Remarks on some sequences of binomial sums

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Abstract
We give simple proofs for the recurrence relations of some sequences of binomial sums which have previously been obtained by other more complicated methods.

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1. Introduction

Modifying an idea of E. Brietzke [2] we give simple proofs for the recurrence relations of sequences of binomial sums of the form $a(n,m,k,z)=\sum_{j\in\mathbb{Z}} z^j \binom{n}{\left\lfloor \frac{n-mj+k}{2} \right\rfloor}$, which have been obtained by other methods in [3].

In order to motivate the method we consider first the well-known special case

$$a(n,5,k,-1)=\sum_{j\in\mathbb{Z}}(-1)^j\binom{n}{\left\lfloor \frac{n-5j+k}{2} \right\rfloor}=(-1)^k\sum_j t(n,k-5j),$$

with $t(n,k)=(-1)^k\binom{n}{\left\lfloor \frac{n+k}{2} \right\rfloor}$.

We use the fact that $t(n,k)=-t(n-1,k-1)-t(n-1,k+1)$ with $t(0,0)=1, t(0,1)=-1$ and $t(0,k)=0$ for all other $k\in\mathbb{Z}$.

Define the operator K by $Kf(n,k)=f(n,k-1)$ and the operator N by $Nf(n,k)=f(n+1,k)$. Then

$$t(n)=Nt(n-1)=-(K+K^{-1})t(n-1)=(-1)^n(K+K^{-1})^n t(0).$$

Let $s(n,k)$ on $\mathbb{N}\times\mathbb{Z}$ be the function which satisfies the same recurrence with initial values $s(0,k)=[k=0]$. Then we have $t(0)=(1-K)s(0).$ Since K is a linear operator we also have $t(n)=(1-K)s(n).$

Let $\mathcal{F}$ be the vector space of all functions on $\mathbb{N}\times\mathbb{Z}$ which are finite linear combinations of functions $K^j s,\ j\in\mathbb{Z}.$ For $f\in\mathcal{F}$ we have $Nf=-(K+K^{-1})f$.

Let T be the linear operator on $\mathcal{F}$ defined by

$$Tf=N^2f-Nf-f=(K+K^{-1})^2f+(K+K^{-1})f-f=(K^{-2}+K^{-1}+1+K+K^2)f.$$

Then $\sum_{j\in\mathbb{Z}} K^{5j} T K^i s(0) = \sum_{j\in\mathbb{Z}} K^j s(0) = 1$ for all $i \in \mathbb{Z}$ since $KT = TK$.

Furthermore

$$\sum_{j\in\mathbb{Z}} K^{5j} T t(n) = \sum_{j\in\mathbb{Z}} K^{5j} T (-1)^n (K+K^{-1})^n (1-K) s(0) = (-1)^n (K+K^{-1})^n (1-K) \sum_{j\in\mathbb{Z}} K^{5j} T s(0) = 0.$$

Since $a(n,5,k,-1) = (-1)^k \sum_j t(n,k-5j)$ is a finite sum for each k, the sequence $\left(a(n,5,k,-1)\right)$ satisfies the recurrence $a(n+2,5,k,-1) - a(n+1,5,k,-1) - a(n,5,k,-1) = 0$ for $n \geq 0$.
Since the Fibonacci numbers F_n satisfy the same recurrence with initial values $F_0 = 0$ and $F_1 = 1$, we get the following results (cf. G.E. Andrews [1]):

Proposition 1

For $k \equiv 0,1 \pmod{10}$ *the initial conditions are* $a(0,5,k) = a(1,5,k) = 1$ *and therefore* $a(n,5,k) = F_{n+1}$.
For $k \equiv 2,9 \pmod{10}$ *we have* $a(0,5,k) = 0, a(1,5,k) = 1$ *and therefore* $a(n,5,k) = F_n$.
For $k \equiv 3,8 \pmod{10}$ *we get* $a(0,5,k) = a(1,5,k) = 0$ *and therefore* $a(n,5,k) = 0$. *Furthermore* $a(n,5,k+5) = -a(n,5,k)$.

It is interesting to observe that this result has first been proved by I. Schur [6] in a strengthened version: Let $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{(1-q^{n-k+1})\cdots(1-q^n)}{(1-q)\cdots(1-q^k)}$ be a $q-$binomial coefficient. Then the following polynomial version of the celebrated Rogers-Ramanujan identity

$$\sum_{k=0}^{n} q^{k^2} \begin{bmatrix} n-k \\ k \end{bmatrix} = \sum_{k\in\mathbb{Z}} (-1)^k q^{\frac{k(5k-1)}{2}} \begin{bmatrix} n \\ \left\lfloor \frac{n+5k}{2} \right\rfloor \end{bmatrix}$$

holds, which for $q = 1$ reduces to

$$\sum_{k=0}^{n} \binom{n-k}{k} = F_{n+1} = \sum_{j\in\mathbb{Z}} (-1)^j \binom{n}{\left\lfloor \frac{n+5j}{2} \right\rfloor}.$$

An elementary proof of this $q-$identity may be found in [5].

2. A useful method

After this example let us consider a more general case.
For $a,b \in \mathbb{R}$ let $s_{a,b}$ be the function on $\mathbb{N} \times \mathbb{Z}$ defined by $s_{a,b}(0,k) = [k=0]$ and the recurrence relation

$$s_{a,b}(n,k) = a s_{a,b}(n-1,k-1) + b s_{a,b}(n-1,k) + a s_{a,b}(n-1,k+1). \tag{1}$$

This can be written in the form

$$s_{a,b}(n)=(aK^{-1}+b+aK)s_{a,b}(n-1)=(aK^{-1}+b+aK)^n s_{a,b}(0).$$

Let $\mathcal{F}$ be the vector space of all functions on $\mathbb{N}$ which are finite linear combinations of functions $K^j s_{a,b},\ j\in\mathbb{Z}.$

For any polynomial $p(x)=\sum_{i=0}^{m} a_i x^i$ we denote by $p(N)$ the linear operator on $\mathcal{F}$ defined by $p(N)f(n)=\sum_{i=0}^{m} a_i f(n+i)$. Then we have $p(N)=p(aK^{-1}+b+aK).$

We are looking for an operator $p(N)$ with analogous properties as T had in the above example.

To this end we define a sequence of polynomials $p_n(x,a,b)=\sum_{k=0}^{n} p_{n,k}(a,b)x^k$ by the recurrence

$$p_n(x,a,b)=(x-b)p_{n-1}(x,a,b)-a^2 p_{n-2}(x,a,b) \tag{2}$$

with initial values $p_0(x,a,b)=1$ and $p_1(x,a,b)=x+a-b.$

Lemma 1

For all $k\in\mathbb{Z}$ the following identity holds

$$p_m(N,a,b)s_{a,b}(0,k)=\sum_{i=0}^{m} p_{m,i}(a,b)s_{a,b}(i,k)=a^m\left[|k|\le m\right]. \tag{3}$$

Proof

It suffices to show that on $\mathcal{F}$

$$p_m(N,a,b)=a^m\sum_{j=-m}^{m} K^j. \tag{4}$$

It is immediately verified that (4) is true for $m=0$ and $m=1$, since $(N+a-b)=(aK+a+aK^{-1}).$

If has already been shown for $m-1$ and $m-2$ we get

$p_m(N,a,b)=(N-b)p_{m-1}(N,a,b)-a^2 p_{m-2}(N,a,b)$

$=a(K+K^{-1})a^{m-1}\sum_{j=-m+1}^{m-1} K^j-a^2a^{m-2}\sum_{j=-m+2}^{m-2} K^j=a^m\sum_{j=-m}^{m} K^j.$

From (3) we get

$$\sum_{i=0}^{m} p_{m,i}(a,b)\sum_{j\in\mathbb{Z}} s_{a,b}(i,k-(2m+1)j)=a^m \tag{5}$$

for each $k\in\mathbb{Z}.$

Application

As an application we consider for each $m\in\mathbb{N}$ the sequence

$$a(n,2m+1,k,-1)=\sum_{j\in\mathbb{Z}}(-1)^j\binom{n}{\left\lfloor \frac{n-(2m+1)j+k}{2}\right\rfloor}=(-1)^k\sum_j t(n,k-(2m+1)j).$$

As shown above we have $t=(1-K)s_{-1,0}$. Therefore by (5) we get

$\sum_{i=0}^{m} p_{m,i}(-1,0)a(0,2m+1,k,-1)=0.$

Formula (1) implies that $t(n)$ is a finite linear combination of functions $K^j t(0)$. Therefore we also get $p_m(N,-1,0)a(n,2m+1,k,-1)=\sum_{i=0}^{m} p_{m,i}(-1,0)a(n,2m+1,k,-1)=0.$

Now we look for an explicit expression for $p_n(x,-1,0)$.

We know that it satisfies the recurrence $p_n(x,-1,0)=xp_{n-1}(x,-1,0)-p_{n-2}(x,-1,0)$ with initial values $p_0(x,-1,0)=1$ and $p_1(x,-1,0)=x-1$.

Recall that the Fibonacci polynomials

$$F_n(x,s)=\sum_{k=0}^{n-1}\binom{n-1-k}{k}s^k x^{n-2k-1}=\frac{1}{\sqrt{x^2+4s}}\left(\left(\frac{x+\sqrt{x^2+4s}}{2}\right)^n-\left(\frac{x-\sqrt{x^2+4s}}{2}\right)^n\right) \tag{6}$$

are characterized by the recurrence

$$F_n(x,s)=xF_{n-1}(x,s)+sF_{n-2}(x,s) \tag{7}$$

with initial conditions $F_0(x,s)=0$ and $F_1(x,s)=1$. Therefore

$p_n(x,-1,0)=F_{n+1}(x,-1)-F_n(x,-1).$

The first values of the polynomials $p_n(x,-1,0)$ are

$1, x-1, x^2-x-1, x^3-x^2-2x+1, x^4-x^3-3x^2+2x+1,\cdots.$

This gives

Theorem 1

The sequence $a(n,2m+1,k,-1)=\sum_{j\in\mathbb{Z}}(-1)^j\binom{n}{\left\lfloor \frac{n-(2m+1)j+k}{2}\right\rfloor}$ *satisfies the recurrence relation of order* m

$$\big(F_{m+1}(N,-1)-F_m(N,-1)\big)a(n,2m+1,k,-1)=0 \tag{8}$$

for each $k\in\mathbb{Z}$.

Remark

This theorem has been proved in [3] with a more complicated method. The recurrence (8) is not for all k the minimal recurrence, because e.g. $a(n,2m+1,m+1,-1)\equiv 0.$ But it is so for $a(n,2m+1,0,-1)$, which has a simple combinatorial interpretation. It is the number of the set of all lattice paths in $\mathbb{R}^2$ which start at the origin, consist of $\left\lfloor \frac{n}{2} \right\rfloor$ northeast steps $(1,1)$ and $\left\lfloor \frac{n+1}{2} \right\rfloor$ southeast steps $(1,-1)$ and which are contained in the strip $-m-1<y<m.$ (cf. e.g. [4], [5]).

It is easy to see that the initial values of $a(n,2m+1,0,-1)$ are $a(j,2m+1,0,-1)=\binom{j}{\left\lfloor \frac{j}{2} \right\rfloor}$ for $0\le j<2m.$

As a special case of Theorem 1 we mention that $a(n,3,0,-1)=1.$ This means

$$\sum_{j\in\mathbb{Z}}(-1)^j\binom{n}{\left\lfloor \frac{n-3j}{2} \right\rfloor}=1 \text{ for all } n\in\mathbb{N}.$$

The generating function of the sequence $\left(a(n,2m+1,0,-1)\right)_{n\ge 0}$ has the form

$\sum_{n\ge 0}a(n,2m+1,0,-1)x^n=\frac{c_m(x)}{d_m(x)},$ where

$$d_m(x)=p_m\left(\frac{1}{x},-1,0\right)x^m=x^m\left(F_{m+1}\left(\frac{1}{x},-1\right)-F_{m+1}\left(\frac{1}{x},-1\right)\right)=F_{m+1}\left(1,-x^2\right)-xF_m\left(1,-x^2\right)$$

and $c_m(x)$ is a polynomial of degree less than m .

The first values of $\left(c_m(x)\right)_{m\ge 1}$ are

$c_1(x)=1,c_2(x)=1,c_3(x)=1-x^2,c_4(x)=1-2x^2,c_5(x)=1-3x^2+x^4,\cdots.$

This suggests that for $m\ge 2$

$$c_m(x)=\sum_{j=0}^{m-1}(-1)^j\binom{m-1-j}{j}x^{2j}=F_m(1,-x^2)\,.$$

This can be proved in the following way: Both $d_m(x)$ and $F_m(1,-x^2)$ satisfy the same recurrence $h_m(x)=h_{m-1}(x)-x^2h_{m-2}(x).$ This implies that for $a_{2m+1}(x)=\sum_{n\ge 0}a(n,2m+1,0,-1)x^n$ we have

$$d_m(x)a_{2m+1}(x)-d_{m-1}(x)a_{2m-1}(x)+x^2d_{m-2}(x)a_{2m-3}(x)=(d_m(x)-d_{m-1}(x)-x^2d_{m-2}(x))a_{2m+1}(x)$$
$$+d_{m-1}(x)(a_{2m+1}(x)-a_{2m-1}(x))+x^2d_{m-2}(x)(a_{2m+1}(x)-a_{2m-3}(x)).$$

Since the coefficients of x^j for $0\le j\le 2m-5$ of $a_{2m-3}(x)$ are the same as those of $a_{2m-1}(x)$ and $a_{2m+1}(x)$ we see that for $2m-4\ge m-1$ the polynomial

$$d_m(x)a_{2m+1}(x)-d_{m-1}(x)a_{2m-1}(x)+x^2d_{m-2}(x)a_{2m-3}(x)$$

which has degree $<m$ must identically vanish. This implies that

$$c_m(x)=d_m(x)a_{2m+1}(x)=F_m(1,-x^2).$$

Corollary 1

For $m \geq 2$ *the generating function for* $a(n,2m+1,0,-1)$ *is given by*

$$\sum_{n\geq 0} a(n,2m+1,0,-1)x^n = \frac{F_m(1,-x^2)}{F_{m+1}\left(1,-x^2\right) - xF_m\left(1,-x^2\right)}. \tag{9}$$

3. A modification of the above method

In order to obtain an analogous result for the sequences $a(n,2m,k,-1)$ we define a sequence of polynomials $q_n(x,a,b) = \sum_{k=0}^{n} q_{n,k}(a,b)x^k$ by the same recurrence

$$q_n(x,a,b) = (x-b)q_{n-1}(x,a,b) - a^2 q_{n-2}(x,a,b), \tag{10}$$

but with initial values $q_0(x,a,b) = 2$ and $q_1(x,a,b) = x - b.$

Lemma 2

For all $k \in \mathbb{Z}$ *the following identity holds*

$$q_m(N,a,b)s_{a,b}(0,k) = \sum_{i=0}^{m} q_{m,i}(a,b)s_{a,b}(i,k) = a^m \left[|k| = m\right]. \tag{11}$$

Proof

It suffices to show that on $\mathcal{F}$

$$q_m(N,a,b) = a^m(K^m + K^{-m}). \tag{12}$$

(12) is true for $m=0$ and $m=1$ by inspection.
If it is already shown for $m-1$ and $m-2$ we get

$$q_m(N,a,b) = a(K+K^{-1})a^{m-1}(K^{m-1} + K^{-(m-1)}) - a^2 a^{m-2}(K^{m-2} + K^{-(m-2)})$$
$$= a^m(K^m + K^{-m}).$$

Application

As an application let $u(n,k) = \binom{n}{\left\lfloor \frac{n+k}{2} \right\rfloor}.$

Then $u(n,k) = u(n-1,k-1) + u(n-1,k+1)$ and $u(0,k) = \left[k \in \{0,1\}\right].$ Therefore $u(n,k) = s_{1,0}(n,k) + s_{1,0}(n,k-1)$ or $u = (1+K)s_{1,0}.$

We have

$$a(n,2m,k,-1)=\sum_{j\in\mathbb{Z}}(-1)^j\binom{n}{\left\lfloor \frac{n-(2m)j+k}{2}\right\rfloor}=\sum_{j\in\mathbb{Z}}\left(\binom{n}{\left\lfloor \frac{n-(2m)2j+k}{2}\right\rfloor}-\binom{n}{\left\lfloor \frac{n-(2m)(2j+1)+k}{2}\right\rfloor}\right)$$

$$=\sum_{j\in\mathbb{Z}}\left(s_{1,0}(n,k-4mj)-s_{1,0}(n,k-2m-4mj)\right)$$

$$+\sum_{j\in\mathbb{Z}}\left(s_{1,0}(n,k-1-4mj)-s_{1,0}(n,k-1-2m-4mj)\right).$$

Here we get $q_m(N,1,0)\sum_{j\in\mathbb{Z}}\left(s_{1,0}(0,i-4mj)-s_{1,0}(0,i-2m-4mj)\right)=0$ for each i, because for $i-4mj=m$ we get $i-4mj-2m=-m$ and the sums cancel and for $i-4mj=-m$ we get $i-4m(j-1)-2m=m$. For other values the sum vanishes.
In the same way as above we conclude that

$$q_m(N,1,0)\sum_{j\in\mathbb{Z}}\left(s_{1,0}(n,i-4mj)-s_{1,0}(n,i-2m-4mj)\right)=0$$

too.
In order to give a concrete representation of $q_m(x,1,0)$ recall that the Lucas polynomials

$$L_n(x,s)=\sum_{k=0}^{n-1}\binom{n-k}{k}\frac{n}{n-k}s^kx^{n-2k}=\left(\frac{x+\sqrt{x^2+4s}}{2}\right)^n+\left(\frac{x-\sqrt{x^2+4s}}{2}\right)^n \tag{13}$$

are characterized by the recurrence

$$L_n(x,s)=xL_{n-1}(x,s)+sL_{n-2}(x,s) \tag{14}$$

with initial conditions $L_0(x,s)=2$ and $L_1(x,s)=x$.

Therefore $q_n(x,1,0)=L_n(x,-1)$.
The first values of the sequence $\left(L_n(x,-1)\right)_{n\geq 1}$ are
$x,x^2-2,x^3-3x,x^4-4x^2+2,\cdots.$

Theorem 2

For $m\geq 1$ *the sequence* $a(n,2m,k,-1)=\sum_{j\in\mathbb{Z}}(-1)^j\binom{n}{\left\lfloor \frac{n-(2m)j+k}{2}\right\rfloor}$ *satisfies the recurrence relation*

$$L_m(N,-1)a(n,2m,k,-1)=0. \tag{15}$$

Remark
It should be noted that $a(n,2m,0,-1)$ has the following combinatorial interpretation. It is the number of the set of all lattice paths in $\mathbb{R}^2$ which start at the origin, consist of $\left\lfloor \frac{n}{2}\right\rfloor$ northeast steps $(1,1)$ and $\left\lfloor \frac{n+1}{2}\right\rfloor$ southeast steps $(1,-1)$ and which are contained in the strip $-m<y<m.$ (cf. e.g. [5]).

The generating function of the sequence $\left(a(n,2m,0,-1)\right)_{n\geq 0}$ is given by $\sum_{n\geq 0} a(n,2m,0,-1)x^n = \frac{c_m(x)}{d_m(x)}$, where $d_m(x) = q_m\left(\frac{1}{x},1,0\right)x^m = x^m L_m\left(\frac{1}{x},-1\right) = L_m(1,-x^2)$ and $c_m(x)$ is a polynomial of degree less than m.

The first values of $\left(c_m(x)\right)_{m\geq 1}$ are

$c_1(x)=1, c_2(x)=1+x, c_3(x)=1+x-x^2, c_4(x)=1+x-2x^2-x^3, c_5(x)=1+x-3x^2-2x^3+x^4, \cdots.$

This implies as above that

$c_m(x) = F_m(1,-x^2) + xF_{m-1}(1,-x^2).$

Corollary 2

For $m \geq 2$ the generating function for $a(n,2m,0,-1)$ is given by

$$\sum_{n\geq 0} a(n,2m,0,-1)x^n = \frac{F_m(1,-x^2)+xF_{m-1}(1,-x^2)}{L_m(1,-x^2)}. \tag{16}$$

4. Further applications

4.a) The same method can be applied to the general sum

$$a(n,m,k,z) = \sum_{j\in\mathbb{Z}} z^j \binom{n}{\left\lfloor \frac{n-mj+k}{2} \right\rfloor} = \sum_{j\in\mathbb{Z}} z^{2j} \binom{n}{\left\lfloor \frac{n-2mj+k}{2} \right\rfloor} + \sum_{j\in\mathbb{Z}} z^{2j-1} \binom{n}{\left\lfloor \frac{n-2mj+k+m}{2} \right\rfloor}.$$

Here we get

$$L_m(N,-1)a(0,m,k,z) = L_m(N,-1)\sum_{j\in\mathbb{Z}} z^{2j} u(0,k-2mj) + L_m(N,-1)\sum_{j\in\mathbb{Z}} z^{2j-1} u(0,k+m-2mj).$$

In this case we have $L_m(N,-1)u(0,k-2mj) = \begin{cases} 1 & \text{if } k = 2mj-m \\ 1 & \text{if } k = 2mj+m \\ 0 & \text{else} \end{cases}$

or $L_m(N,-1)u(0,k-2mj) = u(0,k-m-2mj) + u(0,k+m-2mj).$ This implies

$$L_m(N,-1)a(0,m,k,z) = \sum_{j\in\mathbb{Z}} z^{2j}\left(u(0,k-m-2mj)+u(0,k+m-2mj)\right)$$

$$+\sum_{j\in\mathbb{Z}} z^{2j-1}\left(u(0,k+2m-2mj)+u(0,k-2mj)\right) = \left(z+\frac{1}{z}\right)a(0,m,k,z).$$

Thus we get

$$\left(L_m(N,-1)-\left(z+\frac{1}{z}\right)\right)a(0,m,k,z)=0.$$

Theorem 3

The sequence $a(n,m,k,z)=\sum_{j\in\mathbb{Z}} z^j \binom{n}{\left\lfloor \frac{n-mj+k}{2}\right\rfloor}$ *satisfies the recurrence relation*

$$\left(L_m(N,-1)-\left(z+\frac{1}{z}\right)\right)a(n,m,k,z)=0. \tag{17}$$

Remark

It is easy to see that the initial values of $a(n,m,0,z)$ are $a(n,m,0,z)=\binom{j}{\left\lfloor \frac{j}{2}\right\rfloor}$ for $0\le j<m-1,\ \ a(m-1,m,0,z)=\binom{m-1}{\left\lfloor \frac{m-1}{2}\right\rfloor}+\frac{1}{z},\ a(m,m,0,z)=\binom{m}{\left\lfloor \frac{m}{2}\right\rfloor}+\frac{1}{z}+z.$

The generating function of the sequence $(a(n,m,0,z))$ for $m\ge 1$ has the form $\sum_{n\ge 0} a(n,m,0,z)x^n=\frac{c_m(x,z)}{d_m(x,z)}$ with $d_m(x,z)=x^m\left(L_m\left(\frac{1}{x},-1\right)-\left(z+\frac{1}{z}\right)\right)=d_m(x)-x^m\left(z+\frac{1}{z}\right)$ and $c_m(x,z)=\frac{x^{m-1}}{z}+F_m(1,-x^2)+xF_{m-1}(1,-x^2)$.

Since $d_m(x)=L_m(1,-x^2)$ and $F_m(1,-x^2)+xF_{m-1}(1,-x^2)$ satisfy the same recurrence $h_m(x)=h_{m-1}(x)-x^2h_{m-2}(x)$ we get

$$\left(d_m(x)-x^m\left(z+\frac{1}{z}\right)\right)a_m(x)-\left(d_{m-1}(x)-x^{m-1}\left(z+\frac{1}{z}\right)\right)a_{m-1}(x)+x^2\left(d_{m-2}(x)-x^{m-2}\left(z+\frac{1}{z}\right)\right)a_{m-2}(x)$$
$$=d_{m-1}(x)(a_m(x)-a_{m-1}(x))+x^2d_{m-2}(x)(a_m(x)-a_{m-2}(x))$$
$$-x^m\left(z+\frac{1}{z}\right)a_m(x)+x^{m-1}\left(z+\frac{1}{z}\right)a_{m-1}(x)-x^m\left(z+\frac{1}{z}\right)a_{m-2}(x).$$

Since $d_m(0)=1$ it is easy to verify that for $m\ge 3$

$d_{m-1}(x)(a_m(x)-a_{m-1}(x))=-\frac{x^{m-2}}{z}-x^{m-1}z+x^m(\cdots)$ and

$x^2d_{m-2}(x)(a_m(x)-a_{m-2}(x))=-\frac{x^{m-1}}{z}+x^m(\cdots).$

Therefore we get

$$d_m(x,z)a_m(x)-d_{m-1}(x,z)a_{m-1}(x)+x^2d_{m-2}(x,z)a_{m-2}(x)=-\frac{x^{m-2}}{z}+x^m(\cdots).$$

Now the left hand side must be a polynomial of degree less than m. Therefore we have in fact

$$d_m(x,z)a_m(x)-d_{m-1}(x,z)a_{m-1}(x)+x^2d_{m-2}(x,z)a_{m-2}(x)=-\frac{x^{m-2}}{z}.$$

Now $c_m(x,z)$ satisfies the same recurrence. Since the initial values coincide, we get

Corollary 3

For $m \geq 2$ the generating function for $a(n,m,0,z)$ is given by

$$\sum_{n\geq 0} a(n,m,0,z)x^n = \frac{\frac{x^{m-1}}{z}+F_m(1,-x^2)+xF_{m-1}(1,-x^2)}{L_m(1,-x^2)-x^m\left(z+\frac{1}{z}\right)}. \tag{18}$$

Remark

In the same way we get

$$\sum_{n\geq 0} a(n,2m+1,m+1,z)x^n = \frac{(1+z)x^m\left(F_{m+1}(1,-x^2)+xF_m(1,-x^2)\right)}{L_{2m+1}(1,-x^2)-x^{2m+1}\left(z+\frac{1}{z}\right)}.$$

For $z=-1$ the right hand side vanishes and therefore we get again $a(n,2m+1,m+1,-1)=0.$

4.b) For the special case $z=1$ also simpler recurrences can be found.

It is easy to verify that

$$\left(x+\frac{1}{x}-2\right)F_m\left(x+\frac{1}{x},-1\right)(1+x)=\frac{1}{x^m}-\frac{1}{x^{m-1}}-x^m+x^{m+1}.$$

This implies as above

$$(N-2)F_m(N,-1)u(0)=(K^m-K^{m-1}-K^{-m}+K^{-m-1})s_{1,0}(0).$$

Therefore we get

$$(N-2)F_m(N,-1)\sum_j K^{2jm}u(0)=\sum_j K^{2jm}(K^m-K^{m-1}-K^{-m}+K^{-m-1})s_{1,0}(0)=0.$$

From this we conclude as above

Theorem 4

The sequence $a(n,2m,k,1)=\sum_{j\in\mathbb{Z}}\binom{n}{\left\lfloor \frac{n-(2m)j+k}{2}\right\rfloor}$ *satisfies the recurrence relation*

$$(N-2)F_m(N,-1)a(n,2m,k,1]=0. \tag{19}$$

Corollary 4

For $m\geq 1$ *the generating function for* $a(n,2m,0,1)$ *is given by*

$$\sum_{n\geq 0}a(n,2m,0,1)x^n=\frac{F_m(1,-x^2)-xF_{m-1}(1,-x^2)}{(1-2x)F_m\left(1,-x^2\right)}. \tag{20}$$

4.c) It is again easy to verify that

$$\left(L_m(x+\frac{1}{x},-1)-L_{m-1}(x+\frac{1}{x},-1)\right)(1+x)=\frac{1}{x^m}-\frac{1}{x^{m-2}}-x^{m-1}+x^{m+1}.$$

Therefore we get

$$\left(L_m(K+K^{-1},-1)-L_{m-1}(K+K^{-1},-1)\right)\sum_j K^{(2m-1)j}u(0)$$

$$=\sum_j K^{(2m-1)j}(K^m-K^{m-2}-K^{-m+1}+K^{-m-1})s_{1,0}(0)=0.$$

This implies

Theorem 5

The sequence $a(n,2m-1,k,1)=\sum_{j\in\mathbb{Z}}\binom{n}{\left\lfloor \frac{n-(2m-1)j+k}{2}\right\rfloor}$ *satisfies the recurrence relation*

$$\left(L_m(N,-1)-L_{m-1}(N,-1)\right)a(n,2m-1,k,1)=0. \tag{21}$$

Corollary 5

For $m\geq 2$ *the generating function for* $a(n,2m-1,0,1)$ *is given by*

$$\sum_{n\geq 0}a(n,2m-1,0,1)x^n=\frac{L_{m-1}(1,-x^2)}{L_m(1,-x^2)-xL_{m-1}(1,-x^2)}. \tag{22}$$

Remark

For the special cases $z = \pm 1$ numerator and denumerator of the generating function

$$\frac{\frac{x^{m-1}}{z} + F_m(1,-x^2) + xF_{m-1}(1,-x^2)}{L_m(1,-x^2) - x^m\left(z + \frac{1}{z}\right)}$$

have common divisors which can be cancelled.

This can be verified by using the following identities, which are easily deduced from the representations (6) and (13) (cf. e.g. [3]) :

$$L_{2m}(x,-1) - 2 = (x^2 - 4)\left(F_m(x,-1)\right)^2,$$

$$L_{2m-1}(x,-1) - 2 = \frac{\left(L_m(x,-1) - L_{m-1}(x,-1)\right)^2}{x-2},$$

$$L_{2m}(x,-1) + 2 = \left(L_m(x,-1)\right)^2,$$

$$L_{2m-1}(x,-1) + 2 = (x+2)\left(F_m(x,-1) - F_{m-1}(x,-1)\right)^2.$$

References

[1] G.E. Andrews, *Some formulae for the Fibonacci sequence with generalizations*, Fibonacci Quart. **7** (1969), 113-130

[2] E.H.M. Brietzke, *Generalization of an identity of Andrews*, Fibonacci Quart. **44** (2006), 166-171

[3] J. Cigler, *Recurrences for some sequences of binomial sums*, Sitzungsber. ÖAW **210** (2001), 61-83 in http://hw.oeaw.ac.at/sitzungsberichte_und_anzeiger_collection.

[4] J. Cigler, *A class of Rogers-Ramanujan type recursions*, Sitzungsber. ÖAW **213** (2004), 71-93 in http://hw.oeaw.ac.at/sitzungsberichte_und_anzeiger_collection.

[5] J. Cigler, *Fibonacci-Zahlen, Gitterpunktwege und die Identitäten von Rogers-Ramanujan*, Math. Semesterber. **52** (2005), 97-125

[6] I. Schur, *Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche*, 1917, in Gesammelte Abhandlungen, Bd.2, 117-136, Springer 1973