Addendum to my paper "Hankel determinants for some polynomial sequences"

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Abstract

This preliminary note contains some remarks and supplements to my paper [2]. I am using all notations introduced in [2]. The only exception is that in [2] we considered sequences of determinants of the form $d(n,x)=\det\left(r(i+j)\right)_{i,j=0}^{n}$ *whereas here it is more convenient to change the index and write* $d(n,x)=\det\left(r(i+j)\right)_{i,j=0}^{n-1}$.

Introduction

In the second part of [2] I computed the Hankel determinants $\det\left(a(i+j)x-a(i+j+1)\right)_{i,j=0}^{n-1}$ and $\det\left(a(i+j+1)x-a(i+j+2)\right)_{i,j=0}^{n-1}$ for moments $a(n)$ of symmetric orthogonal polynomials. For the Hankel determinants $\det\left(a(i+j+2)x-a(i+j+3)\right)_{i,j=0}^{n-1}$ a new approach is needed. The purpose of this note is to compute these Hankel determinants for $a(n)=C_n=\frac{1}{n+1}\binom{2n}{n}$ and $a(n)=\binom{2n}{n}$ and derive some $q-$analogues. One possibility is to compute the symmetric orthogonal polynomials corresponding to the shifted sequences $a(n)=C_{n+1}$ and $a(n)=\binom{2n+2}{n+1}$ and apply [2],Theorem 2. This will be done at the end of this note.
But for the above mentioned sequences a simpler proof is possible by a modification of the proof of [2] Theorem 2:

Theorem 1
Let $\left(P_n(x)\right)_{n\geq 0}$ *be a sequence of monic polynomials which are orthogonal with respect to the linear functional* Φ *and with moments* $\Phi\left(x^n\right)=a(n).$ *Let* $r(n,x)=a(n)x-a(n+1).$
Then

$$\det\left(r(i+j,x)\right)_{i,j=0}^{n-1}=\det\left(a(i+j)\right)_{i,j=0}^{n-1}P_n(x). \tag{1.1}$$

Proof

Let $P_n(x)=b(n,0)+b(n,1)x+\cdots+b(n,n-1)x^{n-1}+x^n$
and

$$B_n=\begin{pmatrix} x & -1 & 0 & \cdots & 0 \\ 0 & x & -1 & \cdots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \ddots & -1 \\ b(n,0) & b(n,1) & b(n,2) & \cdots & x+b(n,n-1) \end{pmatrix}.$$

Then

$$\left(r(i+j,x)\right)_{i,j=0}^{n-1}=B_n\left(a(i+j)\right)_{i,j=0}^{n-1} \tag{1.2}$$

because

$$\sum_{i=0}^{n-1}b(n,i)a(i+m)+xa(n+m-1)=\sum_{i=0}^{n-1}b(n,i)a(i+m)+a(n+m)+xa(n+m-1)-a(n+m)$$
$$=\Phi\left(x^mP_n(x)\right)+xa(n+m-1)-a(n+m)=xa(n+m-1)-a(n+m).$$

(1.1) follows because $\det B_n=b(n,0)+b(n,1)x+\cdots+b(n,n-1)x^{n-1}+x^n=P_n(x).$

By searching the literature I found that Theorem 1 is in fact an old result which has been stated without proof as Exercise 5.17 in [1]. A different proof can be found in [7].
But it seems that this result is not well known. Therefore the following straightforward applications may perhaps provide an alternative point of view on some orthogonal polynomials.

1. Some simple examples

For some sequences $\left(a(n)\right)$ it is very easy to find the corresponding orthogonal polynomials.

Let for example $a(n)=\prod_{j=1}^{n}(\alpha+j).$ Then the corresponding orthogonal polynomials are the

Laguerre polynomials $L_n^{(\alpha)}(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{\prod_{j=1}^{n}(\alpha+j)}{\prod_{j=1}^{k}(\alpha+j)}x^k.$

Of course it is well known that $a(n)$ are the moments of the Laguerre polynomials. But we don't need any previous information about these polynomials. We need only the following rather trivial observation:
Let Δ be the difference operator $\Delta f(n)=f(n+1)-f(n).$ Then
$\Delta^n f(0)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}f(k)=0$ for each polynomial $f(x)$ of degree $\deg f<n.$

The quotients $\frac{a(m+k)}{a(k)}=\frac{\prod_{j=1}^{k+m}(\alpha+j)}{\prod_{j=1}^{k}(\alpha+j)}$ are polynomials in k of degree $m.$ Define a linear

functional Φ by $\Phi\left(x^n\right)=a(n).$

Then the above observation implies that $\Phi\left(L_n^{(\alpha)}(x)x^m\right)=0$ for $0\le m<n.$ Therefore the polynomials are orthogonal with respect to Φ with moments $\Phi\left(x^n\right)=a(n).$
Thus we get

Example 1

Let $a(n)=\prod_{j=1}^{n}(\alpha+j)$ *and* $r(n,x)=x\prod_{j=1}^{n}(\alpha+j)-\prod_{j=1}^{n+1}(\alpha+j).$ *Then*

$$\frac{\det\left(r(i+j,x)\right)_{i,j=0}^{n-1}}{\det\left(a(i+j)\right)_{i,j=0}^{n-1}}=L_n^{(\alpha)}(x). \tag{1.3}$$

If we choose $a(n)=\prod_{j=1}^{n}[\alpha+j],$ where $[x]=\dfrac{1-q^x}{1-q},$ the analogous polynomials are

$$L_n^{(\alpha)}(x,q)=\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}\frac{\prod_{j=1}^{n}[\alpha+j]}{\prod_{j=1}^{k}[\alpha+j]}x^k, \tag{1.4}$$

because $\sum_{k=0}^{n}(-1)^{n-k}q^{\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}q^{km}=\left(q^m-1\right)\left(q^m-q\right)\cdots\left(q^m-q^{n-1}\right)=0$ if $0\le m\le n-1.$

Notice that a product of the form $\prod_{j=k+1}^{k+m}[\alpha+j]$ is of the form $\sum_{i=0}^{m}c(i,q)q^{ik}.$

Thus we get

Example 2

Let $a(n)=\prod_{j=1}^{n}[\alpha+j]$ *and* $r(n,x)=x\prod_{j=1}^{n}[\alpha+j]-\prod_{j=1}^{n+1}[\alpha+j].$ *Then*

$$\frac{\det\left(r(i+j,x)\right)_{i,j=0}^{n-1}}{\det\left(a(i+j)\right)_{i,j=0}^{n-1}}=L_n^{(\alpha)}(x,q). \tag{1.5}$$

The same arguments give

Example 3

Let $a(n,m)=(2n+2m-1)!!$ *and* $r(n,x)=x(2n+2m-1)!!-(2n+2m+1)!!.$
Then

$$\frac{\det\left(r(i+j,x)\right)_{i,j=0}^{n-1}}{\det\left(a(i+j)\right)_{i,j=0}^{n-1}}=h_n^{(m)}(x) \tag{1.6}$$

with

$$h_n^{(m)}(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n}{k}\frac{(2n+2m-1)!!}{(2k+2m-1)!!}x^k. \tag{1.7}$$

Remark

Define the Hermite polynomials $H_n(x)$ by

$H_n(x)=xH_{n-1}(x)-(n-1)H_{n-2}(x)$ with initial values $H_{-1}(x)=0$ and $H_0(x)=1.$ Then $h_n^{(0)}(x^2)=H_{2n}(x)$ and $h_n^{(1)}(x^2)=\frac{H_{2n+1}(x)}{x}.$

The polynomials $h_n^{(m)}(x)$ can be expressed by the Laguerre polynomials:

$h_n^{(m)}(x)=2^n L_n^{\left(\frac{2m-1}{2}\right)}\left(\frac{x}{2}\right).$

Example 4

Let $a(n,m)=[2n+2m-1]!!$ *and* $r(n,x)=x[2n+2m-1]!!-[2n+2m+1]!!.$

Then

$$\frac{\det\left(r(i+j,x)\right)_{i,j=0}^{n-1}}{\det\left(a(i+j)\right)_{i,j=0}^{n-1}}=h_n^{(m)}(x,q) \tag{1.8}$$

with

$$h_n^{(m)}(x,q)=\sum_{k=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\begin{bmatrix}n\\k\end{bmatrix}_{q^2}\frac{[2n+2m-1]!!}{[2k+2m-1]!!}x^k. \tag{1.9}$$

Here we must replace q with q^2 because in this case we have polynomials in $q^{2k}=\left(q^2\right)^k$.

2. The main results

Now we come to the examples mentioned in the introduction.

Example 5

Consider the Catalan numbers $C_n=\frac{1}{n+1}\binom{2n}{n}.$ Let $r(n,x)=C_nx-C_{n+1}.$

For the convenience of the reader I repeat some definitions and results from [2]. We need the bivariate Fibonacci polynomials $F_n(x,s)$ which are defined by

$F_n(x,s)=xF_{n-1}(x,s)+sF_{n-2}(x,s)$ with initial values $F_0(x,s)=0$ and $F_1(x,s)=1.$ The polynomials

$p_n(x)=F_{n+1}(x,-1)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}(-1)^k\binom{n-k}{k}x^{n-2k}$

are symmetric orthogonal polynomials as defined in [2] with moments $\Lambda\left(x^{2n}\right)=C_n.$

From [2], Theorem 2, we know that

$$\det\left(C_{i+j}x^2 - C_{i+j+1}\right)_{i,j=0}^{n-1} = p_{2n}(x) = \sum_{k=0}^{n}(-1)^k \binom{2n-k}{k} x^{2n-2k} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}x^{2k}$$

and that

$$\det\left(C_{i+j+1}x^2 - C_{i+j+2}\right)_{i,j=0}^{n-1} = \frac{p_{2n+1}(x)}{x} = \sum_{k=0}^{n}(-1)^k \binom{2n+1-k}{k} x^{2n-2k} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+1+k}{n-k}x^{2k}.$$

Thus

$$\det\left(C_{i+j}x - C_{i+j+1}\right)_{i,j=0}^{n-1} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+k}{n-k}x^{k} \quad (2.1)$$

whose first terms are

$\{1, -1+x, 1-3x+x^2, -1+6x-5x^2+x^3, 1-10x+15x^2-7x^3+x^4\}$

and

$$\det\left(C_{i+j+1}x - C_{i+j+2}\right)_{i,j=0}^{n-1} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+1+k}{n-k}x^{k} \quad (2.2)$$

with first terms

$\{1, -2+x, 3-4x+x^2, -4+10x-6x^2+x^3, 5-20x+21x^2-8x^3+x^4\}$

We want to compute

$$d(n,x) = \det\left(C_{i+j+2}x - C_{i+j+3}\right)_{i,j=0}^{n-1}. \quad (2.3)$$

The first terms are $1, 2x-5, 3x^2-14x+14, 4x^3-27x^2+54x-30, \cdots$.

First we look for a recurrence.

Using the software Guess by Manuel Kauers [4] we get the conjecture that the sequence $\left(d(n,x)\right)$ satisfies the recurrence

$$d(n,x) - 2(x-2)d(n-1,x) + (6-4x+x^2)d(n-2,x) - 2(x-2)d(n-3,x) + d(n-4,x) = 0.$$

Observe that $1-2(x-2)z+(6-4x+x^2)z^2-2(x-2)z^3+z^4 = \left(1-(x-2)z+z^2\right)^2$.

If we compute

$$\left(1-2(x-2)z+(6-4x+x^2)z^2-2(x-2)z^3+z^4\right)\sum_k d(k,x)z^k$$

we get $1-z$.

Therefore our conjecture is equivalent with

$$\sum_{n\ge 0} d(n,x)z^n = \frac{1-z}{\left(1-(x-2)z+z^2\right)^2}. \tag{2.4}$$

The generating function of the Fibonacci polynomials $F_n(x,-1)$ is $\sum_{n\ge 0} F_n(x,-1)z^n = \frac{z}{1-xz+z^2}$ and its derivative is $\frac{d}{dx}\frac{1}{1-xz+z^2} = \frac{z}{(1-xz+z^2)^2}$.

Therefore we get

$$d(n,x) = \frac{d}{dx}\left(F_{n+2}(x-2,-1) - F_{n+1}(x-2,-1)\right). \tag{2.5}$$

Now observe that $xF_n(x^2-2,-1) = F_{2n}(x,-1)$ and that the Lucas polynomials $L_n(x,s) = \sum_{k=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \frac{n}{n-k}\binom{n-k}{k} s^k x^{n-2k}$ satisfy $L_n(x,s) = F_{n+1}(x,s) + sF_{n-1}(x,s)$.

Therefore we get

$xF_{n+2}(x^2-2,-1) - xF_{n+1}(x^2-2,-1) = F_{2n+4}(x,-1) - F_{2n+2}(x,-1) = L_{2n+3}(x,-1)$

and thus

$$F_{n+2}(x-2,-1) - F_{n+1}(x-2,-1) = \frac{L_{2n+3}(\sqrt{x},-1)}{\sqrt{x}} = \sum_{k=0}^{n+1} \frac{2n+3}{2n+3-k}\binom{2n+3-k}{k}(-1)^k x^{n+1-k}$$

$$= \sum_{k=0}^{n+1} \frac{2n+3}{n+2+k}\binom{n+2+k}{2k+1}(-1)^{n+1-k} x^k.$$

The first terms of the sequence $\left(F_{n+2}(x-2,-1) - F_{n+1}(x-2,-1)\right)_{n\ge 0}$ are

$\{-3 + x,\ 5 - 5x + x^2,\ -7 + 14x - 7x^2 + x^3,\ 9 - 30x + 27x^2 - 9x^3 + x^4,$
$-11 + 55x - 77x^2 + 44x^3 - 11x^4 + x^5,\ 13 - 91x + 182x^2 - 156x^3 + 65x^4 - 13x^5 + x^6\}$

Differentiation gives

$$d(n,x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n+2+k}{2k+2}\frac{(2n+3)(1+k)}{3+2k}x^k. \tag{2.6}$$

The coefficient of x^n in $d(n,x)$ is $n+1$. Therefore the monic version is $P_n(x) = \frac{d(n,x)}{n+1}$.

Thus

$$P_n(x) = \sum_{k=0}^{n} (-1)^{n-k}\binom{n+2+k}{n-k}\frac{(2n+3)(k+1)}{(2k+3)(n+1)}x^k. \tag{2.7}$$

Till now all this is only guesswork.

We have yet to show that the polynomials $P_n(x)$ are orthogonal with respect to the linear functional Φ with moments $\Phi\left(x^n\right)=C_{n+2}.$

This means that for $0\le m\le n-1$
$\Phi\left(x^m P_n(x)\right)=0$ or equivalently

$$s(n,m)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+2+k}{n-k}\frac{(2n+3)(k+1)}{(2k+3)(n+1)}\binom{2k+2m+4}{k+m+2}\frac{1}{m+k+3}=0. \qquad (2.8)$$

A computer proof with Zeilberger's algorithm [5] gives

```
Zb[(-1)^k Binomial[n + k + 2, 2 k + 2] (2 n + 3) (k + 1)
    / (2 k + 3) / (n + 1) Binomial[2 k + 2 m + 4, k + m + 2] / (k + m + 3), {k, 0, n}, n, 1]

If `n' is a natural number and `2 (2 + m + n)' is no negative integer, then:
{-(m - n) (3 + n) SUM[n] - (2 + n) (4 + m + n) SUM[1 + n] == 0}
```

Using the certificate $\frac{2k(3+2k)(3+k+m)(2+n)}{(1-k+n)(3+2n)}$ the result can easily be verified.

Therefore $s(m+1,m)=0$ and $s(n,m)=0$ for $n\ge m+1.$

As Michael Schlosser [6] remarked it is also possible to get a "human" proof by using the hypergeometric machinery (cf. e.g. [1]).

Translating $s(n,m)$ into hypergeometric form gives

$$s(n,m)=(-1)^n\frac{(n+2)(2n+3)}{3(m+3)}\binom{2m+4}{m+2}\,{}_3F_2\left(\begin{matrix}-n,n+3,m+\frac{5}{2}\\ m+4,\frac{5}{2}\end{matrix};1\right).$$

The Pfaff-Saalschütz identity ([1],(2.2.8)) states that

$${}_3F_2\left(\begin{matrix}-n,a,b\\ c,1+a+b-c-n\end{matrix};1\right)=\frac{(c-a)_n(c-b)_n}{(c)_n(c-a-b)_n}$$

with $(x)_n=x(x+1)\cdots(x+n-1).$
Since $(c-a)_n=\left((m+4)-(n+3)\right)_n=(m+1-n)_n=(m+1-n)(m+2-n)\cdots m=0$ for $0\le m\le n-1$ we get again (2.8).

Taking into account that $\det\left(C_{i+j+2}\right)_{i,j=0}^{n-1}=n+1$ we get

Theorem 2

The Hankel determinants $\det\left(C_{i+j+2}x - C_{i+j+3}\right)_{i,j=0}^{n-1}$ *are given by*

$$\det\left(C_{i+j+2}x - C_{i+j+3}\right)_{i,j=0}^{n-1} = \sum_{k=0}^{n}(-1)^{n-k}\binom{n+2+k}{n-k}\frac{(2n+3)(k+1)}{(2k+3)}x^k. \qquad (2.9)$$

Example 6

Let now $b_n = \binom{2n}{n}$ and $r(n,x) = b_n x - b_{n+1} = \binom{2n}{n}x - \binom{2n+2}{n+1}$.

In this case we know that the symmetric orthogonal polynomials with moments $\Lambda\left(x^{2n}\right) = \binom{2n}{n}$ are the Lucas polynomials $p_n(x) = L_n(x,-1)$ for $n > 0$ and $p_0(x) = 1$.

This gives for $n > 0$

$$\frac{\det\left(b_{i+j}x^2 - b_{i+j+1}\right)_{i,j=0}^{n-1}}{2^{n-1}} = L_{2n}(x,-1) = L_n(x^2-2,-1) = \sum_{k=0}^{n}(-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{n-k}x^{2k}$$

and

$$\frac{\det\left(b_{i+j+1}x^2 - b_{i+j+2}\right)_{i,j=0}^{n-1}}{2^{n-1}} = \frac{L_{2n+1}(x,-1)}{x} = \sum_{k=0}^{n}(-1)^{n-k}\frac{2n+1}{n+k}\binom{n+1+k}{n-k}x^{2k}$$

$$\frac{\det\left(b_{i+j+1}x^2 - b_{i+j+2}\right)_{i,j=0}^{n-1}}{2^{n}} = \frac{L_{2n+1}(x,-1)}{x} = \sum_{k=0}^{n}(-1)^{n-k}\frac{2n+1}{n+1+k}\binom{n+1+k}{n-k}x^{2k}.$$

Therefore we have

$$\frac{\det\left(b_{i+j}x - b_{i+j+1}\right)_{i,j=0}^{n-1}}{2^{n-1}} = \sum_{k=0}^{n}(-1)^{n-k}\frac{2n}{n+k}\binom{n+k}{n-k}x^k \qquad (2.10)$$

whose first terms are

$\{1,\ -2+x,\ 2-4x+x^2,\ -2+9x-6x^2+x^3,\ 2-16x+20x^2-8x^3+x^4\}$

and

$$\frac{\det\left(b_{i+j+1}x - b_{i+j+2}\right)_{i,j=0}^{n-1}}{2^{n}} = \sum_{k=0}^{n}(-1)^{n-k}\frac{2n+1}{n+1+k}\binom{n+1+k}{n-k}x^k \qquad (2.11)$$

with first terms

$\{1,\ -3+x,\ 5-5x+x^2,\ -7+14x-7x^2+x^3,\ 9-30x+27x^2-9x^3+x^4\}$

Let

$$\frac{\det\left(r(i+j+2,x)\right)_{i,j=0}^{n-1}}{2^n}=d(n,x) \tag{2.12}$$

The same reasoning as above gives the guess

$$\sum_{n\ge 0} d(n,x)z^n=\frac{1+(x-6)z+z^2}{\left(1-(x-2)z+z^2\right)^2}. \tag{2.13}$$

This implies

$$d(n,x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k+1}{2k+1}\frac{(2n+1)(2n+3)}{2k+3}x^k. \tag{2.14}$$

For (2.13) gives as above

$$d(n,x)=\frac{d}{dx}\left(F_{n+1}(x-2,-1)+(x-6)F_{n+2}(x-2,-1)+F_{n+3}(x-2,-1)\right)-F_{n+2}(x-2,-1).$$

Comparing coefficients we get (2.14).

The coefficient of x^n in $d(n,x)$ is $2n+1$. Therefore the monic polynomials are

$$P_n(x)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k+1}{2k+1}\frac{(2n+3)}{2k+3}x^k.$$

In order to prove (2.14) we must show that for $0\le m\le n-1$

$$\sum_{k=0}^{n}(-1)^k\binom{n+k+1}{2k+1}\frac{(2n+3)}{2k+3}\binom{2k+4+2m}{k+2+m}=0. \tag{2.15}$$

This follows again from Zeilberger's algorithm:

```
Zb[(-1)^k Binomial[n + k + 1, 2 k + 1] (2 n + 3) / (2 k + 3) Binomial[2 k + 2 m + 4, k + m + 2], {k, 0, n}, n, 1]

If `n' is a natural number and `2 (2 + m + n)' is no negative integer, then:
{-(m - n) (2 + n) (1 + 2 n) SUM[n] - (1 + n) (3 + m + n) (3 + 2 n) SUM[1 + n] == 0}
```

Here the certificate is $\frac{k(3+2k)(2+k+m)(3+2n)}{1-k+n}$.

The hypergeometric version

$$\sum_{k=0}^{n}(-1)^k\binom{n+k+1}{2k+1}\frac{(2n+3)}{2k+3}\binom{2k+4+2m}{k+2+m}=\frac{(n+1)(2n+3)}{3}\binom{4+2m}{2+m}{}_3F_2\left(\begin{matrix}-n,n+2,m+\frac{5}{2}\\ m+3,\frac{5}{2}\end{matrix};1\right)$$

gives again the factor $(m-n+1)_n$.

Since $\det\left(\left(\binom{2i+2j+4}{i+j+2}\right)\right)_{i,j=0}^{n-1}=2^n(2n+1)$ we get

Theorem 3

The Hankel determinants $\det\left(\left(\binom{2i+2j+4}{i+j+2}x-\binom{2i+2j+6}{i+j+3}\right)\right)_{i,j=0}^{n-1}$ *are given by*

$$\det\left(\left(\binom{2i+2j+4}{i+j+2}x-\binom{2i+2j+6}{i+j+3}\right)\right)_{i,j=0}^{n-1} = 2^n(2n+1)(2n+3)\sum_{k=0}^{n}(-1)^{n-k}\binom{n+k+1}{2k+1}\frac{1}{2k+3}x^k. \tag{2.16}$$

Remark

Karl Dilcher and Kenneth B. Stolarsky [3] have introduced polynomials $Q_n(x)$ whose coefficients are divisible by $2n-1$ with exactly one exception and are divisible by $2n+1$ with exactly one exception if and only if $(2n-1,2n+1)$ is a pair of twin primes. It is interesting that the same polynomials also show up in the context of Hankel determinants.

For let $u(n,x)=\det\left(\left(\binom{2i+2j+4}{i+j+2}x+\binom{2i+2j+6}{i+j+3}\right)\right)_{i,j=0}^{n-2}$.

The first terms of $(u(n,x))_{n\geq 1}$ are

$\{1,\ 2\,(10+3x),\ 4\,(35+28x+5x^2),\ 8\,(84+126x+54x^2+7x^3)\}$

It turns out that $Q_n(x)=\frac{u(n,x)}{2^{n-1}}$. This raises the question if there is some connection between these two facts.

To this end let $M_n=(m_n(i,j))=\left(\left(\binom{2i+2j+4}{i+j+2}x+\binom{2i+2j+6}{i+j+3}\right)\right)_{i,j=0}^{n-2}$ for $n\geq 2$.

If we consider M_n modulo the prime $2n+1$ then all elements below the diagonal $i+j=n-2$ vanish.

For in $\binom{2n+2m}{n+m}=\frac{(2n+2m)\cdots(n+1+m)}{(n+m)!}$ the factor $2n+1$ occurs if and only if $1\leq m\leq n$.

For example M_5 modulo 11 is

$$\begin{pmatrix} 9+6x & 4+9x & 10+4x & 10x \\ 4+9x & 10+4x & 10x & 0 \\ 10+4x & 10x & 0 & 0 \\ 10x & 0 & 0 & 0 \end{pmatrix}$$

Therefore we get that $u(n,x)\equiv c(n)x^{n-1} \bmod (2n+1)$ for some number $c(n)\neq 0$ if $2n+1$ is a prime number.

If we consider M_n modulo the prime $2n-1$ then all elements on and below the diagonal $i+j=n-2$ vanish with the exception of $m(n,n-2,n-2)$.
For in $\binom{2n+2m}{n+m}=\frac{(2n+2m)\cdots(n+1+m)}{(n+m)!}$ the factor $2n-1$ occurs if and only if $0\leq m\leq n-2$.

For example M_6 modulo 11 is

$$\begin{pmatrix} 9+6x & 4+9x & 10+4x & 10x & 0 \\ 4+9x & 10+4x & 10x & 0 & 0 \\ 10+4x & 10x & 0 & 0 & 0 \\ 10x & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 2 \end{pmatrix}$$

We get $u(n,x)\equiv c(n)x^{n-2}$ modulo $2n-1$.

Thus the if-part of the Dilcher – Stolarsky theorem becomes evident by considering the corresponding Hankel matrices.

Finally I want to mention that [3], Theorem 3, which is due to John D'Angelo also is connected to Hankel determinants. The polynomials $J_n(x)=\sum_{k=0}^{n}\frac{2n+1}{n+1+k}\binom{n+1+k}{n-k}x^k$ considered there can be represented as $J_n(x)=\frac{1}{2^n}\det\left(\binom{2i+2j+2}{i+j+1}x+\binom{2i+2j+4}{i+j+2}\right)_{i,j=0}^{n-1}$.

The theorem asserts that $J_n(x)\equiv x^n \bmod (2n+1)$ if and only if $2n+1$ is a prime number.

The if – part is again obvious. For if $2n+1$ is prime then $\binom{2n+2m+2}{n+m+1}\equiv 0 \bmod (2n+1)$ for $0\leq m\leq n-1$.

3. Another method

If $a(n)$ are moments of symmetric orthogonal polynomials $p_n(x,t)$, i.e. polynomials which satisfy $p_n(x,t)=xp_{n-1}(x,t)-t_{n-2}p_{n-2}(x,t)$ with initial values $p_{-1}(x,t)=0$ and $p_0(x,t)=1$ for some sequence $t=(t_n)_{n\geq 0}$ of non-vanishing numbers we have shown in [2] that

$$\det\left(r(i+j,x^2)\right)_{i,j=0}^{n-1}=\det\left(a(i+j)\right)_{i,j=0}^{n-1}p_{2n}(x,t) \tag{3.1}$$

and

$$\det\left(r(i+j+1,x^2)\right)_{i,j=0}^{n-1}=\det\left(a(i+j+1)\right)_{i,j=0}^{n-1}\frac{p_{2n+1}(x,t)}{x}. \tag{3.2}$$

I do not know a simple expression for the determinant $\det\left(r(i+j+2,x)\right)_{i,j=0}^{n-1}$ in terms of $p_n(x,t).$

But we can compute the symmetric orthogonal polynomials with moments $a(n+1)$ and apply (3.2) in this case.

The procedure is well known but for the convenience of the reader we will repeat it. Let us start with the symmetric polynomials $p_n(x,t)$ and define $a(n,k)$ by

$$\sum_{k=0}^{n} a(n,k)p_k(x,t) = x^n. \tag{3.3}$$

Then

$$\begin{aligned} &a(0,j) = [j=0] \\ &a(n,0) = t_0 a(n-1,1) \\ &a(n,j) = a(n-1,j-1) + t_j a(n-1,j+1) \end{aligned} \tag{3.4}$$

By (3.3) it is clear that $a(2n,0) = \Lambda\left(x^{2n}\right) = a(n).$

We observe that the polynomials $P_n(x,t) = p_{2n}(\sqrt{x},t)$ whose first terms are

$\{1,\ x - t[0],\ x^2 - x\,t[0] - x\,t[1] - x\,t[2] + t[0]\,t[2],\ x^3 - x^2\,t[0] - x^2\,t[1] - x^2\,t[2] + x\,t[0]\,t[2] - x^2\,t[3] + x\,t[0]\,t[3] + x\,t[1]\,t[3] - x^2\,t[4] + x\,t[0]\,t[4] + x\,t[1]\,t[4] + x\,t[2]\,t[4] - t[0]\,t[2]\,t[4]\}$

are orthogonal polynomials because they satisfy a three-term recurrence

$P_1(x,t) = \left(x - t_0\right)P_0(x,t)$

and for $n > 1$

$P_n(x,t) = \left(x - t_{2n-2} - t_{2n-3}\right)P_{n-1}(x,t) - t_{2n-3}t_{2n-4}P_{n-2}(x,t)$

with initial values $P_{-1}(x,t) = 0$ and $P_0(x,t) = 1.$

If we define $L\left(P_n\left(\sqrt{x},t\right)\right) = \Lambda\left(p_{2n}(x,t)\right)$ then their moments are $L(x^n) = a(n).$

In the same way the polynomials $Q_n(x,t) = \dfrac{p_{2n+1}(\sqrt{x},t)}{\sqrt{x}}$ whose first terms are

$\{1,\ x - t[0] - t[1],\ x^2 - x\,t[0] - x\,t[1] - x\,t[2] + t[0]\,t[2] - x\,t[3] + t[0]\,t[3] + t[1]\,t[3],\ x^3 - x^2\,t[0] - x^2\,t[1] - x^2\,t[2] + x\,t[0]\,t[2] - x^2\,t[3] + x\,t[0]\,t[3] + x\,t[1]\,t[3] - x^2\,t[4] + x\,t[0]\,t[4] + x\,t[1]\,t[4] + x\,t[2]\,t[4] - t[0]\,t[2]\,t[4] - x^2\,t[5] + x\,t[0]\,t[5] + x\,t[1]\,t[5] + x\,t[2]\,t[5] - t[0]\,t[2]\,t[5] + x\,t[3]\,t[5] - t[0]\,t[3]\,t[5] - t[1]\,t[3]\,t[5]\}$

are orthogonal polynomials because they satisfy

$$Q_n(x,t) = \left(x - t_{2n-2} - t_{2n-1}\right)Q_{n-1}(x,t) - t_{2n-3}t_{2n-2}Q_{n-2}(x,t)$$

with initial values $Q_{-1}(x,t) = 0$ and $Q_0(x,t) = 1.$

From (3.3) we have

$$\sum_{k=0}^{n} a(2n+1,2k+1)\frac{p_{2k+1}(x,t)}{x} = x^{2n} \text{ with } a(2n+1,1) = \frac{a(2n+2,0)}{t_0}.$$

If we define a linear functional M by $M\left(Q_n(x,t)\right) = [n=0]$ then

$$M\left(x^n\right) = a(2n+1,1) = \frac{a(2n+2,0)}{t_0} = \frac{a(n+1)}{t_0}.$$

Now we look for symmetric orthogonal polynomials $\pi_n(x,t)$ such that $Q_n(x,t) = \pi_{2n}(\sqrt{x},t)$.
If

$\pi_n(x,t) = x\pi_{n-1}(x,t) - \tau_{n-2}\pi_{n-2}(x,t),$

then we get

$$\tau_{2n} = \frac{\rho_{n+1}}{\rho_n},$$

$$\tau_{2n+1} = \frac{\rho_n}{\rho_{n+1}} t_{2n+1}t_{2n+2}$$

where ρ_n satisfies $\rho_n = \rho_{n-1}t_{2n-1} + \prod_{j=0}^{n-1} t_{2j}$ with $\rho_0 = 1.$

To prove this we must show that $\tau_{2n}\tau_{2n+1} = t_{2n+1}t_{2n+2}$ which is obvious and that $\tau_0 = t_0 + t_1$ and $\tau_{2n} + \tau_{2n-1} = t_{2n} + t_{2n+1}.$

This clearly holds for $n=0$ since $\rho_1 = t_0 + t_1$. For $n=1$ it is easily seen that $\tau_1 + \tau_2 = t_2 + t_3.$

In the general case we have

$$\tau_{2n} + \tau_{2n-1} = \frac{\rho_{n+1}}{\rho_n} + \frac{\rho_{n-1}}{\rho_n} t_{2n-1}t_{2n} = \frac{1}{\rho_n}\left(\rho_n t_{2n+1} + \rho_{n-1}t_{2n-1}t_{2n} + \prod_{j=0}^{n} t_{2j}\right) = t_{2n+1} + \frac{t_{2n}}{\rho_n}\left(\rho_{n-1}t_{2n-1} + \prod_{j=0}^{n-1} t_{2j}\right)$$
$$= t_{2n+1} + t_{2n}.$$

From (3.2) we now get

$$\det\left(r(i+j+2,x^2)\right)_{i,j=0}^{n-1} = \det\left(a(i+j+2)\right)_{i,j=0}^{n-1} \frac{\pi_{2n+1}(x,t)}{x}. \qquad (3.5)$$

Note that the polynomials

$$R_n(x,t) = \frac{\pi_{2n+1}(\sqrt{x},t)}{\sqrt{x}} \qquad (3.6)$$

satisfy

$$R_n(x,t) = \left(x - \tau_{2n-2} - \tau_{2n-1}\right)R_{n-1}(x,t) - \tau_{2n-3}\tau_{2n-2}R_{n-2}(x,t) \qquad (3.7)$$

with initial values $R_{-1}(x,t) = 0$ and $R_0(x,t) = 1.$

Let us first consider Theorem 2 from this point of view.

Here we have $t_n = 1,$ $\rho_n = n+1$ and thus $\tau_{2n} = \dfrac{n+2}{n+1}$ and $\tau_{2n+1} = \dfrac{n+1}{n+2}.$

This gives

$$R_n(x,t) = \left(x - \frac{n+1}{n} - \frac{n}{n+1}\right)R_{n-1}(x,t) - \frac{n^2-1}{n^2}R_{n-2}(x,t).$$

Comparing coefficients it is easily verified that
$R_n(x,t)=\sum_{k=0}^{n}(-1)^{n-k}\binom{n+2+k}{n-k}\frac{(2n+3)(k+1)}{(2k+3)(n+1)}x^k.$
Thus we have again obtained the orthogonal polynomials (2.7).

Now we consider a $q-$analogue. In [2] we studied the symmetric orthogonal polynomials whose moments are the Andrews $q-$Catalan numbers

$$c_n(q)=\frac{1}{[n+1]}\begin{bmatrix}2n\\n\end{bmatrix}\frac{1+q}{1+q^{n+1}}\frac{1}{\prod_{j=1}^{n}\left(1+q^j\right)^2}. \tag{3.8}$$

These polynomials correspond to the sequence t with

$$t_n=\frac{q^n}{(1+q^{n+1})(1+q^{n+2})}. \tag{3.9}$$

Here we have

$$p_n(x,t)=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k q^{k^2-k}\begin{bmatrix}n-k\\k\end{bmatrix}\frac{x^{n-2k}}{\prod_{j=1}^{k}\left(1+q^j\right)\prod_{j=n+1-k}^{n}\left(1+q^j\right)}=\sum_{k=0}^{\lfloor \frac{n}{2}\rfloor}(-1)^k q^{k^2-k}\begin{bmatrix}n-k\\k\end{bmatrix}_{q^2}\frac{x^{n-2k}}{\prod_{j=n-2k+1}^{n}\left(1+q^j\right)}.$$

This gives

$$p_{2n}(x,t)=\sum_{j=0}^{n}(-1)^k q^{k^2-k}\begin{bmatrix}2n-k\\k\end{bmatrix}_{q^2}\frac{1}{\prod_{j=2n-2k+1}^{2n}\left(1+q^j\right)}x^{2n-2k}$$

$$=\sum_{j=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\begin{bmatrix}n+k\\n-k\end{bmatrix}_{q^2}\frac{1}{\prod_{j=2k+1}^{2n}\left(1+q^j\right)}x^{2k}=\frac{1}{\prod_{j=1}^{2n}\left(1+q^j\right)}\sum_{j=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\prod_{j=1}^{2k}\left(1+q^j\right)\begin{bmatrix}n+k\\n-k\end{bmatrix}_{q^2}x^{2k}.$$

Therefore we get

$$\frac{\det\left(c_{i+j}(q)x-c_{i+j+1}(q)\right)_{i,j=0}^{n-1}}{\det\left(c_{i+j}(q)\right)_{i,j=0}^{n-1}}=\frac{1}{\prod_{j=1}^{2n}\left(1+q^j\right)}\sum_{j=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\prod_{j=1}^{2k}\left(1+q^j\right)\begin{bmatrix}n+k\\n-k\end{bmatrix}_{q^2}x^k \tag{3.10}$$

and

$$\frac{\det\left(c_{i+j+1}(q)x-c_{i+j+2}(q)\right)_{i,j=0}^{n-1}}{\det\left(c_{i+j+1}(q)\right)_{i,j=0}^{n-1}}=\frac{1}{\prod_{j=1}^{2n+1}\left(1+q^j\right)}\sum_{j=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\prod_{j=1}^{2k+1}\left(1+q^j\right)\begin{bmatrix}n+1+k\\n-k\end{bmatrix}_{q^2}x^k. \tag{3.11}$$

To compute $\pi_n(x,t)$ we must determine ρ_n. It is easily verified that

$$\rho_n=q^{2\binom{n}{2}}\frac{[n+1]_{q^2}}{\prod_{j=2}^{2n+1}\left(1+q^j\right)}. \tag{3.12}$$

This gives

$$\tau_{2n}=\frac{[n+2]_{q^2}}{[n+1]_{q^2}}\frac{q^{2n}}{\left(1+q^{2n+2}\right)\left(1+q^{2n+3}\right)} \tag{3.13}$$

and

$$\tau_{2n+1}=\frac{[n+1]_{q^2}}{[n+2]_{q^2}}\frac{q^{2n+3}}{\left(1+q^{2n+3}\right)\left(1+q^{2n+4}\right)}. \tag{3.14}$$

Therefore the corresponding polynomials $R_n(x,t)$ satisfy
$R_n(x,t)=\left(x-\tau_{2n-2}-\tau_{2n-1}\right)R_{n-1}(x,t)-\tau_{2n-3}\tau_{2n-2}R_{n-2}(x,t)$.

It is now an elementary but tedious exercise to verify that

$$R_n(x,t)=\frac{1}{\prod_{j=1}^{2n+3}\left(1+q^j\right)}\sum_{k=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\prod_{j=1}^{2k+3}\left(1+q^j\right)\frac{[2n+3]_{q^2}[k+1]_{q^2}}{[n+1]_{q^2}[2k+3]_{q^2}}\begin{bmatrix}n+2+k\\ n-k\end{bmatrix}_{q^2}x^k.$$

Thus we get

Theorem 4

$$\frac{\det\left(c_{i+j+2}(q)x-c_{i+j+3}(q)\right)_{i,j=0}^{n-1}}{\det\left(c_{i+j+2}(q)\right)_{i,j=0}^{n-1}} =\frac{1}{\prod_{j=1}^{2n+3}\left(1+q^j\right)}\sum_{k=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\prod_{j=1}^{2k+3}\left(1+q^j\right)\frac{[2n+3]_{q^2}[k+1]_{q^2}}{[n+1]_{q^2}[2k+3]_{q^2}}\begin{bmatrix}n+2+k\\ n-k\end{bmatrix}_{q^2}x^k. \tag{3.15}$$

Finally we consider the sequence $b_n(q)=\begin{bmatrix}2n\\ n\end{bmatrix}\frac{1}{\prod_{j=1}^{n}\left(1+q^j\right)^2}$.

Here we have (for details cf. [2])
$t_0=\frac{1}{1+q}$ and $t_n=\frac{q^n}{\left(1+q^n\right)\left(1+q^{n+1}\right)}$ for $n>0$.

In this case

$$\frac{\det\left(b_{i+j}(q)x-b_{i+j+1}(q)\right)_{i,j=0}^{n-1}}{\det\left(b_{i+j}(q)\right)_{i,j=0}^{n-1}}=\frac{1}{\prod_{j=1}^{2n}\left(1+q^j\right)}\sum_{j=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\prod_{j=1}^{2k}\left(1+q^j\right)\frac{[2n]_{q^2}}{[n+k]_{q^2}}\begin{bmatrix}n+k\\ n-k\end{bmatrix}_{q^2}x^k.$$

$$\frac{\det\left(b_{i+j+1}(q)x-b_{i+j+2}(q)\right)_{i,j=0}^{n-1}}{\det\left(b_{i+j+1}(q)\right)_{i,j=0}^{n-1}}=\frac{1}{\prod_{j=1}^{2n+1}\left(1+q^j\right)}\sum_{j=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\prod_{j=1}^{2k+1}\left(1+q^j\right)\frac{[2n+1]_{q^2}}{[n+k+1]_{q^2}}\begin{bmatrix}n+k+1\\n-k\end{bmatrix}_{q^2}x^k.$$

We get $\rho_n=q^{2\binom{n}{2}}\frac{[2n+1]_{q^2}}{\prod_{j=2}^{2n+1}\left(1+q^j\right)}$.

$$\tau_{2n}=\frac{[2n+3]_{q^2}}{[2n+1]_{q^2}}\frac{q^{2n}}{\left(1+q^{2n+2}\right)\left(1+q^{2n+3}\right)}$$

$$\tau_{2n+1}=\frac{[2n+1]_{q^2}}{[2n+3]_{q^2}}\frac{q^{2n+3}}{\left(1+q^{2n+1}\right)\left(1+q^{2n+2}\right)}$$

It is again an elementary exercise to verify that

$$R_n(x,t)=\frac{1}{\prod_{j=1}^{2n+1}\left(1+q^j\right)}\sum_{k=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\begin{bmatrix}n+k+1\\n-k\end{bmatrix}_{q^2}\frac{[2n+3]}{[2k+3]}\prod_{j=1}^{2k+1}\left(1+q^j\right).$$

Therefore we get

Theorem 5

$$\frac{\det\left(b_{i+j+2}(q)x-b_{i+j+3}(q)\right)_{i,j=0}^{n-1}}{\det\left(b_{i+j+2}(q)\right)_{i,j=0}^{n-1}}=\frac{1}{\prod_{j=1}^{2n+1}\left(1+q^j\right)}\sum_{k=0}^{n}(-1)^{n-k}q^{2\binom{n-k}{2}}\begin{bmatrix}n+k+1\\n-k\end{bmatrix}_{q^2}\frac{[2n+3]}{[2k+3]}\prod_{j=1}^{2k+1}\left(1+q^j\right). \quad (3.16)$$

Remark

The reader may ask why we did not consider the case $t_n=q^n$ which leads to the Carlitz $q-$ Catalan numbers $C_n(q)$ as moments. The simple answer is that we do not get beautiful results.

It is true that we get for the first two Hankel determinants simple results

$$\det\left(C_{i+j}(q)x-C_{i+j+1}(q)\right)_{i,j=0}^{n-1}=\sum_{k=0}^{n}(-1)^k q^{2\binom{n-k}{2}}\begin{bmatrix}n+k\\n-k\end{bmatrix}x^k \quad (3.17)$$

and

$$\det\left(C_{i+j+1}(q)x-C_{i+j+2}(q)\right)_{i,j=0}^{n-1}=\sum_{k=0}^{n}(-1)^k q^{2\binom{n-k}{2}}\begin{bmatrix}n+1+k\\n-k\end{bmatrix}x^k. \quad (3.18)$$

But the third Hankel determinant looks ugly.

To compute the third Hankel determinant we first compute $\rho_n = q^{2\binom{n}{2}}[n+1]$.

This gives $\tau_{2n} = q^{2n}\frac{[n+2]}{[n+1]}$ and $\tau_{2n+1} = q^{2n+3}\frac{[n+1]}{[n+2]}$.

Thus

$$R_n(x,t) = \left(x - q^{2n-2}\frac{[n+1]}{[n]} - q^{2n+1}\frac{[n]}{[n+1]}\right)R_{n-1}(x,t) - q^{4n-3}\frac{[n-1][n+1]}{[n]^2}R_{n-2}(x,t).$$

Unfortunately the coefficients of these polynomials do not factor in a simple way.
The first terms of
$\left(\det\left(C_{i+j+2}(q)x - C_{i+j+3}(q)\right)\right)_{n\geq 0}$ are

$$\{1, q^5(-1-2q-q^2-q^3+(1+q)x), q^{19}(q^2(1+q^2)(1+2q+2q^2+q^3+q^4)-(1+q)(1+q+2q^2+q^3+q^4+q^5)x+(1+q+q^2)x^2)\}$$

They reduce for $q=1$ to $1, 2x-5, 3x^2-14x+14, \cdots$ as they should do but I doubt if there is a simple law satisfied by them.

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