q-Fibonacci polynomials and q-Catalan numbers

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The Fibonacci polynomials

$$F_n(x,s)=\sum_{k=0}^{n-1}\binom{n-1-k}{k}s^k x^{n-1-2k} \tag{1}$$

satisfy the recurrence

$$F_n(x,s)=xF_{n-1}(x,s)+sF_{n-2}(x,s) \tag{2}$$

with initial values $F_0(x,s)=0$ and $F_1(x,s)=1.$

These polynomials are intimately related to the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$: The polynomials $F_{n+1}(x,-1), n\geq 0,$ are a basis of the vector space of polynomials. If we define the linear functional L by

$$L(F_{n+1}(x,-1))=[n=0] \tag{3}$$

we get

$L(x^{2n+1})=0$ and

$$L(x^{2n})=C_n. \tag{4}$$

I don't know who has observed this well-known fact for the first time.

In this note I want to give an overview about some more or less known generalizations of this result.

1. The q-Fibonacci polynomials of Leonard Carlitz

A well-known $q-$analogue of the Fibonacci polynomials, which apparently has first been introduced by L. Carlitz [4] and has further been studied in [6] and [8], is

$$F_n(x,s,q)=\sum_{k=0}^{n-1}\begin{bmatrix}n-k-1\\ k\end{bmatrix}q^{2\binom{k}{2}}s^k x^{n-2k-1}. \tag{5}$$

It satisfies the recurrence

$$\begin{aligned}&F_n(x,s,q)=xF_{n-1}(x,s,q)+q^{n-3}sF_{n-2}(x,s,q),\\&F_0(x,s,q)=0, F_1(x,s,q)=1.\end{aligned} \tag{6}$$

The polynomials $F_{n+1}(x,-1,q), n\geq 0,$ are a basis of the vector space $\mathbb{C}(q)[x]$ of all polynomials in x whose coefficients are rational functions in q. We can therefore define a linear functional L on $\mathbb{C}(q)[x]$ by

$$L(F_{n+1}(x,-1,q))=[n=0]. \tag{7}$$

We define coefficients $a(n,k)$ by

$$x^n = \sum_{k=0}^{n} a(n,k) F_{k+1}(x,-1,q). \tag{8}$$

From

$$\sum_k a(n,k) F_{k+1}(x,-1,q) = x \cdot x^{n-1} = \sum_k a(n-1,k)\left(x F_{k+1}(x,-1,q)\right)$$
$$= \sum_k a(n-1,k)\left(F_{k+2}(x,-1,q)\right) + q^{k-1} F_k(x,-1,q))$$
$$= \sum_k F_{k+1}(x,-1,q)\left(a(n-1,k-1) + q^k a(n-1,k+1)\right)$$

we get

$$\begin{aligned} &a(0,k) = [k=0] \\ &a(n,0) = a(n-1,1) \\ &a(n,k) = a(n-1,k-1) + q^k a(n-1,k+1). \end{aligned} \tag{9}$$

From this we get $a(n,0) = L(x^n).$

If we set

$$\hat{F}_n(x) = q^{-\binom{n-1}{2}} F_n(x,-1,q), \tag{10}$$

then we have $x\hat{F}_n(x) = q^{n-1}\hat{F}_{n+1} + \hat{F}_{n-1}.$

Define now the numbers $a_{n,k} = L(x^n \hat{F}_{k+1})$.
They satisfy

$$a_{0,k} = [k=0]$$
$$a_{n,k} = a_{n-1,k-1} + q^k a_{n-1,k+1}$$

where $a_{n,k} = 0$ if $k < 0$. Therefore we have $a_{n,k} = a(n,k),$ in other words

$$a(n,k) = L(x^n \hat{F}_{k+1}). \tag{11}$$

These numbers have an obvious combinatorial interpretation: Consider all nonnegative lattice paths in $\mathbb{R}^2$ which start in $(0,0)$ with upward steps $(1,1)$ und downward steps $(1,-1)$. We associate to each upward step ending on the height k the weight 1 and to each downward step ending on the height k the weight q^k. The weight of the path is the product of the weights of all steps of the path.
Then $a(n,k)$ is the weight of all lattice paths from $(0,0)$ to (n,k) .

It is clear that $a(2n+1,0)=0$.
Let $a(2n,0)=C_n(q)$. Then we have

$$C_n(q)=\sum_{k=0}^{n-1} q^k C_k(q)C_{n-k-1}(q). \tag{12}$$

To obtain this recurrence decompose each lattice path from $(0,0)$ to $(2n,0)$ into the first path which returns to the $x-$axis and the rest path. The first path goes from $(0,0)$ to $(2k+2,0), 0\le k\le n-1,$ and consists of a rising segment followed by a path from $(0,0)$ to $(2k,0)$ (but one level higher) and a falling segment. The weight of each of the k downward steps on this higher level is q times the corresponding weight of the steps of the normal level. This shows that $C_n(q)$ is a $q-$analogue of the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}$ which satisfy the recurrence

$$C_n=\sum_{k=0}^{n-1} C_k C_{n-k-1}. \tag{13}$$

This $q-$analogue of the Catalan numbers has been introduced by Carlitz and Riordan [3].
It is well-known that in the classical case $q=1$ all $a(n,k)$ are given by a closed formula. More precisely we have

$$a(2n,2k)=\frac{2k+1}{n+k+1}\binom{2n}{n-k} \tag{14}$$

and

$$a(2n+1,2k+1)=\frac{2k+2}{n+k+2}\binom{2n+1}{n-k}. \tag{15}$$

This is equivalent with

$$a(n,n-2k)=\frac{n-2k+1}{n-k+1}\binom{n}{k}=\binom{n}{k}-\binom{n}{k-1}. \tag{16}$$

For $q\neq 1$ no such explicit formulas are known. For small values of k it is easy to compute $a(n,n-2k).$ In general this is a polynomial in $\mathbb{C}(q)[q^n]$ of order $k.$ But there are no known formulae for $a(n,k)$. Probably these sequences and especially $\left(C_n(q)\right)$ are not q-holonomic, but I don't know if this has already been proved.

If we write

$$F_{n+1}(x,-1,q)=\sum_{k=0}^{n} p(n,k)x^k, \tag{17}$$

then we have $p(2n,2k+1)=p(2n+1,2k)=0$ and

$$p(2n,2k)=\begin{bmatrix} n+k \\ n-k \end{bmatrix} q^{2\binom{n-k}{2}}(-1)^{n-k} \tag{18}$$

and

$$p(2n+1,2k+1)=\begin{bmatrix} n+k+1 \\ n-k \end{bmatrix} q^{2\binom{n-k}{2}}(-1)^{n-k}. \tag{19}$$

From (8) we see that the matrix

$\left(a(i,j)\right)_{i,j=0}^{n-1}$ is the inverse of $\left(p(i,j)\right)_{i,j=0}^{n-1}$,

i.e.

$$\left(a(i,j)\right)_{i,j=0}^{n-1}\cdot\left(p(j,k)\right)_{i,j=0}^{n-1}=I_n. \tag{20}$$

There is also another connection with Fibonacci polynomials.
Define the generating function

$$A_m(s)=\sum_{n\ge m} a(2n-m,m)s^n. \tag{21}$$

Then Carlitz and Riordan [3] have shown that

$$q^{\binom{m}{2}}A_m(s)=F_{m+1}(1,-s,q)A_0(s)-F_m(1,-qs,q). \tag{22}$$

This can easily be proved by induction.

First we observe that from
$a(2n-m,m)=a(2n-m-1,m-1)+q^m a(2n-m-1,m+1)$
$=a(2(n-1)-(m-1),m-1)+q^m a(2n-(m+1),m+1)$
we get

$A_m(s)=sA_{m-1}(s)+q^m A_{m+1}(s)$
or

$$q^{m-1}A_m(s)=A_{m-1}(s)-sA_{m-2}(s). \tag{23}$$

For small values of m formula (22) is obviously true.
Then we get

$$\begin{aligned}
q^{\binom{m+1}{2}}A_{m+1}(s)&=q^{\binom{m}{2}}\left(A_m(s)-sA_{m-1}(s)\right)=q^{\binom{m}{2}}A_m(s)-q^{m-1}sq^{\binom{m-1}{2}}A_{m-1}(s)\\
&=F_{m+1}(1,-s,q)A_0(s)-F_m(1,-qs,q)-q^{m-1}s\left(F_m(1,-s,q)A_0(s)-F_{m-1}(1,-qs,q)\right)\\
&=\left(F_{m+1}(1,-s,q)-q^{m-1}sF_m(1,-s,q)\right)A_0(s)-\left(F_m(1,-qs,q)-q^{m-1}sF_{m-1}(1,-qs,q)\right)\\
&=F_{m+2}(1,-s,q)A_0(s)-F_{m+1}(1,-qs,q).
\end{aligned}$$

Formula (22) can be stated in the following form

$$\sum_{n=0}^{m-1}(-1)^n q^{n^2}\begin{bmatrix} m-1-n \\ n \end{bmatrix}s^n+\sum_{n\ge m} q^{\binom{m}{2}}a(2n-m,m)s^n=\sum s^n\sum_{k=0}^{n}(-1)^k q^{k^2-k}\begin{bmatrix} m-k \\ k \end{bmatrix}C_{n-k}(q).$$

Comparing coefficients we get for $n=m$

$$\sum_{k=0}^{n}(-1)^k q^{k^2-k}\begin{bmatrix} n-k \\ k \end{bmatrix} C_{n-k}(q)=q^{\binom{n}{2}}. \tag{24}$$

This is also an immediate consequence of (11) if we interpret the left hand side as $L(x^n F_{n+1}(x,-1))$.

Remark

Carlitz [4] has shown formula (8) in the equivalent form $x^n=\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor}(-1)^k q^k c_{n,k} F_{n-2k+1}(x,q)$ for some $q-$polynomials $c_{n,k}$. It is easy to verify that $c_{n,k}=a(n,n-2k)$. He computed the first values but apparently did not notice that $c_{2n,n}=a(2n,0)=C_n(q)$, although some ten years earlier he and Riordan [3] already had studied $C_n(q)$.

2. The q-Catalan numbers of George Andrews

George Andrews [1] has found a $q-$analogue of the Catalan numbers where the $a(n,k)$ have a closed formula and where there exists a $q-$analogue of the recursion (13) too.
There exist corresponding $q-$Fibonacci polynomials (cf.[6]).
Let

$$t(n)=\frac{4q^{n+2}}{(1+q^{n+1})(1+q^{n+2})}. \tag{25}$$

Define polynomials by the recurrence

$$\begin{aligned} &f_n(x,s)=xf_{n-1}(x,s)-t(n-3)sf_{n-2}(x,s),\\ &f_0(x,s)=0,\ f_1(x,s)=1. \end{aligned} \tag{26}$$

It can easily be verified that

$$f_n(x,s)=\sum_{k=0}^{n-1}\begin{bmatrix} n-1-k \\ k \end{bmatrix} x^{n-1-2k}(-1)^k \frac{4^k q^{k^2+k}s^k}{\prod_{j=1}^{k}(1+q^j)(1+q^{n-k+j-1})}. \tag{27}$$

We also need the polynomials $\tilde{f}_n(x,s)$ which satisfy

$$\begin{aligned}&\tilde{f}_n(x,s)=x\tilde{f}_{n-1}(x,s)-t(n-2)s\tilde{f}_{n-2}(x,s),\\&\tilde{f}_0(x,s)=0,\ \tilde{f}_1(x,s)=1.\end{aligned} \tag{28}$$

They are given by

$$\tilde{f}_n(x,s)=\sum_{k=0}^{n-1}\begin{bmatrix}n-1-k\\k\end{bmatrix}x^{n-1-2k}(-1)^k\frac{4^k q^{k^2+2k}s^k}{\prod_{j=1}^{k}(1+q^{j+1})(1+q^{n-k+j})}. \tag{29}$$

Then the polynomials $f_{n+1}(x,1), n\geq 0,$ are a basis of the vector space $\mathbb{C}(q)[x]$ of all polynomials in x whose coefficients are rational functions in q. We can therefore define a linear functional L on $\mathbb{C}(q)[x]$ by

$$L(f_{n+1}(x,1))=[n=0]. \tag{30}$$

Now we define coefficients $a(n,k)$ by

$$x^n=\sum_{k=0}^{n}a(n,k)f_{k+1}(x,1). \tag{31}$$

From

$$\begin{aligned}&\sum_k a(n,k)f_{k+1}(x,1)=x\cdot x^{n-1}=\sum_k a(n-1,k)\left(xf_{k+1}(x,1)\right)\\&=\sum_k a(n-1,k)\left(f_{k+2}(x,1)\right)+t(k-1)f_k(x,1))\\&=\sum_k f_{k+1}(x,1)\left(a(n-1,k-1)+t(k)a(n-1,k+1)\right)\end{aligned}$$

we get

$$\begin{aligned}&a(0,k)=[k=0]\\&a(n,0)=t(0)a(n-1,1)\\&a(n,k)=a(n-1,k-1)+t(k)a(n-1,k+1).\end{aligned} \tag{32}$$

From this we get $a(n,0)=L(x^n)$.

Let $\hat{f}_n(x)=\dfrac{f_n(x,1)}{t(0)t(1)\cdots t(n-2)}$.

Then we have

$$x\hat{f}_n(x)=t(n-1)\hat{f}_{n+1}+\hat{f}_{n-1}.$$

Define now the numbers $a_{n,k} = L(x^n \hat{f}_{k+1}(x))$.

They satisfy

$$a_{0,k} = [k=0]$$
$$a_{n,k} = a_{n-1,k-1} + t(k)a_{n-1,k+1}$$

where $a_{n,k} = 0$ if $k < 0$.

Therefore we get

$$a(n,k) = L(x^n \hat{f}_{k+1}(x)). \tag{33}$$

From (32) it can easily be verified that

$$\begin{aligned} a(2n,2k) &= \frac{[2k+1]}{[n+k+1]}\begin{bmatrix} 2n \\ n-k \end{bmatrix}\frac{1+q^{2k+1}}{1+q^{n+k+1}}\frac{(2q)^{2n-2k}}{\prod_{j=1}^{n-k}(1+q^j)(1+q^{2k+j})} \\ &= \frac{[2k+1]_{q^2}}{[n+k+1]_{q^2}}\begin{bmatrix} 2n \\ n-k \end{bmatrix}_{q^2}\frac{4^{n-k}(q^2)^{n-k}}{\prod_{j=2k+1}^{2n}(1+q^j)}. \end{aligned} \tag{34}$$

and

$$\begin{aligned} a(2n+1,2k+1) &= \frac{[2k+2]}{[n+k+2]}\begin{bmatrix} 2n+1 \\ n-k \end{bmatrix}\frac{1+q^{2k+2}}{1+q^{n+k+2}}\frac{(2q)^{2n-2k}}{\prod_{j=1}^{n-k}(1+q^j)(1+q^{2k+j+1})} \\ &== \frac{[2k+2]_{q^2}}{[n+k+2]_{q^2}}\begin{bmatrix} 2n+1 \\ n-k \end{bmatrix}_{q^2}\frac{4^{n-k}(q^2)^{n-k}}{\prod_{j=2k+2}^{2n+1}(1+q^j)} \end{aligned} \tag{35}$$

If we write

$$f_{n+1}(x,1) = \sum_{k=0}^{n} p(n,k)x^k, \tag{36}$$

then we have $p(2n,2k+1) = p(2n+1,2k) = 0$ and

$$p(2n,2k) = \begin{bmatrix} n+k \\ n-k \end{bmatrix}(-1)^{n-k}\frac{4^{n-k}q^{2\binom{n-k+1}{2}}}{\prod_{j=1}^{n-k}(1+q^j)(1+q^{k+j+n})} = (-1)^{n-k}\begin{bmatrix} n+k \\ n-k \end{bmatrix}_{q^2}\frac{4^{n-k}q^{2\binom{n-k+1}{2}}}{\prod_{j=2k+1}^{2n}(1+q^j)} \tag{37}$$

and

$$p(2n+1,2k+1) = \begin{bmatrix} n+k+1 \\ n-k \end{bmatrix}(-1)^{n-k}\frac{4^{n-k}q^{2\binom{n-k+1}{2}}}{\prod_{j=1}^{n-k}(1+q^j)(1+q^{k+j+n+1})}. \tag{38}$$

From (31) we have again that $\left(a(i,j)\right)_{i,j=0}^{n-1}$ is the inverse of $\left(p(i,j)\right)_{i,j=0}^{n-1}$.

Therefore we know that

$$\sum_{j=0}^{n} a(2n,2k)\,p(2k,2m)=\left[n=m\right]. \tag{39}$$

This means

$$\sum_{k=m}^{n}\frac{\left[2k+1\right]_{q^2}}{\left[n+k+1\right]_{q^2}}\begin{bmatrix}2n\\ n-k\end{bmatrix}_{q^2}\frac{4^{n-k}(q^2)^{n-k}}{\prod\limits_{j=2k+1}^{2n}(1+q^j)}(-1)^{k-m}\begin{bmatrix}k+m\\ k-m\end{bmatrix}_{q^2}\frac{4^{k-m}q^{2\binom{k-m+1}{2}}}{\prod\limits_{j=2m+1}^{2k}(1+q^j)}=[n=m]$$

or

$$\sum_{\ell=0}^{n-m}\frac{\left[2\ell+2m+1\right]_{q^2}}{\left[n+\ell+m+1\right]_{q^2}}\begin{bmatrix}2n\\ n-\ell-m\end{bmatrix}_{q^2}(q^2)^{n-\ell-m}q^{2\binom{\ell+1}{2}}(-1)^{\ell}\begin{bmatrix}\ell+2m\\ \ell\end{bmatrix}_{q^2}=[n-m=0].$$

Replacing n by $n+m$ we get

$$\sum_{\ell=0}^{n}\frac{\left[2\ell+2m+1\right]_{q^2}}{\left[n+\ell+2m+1\right]_{q^2}}\begin{bmatrix}2n+2m\\ n-\ell\end{bmatrix}_{q^2}(q^2)^{n-\ell}q^{2\binom{\ell+1}{2}}(-1)^{\ell}\begin{bmatrix}\ell+2m\\ \ell\end{bmatrix}_{q^2}=[n=0]$$

If we let $q^2\to q$ we get the equivalent formula

$$\sum_{\ell=0}^{n}\frac{\left[2\ell+2m+1\right]}{\left[n+\ell+2m+1\right]}\begin{bmatrix}2n+2m\\ n-\ell\end{bmatrix}q^{n-\ell}q^{\binom{\ell+1}{2}}(-1)^{\ell}\begin{bmatrix}\ell+2m\\ \ell\end{bmatrix}=[n=0]$$

or

$$q^{n}\frac{\left[2n+2m\right]!}{\left[n\right]!\left[2m\right]!}\sum_{\ell=0}^{n}\begin{bmatrix}n\\ \ell\end{bmatrix}\frac{\left[2\ell+2m+1\right]}{\left[n+\ell+2m+1\right]!}(-1)^{\ell}q^{\binom{\ell}{2}}\left[\ell+2m\right]!=[n=0].$$

Therefore (39) is equivalent with

$$\sum_{\ell=0}^{n}(-1)^{\ell}q^{\binom{\ell}{2}}\begin{bmatrix}n\\ \ell\end{bmatrix}(1-q^{2\ell+2m+1})\frac{\left[\ell+2m\right]!}{\left[n+\ell+2m+1\right]!}=[n=0]. \tag{40}$$

Let now

$$c_n(q)=a(2n,0)=\frac{1}{\left[n+1\right]}\begin{bmatrix}2n\\ n\end{bmatrix}\frac{1+q}{1+q^{n+1}}\frac{(2q)^{2n}}{\prod\limits_{j=1}^{n}(1+q^j)^2}=\frac{1}{\left[n+1\right]_{q^2}}\begin{bmatrix}2n\\ n\end{bmatrix}_{q^2}\frac{(2q)^{2n}}{\prod\limits_{j=1}^{2n}(1+q^j)}. \tag{41}$$

This can also be written in the form

$$c_n(q) = (-1)^n q^{n^2} (2q)^{2n} (1+q) \begin{bmatrix} \frac{1}{2} \\ n+1 \end{bmatrix}_{q^2} . \tag{42}$$

The $q-$Vandermonde formula gives

$$\sum_{k=0}^{n} q^k q^{k^2-k} \begin{bmatrix} \frac{1}{2} \\ k \end{bmatrix}_{q^2} q^{(n-k)^2-(n-k)} \begin{bmatrix} \frac{1}{2} \\ n-k \end{bmatrix}_{q^2} = q^{n^2-n} \begin{bmatrix} 1 \\ n \end{bmatrix}_{q^2} .$$

Defining

$$h(z) = \sum_k q^{n^2-n} \begin{bmatrix} \frac{1}{2} \\ n \end{bmatrix}_{q^2} z^n , \tag{43}$$

we conclude that

$$h(z)h(qz) = 1+z. \tag{44}$$

Therefore the generating function $f(z) = \sum_n c_n(q) z^n$ satisfies

$$f(z) = -\frac{1+q}{4qz} \sum_{n \geq 0} q^{(n+1)^2-(n+1)} \begin{bmatrix} \frac{1}{2} \\ n+1 \end{bmatrix} \left(-4qz\right)^{n+1} = \frac{1+q}{4qz}\left(1-h(-4qz)\right). \tag{45}$$

For $q=1$ this reduces to the well-known formula $\sum_{n \geq 0} C_n z^n = \frac{1-\sqrt{1-4z}}{2z}.$

From

$$\left(1-\frac{4qz}{1+q} f(z)\right)\left(1-\frac{4q^2 z}{1+q} f(qz)\right) = 1-4qz$$

we get

$$\frac{f(z)+f(qz)}{1+q} = 1+\frac{4q^2}{(1+q)^2} z f(z) f(qz). \tag{46}$$

Comparing coefficients we obtain the recurrence relation

$$c_n(q) = \frac{4q^2}{(1+q)(1+q^{n+1})} \sum_{k=0}^{n-1} q^k c_k(q) c_{n-k-1}(q) \tag{47}$$

with initial value $c_0(q) = 1.$

Define now the generating function

$$A_m(s)=\sum_{n\ge m} a(2n-m,m)s^n.$$

Then we get as above

$$a(2n-m,m)=a(2n-m-1,m-1)+t(m)a(2n-m-1,m+1)$$
$$=a(2(n-1)-(m-1),m-1)+t(m)a(2n-(m+1),m+1)$$

and therefore

$$A_m(s)=sA_{m-1}(s)+t(m)A_{m+1}(s)$$
or

$$t(m-1)A_m(s)=A_{m-1}(s)-sA_{m-2}(s). \tag{48}$$

This implies
$$t(0)t(1)\cdots t(m-1)A_m(s)=f_{m+1}(1,s)A_0(s)-\tilde{f}_m(1,s). \tag{49}$$

This relation obviously holds for small m. In general we have

$$t(0)t(1)\cdots t(m-1)t(m)A_{m+1}(s)=t(0)t(1)\cdots t(m-1)\left(A_m(s)-sA_{m-1}(s)\right)$$
$$=f_{m+1}(1,s)A_0(s)-\tilde{f}_m(1,s)-t(m-1)s\left(f_m(1,s)A_0(s)-\tilde{f}_{m-1}(1,s)\right)$$
$$=\left(f_{m+1}(1,s)-t(m-1)sf_m(1,s)\right)A_0(s)-\left(\tilde{f}_m(1,s)-t(m-1)s\tilde{f}_{m-1}(1,s)\right)=f_{m+2}(1,s)A_0(s)-\tilde{f}_{m+1}(1,s).$$

Comparing the coefficient of s^m we get

$$\sum_{k=0}^{n}(-1)^k\begin{bmatrix} n-k \\ k \end{bmatrix}\frac{4^k q^{k^2+k}}{\prod_{j=1}^{k}(1+q^j)(1+q^{n-k+j})}c_{n-k}(q)=\prod_{j=0}^{n-1}t(j). \tag{50}$$

This is also a consequence of (33).

3. Another q-analogue of the Fibonacci polynomials

Finally we consider the $q-$analogue $Fib_n(x,s)=\sum_{k=0}^{n-1}q^{\binom{k}{2}}\begin{bmatrix} n-k-1 \\ k \end{bmatrix}s^k x^{n-2k-1}$ which has been studied in [7] and [9].

If we set
$$Fib_n(x,-1)=\sum_{k=0}^{n}p(n,k)x^k,$$

then we have $p(2n,2k+1)=p(2n+1,2k)=0$ and

$$p(2n,2k)=\begin{bmatrix}n+k\\n-k\end{bmatrix}q^{\binom{n-k}{2}}(-1)^{n-k} \tag{51}$$

and

$$p(2n+1,2k+1)=\begin{bmatrix}n+k+1\\n-k\end{bmatrix}q^{\binom{n-k}{2}}(-1)^{n-k}. \tag{52}$$

Let now L be the linear functional defined by

$$L(Fib_{n+1}(x,-1))=[n=0]. \tag{53}$$

Let

$$x^n=\sum_{k=0}^{n}a(n,k)Fib_{k+1}(x,-1). \tag{54}$$

Then we get

$$a(2n,2k)=\frac{[2k+1]}{[n+k+1]}\begin{bmatrix}2n\\n-k\end{bmatrix} \tag{55}$$

and

$$a(2n+1,2k+1)=\frac{[2k+2]}{[n+k+2]}\begin{bmatrix}2n+1\\n-k\end{bmatrix}. \tag{56}$$

As a special case we obtain

$$L(x^{2n})=\frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}. \tag{57}$$

To prove this we observe that (54) implies that

$\left(a(i,j)\right)_{i,j=0}^{n-1}$ is the inverse of $\left(p(i,j)\right)_{i,j=0}^{n-1}$.

In order to show (55) it thus suffices to show that

$$\sum_{k=m}^{n}\frac{[2k+1]}{[n+k+1]}\begin{bmatrix}2n\\n-k\end{bmatrix}\begin{bmatrix}k+m\\k-m\end{bmatrix}q^{\binom{k-m}{2}}(-1)^{k-m}=[m=n]$$

or

$$\sum_{\ell=0}^{n}\frac{[2\ell+2m+1]}{[n+\ell+2m+1]}\begin{bmatrix}2n+2m\\n-\ell\end{bmatrix}q^{\binom{\ell}{2}}(-1)^{\ell}\begin{bmatrix}\ell+2m\\\ell\end{bmatrix}=[n=0].$$

But this is equivalent with (40).

The same argument proves (56).

$$\sum_{k=m}^{n} \frac{[2k+1]}{[n+k+1]} \begin{bmatrix} 2n \\ n-k \end{bmatrix} \begin{bmatrix} k+m \\ k-m \end{bmatrix} q^{\binom{k-m}{2}} (-1)^{k-m} = [m=n]$$

$$\sum_{k=m}^{n} \frac{[2k+2]}{[n+k+2]} \begin{bmatrix} 2n+1 \\ n-k \end{bmatrix} \begin{bmatrix} k+m+1 \\ k-m \end{bmatrix} q^{\binom{k-m}{2}} (-1)^{k-m} = [m=n].$$

It is easy to verify that (54), (55) and (56) are equivalent with

$$\sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix} \right) \frac{1}{q^k} Fib_{n+1-2k}(x,-1) = x^n. \quad (58)$$

This identity follows also from an inversion formula by L. Carlitz [2].
He uses the fact that

$$d(n,k) = \sum_{j=0}^{k} (-1)^j q^{\binom{j+1}{2}} \begin{bmatrix} k \\ j \end{bmatrix} \begin{bmatrix} n-j \\ k \end{bmatrix} = 1 \quad (59)$$

for $0 \le k \le n$.

This is an immediate consequence from the recursion $\begin{bmatrix} n \\ k \end{bmatrix} - q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$, if we write it in the operator form

$(1-q^k U)\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix}$ where $U\begin{bmatrix} n \\ k \end{bmatrix} = \begin{bmatrix} n-1 \\ k \end{bmatrix}$.

If we iterate this we get

$(1-qU)\cdots(1-q^k U) = \begin{bmatrix} n-k \\ 0 \end{bmatrix} = 1.$

Since the left hand side can be written in the form $\sum_{j=0}^{k} (-1)^j q^{\binom{j+1}{2}} \begin{bmatrix} k \\ j \end{bmatrix} U^j$ we get (59).

His inversion formula reads as follows:
Let

$$u_n = \sum_{2k \le n} \left(\begin{bmatrix} n \\ k \end{bmatrix} - \begin{bmatrix} n \\ k-1 \end{bmatrix} \right) v_{n-2k}, \quad (60)$$

then

$$v_n = \sum_{2k \le n} (-1)^k q^{\binom{k+1}{2}} \begin{bmatrix} n-k \\ k \end{bmatrix} u_{n-2k}. \quad (61)$$

To prove this observe that

$$\sum_{2k\le n}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix}n-k\\k\end{bmatrix}\sum_{2j\le n-2k}\left(\begin{bmatrix}n-2k\\j\end{bmatrix}-\begin{bmatrix}n-2k\\j-1\end{bmatrix}\right)v_{n-2k-2j}$$
$$=\sum_{2m\le n}v_{n-2m}\sum_{k=0}^{m}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix}n-k\\k\end{bmatrix}\left(\begin{bmatrix}n-2k\\m-k\end{bmatrix}-\begin{bmatrix}n-2k\\m-k-1\end{bmatrix}\right)=v_n. \tag{62}$$

Now for $0<2m\le n$

$$\sum_{j=0}^{m}(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}n-j\\j\end{bmatrix}\begin{bmatrix}n-2j\\m-j\end{bmatrix}=\sum_{j=0}^{m}(-1)^j q^{\binom{j+1}{2}}\frac{[n-j]![m]!}{[j]![m-j]![n-m-j]![m]!}$$
$$=\sum_{j=0}^{m}(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}m\\j\end{bmatrix}\begin{bmatrix}n-j\\m\end{bmatrix}=1$$

and

$$\sum_{j=0}^{m}(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}n-j\\j\end{bmatrix}\begin{bmatrix}n-2j\\m-j-1\end{bmatrix}=\sum_{j=0}^{m}(-1)^j q^{\binom{j+1}{2}}\frac{[n-j]![m-1]!}{[j]![m-j-1]![n-m-j+1]![m-1]!}$$
$$=\sum_{j=0}^{m}(-1)^j q^{\binom{j+1}{2}}\begin{bmatrix}m-1\\j\end{bmatrix}\begin{bmatrix}n-j\\m-1\end{bmatrix}=1.$$

Therefore all sums $\sum_{k=0}^{m}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix}n-k\\k\end{bmatrix}\left(\begin{bmatrix}n-2k\\m-k\end{bmatrix}-\begin{bmatrix}n-2k\\m-k-1\end{bmatrix}\right)$ with $m>0$ vanish and (62) is proved.

If we choose $u_n=x^n$, then $v_n=\sum_{2k\le n}(-1)^k q^{\binom{k+1}{2}}\begin{bmatrix}n-k\\k\end{bmatrix}x^{n-2k}=Fib_{n+1}(x,q)$ and we get

$$\sum_{2k\le n}\left(\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n\\k-1\end{bmatrix}\right)Fib_{n+1-2k}(x,-q)=x^n.$$

By comparing coefficients we see that for all s

$$\sum_{2k\le n}\left(\begin{bmatrix}n\\k\end{bmatrix}-\begin{bmatrix}n\\k-1\end{bmatrix}\right)Fib_{n+1-2k}(x,-s)\frac{s^k}{q^k}=x^n.$$

This proves (58).

Remark

In the case of the $q-$Catalan numbers of Carlitz and Andrews the corresponding $q-$Fibonacci polynomials are orthogonal with respect to the linear functional L. This follows immediately from (11) and (33).

For in the first case we get

$$L(F_{n+1}(x,-1,q)F_{m+1}(x,-1,q)) = L\left(\sum_{k=0}^{n} p(n,k)q^{\binom{m}{2}}x^k\hat{F}_{m+1}(x)\right) = q^{\binom{m}{2}}\sum_{k=0}^{n} a(m,k)p(n,k) = q^{\binom{m}{2}}[m=n]$$

and in the second one

$$L(f_{n+1}(x,1)f_{m+1}(x,1)) = L\left(\prod_{i=0}^{m-1} t(i)\sum_{k=0}^{n} p(n,k)x^k\hat{f}_{m+1}(x)\right) = \prod_{i=0}^{m-1} t(i)\sum_{k=0}^{n} a(m,k)p(n,k) = \prod_{i=0}^{m-1} t(i)[m=n].$$

With respect to the functional L defined in (53) the polynomials $Fib_n(x,-1)$ are not orthogonal, since e.g. $L(Fib_2(x,-1)Fib_4(x,-1)) = L(xFib_4(x,-1)) = (q-1)q \neq 0.$

More generally we have $L(xFib_{2n}(x,-1)) = (q-1)q^{n^2-n-1}$ for $n \geq 2.$

This follows from the recurrence

$Fib_n(x,-1) = xFib_{n-1}(x,-1) - q^{n-3}xFib_{n-3}(x,-1) + q^{n-4}Fib_{n-4}(x,-1),$

which has been proved in [9].

For this implies

$L(xFib_{2n+2}(x,1)) = q^{2n}L(xFib_{2n}(x,1))$

for $n \geq 2.$

There are many other $q-$analogues of the Catalan numbers (cf. [10]).

A simple class are the Pólya-Gessel $q-$Catalan numbers $c(n,q,s)$ defined by

$$c(n,q,s) = c(n-1,q,s) + s\sum_{k=0}^{n-2} q^k c(k,q,s)c(n-1-k,q,s)$$

with initial value $c(0,q,s) = 1.$

For these numbers we define Fibonacci-like polynomials $\varphi(n,x,s)$ by

$\varphi(n,x,s) = x\varphi(n-1,x,s) - t(n-3)\varphi(n-2,x,s)$ with initial values $\varphi(0,x,s) = 0$ and $\varphi(1,x,s) = 1.$ Here $t(n)$ is defined by $t(2n) = q^n, t(2n+1) = q^n s.$

As is shown in [5] the orthogonal polynomials $\Phi_n(x,s)$ with respect to the linear functional L defined by the moments $L(x^n) = c(n,q,s)$ are

$$\Phi_n(x,s) = \varphi(2n+1,\sqrt{x},s) = \sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}x^k\sum_{j=0}^{n-k}\begin{bmatrix} n-j \\ k \end{bmatrix}\begin{bmatrix} k+j-1 \\ j \end{bmatrix}s^j.$$

For $q=1$ and $s=1$ this reduces to $\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}x^k.$

For other $q-$analogues I don't know if there are analogues of the Fibonacci polynomials which are in the same relation as in the above cases.

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