

Recurrences for Some Sequences of Binomial Sums

By

J. Cigler

(Eingereicht in der Sitzung der math.-nat. Klasse am 11. Oktober 2001
durch das w. M. Johann Cigler)

Abstract

We prove that the sequences

$$a(n, i, l, m, z) = \sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{m} \rfloor} z^k \in \mathbb{Q}[z, z^{-1}]$$

satisfy a linear recurrence of order i with constant coefficients and show how these coefficients can be computed.

Mathematics Subject Classifications: 05A10, 11B37, 11B39, 39A10.

Key words: Recurrence, Lucas polynomial, Fibonacci polynomial, binomial coefficient, binomial sum, difference operator.

0. Introduction

Our starting point are the following remarkable identities for the Fibonacci numbers F_n (cf. [1] and [5]):

$$F_n = \sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{\lfloor \frac{n+5k+2}{2} \rfloor} \quad (0.1)$$

and

$$F_{n+1} = \sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{\lfloor \frac{n+5k}{2} \rfloor} \quad (0.2)$$

The purpose of this paper is to put these identities into a general context in order to give an “explanation” of these formulas. We prove that the sequences $a(n, i, l, m, z) = \sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{m} \rfloor} z^k$ satisfy a linear recurrence of order i with constant coefficients and show how these coefficients can be computed. Some special cases have previously been obtained by G. E. Andrews [1] with other methods. I want to thank G. Kowol for some useful hints and Professor Andrzej Schinzel for providing a proof of Lemma 7.2.

1. Sums of the Form $\sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{2} \rfloor} z^k$

We study first the polynomials

$$a(n, i, l, z) = a(n, i, l, 2, z) = \sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{2} \rfloor} z^k \in \mathbb{Q}[z, z^{-1}], \quad (1.1)$$

where $i \geq 1$ and l are integers.

It is easy to see that

$$a(n, 2, l, z) = (1+z)^n z^{-\lfloor \frac{n+l}{2} \rfloor} \quad (1.2)$$

There are some obvious relations between some of these polynomials. E.g. we have

$$a(n, i, l, z) = a(n, 2i, l, z^2) + za(n, 2i, l+i, z^2) \quad (1.3)$$

For

$$\begin{aligned} a(n, i, l, z) &= \sum \binom{n}{\lfloor \frac{n+l+2ki}{2} \rfloor} z^{2k} + \\ &+ \sum \binom{n}{\lfloor \frac{n+l+(2k+1)i}{2} \rfloor} z^{2k+1} = \\ &= a(n, 2i, l, z^2) + za(n, 2i, l+i, z^2). \end{aligned}$$

A trivial relation is

$$a(n, i, l - i, z) = za(n, i, l, z). \tag{1.4}$$

Therefore we only need to consider $l \in \{0, 1, \dots, i - 1\}$.

It follows that

$$\begin{aligned} a(n, 1, l, z) &= a(n, 2, l, z^2) + za(n, 2, l + 1, z^2) = \\ &= (1 + z^2)^n (z^{-2\lfloor \frac{n+l}{2} \rfloor} + z^{1-2\lfloor \frac{n+l+1}{2} \rfloor}) = \\ &= (1 + z)(1 + z^2)^n z^{-n-l}. \end{aligned}$$

Thus

$$a(n, 1, l, z) = (1 + z)(1 + z^2)^n z^{-n-l}. \tag{1.5}$$

In other cases we cannot hope to find explicit formulas. But it turns out that they satisfy simple recurrence relations.

We shall prove the following

Theorem 1.1. *The sequence of the polynomials $(a(n, i, l, z))_{n \geq 0}$ satisfies the homogeneous recurrence*

$$\sum_{j \leq \frac{i}{2}} (-1)^j \binom{i-j}{j} \frac{i}{i-j} a(n+i-2j, i, l, z) = \left(z + \frac{1}{z}\right) a(n, i, l, z) \tag{1.6}$$

with constant coefficients.

Before proving this theorem let us make some remarks.

For $i = 1$ we get the recurrence $a(n + 1, 1, l, z) = (z + \frac{1}{z}) \cdot a(n, 1, l, z)$. Since $a(0, 1, l, z) = \binom{0}{\lfloor \frac{l}{2} \rfloor} z^{-l} + \binom{0}{\lfloor \frac{l+1}{2} \rfloor} z^{-l+1} = z^{-l}(1 + z)$ we get (1.5).

For $i = 2$ the recurrence reduces to

$$a(n + 2, 2, l, z) = \frac{(z + 1)^2}{z} a(n, 2, l, z).$$

This is a recurrence of order 2. Therefore we need 2 initial values. These are $a(0, 2, l, z) = z^{-\lfloor \frac{l}{2} \rfloor}$, $a(1, 2, l, z) = (1 + z)z^{-\lfloor \frac{l+1}{2} \rfloor}$. Hence (1.2) holds.

Everyone familiar with Fibonacci numbers recognizes the coefficients on the left hand side of (1.6). These occur in the Lucas polynomials $L_n(x, s)$ which are defined by

$$L_n(x, s) = xL_{n-1}(x, s) + sL_{n-2}(x, s), \quad n \geq 2, \tag{1.7}$$

with initial values $L_0(x, s) = 2$, $L_1(x, s) = x$.

They are explicitly given by $L_n(x, s) = \sum_{j < n} \binom{n-j}{j} \frac{n}{n-j} x^{n-2j} s^j$. From (1.7) we see by induction that

$$L_n(x, 1-x) = 1 + (x-1)^n \quad (1.8)$$

holds.

Consider now the vector space of all functions f on the nonnegative integers and define the translation operator E and the finite difference operator $\Delta = E - I$ as usual by $Ef(n) = f(n+1)$ and $\Delta f(n) = f(n+1) - f(n)$. Then (1.8) implies

$$L_i(E, -\Delta)f(n) = f(n) + \Delta^i f(n). \quad (1.9)$$

For $f(n) = \binom{n}{r}$ this reduces to

$$\begin{aligned} \binom{n}{r} + \binom{n}{r-i} &= L_i(E, -\Delta) \binom{n}{r} = \\ &= \sum_{j < i} (-1)^j \binom{i-j}{j} \frac{i}{i-j} \binom{n+i-2j}{r-j}. \end{aligned} \quad (1.10)$$

This holds for all values $r \in \mathbb{Z}$. As a special case we get

$$\begin{aligned} \left(\left\lfloor \frac{n + i(k+1) + l}{2} \right\rfloor \right) + \left(\left\lfloor \frac{n + i(k-1) + l}{2} \right\rfloor \right) &= \\ = \sum_{j < i} (-1)^j \binom{i-j}{j} \frac{i}{i-j} \left(\left\lfloor \frac{n + i(k+1) - 2j + l}{2} \right\rfloor \right) &= \\ = \sum_{j < i} (-1)^j \binom{i-j}{j} \frac{i}{i-j} E^{i-2j} \left(\left\lfloor \frac{n + ik + l}{2} \right\rfloor \right). \end{aligned}$$

If we multiply both sides by z^k and sum over all $k \in \mathbb{Z}$ we get (1.6). Thus our theorem is proved.

In order to give some impression of the operators

$$A_i(E) = \sum_{j < i} (-1)^j \binom{i-j}{j} \frac{i}{i-j} E^{i-2j} = L_i(E, -1)$$

we explicitly state the first ones:

$$\begin{aligned} A_1(E) &= E \\ A_2(E) &= E^2 - 2 \\ A_3(E) &= E^3 - 3E \\ A_4(E) &= E^4 - 4E^2 + 2 \\ A_5(E) &= E^5 - 5E^3 + 5E \\ A_6(E) &= E^6 - 6E^4 + 9E^2 - 2 \end{aligned}$$

For $i = 5$ and $z = -1$ the theorem tells us that sequence $b(n, l) = a(n, 5, l, -1)$ satisfies the recurrence $(E^5 - 5E^3 + 5E + 2)b(n, l) = 0$. From $E^5 - 5E^3 + 5E + 2 = (E + 2)(E^2 - E - 1)^2$ it is clear that the Fibonacci numbers satisfy the same recurrence.

Computing the initial values immediately gives $b(n, 0) = b(n, 1) = F_{n+1}$ and $b(n, 2) = F_n$, thus proving (0.1) and (0.2).

Remark:

If we only want to prove that the sequence of polynomials $(a(n, i, l, z))_{n \geq 0}$ satisfies a homogeneous linear recurrence with constant coefficients of order i , it suffices to show the following

Lemma 1.1. *For each integer $n \geq 1$ there exist integers such that $1 + x^n = \sum a_{n,j}(x + 1)^{n-2j}x^j$ holds.*

For this implies that $1 + \Delta^n = \sum a_{n,j}E^{n-2j}\Delta^j$ and the argument continues as above.

In order to prove the lemma observe that for each $j, 0 \leq j \leq \frac{n}{2}$, the polynomial $(x + 1)^{n-2j}x^j$ is symmetric about $\frac{n}{2}$. If we eliminate the coefficients of $x^j, 1 \leq j \leq \lfloor \frac{n}{2} \rfloor$, in $(x + 1)^n$ we get therefore $1 + x^n$.

It turns out that for $z = 1$ and $z = -1$ there are simpler recurrences. We consider first the case $z = -1$.

First we notice that because of (1.4) we have $a(n, i, l + i, -1) = -a(n, i, l, -1)$. Therefore we need only consider $l \in \{0, 1, \dots, i - 1\}$. For small values of i we can find explicit values for these sums. Thus we have $a(n, 1, l, -1) \equiv 0$ for all $l \in \mathbb{Z}$ and $a(n, 2, 0, -1) = a(n, 2, 1, -1) = [n = 0]$, where the symbol $[P]$ equals 1 if the assertion P holds and it is 0 if P does not hold.

Further we have $a(n, 3, 0, -1) = a(n, 3, 1, -1) \equiv 1$ and $a(n, 3, 2, -1) \equiv 0, a(n, 4, 0, -1) = a(n, 4, 1, -1) = 2^{\lfloor \frac{n}{2} \rfloor}, a(n, 4, 2, -1) = -a(n, 4, 3, -1) = 0$ for even $n, a(n, 4, 2, -1) = -a(n, 4, 3, -1) = 2^{\lfloor \frac{n-1}{2} \rfloor}$ for n odd, and finally $a(n, 5, 0, -1) = a(n, 5, 1, -1) = F_{n+1}, a(n, 5, 2, -1) = -a(n, 5, 4, -1) = F_n$, and $a(n, 5, 3, -1) \equiv 0$.

It turns out that our results differ for even and odd values of i .

2. Sums of the Form $\sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{\lfloor \frac{n+2ik+l}{2} \rfloor}$

We first prove

Theorem 2.1. *The sequences $(a(n, 2i, l, -1))$ satisfy the homogeneous linear recurrence*

$$\sum_{j < i} (-1)^j \binom{i-j}{j} \frac{i}{i-j} E^{i-2j} a(n, 2i, l, -1) = 0 \quad (2.1)$$

Remark. If we denote the operator on the lefthand side by

$$B_{2i}(E) = \sum_{j < i} (-1)^j \binom{i-j}{j} \frac{i}{i-j} E^{i-2j} = L_i(E, -1), \text{ then}$$

$$B_{2i}(E) = A_i(E).$$

The sequence $(a(n, 2i, l, -1))$ satisfies therefore both the recurrences

$$A_i(E)a(n, 2i, l, -1) = 0 \text{ and } (A_{2i}(E) + 2I)a(n, 2i, l, -1) = 0.$$

In order to understand this situation note that

$$L_{2i}(x, -1) + 2 = (L_i(x, -1))^2.$$

This follows immediately from the well-known formula (cf. e.g. [3])

$$L_n(x, s) = \left(\frac{x + \sqrt{x^2 + 4s}}{2} \right)^n + \left(\frac{x - \sqrt{x^2 + 4s}}{2} \right)^n. \quad (2.2)$$

In order to prove (2.1) we observe that

$$\begin{aligned} (I + \Delta^i) \binom{n}{\lfloor \frac{r + 2ik + l}{2} \rfloor} &= \binom{n}{\lfloor \frac{r + 2ik + l}{2} \rfloor} + \\ &+ \binom{n}{\lfloor \frac{r + 2i(k-1) + l}{2} \rfloor}. \end{aligned}$$

This implies $(I + \Delta^i)a(n, 2i, l, -1) = 0$, because it turns out to be a telescoping sum.

Now we know that $1 + \Delta^i = \sum a_{i,j} E^{i-2j} \Delta^j$ for some coefficients $a_{i,j}$. If we apply $E^{i-2j} \Delta^j$ to $\binom{n}{\lfloor \frac{r+l+2ik}{2} \rfloor}$ we get

$$\binom{n+i-2j}{\lfloor \frac{r+i-2j+l-i+2ik}{2} \rfloor}.$$

Hence we get $\sum a_{i,j} a(n, 2i, l-i, -1) = 0$. Since this holds for each l , our theorem is proved.

3. Sums of the Form $\sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{\lfloor \frac{n+(2i-1)k+l}{2} \rfloor}$

In order to study $a(n, 2i-1, l, -1)$ we observe that

$$a(n, 2i-1, i+l, -1) = -a(n, 2i-1, i-l, -1). \quad (3.1)$$

Since $n = \lfloor \frac{n}{2} \rfloor + \lfloor \frac{n+1}{2} \rfloor$ we have

$$n - \left\lfloor \frac{n + (2i-1)k - l + i}{2} \right\rfloor = \left\lfloor \frac{n - (2i-1)(k+1) - i + l}{2} \right\rfloor.$$

This means that the term with index k in $a(n, 2i-1, i-l, -1)$ is the opposite of the term with index $-(k+1)$ in $a(n, 2i-1, i+l, -1)$.

As special cases we get

$$a(n, 2i-1, i, -1) = 0 \quad (3.2)$$

and

$$a(n, 2i-1, 1, -1) = -a(n, 2i-1, 2i, -1) = a(n, 2i-1, 0, -1) \quad (3.3)$$

Therefore we get

$$\sum_{j=1}^{2i-1} a(n, 2i-1, j, -1) = 0. \quad (3.4)$$

But we also have

$$\sum_{j=1}^{i-1} a(n, 2i-1, 2j, -1) = 0, \quad (3.5)$$

because

$$\begin{aligned} a(n, 2i-1, 2i-2j, -1) &= a(n, 2i-1, i+(i-2j), -1) = \\ &= -a(n, 2i-1, i-(i-2j), -1) = -a(n, 2i-1, 2j, -1). \end{aligned}$$

Thus also

$$\sum_{j=1}^i a(n, 2i-1, 2j-1, -1) = - \sum_{j=1}^i a(n, 2i-1, 2j-1+2i-1, -1) = 0$$

which implies $\sum_{j=1}^{2i-1} a(n, 2i-1, 2j, -1) = 0$.

As a consequence we get that the sum over $2i-1$ consecutive values of the form $a(n, 2i-1, l-2j, -1)$ vanishes and therefore we have

$$\sum_{j=-i+1}^{i-1} a(n, 2i-1, l-2j, -1) = 0 \quad (3.6)$$

for each $l \in \mathbb{Z}$.

This is equivalent with

$$\sum_{j=0}^{i-1} a(n, 2i-1, l-2j, -1) - \sum_{j=1}^{i-1} a(n, 2i-1, l+1-2j, -1) = 0 \quad (3.7)$$

Each term in $a(n, 2i-1, l, -1)$ has the form $f(n, r, k, l) = (-1)^k \binom{n}{\lfloor \frac{r+l+(2i-1)k}{2} \rfloor}$ with $r = n$ and $a(n, 2i-1, l, -1)$ is a finite sum of expressions $f(n, n, k, l) - f(n, n, k+1, l)$.

Given r and l we may choose one of two consecutive k 's such that $r+l+(2i-1)k$ is even.

Then we have

$$\begin{aligned} f(n, r, k, l-2j) - f(n, r, k+1, l-2j) &= \\ &= \Delta^j (f(n, r, k, l) - f(n, r, k+1, l)) \end{aligned} \quad (3.8)$$

If we set

$$b(n, i, l, r) = \sum_{k \in \mathbb{Z}} (-1)^k \binom{n}{\lfloor \frac{r+ik+l}{2} \rfloor},$$

then we have

$$\begin{aligned} 0 &= \sum_{j=-i+1}^{i-1} a(n, 2i-1, l-2j, -1) = \\ &= (1 + \Delta + \cdots + \Delta^{i-1}) b(n, 2i-1, l, r)_{r=n} - \\ &\quad - \Delta(1 + \Delta + \cdots + \Delta^{i-2}) b(n, 2i-1, l+1, r)_{r=n} \end{aligned}$$

If we define the Filbonacci polynomials $F_n(x, s)$ by the recurrence $F_n(x, s) = xF_{n-1}(x, s) + sF_{n-2}(x, s)$ with the initial values $F_0(x, s) = 0$ and $F_1(x, s) = 1$ then we have

$$F_n(x, s) = \sum_{j \leq \frac{n-1}{2}} \binom{n-1-j}{j} x^{n-1-j} s^j.$$

It is immediately verified that

$$F_n(x, 1-x) = 1 + (x-1) + \dots + (x-1)^{n-1}$$

holds.

Therefore (3.7) may be formulated as

$$(F_i(E, -\Delta)b(n, 2i-1, l, r) - \Delta F_{i-1}(E, -\Delta) \cdot b(n, 2i-1, l+1, r))_{r=n} = 0 \tag{3.9}$$

Now we have

$$\begin{aligned} E^{-2j} \Delta^j \left(\left\lfloor \frac{r+l+(2i-1)k}{2} \right\rfloor \right)_{r=n} &= \\ &= \left(\left\lfloor \frac{n-2j+l+(2i-1)k}{2} \right\rfloor \right) = \\ &= E^{-2j} \left(\left\lfloor \frac{n+l+(2i-1)k}{2} \right\rfloor \right) \end{aligned} \tag{3.10}$$

and

$$\begin{aligned} E^{-2j-1} \Delta^{j+1} \left(\left\lfloor \frac{r+l+1+(2i-1)k}{2} \right\rfloor \right)_{r=n} &= \\ &= \left(\left\lfloor \frac{n-2j-1+l+(2i-1)k}{2} \right\rfloor \right) = \\ &= E^{-2j-1} \left(\left\lfloor \frac{n+l+(2i-1)k}{2} \right\rfloor \right) \end{aligned} \tag{3.11}$$

Therefore (3.9) is equivalent with

$$(F_i(E, -1) - F_{i-1}(E, -1))a(n, 2i-1, l, -1) = 0.$$

This implies

Theorem 3.1. *The sequences $(a(n, 2i - 1, l, -1))$ satisfy the homogeneous linear recurrence*

$$(F_i(E, -1) - F_{i-1}(E, -1))a(n, 2i - 1, l, -1) = 0 \quad (3.12)$$

The operators $C_i(E) = F_i(E, -1) - F_{i-1}(E, -1)$ are for small values i explicitly given by

$$\begin{aligned} C_1(E) &= 1 \\ C_2(E) &= E - 1 \\ C_3(E) &= E^2 - E - 1 \\ C_4(E) &= E^3 - E^2 - 2E + 1 \\ C_5(E) &= E^4 - E^3 - 3E^2 + 2E - 1 \end{aligned}$$

As an example consider $i = 3$. We get

$$a(n + 2, 5, l, -1) - a(n + 1, 5, l, -1) - a(n, 5, l, -1) = 0.$$

Thus $a(n, 5, l, -1)$ satisfies the recurrence of the Fibonacci numbers. By choosing the appropriate initial values we get again formulas (0.1) and (0.2).

To understand the relation between Theorem 1 and Theorem 3 we note that

$$(F_n(x, -1) - F_{n-1}(x, -1))^2(x + 2) = L_{2n-1}(x, -1) + 2$$

holds. This follows from the well-known formula (cf. [3])

$$F_n(x, s) = \frac{1}{\sqrt{x^2 + 4s}} \left(\left(\frac{x + \sqrt{x^2 + 4s}}{2} \right)^n - \left(\frac{x - \sqrt{x^2 + 4s}}{2} \right)^n \right) \quad (3.13)$$

in connection with (2.2).

4. Sums of the Form $\sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+2k+l}{2} \rfloor}$

Simpler recurrences than in the general case are also possible for $z = 1$.

We must again distinguish between even and odd i .

Let us first consider the sums $a(n, 2i, l, 1)$. Here we have

$$(\Delta^i - 1) \sum \binom{n}{r + ik} = 0.$$

We know already that $F_n(x, 1 - x) = 1 + (x - 1) + \dots + (x - 1)^{n-1}$ holds and therefore we may write

$$\Delta^i - 1 = (E - 2)F_i(E, -\Delta).$$

For $n + l \equiv 1 \pmod{2}$ we have

$$E^{-2j} \binom{n}{\lfloor \frac{n + 2ik + l}{2} \rfloor} = E^{-2j} \Delta^j \binom{n}{r}_{r = \lfloor \frac{n + 2ik + l}{2} \rfloor}$$

and

$$E^{-2j-1} \binom{n}{\lfloor \frac{n + 2ik + l}{2} \rfloor} = E^{-2j-1} \Delta^j \binom{n}{r}_{r = \lfloor \frac{n + 2ik + l}{2} \rfloor}.$$

If $n + l \equiv 0 \pmod{2}$ we have

$$\binom{n}{\lfloor \frac{n + 2ik + l}{2} \rfloor} = \binom{n}{\lfloor \frac{n + 2ik - l + 1}{2} \rfloor}.$$

Thus in each case we get

$$(E - 2)F_i(E, -1)a(n, 2i, l, 1) = 0.$$

This gives

Theorem 4.1. *The sequences $(a(n, 2i, l, 1))$ satisfy the homogeneous linear recurrence*

$$(E - 2)F_i(E, -1)a(n, 2i, l, 1) = 0 \tag{4.1}$$

These operators are for small values of i given by

$$\begin{aligned} D_1(E) &= E - 2 \\ D_2(E) &= E^2 - 2E \\ D_3(E) &= E^3 - 2E^2 - E - 2 \\ D_4(E) &= E^4 - 2E^3 - 2E^2 + 4E \end{aligned}$$

Again this is connected to the formula

$$L_{2i}(x, -1) - 2 = (x^2 - 4)(F_i(x, -1))^2$$

5. Sums of the Form $\sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+(2i-1)k+l}{2} \rfloor}$

Now remains the case $a(n, 2i - 1, l, 1)$.

Our aim is the following theorem.

Theorem 5.1. *The sequences $(a(n, 2i - 1, l, 1))$ satisfy the homogeneous linear recurrence*

$$(L_i(E, -1) - L_{i-1}(E, -1))a(n, 2i - 1, l, 1) = 0.$$

In order to prove this observe that

$$a(n, 2i - 1, l, 1) = a(n, 2i - 1, l + 2i - 1, 1)$$

and therefore

$$\begin{aligned} a(n, 2i - 1, l, 1) + a(n, 2i - 1, l - 2i, 1) &= \\ &= (a(n, 2i - 1, l - 1, 1) + a(n, 2i - 1, l + 1 - 2i, 1)) \end{aligned} \quad (5.1)$$

The same reasoning as in (3.8) gives

$$(1 + \Delta^i)a(n, 2i - 1, l, 1) = \Delta(1 + \Delta^{i-1})a(n, 2i - 1, l + 1, 1). \quad (5.2)$$

From (3.10) and (3.11) and (1.9) we see that (5.2) means

$$(L_i(E, -1) - L_{i-1}(E, -1))a(n, 2i - 1, l, 1) = 0.$$

Thus our theorem is proved.

We note again the corresponding formula for the Lucas polynomials

$$(x - 2)(L_{2n-1}(x, -1) - 2) = (L_n(x, -1) - L_{n-1}(x, -1))^2.$$

6. Sums of the Form $\sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{3} \rfloor} z^k$

We first prove the

Lemma 6.1. *For each integer $n \geq 1$ there exist integers such that*

$$1 + (-1)^{n-1}x^n = \sum a_{n,j}(x+1)^{n-3j}x^j + \sum b_{n,j}(x+1)^{2n-3j}x^j \quad (6.1)$$

holds.

Let us make some comments. We are looking for a term of the form $1 + a(n)x^n$ which occurs on the right hand side, i.e. which is a linear combination of the polynomials $(x+1)^{n-3j}x^j$, $0 \leq j \leq \frac{n}{3}$ and

$(x + 1)^{2n-3j}x^j$, $\frac{n}{2} \leq j \leq \frac{2n}{3}$. The only polynomial of this form which contains 1 is $(x + 1)^n$. For odd n this is also the only one containing x^n . Thus $a(n)$ must be 1 in this case.

If $n = 2m$ there is another polynomial of this form which contains x^n , namely $(x + 1)^m x^m = (x + 1)^{4m-3m} x^m$. In order to obtain a linear combination of the form $1 + a(n)x^n$ we must find b such that the coefficient of x^{n-1} in $(x + 1)^{2m} + b((x + 1)^m x^m)$ equals 0. Thus $b = -2$ and the coefficient of x^n must therefore be -1 .

Proof of Lemma 6.1:

To prove this define polynomials

$$v_n(x, s) = xv_{n-1}(x, s) - sv_{n-3}(x, s) \tag{6.2}$$

with initial values

$$v_0(x, s) = 3, v_1(x, s) = x, v_2(x, s) = x^2 \tag{6.3}$$

Then $v_n(x, s) = \sum_{3j \leq n} a_{n,j} x^{n-3j} s^j$ for some coefficients $a_{n,j}$.

This is easily proved by induction.

It is not difficult to determine these polynomials explicitly: For $n > 0$ we have

$$v_n(x, s) = \sum_{3j \leq n} (-1)^j \binom{n-2j}{j} \frac{n}{n-2j} x^{n-3j} s^j. \tag{6.4}$$

For the recurrence is easily verified and the initial values coincide for $n = 1, 2, 3$.

Consider now

$$v_n(x + 1, x) = \sum_{3j \leq n} a_{n,j} (x + 1)^{n-3j} x^j.$$

These polynomials satisfy the recurrence

$$v_n(x + 1, x) = (x + 1)v_{n-1}(x + 1, x) - xv_{n-3}(x + 1, x).$$

The characteristic polynomial of this recurrence is $\lambda^3 - (x + 1)\lambda^2 + x$ with $\lambda = 1$ as one root. Dividing this polynomial by $\lambda - 1$ we get $\lambda^2 - x\lambda - x$.

Now we know that the Lucas polynomials

$$L_n(x, x) = \sum \binom{n-j}{j} \frac{n}{n-j} x^{n-j} = \sum_i \binom{i}{n-i} \frac{n}{i} x^i$$

satisfy the recurrence $L_n(x, x) = xL_{n-1}(x, x) + xL_{n-2}(x, x)$. Therefore they also satisfy $L_n(x, x) = (x + 1)L_{n-1}(x, x) - xL_{n-3}(x, x)$.

The initial values are $L_0(x, x) = 2$, $L_1(x, x) = x$.

Since the constant polynomials also satisfy this recurrence we see that $v_n(x + 1, x) = L_n(x, x) + 1$.

Next we define polynomials

$$w_n(x, s) = xsw_{n-2}(x, s) + s^2w_{n-3}(x, s) \quad (6.5)$$

with initial values

$$w_0(x, s) = 3, \quad w_1(x, s) = 0, \quad w_2(x, s) = 2xs \quad (6.6)$$

Then $w_n(x, s) = \sum_{3j \leq 2n} b_{n,j} x^{2n-3j} s^j$ for some coefficients $b_{n,j}$. This may be proved by induction.

For $n > 0$ these polynomials are explicitly given by

$$w_n(x, s) = \sum_{3j \leq 2n} \binom{n-j}{2n-3j} \frac{n}{n-j} x^{2n-3j} s^j. \quad (6.7)$$

This is again easily verified.

Consider now

$$w_n(x + 1, x) = \sum_{3j \leq 2n} b_{n,j} (x + 1)^{2n-3j} x^j.$$

These polynomials satisfy the recurrence

$$w_n(x + 1, x) = x(x + 1)w_{n-2}(x + 1, x) + x^2w_{n-3}(x + 1, x).$$

The characteristic polynomial of this recurrence is $\lambda^3 - x(x + 1)\lambda - x^2$ with $\lambda = -x$ as one root. Dividing this polynomial by $\lambda + x$ we get again $\lambda^2 - x\lambda - x$.

Thus $w_n(x + 1, x) = L_n(x, x) + a(-x)^n$.

Comparing the initial values we see that $a = 1$.

Therefore we get

$$v_n(x + 1, x) - w_n(x + 1, x) = 1 - (-x)^n.$$

Thus our lemma is proved.

Therefore we also have

$$v_i(E, \Delta) - w_i(E, \Delta) = 1 + (-1)^{i-1} \Delta^i \quad (6.8)$$

for all i .

If we apply this to $\binom{n}{r}$ we get

$$\binom{n}{r} + (-1)^{i-1} \binom{n}{r-i} = \sum a_{i,j} \binom{n+i-3j}{r-j} + \sum b_{i,j} \binom{n+2i-3j}{r-j}$$

Now let $r = \lfloor \frac{n+i(k+1)}{3} \rfloor$.

Then we have

$$\begin{aligned} & \left(\binom{n}{\lfloor \frac{n+i(k+1)}{3} \rfloor} \right) + (-1)^{i-1} \left(\binom{n}{\lfloor \frac{n+i(k-2)}{3} \rfloor} \right) = \\ & = \sum a_{i,j} \binom{n+i-3j}{\lfloor \frac{n-3j+i(k+1)}{3} \rfloor} + \\ & + \sum b_{i,j} \binom{n+2i-3j}{\lfloor \frac{n-3j+i(k+1)}{3} \rfloor} \end{aligned}$$

Multiplying this identity by z^k and summing over all k we obtain

Theorem 6.1. *The sequence*

$$a(n, i, l, 3, z) := \sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{3} \rfloor} z^k$$

satisfies the recursion

$$(v_i(E, 1) + zw_i(E, 1))b(n, i, l, 3, z) = \left(\frac{1}{z} + z^2 \right) a(n, i, l, 3, z) \quad (6.9)$$

7. Sums of the Form $\sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{m} \rfloor} z^k$

Now we want to prove a general result on sums of the form

$$a(n, i, l, m, z) = \sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{m} \rfloor} z^k.$$

We need the following

Lemma 7.1. *For each pair of integers $n \geq 1$, $m \geq 1$ there exist uniquely determined integers $a_{n,m,j}$ such that*

$$(1+x)^n + \sum_{k=1}^{m-1} \sum_{j=\lceil \frac{(k-1)n}{m-1} \rceil}^{\lfloor \frac{kn}{m} \rfloor} a_{n,m,j} (x+1)^{kn-mj} x^j = 1$$

or equivalently

$$(1+x)^n + \sum_{j=1}^n a_{n,m,j} (x+1)^{(-mj) \pmod n} x^j = 1, \quad (7.1)$$

where $x \pmod n$ denotes the least nonnegative representative of the residue class modulo n and $a_{n,m,j} = 0$ if $(-mj) \pmod n + j > n$.

There is a useful reformulation which has been inspired by a remark of G. Kowol (private communication): If (7.1) is true and we replace $x+1$ by ζ_n , a primitive root of unity of order n , then we get $\sum_{j=1}^n a_{n,m,j} \zeta_n^{-mj} (\zeta_n - 1)^j = 0$. Thus $\zeta_n^{-m} (\zeta_n - 1)$ is a root of the polynomial $\sum_{j=1}^n a_{n,m,j} x^j$. The most obvious polynomial having $\zeta_n^{-m} (\zeta_n - 1)$ as a root is $\sum_{j=1}^n b_{n,m,j} x^j = \prod_{k=1}^n (x - \zeta_n^{-mk} (\zeta_n^k - 1))$. If $b_{n,m,j} = 0$ for all j with $(-mj) \pmod n + j > n$ then the polynomials

$$\sum_{j=1}^n b_{n,m,j} (x+1)^{(-mj) \pmod n} x^j$$

and $(1+x)^n - 1$ both have degree n and the roots 0 and $\lambda_k = \zeta_n^k - 1$, $1 \leq k \leq n-1$. Therefore they are proportional. Now we have

$$\begin{aligned} b_{n,m,1} &= (-1)^{n-1} \prod_{k=1}^{n-1} \zeta_n^{-mk} (\zeta_n^k - 1) = \zeta_n^{-m \binom{n}{2}} \prod_{k=1}^{n-1} (1 - \zeta_n^k) = \\ &= (-1)^{m(n-1)} \prod_{k=1}^{n-1} (1 - \zeta_n^k) \end{aligned}$$

From $\prod_{k=1}^{n-1} (x - \zeta_n^k) = \frac{x^n - 1}{x - 1}$ we get $\prod_{k=1}^{n-1} (1 - \zeta_n^k) = n$ and thus we have

$$b_{n,m,1} = (-1)^{m(n-1)} n.$$

Thus we get

$$(1+x)^n - (-1)^{m(n-1)} \sum_{1 \leq j \leq n} b_{n,m,j} (x+1)^{(-mj) \pmod n} x^j = 1 \quad (7.2)$$

and Lemma 7.1 is proved.

Therefore our assertion is reduced to

Lemma 7.2. *Let $m, n \in \mathbb{Z}$, $n > 0$ and let ζ_n be a primitive root of unity of order n . If we set*

$$\sum_{j=1}^n b_{n,m,j} x^j = \prod_{k=1}^n (x - \zeta_n^{-mk} (\zeta_n^k - 1)), \quad (7.3)$$

then $b_{n,m,j} = 0$ for all j satisfying $\left\{ \frac{-mj}{n} \right\} + \frac{j}{n} > 1$ where $\{x\}$ denotes $\{x\} = x - \lfloor x \rfloor$.

The following proof is due to Prof. Andrzej Schinzel [4].

Let s_j be the j 'th elementary symmetric function of

$$\zeta_n^{-mk} (\zeta_n^k - 1), \quad 1 \leq k \leq n,$$

and p_j the j 'th powersum of those numbers.

Lemma 7.3. *If $\left\{ \frac{mj}{n} \right\} > \frac{j}{n}$, then $p_j = 0$, $1 \leq j < n$.*

Proof: We have

$$p_j = \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \sum_{k=1}^n \zeta_n^{k(i-mj)} \quad (7.4)$$

since

$$\begin{aligned} p_j &= \sum_{k=1}^n \zeta_n^{-mkj} (\zeta_n^k - 1)^j = \sum_{k=1}^n \zeta_n^{-mkj} \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \zeta_n^{ki} = \\ &= \sum_{i=0}^j (-1)^{j-i} \binom{j}{i} \sum_{k=1}^n \zeta_n^{k(i-mj)}. \end{aligned}$$

The inner sum is different from 0 only if $i - mj \equiv 0 \pmod{n}$.

But since $i \leq j$ this implies $\left\{ \frac{mj}{n} \right\} \leq \frac{j}{n}$, contrary to the assumption.

Proof of Lemma 7.2:

We shall prove by induction on $j \leq n$ that

$$\text{either } \left\{ \frac{mj}{n} \right\} \leq \frac{j}{n} \text{ or } s_j = 0.$$

This is true for $j = 0$.

Assume now that it is true for all $i < j$ and that $\left\{ \frac{mj}{n} \right\} > \frac{j}{n}$.

Then by Lemma 7.3 $p_j = 0$.

Newton's formula gives

$$\sum_{i=0}^{j-1} (-1)^i p_{j-i} s_i + (-1)^j j s_j = 0. \quad (7.5)$$

Since $\left\{\frac{mj}{n}\right\} > \frac{j}{n}$ we have for each $i < j$ either $\left\{\frac{mi}{n}\right\} > \frac{i}{n}$ or $\left\{\frac{m(j-i)}{n}\right\} > \frac{j-i}{n}$.

This implies by the inductive assumption either $s_i = 0$ or by the Lemma 7.3 $p_{j-i} = 0$.

Thus $\sum_{i=0}^{j-1} (-1)^i p_{j-i} s_i = 0$ and by Newton's formula $s_j = 0$.

The inductive proof is complete.

Now if $\left\{\frac{-mj}{n}\right\} + \frac{j}{n} > 1$ then $\left\{\frac{m(n-j)}{n}\right\} > \frac{n-j}{n}$.

Hence $s_{n-j} = 0$ and $b_{n,m,j} = (-1)^{n-j} s_{n-j} = 0$.

Remark:

G. Kowol pointed out to me that for $\gcd(m, n) = 1$ formula (7.1) may be written in the following form

$$(1+x)^n + \sum_{j=1}^{n-1} a_{n,m,j} (x+1)^k x^{-m^{-1}k(\bmod n)} = 1 + (-1)^{m(n-1)} x^n,$$

if we put $k = -mj(\bmod n)$.

Changing x into $-1-x$ we get

$$\begin{aligned} (-x)^n + \sum_{j=1}^{n-1} a_{n,m,j} (-1)^{k-m^{-1}k(\bmod n)} x^k (1+x)^{-m^{-1}k(\bmod n)} = \\ = 1 + (-1)^{m(n-1)+n} (1+x)^n \end{aligned}$$

This gives the corresponding formula for $m^{-1}(\bmod n)$.

More precisely we have in this case

$$a_{n,m^{-1}(\bmod n),k} = (-1)^{j+(-mj)(\bmod n)} a_{n,m,j},$$

with $k = -mj(\bmod n)$.

For example we have $(a_{7,2,j})_{j=1}^6 = (-7, 14, -7, 0, 0, 0)$. Now $2^{-1}(\bmod 7) = 4$ and therefore we get $(a_{7,4,j})_{j=1}^6 = (-7, 0, -14, 0, -7, 0)$.

With the same notation as in Lemma 7.2 we can now prove

Lemma 7.4. For all $n, i, r \in \mathbb{Z}$ we have the identity

$$\binom{n+i}{r} - (-1)^{m(i-1)} \sum_{j=1}^i b_{i,m,j} \binom{n+(-mj)(\bmod i)}{r-j} = \binom{n}{r}$$

or

$$\begin{aligned} \binom{n+i}{r} - (-1)^{m(i-1)} \sum_{j=1}^{i-1} b_{i,m,j} \binom{n+(-mj)(\text{mod } i)}{r-j} &= \\ &= \binom{n}{r} + (-1)^{m(i-1)} \binom{n}{r-i} \end{aligned}$$

Proof: This may be derived by multiplying (7.2) by a power of $1+x$ and comparing coefficients or by considering the operator identity

$$E^i - (-1)^{m(i-1)} \sum_{1 \leq j \leq i} b_{i,m,j} E^{(-mj)(\text{mod } i)} \Delta^j = 1,$$

which follows from (7.2) by the homomorphism $x \rightarrow \Delta$ and applying it to $\binom{n}{r}$.

Now let $r = \lfloor \frac{n+i(k+1)}{m} \rfloor$.

Then we have

$$\begin{aligned} \left(\binom{n}{\lfloor \frac{n+i(k+1)}{m} \rfloor} \right) + (-1)^{m(i-1)} \left(\binom{n}{\lfloor \frac{n+i(k-m+1)}{m} \rfloor} \right) &= \\ &= \left(\binom{n+i}{\lfloor \frac{n+i(k+1)}{m} \rfloor} \right) - (-1)^{m(i-1)} \cdot \\ &\quad \cdot \sum_{j=1}^{i-1} b_{i,m,j} \left(\binom{n+(-mj)(\text{mod } i)}{\lfloor \frac{n+i(k+1)}{m} \rfloor - j} \right) \end{aligned}$$

Multiplying this identity by z^k and summing over all k we see that the sequence $a(n, i, l, m, z) = \sum_{k \in \mathbb{Z}} \binom{n}{\lfloor \frac{n+ik+l}{m} \rfloor} z^k$ satisfies the recursion

$$\begin{aligned} \sum_{k \in \mathbb{Z}} \left(\binom{n}{\lfloor \frac{n+l+i(k+1)}{m} \rfloor} \right) z^k + (-1)^{m(i-1)} \cdot \\ \cdot \sum_{k \in \mathbb{Z}} \left(\binom{n}{\lfloor \frac{n+l+i(k-m+1)}{m} \rfloor} \right) z^k = \end{aligned}$$

$$\begin{aligned}
&= \sum_{k \in \mathbb{Z}} \left(\left\lfloor \frac{n+i}{m} \right\rfloor \right) z^k - \\
&\quad - (-1)^{m(i-1)} \sum_{j=1}^{i-1} b_{i,m,j} \sum_{k \in \mathbb{Z}} \left(\left\lfloor \frac{n+(-mj)(\bmod i)}{m} \right\rfloor \right) z^k
\end{aligned}$$

This gives

Theorem 7.1. Define $h(j)$ by $h(j)i = mj + (-mj) \bmod i - i$. The sequence

$$a(n, i, l, m, z) = \sum_{k \in \mathbb{Z}} \left(\left\lfloor \frac{n}{m} \right\rfloor \right) z^k$$

satisfies the recursion

$$\begin{aligned}
&(E^i - (-1)^{m(i-1)} \sum_{j=1}^{i-1} b_{i,m,j} z^{h(j)} E^{(-mj) \bmod i} - \frac{1}{z} - \\
&\quad - (-1)^{m(i-1)} z^{m-1}) a(n, i, l, m, z) = 0.
\end{aligned}$$

As an example consider $a(n, 8, 0, 6, z) = \sum_{k \in \mathbb{Z}} \left(\left\lfloor \frac{n}{6} \right\rfloor \right) z^k$.

In this case we have

$$1 + x^8 = (1+x)^8 - 8x(1+x)^2 - 12x^2(1+x)^4 + 2x^4 - 8x^5(1+x)^2$$

and therefore we get the recurrence

$$\left(E^8 - 8E^2 - 12zE^4 + 2z^2 - 8z^3E^2 - \frac{1}{z} - z^5 \right) a(n, 8, 0, 6, z) = 0.$$

8. Concluding Remarks

We have seen that there are integers $a_{n,m,j}$ such that

$$(1+x)^n + \sum_{j=1}^n a_{n,m,j} (x+1)^{(-mj) \bmod n} x^j = 1$$

holds.

For $m = 1, 2, 3$ we know the explicit values of the coefficients $a_{n,m,j}$. Professor Schinzel has informed me that for n odd and $m = \frac{n+1}{2}$ explicit values may be obtained from formula (8) in [2]. There it

is shown that $(x + 1)^{2n+1} - 1 = \sum_{j=0}^n s_j x^{2j+1} (x + 1)^{n-j}$ where $\sum_{j=0}^n s_j z^j = \prod_{j=1}^n (z + 2(1 - \cos \frac{2\pi j}{2n+1}))$ and $s_j = \frac{2n+1}{n+j+1} \binom{n+j+1}{2j+1}$.

Here we have $m = n + 1$ since

$$n - j \equiv -(n + 1)(2j + 1) \pmod{2n + 1}.$$

This is a special case of our results because we have

$$\begin{aligned} \sum_{j=0}^n s_j z^{2j} &= \prod_{k=1}^{2n} (z + \zeta^{-(n+1)k} (1 - \zeta^k)) = \\ &= \prod_{j=1}^{2n} (z + \zeta^{nk} - \zeta^{-nk}) = \prod_{j=1}^n (z + \zeta^{nk} - \zeta^{-nk}) \prod_{j=1}^n (z - (\zeta^{nk} - \zeta^{-nk})) = \\ &= \prod_{k=1}^n (z^2 - (\zeta^{nk} - \zeta^{-nk})^2) = \prod_{k=1}^n \left(z^2 - 2 \cos \frac{2\pi nk}{2n+1} + 2 \right). \end{aligned}$$

From $-nk \equiv (n + 1)k \pmod{2n + 1}$ we see that this also equals $\prod_{k=1}^n (z^2 - 2 \cos \frac{2\pi k}{2n+1} + 2)$.

In the general case we have only some special explicit results.

First we show that Newton's formula gives a recursion for the coefficients $b_{n,m,j}$.

Lemma 8.1. Let $d_{n,m,j} = (-1)^{j-(mj) \pmod{n}} n \binom{j}{(mj) \pmod{n}}$. For $j < n$ we have

$$b_{n,m,n-j} = -\frac{1}{j} \sum_{i=0}^{j-1} b_{n,m,n-i} d_{n,m,j-i} \tag{8.1}$$

with initial value $b_{n,m,n} = 1$.

Proof: (7.4) gives $p_j = d_{n,m,j}$, $j < n$.

Finally we want to show how the formulas (6.4) and (6.7) for $m = 3$ can be generalized to arbitrary m to give explicit values for some coefficients.

To do this it is convenient to define $0 \pmod{n} = n$ and $a_{n,m,0} = 1$.

Lemma 8.2. $a_{n,m,j} = (-1)^j \binom{n-(m-1)j}{j} \frac{n}{n-(m-1)j}$ for $0 \leq j \leq \frac{n}{m}$.

Proof:

It suffices to show that in

$$\sum_{j=0}^{\frac{n}{m}} (-1)^j \binom{n-(m-1)j}{j} \frac{n}{n-(m-1)j} (x+1)^{n-mj} x^j$$

all coefficients of x^j up to $j = \lfloor \frac{n}{m} \rfloor$ vanish.

The coefficient of x^r in this sum is given by

$$\sum_{j=0}^r (-1)^j \binom{n - (m-1)j}{j} \frac{n}{n - (m-1)j} \binom{n - mj}{r - j}$$

for all r such that $r \leq \lfloor \frac{n}{m} \rfloor$.

Now

$$\begin{aligned} \sum_{j=0}^r (-1)^j \binom{n - (m-1)j}{j} \frac{n}{n - (m-1)j} \binom{n - mj}{r - j} &= \\ &= \frac{n}{r!} \sum_{j=0}^r (-1)^j \binom{r}{j} (n - (m-1)j - 1) \cdots (n - (m-1)j - r + 1) = \\ &= \Delta^r p(j) = 0 \end{aligned}$$

for some polynomial with $\deg p(x) < r$ and the lemma is proven.

Remark. It is easily verified that for $n > 0$

$$v(n, m, x, s) = \sum_{j=0}^{\frac{n}{m}} (-1)^j \binom{n - (m-1)j}{j} \frac{n}{n - (m-1)j} (x+1)^{n-mj} x^j$$

coincides with the sequence defined by the recurrence

$$v(n+m, m, x, s) = xv(n+m-1, m, x, s) + sv(n, m, x, s)$$

with initial values $v(0, m, x, s) = m$, $v(i, m, x, s) = x^i$, $0 < i < m$.

Lemma 8.3. Let $w(n, m, x, s) = (-1)^m t(n, m, x, (-1)^{m-1} s)$ with

$$t(n, m, x, s) = \sum \binom{n-j}{(m-1)n-mj} \frac{n}{n-j} x^{(m-1)n-mj} s^j.$$

Then $\sum_{j \geq \frac{(m-2)n}{m-1}} a_{n,m,j} x^{(-mj) \bmod n} s^j = w(n, m, x, s)$ for $m > 2$.

Proof: We know already that $a_{n,m,j} \neq 0$ for these values of j only if

$$\frac{(m-2)n}{m-1} \leq j \leq \frac{(m-1)n}{m} \quad \text{or} \quad \frac{n}{m} \leq n-j \leq \frac{n}{m-1}.$$

Let $k = n - j$.

For $m > 2$ it is evident that if $0 < k_1 + k_2 \leq \frac{n}{m-1}$ for positive numbers k_i there is at most one $k_i \geq \frac{n}{m}$. For $k < \frac{n}{m}$ we have $\left\{ \frac{km}{n} \right\} = \frac{km}{n} > \frac{k}{n}$ and therefore $p_k = 0$ by Lemma 7.3.

Thus $\sum_{i=1}^{k-1} b_{n,m,n-i} p_{k-i} = 0$ for each $k < \lfloor \frac{n}{m-1} \rfloor$.

Newton's formula now tells us that $p_k = -kb_{n,m,n-k}$.

By Lemma 7.3 we have $p_k = (-1)^{k-mk+n} \binom{k}{mk-n}$. This implies

$$b_{n,m,j} = (-1)^{(m-1)(n-j)+n+1} \binom{n-j}{(m-1)n-mj} \frac{n}{n-j} \text{ as asserted.}$$

Remark:

It is easy to verify that $t(n, m, x, s)$ may be characterized for $n > 0$ by the recursion

$$t(n, m, x, s) = xs^{m-2}t(n - m + 1, m, x, s) + s^{m-1}t(n - m, m, x, s)$$

with the initial values

$$t(0, m, x, s) = m, t(1, m, x, s) = \cdots t(m - 2, m, x, s) = 0, \\ t(m - 1, m, x, s) = (m - 1)s^{m-2}$$

References

[1] Andrews, G. E.: Some formulae for the Fibonacci sequence with generalization. *Fibonacci Q.* **7**, 113–130 (1969).
 [2] Browkin, J., Brzezinski, J.: Some remarks on the abc-conjecture. *Math. Comp.* **62**, 931–939 (1994).
 [3] Graham, R. L., Knuth, D. E., Patashnik, O.: *Concrete mathematics*. Addison-Wesley, Reading 1988.
 [4] Schinzel, A.: Letter from July 2, 2001.
 [5] Schur, I.: Ein Beitrag zur additiven Zahlentheorie und zur Theorie der Kettenbrüche. In: I. Schur, *Gesammelte Abhandlungen*, Vol. 2, pp. 117–136. Springer, Berlin 1973.

Author's address: Prof. Dr. J. Cigler, Institut für Mathematik, Universität Wien, Strudlhofgasse 4, A-1090 Wien.