$ , which

implies that  $B(z) + \frac{z}{2} = \frac{z}{2} \frac{e^{\frac{z}{2}} + e^{-\frac{z}{2}}}{e^{\frac{z}{2}} - e^{-\frac{z}{2}}}$  is an even function.

To simplify the final step it is convenient to consider the linear functional  $V$  on the polynomials defined by  $V(x^n) = B_n$  for  $n \neq 1$  and  $V(x) = \frac{1}{2}$ . This has the effect that in place of (1.2) we get

$$V((1-x)^n) = V(x^n) \tag{2.8}$$

for all  $n \in \mathbb{N}$ . Therefore by linearity

$$V(f(1-x)) = V(f(x)) \tag{2.9}$$

for each polynomial  $f(x)$ .

Choosing  $f(x) = (1-x)^{n+1}(-x)^n$  we get  $V(x^{n+1}(x-1)^n) = V((1-x)^{n+1}(-x)^n)$ , which is the same as (2.7).

### 3. The Seidel triangle for Genocchi numbers

Seidel [9] has given a ‘‘Treppen-Schema’’ for the computation of the Genocchi numbers  $G_{2n}$ , which he called ‘‘Bernoulli'sche Zähler’’. We use a slightly changed version as in [3] and [10] and define the ‘‘Seidel triangle’’ for the Genocchi numbers as an array of integers

$(g_{i,j})_{i,j \geq 1}$  such that  $g_{1,1} = g_{2,1} = 1$ ,  $g_{i,j} = 0$  if  $j < 0$  or  $j > \left\lceil \frac{i}{2} \right\rceil$  and

$$g_{2i+1,j} = g_{2i+1,j-1} + g_{2i,j} \tag{3.1}$$

for  $j = 1, 2, \dots, i+1$  and

$$g_{2i,j} = g_{2i,j+1} + g_{2i-1,j} \tag{3.2}$$

for  $j = i, i-1, \dots, 1$ .

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 3 & 0 & 0 & 0 & 0 & 0 \\ 8 & 6 & 3 & 0 & 0 & 0 & 0 & 0 \\ 8 & 14 & 17 & 17 & 0 & 0 & 0 & 0 \\ 56 & 48 & 34 & 17 & 0 & 0 & 0 & 0 \end{pmatrix}$$

To compute this triangle note that the odd rows (3.1) are computed from left to right and the even rows (3.2) from right to left.

From (3.2) we see that

$$g_{2i,j} = \sum_{\ell \geq j} g_{2i-1,\ell} \quad (3.3)$$

and from (3.1) we have

$$g_{2i+1,j} = \sum_{\ell \leq j} g_{2i,\ell}. \quad (3.4)$$

We show now that the Seidel triangle is also closely related to the Fibonacci polynomials.

**Theorem 3.1**

For  $k = 1, 2, \dots, n$

$$g_{2n,k} = (-1)^n L(s^{n+1-k} F_{2k-1}) \quad (3.5)$$

and for  $k = 1, 2, \dots, n+1$

$$g_{2n+1,k} = (-1)^n L(s^{n+1-k} F_{2k}). \quad (3.6)$$

**Proof**

This is easily verified for  $g_{1,1} = 1$  and  $g_{2,1} = 1$ .

The general case follows from

$$g_{2n,k} = (-1)^n L(s^{n+1-k} F_{2k-1}) = (-1)^n L(s^{n-k} F_{2k+1}) - (-1)^n L(s^{n-k} F_{2k}) = g_{2n,k+1} + g_{2n-1,k}$$

and

$$g_{2n+1,k} = (-1)^n L(s^{n+1-k} F_{2k}) = (-1)^n L(s^{n-k+2} F_{2k-2}) + (-1)^n L(s^{n-k+1} F_{2k-1}) = g_{2n+1,k-1} + g_{2n,k}.$$

As a special case we get that the ‘‘median Genocchi numbers’’  $H_{2n+1} := g_{2n+1,1}$  are given by

$$H_{2n+1} = (-1)^n L(s^n). \quad (3.7)$$

As another application we have

$$\sum_{j=0}^k \binom{k}{j} g_{n+j} = (-1)^{n+k-1} L\left(\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} F_{n+j}(s)\right).$$

Observing that

$$\sum_{j=0}^k (-1)^{k-j} \binom{k}{j} F_{n+j}(s) = \frac{\alpha^n (\alpha-1)^k - \beta^n (\beta-1)^k}{\alpha - \beta} = s^k F_{n-k}(s)$$

we get

$$\sum_{j=0}^k \binom{k}{j} g_{n+j} = (-1)^{n+k-1} L(s^k F_{n-k}(s)). \quad (3.8)$$

**Corollary 3.2 (Seidel identity, L. Seidel [9], XIII )**

$$\sum_{j=0}^n \binom{n}{j} g_{n+j} = \sum_{k=0}^n \binom{n}{2k} (-1)^k G_{2n-2k} = 0. \quad (3.9)$$

This is a counterpart to the v. Ettingshausen-Seidel-Kaneko identity (2.4).

**Corollary 3.3**

$$H_{2n+1} = (-1)^n \sum_{k=0}^{n+1} \binom{n+1}{k} g_{n+k} = \sum_{k=0}^{\frac{n}{2}} (-1)^k \binom{n+1}{2k+1} G_{2n-2k}. \quad (3.10)$$

By (3.8) and (3.7) and using  $F_{-1}(s) = \frac{1}{s}$  we get

$$\sum_{k=0}^{n+1} \binom{n+1}{k} g_{n+k} = L(s^{n+1} F_{-1}(s)) = L(s^n) = (-1)^n H_{2n+1}.$$

These considerations also cast new light on [4], Theorem 1:

Apply  $L$  to the generating function

$$\sum_{n \geq 0} F_{n+1}(s) z^n = \frac{1}{1-z-sz^2} = \frac{1}{1-z} \sum_{n \geq 0} s^n \left( \frac{z^2}{1-z} \right)^n.$$

This gives

$$1 + \sum_{n \geq 1} (-1)^{n-1} G_{2n} z^{2n-1} = \frac{1}{1-z} \sum_{n \geq 0} (-1)^n H_{2n+1} \left( \frac{z^2}{1-z} \right)^n. \quad (3.11)$$

which gives [4], Theorem 1, by multiplying with  $z^2$  and  $z \rightarrow -z$ .

#### 4. $q$ -analogues

Next we show that the Seidel generation of the  $q$  – Genocchi numbers introduced in [10] is intimately related to the (Carlitz-)  $q$  – Fibonacci polynomials (cf. e.g. [1]).

Let

$$F_n(s, q) = \sum_{k=0}^{\frac{n-1}{2}} q^{2\binom{k}{2}} \begin{bmatrix} n-1-k \\ k \end{bmatrix} s^k. \quad (4.1)$$

Recall that

$$F_n(s, q) = F_{n-1}(s, q) + q^{n-3} s F_{n-2}(s, q). \quad (4.2)$$

Define a linear functional  $L$  by

$$L\left(F_{2n+1}\left(s, \frac{1}{q}\right)\right) = [n = 0]. \quad (4.3)$$

Now define a  $q$  – Seidel triangle  $g_{n,k}(q)$  by

$$g_{2n,k}(q) = (-1)^n q^{2\binom{k-1}{2}} L\left(s^{n+1-k} F_{2k-1}\left(s, \frac{1}{q}\right)\right) \quad (4.4)$$

for  $1 \leq k \leq n$   
and

$$g_{2n+1,k}(q) = (-1)^n q^{(k-1)^2} L\left(s^{n+1-k} F_{2k}\left(s, \frac{1}{q}\right)\right) \quad (4.5)$$

for  $1 \leq k \leq n+1$ .

All other values should be 0.

Then  $g_{1,1}(q) = g_{2,1}(q) = 1$

and

$$g_{2n,k}(q) = g_{2n,k+1} + q^{k-1} g_{2n-1,k} \quad (4.6)$$

and

$$g_{2n+1,k}(q) = q^{k-1} g_{2n,k}(q) + g_{2n+1,k-1}(q). \quad (4.7)$$

This is precisely the definition given in [10].

The proof follows from

$$\begin{aligned}
g_{2n,k}(q) &= (-1)^n q^{2\binom{k-1}{2}} L\left(s^{n+1-k} F_{2k-1}\left(s, \frac{1}{q}\right)\right) = (-1)^n q^{2\binom{k-1}{2}} L\left(q^{2(k-1)} \left(s^{n-k} F_{2k+1}\left(s, \frac{1}{q}\right) - s^{n-k} F_{2k}\left(s, \frac{1}{q}\right)\right)\right) \\
&= (-1)^n q^{2\binom{k}{2}} L\left(s^{n-k} F_{2k+1}\left(s, \frac{1}{q}\right)\right) + (-1)^{n-1} q^{(k-1)^2 + (k-1)} L\left(s^{n-k} F_{2k}\left(s, \frac{1}{q}\right)\right) = g_{2n,k+1} + q^{k-1} g_{2n-1,k}
\end{aligned}$$

and

$$\begin{aligned}
g_{2n+1,k}(q) &= (-1)^n q^{(k-1)^2} L\left(s^{n+1-k} F_{2k}\left(s, \frac{1}{q}\right)\right) \\
&= (-1)^n q^{(k-1)^2} L\left(s^{n+1-k} F_{2k-1}\left(s, \frac{1}{q}\right)\right) + (-1)^n q^{(k-1)^2 - 2k + 3} L\left(s^{n+2-k} F_{2k-2}\left(s, \frac{1}{q}\right)\right) \\
&= q^{k-1} g_{2n,k}(q) + g_{2n+1,k-1}(q).
\end{aligned}$$

The first terms are

$$\begin{pmatrix} 1 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 \\ 1 + q & q & 0 & 0 \\ 1 + q & 1 + q + q^2 & 1 + q + q^2 & 0 \\ (1 + q)^2 & q(1 + q) & q^2(1 + q + q^2) & 0 \end{pmatrix}$$

Observe that from (4.7) we have

$$g_{2n+1,k}(q) = \sum_{\ell \geq 0} q^{k-1-\ell} g_{2n,k-\ell}(q) \quad (4.8)$$

and from (4.6)

$$g_{2n,k}(q) = \sum_{\ell \geq 0} q^{k-1+\ell} g_{2n-1,k+\ell}(q). \quad (4.9)$$

Now we follow [10] and define

$$G_{2n}(q) = g_{2n-1,n}(q) \quad (4.10)$$

and

$$H_{2n-1}(q) = q^{n-2} g_{2n-1,1}(q). \quad (4.11)$$

Then we get

$$L\left(s F_{2n}\left(s, \frac{1}{q}\right)\right) = (-1)^n g_{2n+1,n}(q) q^{-(n-1)^2} = (-1)^n q^{-(n-1)^2} G_{2n+2}(q), \quad (4.12)$$

$$L\left(F_{2n}\left(s, \frac{1}{q}\right)\right) = (-1)^{n-1} q^{-(n-1)^2} G_{2n}(q) \quad (4.13)$$

and

$$L(s^n) = (-1)^n g_{2n+1,1}(q) = (-1)^n \frac{H_{2n+1}(q)}{q^{n-1}}. \quad (4.14)$$

Next we prove

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} F_{2n-k}(s, q) = 0. \quad (4.15)$$

By changing  $q \rightarrow \frac{1}{q}$  we get

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k+1}{2} - kn} F_{2n-k}\left(s, \frac{1}{q}\right) = 0. \quad (4.16)$$

If we apply the linear functional  $L$  and observe (4.13) we get

$$(-1)^{2n-1} \begin{bmatrix} n \\ 2n-1 \end{bmatrix} q^{\binom{2n}{2} - (2n-1)n} F_1\left(s, \frac{1}{q}\right) + \sum_{k=0}^n (-1)^{n-k-1} \begin{bmatrix} n \\ 2k \end{bmatrix} q^{\binom{2k+1}{2} - 2kn - (n-k-1)^2} G_{2n-2k}(q) = 0.$$

Therefore we get

**Theorem 4.1 (q-Seidel identity)**

$$\sum_{k=0}^n (-1)^k \begin{bmatrix} n \\ 2k \end{bmatrix} q^{2\binom{k}{2}} G_{2n-2k}(q) = [n = 1]. \quad (4.17)$$

This formula can be used to compute the polynomials  $G_{2n}(q)$ .

In order to prove (4.15) we show more generally

$$\sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} F_{2n+m-k}(s, q) = q^{2\binom{n}{2} + (m-1)n} s^n F_m(s, q). \quad (4.18)$$

For  $n = 0$  this is trivial.

For  $n = 1$  (4.18) reduces to the recursion

$$F_{m+2}(s, q) - F_{m+1}(s, q) = q^{m-1} s F_m(s), \quad (4.19)$$

which holds for  $m \in \mathbb{Z}$ .

Suppose that (4.18) is already known for  $i < n$  and all  $m \in \mathbb{Z}$ .

Then

$$\begin{aligned}
& \sum_k (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n \\ k \end{bmatrix} F_{2n+m-k}(s, q) = \sum_k (-1)^k q^{\binom{k}{2}} q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} F_{2n+m-k}(s, q) + \sum_k (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} F_{2n+m-k}(s, q) \\
& = \sum_k (-1)^{k-1} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} F_{2n+m-k+1}(s, q) + \sum_k (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} F_{2n+m-k}(s, q) \\
& = \sum_k (-1)^{k-1} q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} (F_{2n+m-k+1}(s, q) - F_{2n+m-k}(s, q)) \\
& = \sum_k (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix} q^{2n+m-k-3} s F_{2n+m-k-2}(s, q) = q^{2n+m-3} s \sum_k (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n-1 \\ k \end{bmatrix} F_{2(n-1)+m-k}(s, q) \\
& = q^{2n+m-3} q^{2\binom{n-1}{2} + (m-1)(n-1)} s s^{n-1} F_m(s, q) = q^{2\binom{n}{2} + (m-1)n} s^n F_m(s, q).
\end{aligned}$$

A  $q$ -analogue of (3.10) is

$$H_{2n+1}(q) = \sum_{k=0}^n (-1)^k \begin{bmatrix} n+1 \\ 2k+1 \end{bmatrix} q^{k^2-k+n-2} G_{2n-2k}(q) \quad (4.20)$$

for  $n \geq 2$ .

If we choose  $m = -1$  in (4.18) and replace  $n$  by  $n+1$  we get

$$\begin{aligned}
& \sum_{k=0}^n (-1)^k q^{\binom{k}{2}} \begin{bmatrix} n+1 \\ k \end{bmatrix} F_{2(n+1)-1-k}(s, q) = q^{2\binom{n+1}{2} + (-1-1)(n+1)} s^{n+1} F_{-1}(s, q) \\
& = q^{n^2-n-2} s^{n+1} \frac{q^2}{s} = q^{2\binom{n}{2}} s^n.
\end{aligned}$$

If we now change  $q \rightarrow \frac{1}{q}$  we have

$$s^n = q^{2\binom{n}{2}} \sum_{k=0}^n (-1)^k q^{\binom{k}{2}-kn} \begin{bmatrix} n+1 \\ k \end{bmatrix} F_{2n+1-k}\left(s, \frac{1}{q}\right).$$

By applying the linear functional  $L$  we obtain the desired result

$$\begin{aligned}
H_{2n+1}(q) & = (-1)^n q^{n-1} L(s^n) = (-1)^n q^{n-1} q^{2\binom{n}{2}} \sum_{k=0}^n (-1)^k q^{\binom{k}{2}-kn} \begin{bmatrix} n+1 \\ k \end{bmatrix} L\left(F_{2n+1-k}\left(s, \frac{1}{q}\right)\right) \\
& = (-1)^{n-1} q^{n-1} q^{2\binom{n}{2}} \sum_{k=0}^n q^{\binom{2k+1}{2}-(2k+1)n} \begin{bmatrix} n+1 \\ 2k+1 \end{bmatrix} L\left(F_{2n+1-2k-1}\left(s, \frac{1}{q}\right)\right) \\
& = \sum_{k=0}^n (-1)^k q^{k^2-k+n-2} \begin{bmatrix} n+1 \\ 2k+1 \end{bmatrix} G_{2n-2k}(q).
\end{aligned}$$

There is also a  $q$ -analogue of (3.11).

In the generating function

$$\sum_n F_{n+1}(s, q) z^n = \sum_n z^n \sum_{k=0}^n q^{\binom{k}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix} s^k = \sum_k q^{\binom{k}{2}} s^k z^{2k} \sum_j \begin{bmatrix} k+j \\ k \end{bmatrix} z^j = \sum_k q^{\binom{k}{2}} \frac{s^k z^{2k}}{(1-z) \cdots (1-q^k z)}$$

change  $q$  to  $\frac{1}{q}$ . We obtain

$$\sum_n F_{n+1}\left(s, \frac{1}{q}\right) z^n = \sum_k (-1)^{k+1} q^{-2\binom{k}{2} + \binom{k+1}{2}} \frac{s^k z^{2k}}{(z-1) \cdots (z-q^k)}.$$

By applying  $L$  we get

$$\begin{aligned} 1 + \sum_{n \geq 1} (-1)^{n-1} q^{-(n-1)^2} G_{2n}(q) z^{2n-1} &= - \sum_k q^{1-\binom{k}{2}} \frac{z^{2k}}{(z-1) \cdots (z-q^k)} H_{2k+1}(q) \\ &= \sum_k q^{k-\binom{k}{2}} (-1)^k \frac{z^{2k}}{(1-z) \cdots (q^k - z)} g_{2k+1,1}(q). \end{aligned} \quad (4.21)$$

### Remark

For  $q=1$  some of these results have been obtained by using Seidel matrices instead of the above Seidel triangle.

The Seidel matrix  $(a_{n,k})$  for a given sequence  $(c_n)$  is defined by  $a_{n,0} = c_n$  and

$$a_{n,k} = a_{n,k-1} + a_{n+1,k-1} \text{ for } k \geq 1. \text{ Then } a_{n,k} = \sum_{i=0}^k \binom{k}{i} a_{n+i,0}.$$

To obtain a useful  $q$ -analogue define instead

$$a_{n,k} = q^{n-1} (a_{n,k-1} + a_{n+1,k-1}). \quad (4.22)$$

Then

$$a_{n,k} = q^{k(n-1)} \sum_{j=0}^k q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} a_{n+j,0}. \quad (4.23)$$

This holds for  $k=1$ .

By induction we get

$$\begin{aligned} a_{n,k+1} &= q^{n-1} (a_{n,k} + a_{n+1,k}) = q^{n-1} \left( q^{k(n-1)} \sum_{j=0}^k q^{\binom{j}{2}} \begin{bmatrix} k \\ j \end{bmatrix} a_{n+j,0} + q^{kn} \sum_{j=0}^k q^{\binom{j-1}{2}} \begin{bmatrix} k \\ j-1 \end{bmatrix} a_{n+j,0} \right) \\ &= q^{n-1+kn-k} \sum_j q^{\binom{j}{2}} a_{n+j,0} \left( \begin{bmatrix} k \\ j \end{bmatrix} + q^{k-j+1} \begin{bmatrix} k \\ j-1 \end{bmatrix} \right) = q^{(k+1)(n-1)} \sum_j q^{\binom{j}{2}} \begin{bmatrix} k+1 \\ j \end{bmatrix} a_{n+j,0}. \end{aligned}$$

By choosing

$$c_n = a_{n,0} = L \left( (-1)^{n-1} F_n \left( s, \frac{1}{q} \right) \right) \quad (4.24)$$

we get

$$a_{n,k} = L \left( (-1)^{n-k-1} q^{\binom{k+1}{2}} s^k F_{n-k} \left( s, \frac{1}{q} \right) \right). \quad (4.25)$$

I want also mention a  $q$  – analogue of the v. Ettingshausen- Seidel-Kaneko identity.

If we apply the linear functional  $M$  defined by  $M(F_{2n+2}(s, q)) = [n = 0]$  to (4.15) we get

$$\sum_{k=0}^{n+1} (-1)^k \begin{bmatrix} n+1 \\ k \end{bmatrix} q^{\binom{k}{2}} M(F_{2n+2-k}(s, q)) = 0. \quad (4.26)$$

This can be used to compute the sequence  $(M(F_{2k+1}(s, q)))$  which begins with

$$\left\{ 1, \frac{q}{1+q}, -\frac{q^4}{(1+q)(1+q+q^2)}, \frac{q^7}{(1+q)(1+q+q^2)}, \right. \\ \left. -\frac{q^{10}(1+q+2q^2+2q^3+q^4+q^5+q^6)}{(1+q)(1+q+q^2)(1+q+q^2+q^3+q^4)}, \frac{q^{13}(1+q+3q^2+3q^3+2q^4+2q^5+q^6+q^7+q^8)}{(1+q)(1+q+q^2)^2} \right\}$$

## 5. Some identities

Each identity for the  $q$  – Fibonacci polynomials gives an identity for the entries of the  $q$  – Seidel triangle  $(g_{i,j}(q))$ .

I shall give some examples.

1) From the definition of the  $q$  – Fibonacci polynomials we get

$$F_n \left( s, \frac{1}{q} \right) = \sum_{k=0}^{\frac{n-1}{2}} q^{k^2+2k-nk} \begin{bmatrix} n-1-k \\ k \end{bmatrix} s^k.$$

Applying the linear functional  $L$  gives

$$\sum_{k=0}^n (-1)^k q^{k^2+k-2nk} \begin{bmatrix} 2n-k \\ k \end{bmatrix} g_{2k+1,1}(q) = 0 \quad (5.1)$$

and

$$q^{(n-1)^2} \sum_{k=0}^{n-1} (-1)^{n-k-1} q^{k^2+2k-2nk} \begin{bmatrix} 2n-1-k \\ k \end{bmatrix} g_{2k+1,1}(q) = G_{2n}(q). \quad (5.2)$$

These identities are  $q$  – analogues of [3], Corollary 1.

2) It is easily verified (cf. [2]) that

$$F_{m+2n}(s, q) = \sum_{k=0}^n \begin{bmatrix} n \\ k \end{bmatrix} q^{k(m+n-2)} s^k F_{m+n-k}(s, q). \quad (5.3)$$

Therefore

$$F_{2n+1}(s, q) = \sum_{j=0}^{\frac{n}{2}} \begin{bmatrix} n \\ n-2j \end{bmatrix} q^{(n-2j)(n-1)} s^{n-2j} F_{2j+1}(s, q) + \sum_{j=0}^{\frac{n+1}{2}} \begin{bmatrix} n \\ n-2j+1 \end{bmatrix} q^{(n-2j+1)(n-1)} s^{n-2j+1} F_{2j}(s, q)$$

By changing  $q \rightarrow \frac{1}{q}$  we get

$$F_{2n+1}\left(s, \frac{1}{q}\right) = \sum_j \begin{bmatrix} n \\ 2j \end{bmatrix} q^{2\binom{2j}{2} - 2\binom{n}{2}} s^{n-2j} F_{2j+1}\left(s, \frac{1}{q}\right) + \sum_j \begin{bmatrix} n \\ 2j-1 \end{bmatrix} q^{2\binom{2j-1}{2} - 2\binom{n}{2}} s^{n-2j+1} F_{2j}\left(s, \frac{1}{q}\right)$$

If we apply the linear functional  $L$  we get

$$0 = \sum_j (-1)^j \begin{bmatrix} n \\ 2j \end{bmatrix} q^{2\binom{2j}{2} - 2\binom{j}{2}} g_{2n-2j, j+1}(q) + \sum_j (-1)^j \begin{bmatrix} n \\ 2j-1 \end{bmatrix} q^{2\binom{2j-1}{2} - (j-1)^2} g_{2n-2j+1, j}(q)$$

or

$$\sum_j (-1)^j \begin{bmatrix} n \\ 2j \end{bmatrix} q^{3j^2 - j} g_{2n-2j, j+1}(q) = \sum_j (-1)^{j-1} \begin{bmatrix} n \\ 2j-1 \end{bmatrix} q^{3j^2 - 4j + 1} g_{2n-2j+1, j}(q). \quad (5.4)$$

3) Now consider the following identities which in the special case  $m = -1$  have been proved in [8]:

$$\sum_{k=0}^n (-s)^k \begin{bmatrix} 2k+m \\ k \end{bmatrix} q^{-\binom{k+m+2}{2}} = \sum_{k=0}^n (-s)^{n-k} \begin{bmatrix} 2n+m+2 \\ n-k \end{bmatrix} q^{-\binom{n+2+m-k}{2}} F_{2k+2}(s, q) \quad (5.5)$$

and

$$\sum_{k=0}^n (-s)^k \begin{bmatrix} 2k+m \\ k \end{bmatrix} q^{-\binom{k+m+2}{2}} = \sum_{k=0}^n (-s)^{n-k} \begin{bmatrix} 2n+m+1 \\ n-k \end{bmatrix} q^{-\binom{n+2+m-k}{2}} F_{2k+1}(s, q). \quad (5.6)$$

Comparing coefficients equation (5.5) is equivalent with

$$(-1)^k \begin{bmatrix} 2k+m \\ k \end{bmatrix} q^{-\binom{k+m+2}{2}} = \sum_{i+j=k} (-1)^j \begin{bmatrix} 2n+m+2 \\ j \end{bmatrix} q^{-\binom{j+2+m}{2}} \begin{bmatrix} 2n-2j+1-i \\ i \end{bmatrix} q^{i^2-i}$$

Observing that  $\begin{bmatrix} -r \\ k \end{bmatrix} = (-1)^k q^{-\binom{kr+\binom{k}{2}}{2}} \begin{bmatrix} r+k-1 \\ k \end{bmatrix}$  we get

$$\begin{aligned} & \sum_{i+j=k} (-1)^j \begin{bmatrix} 2n+m+2 \\ j \end{bmatrix} q^{-\binom{j+2+m}{2}} \begin{bmatrix} 2n-2j+1-i \\ i \end{bmatrix} q^{i^2-i} \\ &= (-1)^k \sum_j \begin{bmatrix} 2n+m+2 \\ j \end{bmatrix} q^{-\binom{j+2+m}{2}} q^{2\binom{k-j}{2}} \begin{bmatrix} 2k-2n-2 \\ k-j \end{bmatrix} q^{(k-j)(2n-j-k+1) - \binom{k-j}{2}}. \end{aligned}$$

This can be simplified to  $\begin{bmatrix} 2k+m \\ k \end{bmatrix} = \sum_j q^{(k-j)(2n+m+2-j)} \begin{bmatrix} 2n+m+2 \\ j \end{bmatrix} \begin{bmatrix} 2k-2n-2 \\ k-j \end{bmatrix}$  which is true by the  $q$ -Vandermonde theorem.

In the same way we can prove (5.6).

By changing  $q \rightarrow \frac{1}{q}$  we get

$$\sum_{k=0}^n (-s)^k \begin{bmatrix} 2k+m \\ k \end{bmatrix} \frac{1}{q^{\frac{k(k-3)}{2}}} = \frac{1}{q^{\binom{n+1}{2}}} \sum_{k=0}^n (-s)^{n-k} \begin{bmatrix} 2n+m+2 \\ n-k \end{bmatrix} q^{\frac{k(3k+1)}{2}} q^{-kn} F_{2k+2} \left( s, \frac{1}{q} \right) \quad (5.7)$$

and

$$\sum_{k=0}^n (-s)^k \begin{bmatrix} 2k+m \\ k \end{bmatrix} \frac{1}{q^{\frac{k(k-3)}{2}}} = \frac{1}{q^{\binom{n}{2}}} \sum_{k=0}^n (-s)^{n-k} \begin{bmatrix} 2n+m+1 \\ n-k \end{bmatrix} q^{\frac{k(3k-1)}{2}} q^{-kn} F_{2k+1} \left( s, \frac{1}{q} \right). \quad (5.8)$$

Applying  $L$  we get the identity

$$\begin{aligned} \sum_{k=0}^n \begin{bmatrix} 2k+m \\ k \end{bmatrix} \frac{1}{q^{\frac{k(k-3)}{2}}} g_{2k+1,1}(q) &= \frac{1}{q^{\binom{n+1}{2}}} \sum_{k=0}^n (-1)^k \begin{bmatrix} 2n+m+2 \\ n-k \end{bmatrix} q^{\binom{k+1}{2}-kn} g_{2n+1,k+1}(q) \\ &= \frac{1}{q^{\binom{n}{2}}} \sum_{k=0}^n (-1)^k \begin{bmatrix} 2n+m+1 \\ n-k \end{bmatrix} q^{\binom{k+1}{2}-kn} g_{2n,k+1}(q). \end{aligned} \quad (5.9)$$

For  $n=1$  this reduces to

$$1 + q[m+2] = \frac{[m+4]-1}{q} = [m+3]$$

and for  $n=2$  we get

$$1 + q[m+2] + q \begin{bmatrix} m+4 \\ 2 \end{bmatrix} [2] = \begin{bmatrix} m+6 \\ 2 \end{bmatrix} \frac{[2]}{q^3} - \begin{bmatrix} m+6 \\ 1 \end{bmatrix} \frac{[3]}{q^4} + \frac{[3]}{q^4} = \begin{bmatrix} m+5 \\ 2 \end{bmatrix} \frac{[2]}{q} - \begin{bmatrix} m+5 \\ 1 \end{bmatrix} \frac{1}{q}.$$

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