A CURIOUS q-ANALOGUE OF HERMITE POLYNOMIALS

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Abstract. Two well-known q-Hermite polynomials are the continuous and discrete q-Hermite polynomials. In this paper we consider a new family of q-Hermite polynomials and prove several curious properties about these polynomials. One striking property is the connection with q-Fibonacci and q-Lucas polynomials. The latter relation yields a generalization of the Touchard-Riordan formula.

1. Introduction

The classical Hermite polynomials have two important properties: (i) they form a family of orthogonal polynomials and (ii) are intimately connected with the commutation properties between the multiplication operator x and the differentiation operator D. In contrast to the discrete q-Hermite polynomials, which generalize both aspects, the continuous q-Hermite polynomials generalize only the first one. The purpose of this paper is to introduce a q-analogue which generalizes the second property and establish the missing link with the continuous q-Hermite polynomials. It turns out that these new polynomials are in some sense dual to the continuous q-Hermite polynomials. Moreover, they provide interesting connections with q-Fibonacci and q-Lucas polynomials and the Touchard-Riordan formula for the moments of the continuous q-Hermite polynomials. In order to provide the reader with the necessary background we first collect some well-known results about the classical Hermite polynomials and their known q-analogues.

The normalized Hermite polynomials $H_n(x,s) = s^{n/2} H_n(x/\sqrt{s}, 1)$ $(n \geq 0)$ may be defined by the recurrence relation:

$$H_{n+1}(x,s) = xH_n(x,s) - nsH_{n-1}(x,s), \tag{1.1}$$

with initial values $H_0(x,s) = 1$ and $H_{-1}(x,s) = 0$. By induction, we have

$$H_n(x,s) = (x - s\mathcal{D})^n \cdot 1, \tag{1.2}$$

where $\mathcal{D} = \frac{d}{dx}$ denotes the differentiation operator. It follows that

$$\mathcal{D}\, H_n(x,s) = nH_{n-1}(x,s). \tag{1.3}$$

The Hermite polynomials have the explicit formula (see [1, Chapter 6])

$$H_n(x,s) = \sum_{k=0}^{n} \binom{n}{2k} (-s)^k (2k-1)!!x^{n-2k}.$$

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The first few polynomials are

$$1,\ x,\ -s+x^2,\ -3sx+x^3,\ 3s^2-6sx^2+x^4,\ 15s^2x-10sx^3+x^5.$$

The Hermite polynomials are orthogonal with respect to the linear functional defined by the moments

$$\mu_n = \frac{1}{\sqrt{2\pi}}\int_{-\infty}^{\infty} x^n e^{-x^2/2}dx = \begin{cases} (n-1)!! & \text{if } n \text{ is even,} \\ 0 & \text{otherwise.}\end{cases}$$

In other words, the n-th moment μ_n of the measure of the Hermite polynomials is the number of the *complete matchings* on $[n] := \{1,\dots,n\}$, i.e., $\mu_{2n} = (2n-1)!!$ and $\mu_{2n+1} = 0$.

Consider the rescaled Hermite polynomials $p_n(z,x,s) = H_n(z-x,-s)$, also determined by

$$p_{n+1}(z,x,s) = (z-x)p_n(z,x,s) + snp_{n-1}(z,x,s) \tag{1.4}$$

with initial values $p_0(z,x,s) = 1$ and $p_{-1}(z,x,s) = 0$. Let $\mathcal{F}$ be the linear functional on polynomials in z defined by $\mathcal{F}(p_n(z,x,s)) = \delta_{n,0}$. Then the moments $\mathcal{F}(z^n)$ are again the Hermite polynomials

$$\mathcal{F}(z^n) = (\sqrt{-s})^n \sum_{k=0}^{n} \binom{n}{k} (x/\sqrt{-s})^{n-k}\mu_k = H_n(x,s). \tag{1.5}$$

This is equivalent to saying that the generating function of the Hermite polynomials $H_n(x,s)$ has the following continued fraction expansion:

$$H(z,x,s) = \sum_{n\geq 0} H_n(x,s)z^n = \cfrac{1}{1-xz+\cfrac{sz^2}{1-xz+\cfrac{2sz^2}{1-xz+\cfrac{3sz^2}{\cdots}}}}. \tag{1.6}$$

Two important classes of orthogonal q-analogues of $H_n(x,s)$ are the continuous and the discrete q-Hermite I polynomials, which are both special cases of the Al-Salam–Chihara polynomials. Before we describe these q-Hermite polynomials, we introduce some standard q-notations (see [6]). For $n \geq 1$ let

$$[n] := [n]_q = \frac{1-q^n}{1-q}, \quad [n]_q! = \prod_{k=1}^{n}[k]_q, \quad [2n-1]_q!! = \prod_{k=1}^{n}[2k-1]_q,$$

and $(a;q)_n = (1-a)(1-aq)\cdots(1-aq^{n-1})$ with $(a;q)_0 = 1$. The q-binomial coefficient is defined by

$$\begin{bmatrix} n \\ k \end{bmatrix} := \begin{bmatrix} n \\ k \end{bmatrix}_q = \frac{(q;q)_n}{(q;q)_k(q;q)_{n-k}}$$

for $0 \leq k \leq n$ and zero otherwise.

Recall [8] that the Al-Salam–Chihara polynomials $P_n(x;a,b,c)$ satisfy the three term recurrence:

$$P_{n+1}(x;a,b,c) = (x-aq^n)P_n(x;a,b,c) - (c+bq^{n-1})[n]_q P_{n-1}(x;a,b,c) \tag{1.7}$$

with initial values $P_{-1}(x;a,b,c) = 0$ and $P_0(x;a,b,c) = 1$.

Definition 1. *Let $\mathcal{F}_{a,b,c}$ be the unique linear functional acting on the polynomials in z that satisfies*

$$\mathcal{F}_{a,b,c}(P_n(z;a,b,c)) = \delta_{n,0}. \tag{1.8}$$

Then the continuous q-Hermite polynomials are

$$\tilde{H}_n(x,s|q) = P_n(x;0,0,s) \tag{1.9}$$

and are also the moments (see [8] and Proposition 16):

$$\tilde{H}_n(x,s|q) = \mathcal{F}_{x,-s,0}(z^n). \tag{1.10}$$

The discrete q-Hermite polynomials I are

$$\tilde{h}_n(x,s;q) = P_n(x;0,(1-q)s,0), \tag{1.11}$$

and the discrete q-Hermite polynomials II are

$$\tilde{h}_n(x;q) = (-i)^n \tilde{h}_n(ix,1;q^{-1}). \tag{1.12}$$

It is also convenient to introduce the polynomials

$$h_n(x,s;q) := P_n(0;-x,0,s), \tag{1.13}$$

which are actually a rescaled version of $\tilde{h}_n(x;q)$ (see Section 4). The main purpose of this paper is to study another q-analogue of Hermite polynomials.

Definition 2. *The q-Hermite polynomials $H_n(x,s|q)$ are defined by*

$$H_n(x,s|q) := \mathcal{F}_{x,0,-s}(z^n). \tag{1.14}$$

The q-Hermite polynomials $\tilde{H}_n(x,s|q)$ have, amongst other facts,

(1) orthogonality with an explicit measure,
(2) an explicit 3-term recurrence relation,
(3) explicit expressions,
(4) a combinatorial model using matchings,
(5) are moments for other orthogonal polynomials,
(6) a closed form expression for Hankel determinants,
(7) an explicit Jacobi continued fraction as generating function.

The new q-Hermite polynomials $H_n(x,s|q)$ are not orthogonal, i.e., they do not have (1) and (2). Instead they have a nice q-analogue of the operator formula (1.2) for the ordinary Hermite polynomials (see Theorem 5), the coefficients of the $H_n(x,s|q)$ appear in the inverse matrix of the coefficients in the continuous q-Hermite polynomials (cf. Theorem 6), they have simple connection coefficients with q-Lucas and q-Fibonacci polynomials (cf. Theorem 12). The discrete q-Hermite polynomials $h_n(x,s;q)$ also have (1)–(4), and we will show in Theorem 7 that they are also moments. Moreover, the quotients of two consecutive polynomials $h_n(x,s;q)$ (see Eq.(4.21)) appear as coefficients in the expansion of the S-continued fraction of the generating function of the $H_n(x,s|q)$'s, which leads to a second proof of Theorem 5.

This paper is organized as follows: in Section 2, we recall some well-known facts about the general theory of orthogonal polynomials and show how to prove (1.10) by using this theory; we prove the main properties of $H_n(x,s|q)$ and $h_n(x,s;q)$ in Section 3 and Section 4, respectively; in Section 5 we shall establish the connection between our new q-Hermite polynomials and the

q-Fibonacci and q-Lucas polynomials. This yields, in particular, a generalization of Touchard-Riordan's formula for the moments of continuous q-Hermite polynomials (cf. Proposition 15), first obtained by Josuat-Vergès [10].

2. Some well-known facts

In this section we recall some well-known facts about orthogonal polynomials (see [2, 18, 17]). Let $p_n(x)$ be a sequence of polynomials which satisfies the three term recurrence relation

$$p_{n+1}(x) = (x - b_n)p_n(x) - \lambda_n p_{n-1}(x) \tag{2.1}$$

with initial values $p_0(x) = 1$ and $p_{-1}(x) = 0$.

Define the coefficients $a(n, k)$ $(0 \le k \le n)$ by

$$\sum_{k=0}^{n} a(n, k)p_k(x) = x^n. \tag{2.2}$$

These are characterized by the *Stieltjes tableau*:

$$\begin{aligned} a(0, k) &= \delta_{k,0}, \\ a(n, 0) &= b_0 a(n - 1, 0) + \lambda_1 a(n - 1, 1), \\ a(n, k) &= a(n - 1, k - 1) + b_k a(n - 1, k) + \lambda_{k+1} a(n - 1, k + 1). \end{aligned} \tag{2.3}$$

If $\mathcal{F}$ is the linear functional such that $\mathcal{F}(p_n(x)) = \delta_{n,0}$, then

$$\mathcal{F}(x^n) = a(n, 0). \tag{2.4}$$

The generating function of the moments has the continued fraction expansion

$$\sum_{n\ge 0} \mathcal{F}(x^n)z^n = \cfrac{1}{1 - b_0 z - \cfrac{\lambda_1 z^2}{1 - b_1 z - \cfrac{\lambda_2 z^2}{1 - \cdots}}}. \tag{2.5}$$

The Hankel determinants for the moments are

$$d(n, 0) = \det(\mathcal{F}(z^{i+j}))_{i,j=0}^{n-1} = \prod_{i=1}^{n-1}\prod_{k=1}^{i} \lambda_k, \tag{2.6}$$

and

$$d(n, 1) = \det(\mathcal{F}(z^{i+j+1}))_{i,j=0}^{n-1} = d(n, 0)(-1)^n p_n(0). \tag{2.7}$$

By using the Stieltjes tableau we can give a simple proof of (1.10).

Proposition 3. *The continuous q-Hermite polynomials $\tilde{H}_n(x, s|q)$ defined by* (1.9)*, i.e.,*

$$\tilde{H}_{n+1}(x, s|q) = x\tilde{H}_n(x, s|q) - s[n]_q \tilde{H}_{n-1}(x, s|q), \tag{2.8}$$

are the moments of the measure of the orthogonal polynomials $p_n(z) := P_n(z; x, -s, 0)$ defined by the recurrence

$$p_{n+1}(z) = (z - xq^n)p_n(z) + sq^{n-1}[n]_q p_{n-1}(z). \tag{2.9}$$

Proof. Let $b_n = q^n x$ and $\lambda_{n+1} = (-s)q^n[n+1]_q$ for $n \geq 0$. It is sufficient to verify that in this case (2.3) is satisfied with

$$a(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} \tilde{H}_{n-k}(z,s|q). \tag{2.10}$$

This is clearly equivalent to (2.8). ∎

As a consequence of the previous proposition, and in view of (2.6) and (2.7), we can derive immediately the Hankel determinants

$$d(n,0) = (-s)^{\binom{n}{2}} q^{\binom{n}{3}} \prod_{j=0}^{n-1} [j]_q!, \tag{2.11}$$

and

$$d(n,1) = d(n,0)r(n), \tag{2.12}$$

where $r(n) = (-1)^n p_n(0; x, -s, 0)$.

Note that the polynomials $r(n)$ satisfy

$$r(n) = q^{n-1} x r(n-1) + q^{n-2} s [n-1]_q r(n-2).$$

This implies that

$$r(n) = q^{\frac{n(n-2)}{2}} \tilde{H}_n \left(x\sqrt{q}, -s | \frac{1}{q} \right). \tag{2.13}$$

The first few polynomials of the sequence $\tilde{H}_n(x,s|q)$ are

$$1,\ x,\ -s + x^2,\ x(-(2+q)s + x^2),\ (1+q+q^2)s^2 - (3+2q+q^2)sx^2 + x^4,$$
$$x((3+4q+4q^2+3q^3+q^4)s^2 - (4+3q+2q^2+q^3)sx^2 + x^4).$$

From their recurrence relation we see that

$$\tilde{H}_{2n}(0,s|q) = (-s)^n [2n-1]_q!! \quad \text{and} \quad \tilde{H}_{2n+1}(0,s|q) = 0.$$

3. The q-Hermite polynomials $H_n(x,s|q)$

By (1.8) the q-Hermite polynomials $H_n(x,s|q)$ are the moments of the measure of the orthogonal polynomials $P_n(z)$ satisfying the recurrence:

$$P_{n+1}(z) = (z - xq^n)P_n(z) + s[n]_q P_{n-1}(z). \tag{3.1}$$

Recall [13, p.80] that the Al-Salam–Chihara polynomials $Q_n(x) := Q_n(x; \alpha, \beta)$ satisfy the three term recurrence:

$$Q_{n+1}(x) = (2x - (\alpha+\beta)q^n)Q_n(x) - (1-q^n)(1-\alpha\beta q^{n-1})Q_{n-1}(x), \tag{3.2}$$

with $Q_0(x) = 1$ and $Q_{-1}(x) = 0$. They have the following explicit formulas:

$$Q_n(x;\alpha,\beta|q) = (\alpha e^{i\theta};q)_n e^{-i\theta} {}_2\phi_1 \left(\begin{matrix} q^{-n}, & \beta e^{-i\theta} \\ \alpha^{-1}q^{-n+1}e^{-i\theta} & \end{matrix} |q; \alpha^{-1} q e^{i\theta} \right), \tag{3.3}$$

where $x = \cos\theta$.

Comparing (3.1) and (3.2) we have $P_n(z) = \frac{1}{(2a)^n} Q_n(az; \alpha, 0)$ with

$$a = \frac{1}{2}\sqrt{\frac{q-1}{s}} \quad \text{and} \quad \alpha = x\sqrt{\frac{q-1}{s}}. \tag{3.4}$$

Using the known formula for Al-Salam–Chihara polynomials we obtain

$$\begin{aligned} P_n(z) &= \frac{1}{(2a\alpha)^n}\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}q^k\prod_{i=0}^{k-1}(1+\alpha^2q^{2i}-2q^ia\alpha z) \\ &= \left(\frac{s}{x(q-1)}\right)^n\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}\left(\frac{-q}{s}\right)^k\prod_{i=0}^{k-1}\left((q-1)q^ixz-s-(q-1)q^{2i}x^2\right). \end{aligned} \tag{3.5}$$

The first few polynomials $P_n(z)$ are

$$\begin{aligned} P_1(z) &= z - x, \\ P_2(z) &= z^2 - x(1+q)z + (s+qx^2), \\ P_3(z) &= z^3 - x[3]_q z^2 + (2s+qs+q[3]_q x^2)z - (s+qs+q^2s+q^3x^2)x. \end{aligned}$$

A matching m of $\{1, 2, \ldots, n\}$ is a set of pairs (i, j) such that $i < j$ and $i, j \in [n]$. Each pair (i, j) is called an edge of the matching. Let $\mathrm{ed}(m)$ be the number of edges of m, so $n - 2\mathrm{ed}(m)$ is the number of unmatched vertices. Two edges (i, j) and (k, l) have a crossing if $i < k < j < l$ or $k < i < l < j$. Let $\mathrm{cr}(m)$ be the number of crossing numbers in the matching m. Using the combinatorial theory of Viennot [17], Ismail and Stanton [8, Theorem 6] gave a combinatorial interpretation of the moments of Al-Salam–Chihara polynomials. In particular we derive the following result from [8, Theorem 6].

Lemma 4. *The moments of the measure of the orthogonal polynomials $\{P_n(x)\}$ are the generating functions for all matchings m of $[n]$:*

$$\mathcal{F}_{x,0,-s}(z^n) = \sum_m x^{n-2\mathrm{ed}(m)}(-s)^{\mathrm{ed}(m)}q^{\mathrm{c}(m)+\mathrm{cr}(m)}, \tag{3.6}$$

where $c(m) = \sum_{\text{a−vertices}} |\{\text{edges } i < j : \ i < a < j\}|$ and the sum extends over all matchings m of $[n]$.

Let $M(n, k)$ be the set of matchings of $\{1, \ldots, n\}$ with k unmatched vertices. Then

$$\mathcal{F}_{x,0,-s}(z^n) = \sum_k c(n,k,q)x^k(-s)^{\frac{n-k}{2}}, \tag{3.7}$$

where

$$c(n,k,q) = \sum_{m\in M(n,k)} q^{\mathrm{c}(m)+\mathrm{cr}(m)}. \tag{3.8}$$

It is easy to verify that

$$c(n,k,q) = c(n-1,k-1,q) + [k+1]_q c(n-1,k+1,q) \tag{3.9}$$

with $c(0, k, q) = \delta_{k,0}$ and $c(n, 0, q) = c(n-1, 1, q)$. Indeed, if n is an unmatched vertex then for the restriction m_0 of m to $[n-1]$ we get $c(m_0) = c(m)$ and $\mathrm{cr}(m_0) = \mathrm{cr}(m)$. If n is matched with $m(n)$, such that there are i unmatched vertices and j endpoints of edges which cross the edge $(m(n),\ n)$ between $m(n)$ and n, then $c(m) = c(m_0) + i - j$ and $\mathrm{cr}(m) = \mathrm{cr}(m_0) + j$. Thus $c(m) + \mathrm{cr}(m) = c(m_0) + \mathrm{cr}(m_0) + i$. Since each i with $0 \le i \le k$ can occur we get (3.9).

Let now $\mathcal{D}_q$ be the q-derivative operator defined by

$$\mathcal{D}_q f(z) = \frac{f(z) - f(qz)}{(1-q)z}.$$

We have then the following q-analogue of (1.2).

Theorem 5. *The q-Hermite polynomials $H_n(x, s|q)$, defined as moments $\mathcal{F}_{x,0,-s}(z^n)$, have the following operator formula:*

$$H_n(x, s|q) = (x - s\mathcal{D}_q)^n \cdot 1. \tag{3.10}$$

Proof. We know that

$$H_n(x, s|q) = \sum_k c(n, k, q) x^k (-s)^{\frac{n-k}{2}}, \tag{3.11}$$

where $c(n, k, q)$ satisfies (3.9). Therefore

$$\begin{aligned} H_n(x, s|q) &= \sum_k c(n-1, k-1, q) x^k (-s)^{\frac{n-k}{2}} + \sum_k [k+1]_q c(n-1, k+1, q) x^k (-s)^{\frac{n-k}{2}} \\ &= x H_{n-1}(x, s|q) - s\mathcal{D}_q H_{n-1}(x, s|q). \end{aligned}$$

The result then follows by induction on n. ∎

Remark. It should be noted that the method of Varvak [16] (see also [10]) can also be applied to prove Theorem 5. In fact her method proves first that $(x - sD_q)^n \cdot 1$ is a generating function of some rook placements, which is then shown to count involutions with respect to the statistic $\mathrm{c}(m) + \mathrm{cr}(m)$ (see [16, Theorem 6.4]). We will give another proof of (3.10) by using continued fractions, see the remark after Theorem 9.

The first few polynomials $H_n(x, s|q)$ are

$$\begin{gathered} 1,\ x,\ -s + x^2,\ x(-(2+q)s + x^2),\ (2+q)s^2 - (3 + 2q + q^2)sx^2 + x^4, \\ x((5 + 6q + 3q^2 + q^3)s^2 - (4 + 3q + 2q^2 + q^3)sx^2 + x^4), \ldots \end{gathered}$$

Let

$$\tilde{H}_n(x, s|q) = \sum_k b(n, k, q) x^k (-s)^{\frac{n-k}{2}}. \tag{3.12}$$

Theorem 6. *The matrices $(c(i, j, q))_{i,j=0}^{n-1}$ and $(b(i, j, q)(-1)^{\frac{i-j}{2}})_{i,j=0}^{n-1}$ are mutually inverse.*

Proof. We first show by induction that

$$\tilde{H}_n(x + s\mathcal{D}_q, s|q) \cdot 1 = x^n. \tag{3.13}$$

For this is obvious for $n = 0$. If it is already shown for n we get

$$\begin{aligned} \tilde{H}_{n+1}(x + s\mathcal{D}_q, s|q) \cdot 1 &= (x + s\mathcal{D}_q)\tilde{H}_n(x + s\mathcal{D}_q, s|q) \cdot 1 - s[n]_q \tilde{H}_{n-1}(x + s\mathcal{D}_q, s|q) \cdot 1 \\ &= (x + s\mathcal{D}_q)x^n - s[n]_q x^{n-1} = x^{n+1}. \end{aligned}$$

On the other hand we have

$$\begin{aligned}\tilde{H}_n(x+s\mathcal{D}_q,s|q)\cdot 1 &= \sum_{k=0}^{n} b(n,k,q)(-s)^{\frac{n-k}{2}}(x+s\mathcal{D}_q)^k\cdot 1\\ &= \sum_{k=0}^{n} b(n,k,q)(-s)^{\frac{n-k}{2}}\sum_{j=0}^{k} c(k,j,q)s^{\frac{k-j}{2}}x^j\\ &= \sum_{j=0}^{n} s^{\frac{n-j}{2}}x^j\sum_{k=j}^{n} b(n,k,q)(-1)^{\frac{n-k}{2}}c(k,j,q). \end{aligned} \tag{3.14}$$

The result then follows by comparing (3.13) and (3.14). ∎

Remark. If we set $q=0$ then (3.9) reduces to the well-known Catalan triangle (see [2, Chap. 7]), which implies

$$\begin{aligned} c(2n,0,0) &= C_n = \frac{1}{n+1}\binom{2n}{n},\\ c(2n,2k,0) &= \frac{2k+1}{n+k+1}\binom{2n}{n-k} = \binom{2n}{n-k}-\binom{2n}{n-k-1},\\ c(2n+1,2k+1,0) &= \frac{2k+2}{n+k+2}\binom{2n+1}{n-k} = \binom{2n+1}{n-k}-\binom{2n+1}{n-k-1}. \end{aligned}$$

The recurrence (3.1) implies that the Hankel determinants of $H_n(x,s|q)$ are

$$\det(H_{i+j}(x,s|q))_{i,j}^{n-1} = (-s)^{\binom{n}{2}}\prod_{j=0}^{n-1}[j]_q! \tag{3.15}$$

and

$$\det(H_{i+j+1}(x,s;q))_{i,j}^{n-1} = h_n(x,-s;q)(-s)^{\binom{n}{2}}\prod_{j=0}^{n-1}[j]_q!, \tag{3.16}$$

where

$$h_n(x,-s;q) = (-1)^n P_n(0) = \left(\frac{s}{x(1-q)}\right)^n\sum_{k=0}^{n}\frac{(q^{-n};q)_k}{(q;q)_k}q^k\prod_{i=0}^{k-1}(1+x^2(q-1)q^{2i}/s).$$

4. The rescaled discrete q-Hermite polynomials II

By definition (1.13) and (1.7) we have

$$h_{n+1}(x,s;q) = q^n x h_n(x,s;q) - [n]_q s h_{n-1}(x,s;q). \tag{4.1}$$

Comparing with the three-term recurrence relation for the discrete q-Hermite polynomials II (see (1.12) and (1.7)), we derive

$$h_n(x,s;q) = q^{\binom{n}{2}}\sqrt{s^n}\tilde{h}_n\left(\frac{x}{\sqrt{s}};q\right) \tag{4.2}$$

$$= \sum_{k=0}^{n} q^{\binom{n-2k}{2}}\begin{bmatrix} n \\ 2k \end{bmatrix}[2k-1]_q!!(-s)^k x^{n-2k}, \tag{4.3}$$

where the last expression follows from the known formula for $\tilde{h}_n(x;q)$.

Since $\mathcal{D}_q(fg) = \mathcal{D}_q(f)g + f(qx)\mathcal{D}_q(g)$ and $\mathcal{D}_q(x) = 1$, we see that

$$\mathcal{D}_q(h_{n+1}(x)) = q^n x\mathcal{D}_q(h_n(x)) + q^n h_n(qx) - [n]_q s\mathcal{D}_q(h_{n-1}(x)).$$

We find by induction on n that

$$\mathcal{D}_q h_n(x,s;q) = [n]_q h_{n-1}(qx,s;q). \tag{4.4}$$

The first few polynomials $h_n(x,s;q)$ are

$$1, \quad x, \quad qx^2 - s, \quad q^3x^3 - s[3]_q x, \quad q^6x^4 - s(q^5+q^4+2q^3+q^2+q)x^2 + s^2[3]_q.$$

The following result shows that the polynomials $h_n(x,s;q)$ are moments of some orthogonal polynomials.

Theorem 7. *The generating function of $h_n(x,s;q)$ has the continued fraction expansion:*

$$\sum_{m\geq 0} h_n(x,s;q)t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1 t^2}{1-b_1t-\cfrac{\lambda_2 t^2}{1-b_2t-\cfrac{\lambda_3 t^2}{1-\cdots}}}},$$

with

$$b_n = q^{n-1}(q^n+q^{n+1}-1)x \quad \text{and} \quad \lambda_n = -q^{n-1}[n]_q(s+q^{2n-2}(1-q)x^2). \tag{4.5}$$

Proof. To prove this it suffices to show that the Stieltjes tableau (2.3) is satisfied with

$$a(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} h_{n-k}(q^k x,s;q).$$

This is easily verified. ∎

Using (2.6) and (2.7), Theorem 7 implies the following Hankel determinant evaluations:

$$\det(h_{i+j}(x,s;q))_{i,j}^{n-1} = (-1)^{\binom{n}{2}}q^{\binom{n}{3}}\prod_{j=0}^{n-1}\left([j]_q!(s+q^{2j}(1-q)x^2)^{n-1-j}\right) \tag{4.6}$$

and

$$\frac{\det(h_{i+j+1}(x,s;q))_{i,j}^{n-1}}{\det(h_{i+j}(x,s;q))_{i,j}^{n-1}} = w(n), \tag{4.7}$$

where $w(n)$ satisfies

$$w(n+1) = q^{n-1}(q^n + q^{n+1} - 1)xw(n) + q^{n-1}[n]_q(s + q^{2n-2}(1-q)x^2)w(n-1).$$

It is easily verified that

$$w(n) = \sum_{k=0}^{n} q^{2\binom{n-k}{2}} \begin{bmatrix} n \\ 2k \end{bmatrix} [2k-1]_q!! s^k x^{n-2k} \tag{4.8}$$

satisfies the same recurrence with the same initial values.

Lemma 8. *Let $L_n(x) := h_n(x, (1-q)s; q)$. Then*

$$sL_n(x) + xL_{n+1}(x) = (x^2 + s)L_n(qx). \tag{4.9}$$

Proof. First we note that the constant terms of both sides of (4.9) are equal to $sL_n(0)$. So it suffices to show that the derivatives of the two sides are equal. Applying $\mathcal{D}_q$ to (4.9) and using (4.4) we obtain, after replacing x by x/q,

$$s[n]L_{n-1}(x) + xq[n-1]L_n(x) + L_{n+1}(x) = (x^2 + s)q[n]L_{n-1}(qx).$$

Since $L_{n+1}(x) = q^n xL_n(x) - (1-q^n)sL_{n-1}(x)$, we can rewrite the above equation as follows:

$$sL_{n-1}(x) + xL_n(x) = (x^2 + s)L_{n-1}(qx). \tag{4.10}$$

The proof is thus completed by induction on n. ∎

We shall prove the following Jacobi continued fraction expansion for the generating function of $(x + (1-q)s\mathcal{D}_q)^n \cdot 1$. This is equivalent to Theorem 5.

Theorem 9. *Let $T_n(x, s) = (x + (1-q)s\mathcal{D}_q)^n \cdot 1$. Then*

$$\sum_{n\geq 0} T_n(x, s)t^n = \cfrac{1}{1 - b_0 t - \cfrac{\lambda_1 t^2}{1 - b_1 t - \cfrac{\lambda_2 t^2}{1 - \cdots}}}, \tag{4.11}$$

where the coefficients are

$$b_n = q^n x, \quad \textit{for } n \geq 0; \quad \textit{and} \quad \lambda_n = (1-q^n)s, \quad \textit{for } n \geq 1. \tag{4.12}$$

Proof. Since $T_n(x, s) = (x + (1-q)s\mathcal{D}_q)T_{n-1}(x, s)$, we have

$$T_n(x, s) = (x + \frac{s}{x})T_{n-1}(x, s) - \frac{s}{x}T_{n-1}(qx, s).$$

Equivalently the generating function $G(x, t) = \sum_{n\geq 0} T_n(x, s)t^n$ satisfies the functional equation:

$$\left(1 - \frac{x^2 + s}{x}t\right) G(x, t) = 1 - \frac{s}{x}tG(qx, t). \tag{4.13}$$

Suppose that

$$G(x,t) = \cfrac{1}{1 - \cfrac{c_1 t}{1 - \cfrac{c_2 t}{1 - \cfrac{c_3 t}{1 - \cdots}}}}, \tag{4.14}$$

where $c_n = (g_n - 1)g_{n-1}A$ with $A := A(x) = -\frac{x^2+s}{x}$ and $g_i := g_i(x)$.

Substituting (4.14) in (4.13) and then replacing t by t/A we obtain

$$\cfrac{1+t}{1 - \cfrac{(g_1-1)t}{1 - \cfrac{(g_2-1)g_1 t}{1 - \cfrac{(g_3-1)g_2 t}{1 - \cfrac{(g_4-1)g_3 t}{1 - \dots}}}}} = 1 + \cfrac{\frac{s}{x^2+s} t}{1 - \cfrac{(g_1'-1)\frac{A'}{A} t}{1 - \cfrac{(g_2'-1)g_1' \frac{A'}{A} t}{1 - \cfrac{(g_3'-1)g_2' \frac{A'}{A} t}{1 - \dots}}}}, \tag{4.15}$$

where $A' := A(qx)$ and $g_i' := g_i(qx)$. Comparing this with Wall's formula (see [12]):

$$\cfrac{1+z}{1 - \cfrac{(g_1-1)z}{1 - \cfrac{(g_2-1)g_1 z}{1 - \cfrac{(g_3-1)g_2 z}{1 - \cfrac{(g_4-1)g_3 z}{1 - \dots}}}}} = 1 + \cfrac{g_1 z}{1 - \cfrac{(g_1-1)g_2 z}{1 - \cfrac{(g_2-1)g_3 z}{1 - \cfrac{(g_3-1)g_4 z}{1 - \dots}}}}, \tag{4.16}$$

we derive that $g_0 = 1$ and for $n \geq 1$,

$$\begin{cases} g_{2n} = \dfrac{A'}{A} \dfrac{g_{2n-1}' - 1}{g_{2n-1} - 1} g_{2n-2}', \\ g_{2n+1} = \dfrac{A'}{A} \dfrac{g_{2n}' - 1}{g_{2n} - 1} g_{2n-1}'. \end{cases} \tag{4.17}$$

For example,

$$g_1 = \frac{s}{x^2+s}, \qquad g_3 = \frac{A'}{A}\frac{g_2'-1}{g_2-1} g_1' = \frac{s}{x^2+s}\frac{1}{q},$$

$$g_2 = \frac{A'}{A}\frac{g_1'-1}{g_1-1} = q, \qquad g_4 = \frac{A'}{A}\frac{g_3'-1}{g_3-1} g_2' = \frac{-s+qs+q^3x^2}{-s+qs+qx^2}.$$

In general we have the following result

$$\begin{cases} g_{2n} = \dfrac{sL_n(x) + xL_{n+1}(x)}{(x^2+s)L_n(x)}, \\ g_{2n+1} = \dfrac{sL_n(x)}{sL_n(x) + xL_{n+1}(x)}, \end{cases} \qquad (n \geq 0). \tag{4.18}$$

This can be verified by induction on n. Suppose that the formula (4.18) is true for $n \geq 0$. We prove that the formula holds for $n+1$. By (4.17) we have

$$g_{2n+2} = \frac{A'}{A}\frac{g'_{2n+1}-1}{g_{2n+1}-1}g'_{2n} = \frac{sL_n(x)+xL_{n+1}(x)}{(x^2+s)L_{n+1}(x)}\frac{L_{n+1}(qx)}{L_n(qx)}.$$

It follows from Lemma 1 that

$$g_{2n+2} = \frac{sL_{n+1}(x)+xL_{n+2}(x)}{(x^2+s)L_{n+1}(x)}. \tag{4.19}$$

Since

$$L_{n+1}(x) - xL_n(x) = (q^n-1)(xL_n(x)+sL_{n-1}(x)), \tag{4.20}$$

the verification for g_{2n+3} is then straightforward. We derive from (4.14) and (4.18) that

$$\begin{cases} c_{2n} = (g_{2n}-1)g_{2n-1}A = (1-q^n)s\dfrac{L_{n-1}(x)}{L_n(x)}, & \text{for } n \geq 1; \\ c_{2n+1} = (g_{2n+1}-1)g_{2n}A = \dfrac{L_{n+1}(x)}{L_n(x)}, & \text{for } n \geq 0. \end{cases} \tag{4.21}$$

Invoking the *contraction formula* (see [19]), which transforms a S-continued fraction to a J-continued fraction,

$$\cfrac{1}{1-\cfrac{c_1z}{1-\cfrac{c_2z}{1-\cfrac{c_3z}{1-\cfrac{c_4z}{\cdots}}}}} = \cfrac{1}{1-c_1z-\cfrac{c_1c_2z^2}{1-(c_2+c_3)z-\cfrac{c_3c_4z^2}{\cdots}}}, \tag{4.22}$$

we obtain

$$\begin{cases} b_n = \dfrac{h_{n+1}(x,(1-q)s;q)}{h_n(x,(1-q)s;q)} + (1-q^n)s\dfrac{h_{n-1}(x,(1-q)s;q)}{h_n(x,(1-q)s;q)} = q^nx, \\ \lambda_n = \dfrac{h_n(x,(1-q)s;;q)}{h_{n-1}(x,(1-q)s;q)}\cdot(1-q^n)s\dfrac{h_{n-1}(x,(1-q)s;;q)}{h_n(x,(1-q)s;;q)} = (1-q^n)s. \end{cases} \tag{4.23}$$

This completes the proof. ∎

Remark. Instead of the contraction formula (4.22), we can also proceed as follows. Define a table $(A(n,k))_{n,k\geq 0}$ by

$$\begin{aligned} &A(0,k) = \delta_{k,0}, \\ &A(n,0) = c_1A(n-1,1), \\ &A(n,k) = A(n-1,k-1) + c_{k+1}A(n-1,k+1). \end{aligned} \tag{4.24}$$

In this case $A(2n,2k+1) = A(2n+1,2k) = 0$ for all n,k. If we define

$$a(n,k) = A(2n,2k),$$

then it is easily verified that $a(n,k)$ satisfy (2.3) with

$$b_0 = c_1, \quad b_n = c_{2n} + c_{2n+1}, \quad \lambda_n = c_{2n}c_{2n-1}. \tag{4.25}$$

Substituting the values in (4.21) for c_n we obtain (4.23). Therefore

$$\sum_n A(2n,0)t^n = \sum_n a(n,0)t^n = \sum_n T_n(x,s)t^n.$$

As another application of this remark we prove the following result.

Proposition 10. *Let* $w_n(m,q) = q^{\frac{n((2m+1)n+1)}{2}}$. *Then*

$$\sum_{m\geq 0} w_n(m,q)t^n = \cfrac{1}{1-b_0t-\cfrac{\lambda_1t^2}{1-b_1t-\cfrac{\lambda_2t^2}{1-b_2t-\cfrac{\lambda_3t^2}{1-\cdots}}}},$$

where

$$b_n = q^{(2m+1)n-m}(q^{(2m+1)n}-1) + q^{(2m+1)(2n+1)-m},$$
$$\lambda_n = q^{(2m+1)(3n-1)-2m}(q^{(2m+1)n}-1).$$

Proof. Let

$$A(2n,2k) = \frac{w_n(m,q)}{w_k(m,q)}\begin{bmatrix} n \\ k \end{bmatrix}_{q^{2m+1}} \quad \text{and} \quad A(2n+1,2k+1) = \frac{w_{n+1}(m,q)}{w_{k+1}(m,q)}\begin{bmatrix} n \\ k \end{bmatrix}_{q^{2m+1}}.$$

Then it is easily verified that the table (4.24) holds with $c_{2n} = q^{(2m+1)n-m}(q^{(2m+1)n}-1)$ and $c_{2n+1} = q^{(2m+1)(2n+1)-m}$. Therefore

$$\sum_n A(2n,0)t^n = \sum_n a(n,0)t^n = \sum_n w_n(m,q)t^n. \blacksquare$$

5. Connection with q-Fibonacci polynomials and q-Lucas polynomials

In this section we derive some explicit expansion formulae for the q-Hermite polynomials $H_n(x,s|q)$ in terms of q-Fibonacci polynomials and q-Lucas polynomials. We first recall some basic results about the latter polynomials in the $q=1$ case and then define their q-analogue with the ordinary Fibonacci and Lucas polynomials and q-operator $\mathcal{D}_q$.

The Lucas polynomials are defined by the recurrence

$$l_n(x,s) = xl_{n-1}(x,s) + sl_{n-2}(x,s) \quad \text{for} \quad n>2,$$

with initial values $l_1(x,s)=x$ and $l_2(x,s)=x^2+2s$. They have the explicit formula

$$l_n(x,s) = \sum_{2k\leq n} \frac{n}{n-k}\binom{n-k}{k} s^k x^{n-2k} \quad (n>0). \tag{5.1}$$

Furthermore we define $l_0(x,s)=1$. Note that this definition differs from the usual one in which $l_0(x,s)=2$.

The Fibonacci polynomials are defined by

$$f_n(x,s) = xf_{n-1}(x,s) + sf_{n-2}(x,s)$$

with $f_0(x,s)=0$ and $f_1(x,s)=1$. They have the explicit formula

$$f_n(x,s)=\sum_{k=0}^{\lfloor\frac{n-1}{2}\rfloor}\binom{n-1-k}{k}s^kx^{n-1-2k}. \tag{5.2}$$

We first establish the following inversion of (5.1) and (5.2), which will be used in the proof of Theorem 12.

Lemma 11.

$$x^n=\sum_{2k\le n}\binom{n}{k}s^kl_{n-2k}(x,-s), \tag{5.3}$$

$$x^n=\sum_{2k\le n+1}\left(\binom{n}{k}-\binom{n}{k-1}\right)s^kf_{n+1-2k}(x,-s). \tag{5.4}$$

Proof. Recall the Tchebyshev inverse relations [15, p. 54-62]:

$$b_n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\frac{n}{n-k}\binom{n-k}{k}a_{n-2k}\Longleftrightarrow a_n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\binom{n}{k}b_{n-2k}, \tag{5.5}$$

where $a_0=b_0=1$, and

$$b_n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}(-1)^k\binom{n-k}{k}a_{n-2k}\Longleftrightarrow a_n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\left[\binom{n}{k}-\binom{n}{k-1}\right]b_{n-2k}. \tag{5.6}$$

We derive immediately (5.3) from (5.1) and (5.5). Clearly (5.2) is equivalent to the left identity in (5.6) with $a_n=\left(\frac{x}{\sqrt{s}}\right)^n$ and $b_n=\frac{f_{n+1}(x,-s)}{(\sqrt{s})^n}$. By inversion we find

$$x^n=\sum_{k=0}^{\lfloor\frac{n}{2}\rfloor}\left(\binom{n}{k}-\binom{n}{k-1}\right)s^kf_{n+1-2k}(x,-s). \tag{5.7}$$

Now, noticing that

- if n is odd, then $\binom{n}{k}=\binom{n}{k-1}$ for $k=\lfloor\frac{n+1}{2}\rfloor$,
- if n is even, then $\lfloor\frac{n+1}{2}\rfloor=\lfloor\frac{n}{2}\rfloor$,

we see the equivalence of (5.4) and (5.7). ∎

Define the q-Lucas and q-Fibonacci polynomials by

$$L_n(x,s)=l_n(x+(q-1)s\mathcal{D}_q,s)\cdot 1, \tag{5.8}$$

$$F_n(x,s)=f_n(x+(q-1)s\mathcal{D}_q,s)\cdot 1. \tag{5.9}$$

It is known (see [3] and [4]) that they have the explicit formulae

$$L_n(x,s) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} q^{\binom{k}{2}} \frac{[n]}{[n-k]} \begin{bmatrix} n-k \\ k \end{bmatrix} s^k x^{n-2k}, \tag{5.10}$$

$$F_n(x,s) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} q^{\binom{k+1}{2}} \begin{bmatrix} n-1-k \\ k \end{bmatrix} s^k x^{n-1-2k}, \tag{5.11}$$

for $n > 0$, with $L_0(x,s) = 1$ and $F_0(x,s) = 0$.

Theorem 12. *We have*

$$H_n(x,(q-1)s|q) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{n}{k} s^k L_{n-2k}(x,-s) \tag{5.12}$$

$$= \sum_{k=0}^{\lfloor \frac{n+1}{2} \rfloor} \left(\binom{n}{k} - \binom{n}{k-1} \right) s^k F_{n+1-2k}(x,-s). \tag{5.13}$$

Proof. Since

$$L_n(x,-s) = l_n(x-(q-1)s\mathcal{D}_q, s) \cdot 1,$$
$$F_n(x,-s) = f_n(x-(q-1)s\mathcal{D}_q, s) \cdot 1,$$

the theorem follows by applying the homomorphism $x \mapsto x-(q-1)s\mathcal{D}_q$ to (5.3) and (5.4). ∎

We derive some consequences of the formula (5.13).

Corollary 13. *We have*

$$H_n(1,q-1|q) = \sum_{k=-\lfloor (n+1)/3 \rfloor}^{\lfloor n/3 \rfloor} (-1)^k q^{\frac{k(3k+1)}{2}} \binom{n}{\lfloor \frac{n-3k}{2} \rfloor}. \tag{5.14}$$

Proof. Let $r(j) = \frac{j(3j+1)}{2}$. Then, it follows from [3] that

$$F_{3n}(1,-1) = \sum_{j=-n}^{n-1} (-1)^j q^{r(j)}, \quad F_{3n+1}(1,-1) = F_{3n+2}(1,-1) = \sum_{j=-n}^{n} (-1)^j q^{r(j)},$$

or

$$F_n(1,-1) = \sum_{-n \le 3j \le n-1} (-1)^j q^{r(j)}.$$

Let $w(n) = \sum_{k\ge 0} \left(\binom{n}{k} - \binom{n}{k-1} \right) F_{n+1-2k}(1,-1)$. Consider a fixed term $(-1)^j q^{r(j)}$. This term occurs in $F_n(1,-1)$ if $-\frac{n}{3} \le j \le \frac{n-1}{3}$. We are looking for all k, such that this term occurs in $F_{n+1-2k}(1,-1)$. If $j \ge 0$ then the largest such number is $k_0 = \left\lfloor \frac{n-3j}{2} \right\rfloor$. For $j \le \frac{n-2k}{3}$ is equivalent with $k \le k_0$. Therefore the coefficient of $(-1)^j q^{r(j)}$ in $w(n)$ is $\sum_{k=0}^{k_0}\left(\binom{n}{k}-\binom{n}{k-1}\right)=\binom{n}{k_0}$. If $j<0$ then $-\frac{n+1-2k}{3}\le j$ is equivalent with $k\le\lfloor\frac{n+1+3j}{2}\rfloor$. This gives

$$H_n(1,q-1|q)=\sum_{j=0}^{\lfloor n/3\rfloor}(-1)^j q^{\frac{j(3j+1)}{2}}\binom{n}{\left\lfloor\frac{n-3j}{2}\right\rfloor}+\sum_{j=1}^{\lfloor (n+1)/3\rfloor}(-1)^j q^{\frac{j(3j-1)}{2}}\binom{n}{\left\lfloor\frac{n-3j+1}{2}\right\rfloor}. \tag{5.15}$$

Now, we have

$$\binom{n}{\left\lfloor\frac{n-3j+1}{2}\right\rfloor}=\binom{n}{\left\lfloor\frac{n+3j}{2}\right\rfloor}$$

because $\left\lfloor\frac{n-3j+1}{2}\right\rfloor+\left\lfloor\frac{n+3j}{2}\right\rfloor=n$. So (5.15) is equivalent to (5.14). ∎

Corollary 14. *We have*

$$H_{2n}\left(1,\frac{q-1}{q}|q\right)=q^{-n}\sum_{j=-n}^{n}\left(\binom{2n}{n-3j}-\binom{2n}{n-3j-1}\right)q^{2j(3j+1)}, \tag{5.16}$$

and

$$H_{2n+1}(1,\frac{q-1}{q}|q)=q^{-n}\sum_{j=-n}^{n}\left(\binom{2n+1}{n-3j}-\binom{2n+1}{n-3j-1}\right)q^{2j(3j+2)}. \tag{5.17}$$

Proof. Note that

$$H_{2n}(1,\frac{q-1}{q}|q)=\frac{1}{q^n}\sum_{k=0}^{n}\left(\binom{2n}{n-k}-\binom{2n}{n-k-1}\right)q^k F_{2k+1}(1,-\frac{1}{q}), \tag{5.18}$$

$$H_{2n+1}(1,\frac{q-1}{q}|q)=\frac{1}{q^n}\sum_{k=0}^{n+1}\left(\binom{2n+1}{n+1-k}-\binom{2n+1}{n-k}\right)q^{k-1} F_{2k}(1,-\frac{1}{q}). \tag{5.19}$$

Recall (see [3]) that

$$F_{3n}(1,-\frac{1}{q})=0,\quad F_{3n+1}(1,-\frac{1}{q})=(-1)^n q^{r(n)},\quad F_{3n+2}(1,-\frac{1}{q})=(-1)^n q^{r(-n)}. \tag{5.20}$$

Hence

- if $k=3j$ then $2k+1=6j+1$ and $q^k F_{2k+1}(1,-\frac{1}{q})=q^{3j}F_{6j+1}(1,-\frac{1}{q})=q^{2j(3j+1)}$.
- if $k=3j+1$ then $2k+1=6j+3$ and $q^k F_{2k+1}(1,-\frac{1}{q})=0$.
- If $k=3j+2$ then $2k+1=6j+5$ and $q^k F_{2k+1}(1,-\frac{1}{q})=q^{3j+2}$.
- if $k=3j$ then $2k=6j$ and $q^{k-1}F_{2k}(1,-\frac{1}{q})=0$.
- if $k=3j+1$ then $2k=6j+2$ and $q^{k-1}F_{2k}(1,-\frac{1}{q})=q^{2j(3j+2)}$.
- If $.k=3j+2$ then $2k=6j+4$ and $q^{k-1}F_{2k}(1,-\frac{1}{q})=-q^{(3j+1)(2j+2)}$.

Substituting the above values into (5.18) and (5.19) yields (5.16) and (5.17). ∎

Finally, from (5.12) and (3.8) we derive two explicit formulae for the coefficient $c(n,k,q)$.

Proposition 15. *If $k \equiv n \pmod 2$ then*

$$\begin{aligned} c(n,k,q) &= \sum_{m\in M(n,k)} q^{c(m)+\mathrm{cr}(m)} \\ &= (1-q)^{-\frac{n-k}{2}} \sum_{j\geq 0} \binom{n}{\frac{n-k-2j}{2}} (-1)^j q^{\binom{j}{2}} \frac{[k+2j]}{[k+j]} \begin{bmatrix} k+j \\ j \end{bmatrix} \qquad (5.21) \\ &= (1-q)^{-\frac{n-k}{2}} \sum_{j\geq 0} \left(\binom{n}{\frac{n-k-2j}{2}} - \binom{n}{\frac{n-k-2j-2}{2}} \right) (-1)^j q^{\binom{j+1}{2}} \begin{bmatrix} k+j \\ k \end{bmatrix}. \qquad (5.22) \end{aligned}$$

We now give a second proof of Proposition 15 using Theorem 6 and the orthogonality of the continuous q-Hermite polynomials.

Proof. Clearly Theorem 6 is equivalent to

$$x^n = \sum_{k\equiv n \pmod 2} c(n,k,q) s^{(n-k)/2} \tilde{H}_k(x,s|q). \qquad (5.23)$$

To compute $c(n,k,q)$ we can take $s=1$ and let $\tilde{H}_n(x|q) = \tilde{H}_n(x,s|q)$. It is known (see [9]) that the continuous q-Hermite polynomials $(\tilde{H}_n(x|q))$ are orthogonal with respect to the linear functional φ defined by

$$\varphi(x^n) = \int_{-2/\sqrt{1-q}}^{2/\sqrt{1-q}} x^n v(x,q) dx, \qquad (5.24)$$

where

$$v(x,q) = \frac{\sqrt{(1-q)}(q)_\infty}{\sqrt{1-(1-q)x^2/4}4\pi} \prod_{k=0}^{\infty} \{1 + (2-(1-q)x^2)q^k + q^{2k}\}.$$

Since $\varphi((\tilde{H}_k(x|q))^2) = [k]_q!$, it follows from (5.23) that, for $k \equiv n \pmod 2$,

$$c(n,k,q) = \frac{1}{[k]_q!} \varphi(x^n \tilde{H}_k(x|q)). \qquad (5.25)$$

Recall the well-known formula (see [9])

$$x^{2n} = \sum_{j=-n}^{n} \binom{2n}{n+j} T_{2j}(x/2), \qquad (5.26)$$

where $T_n(\cos\theta) = \cos(n\theta) = T_{-n}(\cos\theta)$ is the nth Chybyshev polynomial of the first kind. By using the Jacobi triple product formula and the terminating q-binomial formula, we can prove (see [7, p. 307]) that, for any integer j and $a = \sqrt{1-q}$,

$$\varphi(T_{n-2j}(ax/2)\tilde{H}_n(x|q)) = \frac{(-1)^{n+j}}{2a^n} q^{\binom{n-j}{2}} \{(q^{-n+j+1};q)_n + q^{n-j}(q^{-n+j};q)_n\}. \qquad (5.27)$$

It follows from (5.25), (5.26) and (5.27) that

$$\begin{aligned}
c(2n,2k,q) &= \frac{a^{-2n}}{[2k]_q!}\sum_{j=-n}^{n}\binom{2n}{n+j}\varphi(T_{2j}(ax/2)\tilde{H}_{2k}(x|q))\\
&= \frac{(1-q)^{-(n-k)}}{(q;q)_{2k}}\sum_{j=-n}^{n}\binom{2n}{n+j}\frac{(-1)^{k+j}}{2}q^{\binom{k+j}{2}}\{(q^{-k-j+1};q)_{2k}+q^{k+j}(q^{-k-j};q)_{2k}\}.
\end{aligned}$$

Since $(q^{-k-j+1};q)_{2k}$ is zero if $j\neq -n,\dots,-k$ and $j\neq k+1,\dots,n$, and $(q^{-k-j};q)_{2k}$ is zero if $j\neq -n,\dots,-k-1$ or $j\neq k,\dots,n$, we can split the last summation into the following four summations:

$$\begin{aligned}
S_1 &= \sum_{j=-n}^{-k}\binom{2n}{n+j}\frac{(-1)^{k+j}}{2}q^{\binom{k+j}{2}}(q^{-k-j+1};q)_{2k},\\
S_2 &= \sum_{j=k+1}^{n}\binom{2n}{n+j}\frac{(-1)^{k+j}}{2}q^{\binom{k+j}{2}}(q^{-k-j+1};q)_{2k},\\
S_3 &= \sum_{j=-n}^{-k-1}\binom{2n}{n+j}\frac{(-1)^{k+j}}{2}q^{\binom{k+j}{2}}q^{k+j}(q^{-k-j};q)_{2k},\\
S_4 &= \sum_{j=k}^{n}\binom{2n}{n+j}\frac{(-1)^{k+j}}{2}q^{\binom{k+j}{2}}q^{k+j}(q^{-k-j};q)_{2k}.
\end{aligned}$$

It is readily seen, by replacing j by $-j$ in S_1 and S_3, that $S_1=S_4$ and $S_2=S_3$. Therefore,

$$\begin{aligned}
c(2n,2k,q) &= \frac{(1-q)^{-(n-k)}}{(q;q)_{2k}}(S_2+S_4)\\
&= (1-q)^{-(n-k)}\sum_{j\geq 0}\binom{2n}{n+k+j}(-1)^j q^{\binom{j}{2}}\frac{[2k+2j]}{[2k+j]}\begin{bmatrix}2k+j\\ j\end{bmatrix}. \qquad (5.28)
\end{aligned}$$

This corresponds to (5.21) for even indices. To derive the formula for odd indices we can use (3.9) to get

$$c(2n+1,2k+1,q) = [2k+2]_q c(2n,2k+2,q) + c(2n,2k,q),$$

and then apply (5.28). ∎

Some remarks about the above formula are in order.

(a) Formula (5.22) has been obtained by different means by Josuat-Vergès [10, Proposition 12] and is also used in [5]. It is easy to see that (5.21) and (5.22) are equal by writing

$$\frac{[k+2j]}{[k+j]} = q^j + \frac{[j]}{[k+j]}.$$

(b) When $k=0$, we recover a formula of Touchard-Riordan (see [2, 9, 14]):

$$c(2n,0,q) = \sum_{m\in M(2n,0)} q^{\mathrm{cr}(m)} = \frac{1}{(1-q)^n}\sum_{j=-n}^{n}\binom{2n}{n+j}(-1)^j q^{\binom{j}{2}}. \qquad (5.29)$$

(c) Notice that $H_{2n}(0,-1|q) = c(2n,0,q)$ and $H_{2n+1}(0,-1|q) = c(2n+1,0,q) = 0$. Hence

$$\sum_{n\geq 0} c(n,0,q)t^n = \cfrac{1}{1-\cfrac{t^2}{1-\cfrac{[2]_q t^2}{1-\cfrac{[3]_q t^2}{1-\cdots}}}}.$$

We derive a known result (see [9]): the coefficient $c(n,0,q)$ coincides with the n-th moment of the continuous q-Hermite polynomials $\tilde{H}(x,1|q)$, i.e.,

$$\mathcal{F}(z^n) = c(n,0,q),$$

where $\mathcal{F}$ is the linear functional acting on the polynomials in z defined by $\mathcal{F}(\tilde{H}_n(z,1|q)) = \delta_{n,0}$.

As in [11] we can derive another double sum expression for $H_n(x,s|q)$. The proof is omitted.

Proposition 16. *We have*

$$H_n(x,s|q) = \sum_{k=0}^{n}(-1)^k q^{-\binom{k}{2}} \sum_{i=0}^{k} \left(\frac{s}{x(q-1)}q^{-i} + xq^i\right)^n \times \prod_{j=0, j\neq i}^{k} \frac{1}{q^{-i} - q^{-j} + x^2\frac{q-1}{s}(q^i - q^j)}. \tag{5.30}$$

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