q-Abel polynomials

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Abstract

This note gives a simple approach to $q-$ analogues of some results associated with Abel polynomials.

0. Introduction

In this note I want to give a simple approach to $q-$ analogues of the following well-known results about Abel polynomials:
Let $a_n(x,a)=x(x-na)^{n-1}$ be the Abel polynomials.
N.H. Abel [1] has found the beautiful formula

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}a_k(x,a)(y+ka)^{n-k}. \tag{0.1}$$

We want to state it in a slightly more general form by changing $x\to x-b, y\to y+b$ and defining

$$a_n(x,a,b)=a_n(x-b,a)=(x-b)(x-b-na)^{n-1}. \tag{0.2}$$

Then we get

$$(x+y)^n=\sum_{k=0}^{n}\binom{n}{k}a_k(x,a,b)(y+ka+b)^{n-k}. \tag{0.3}$$

If we denote by ∂ the differentiation operator we see that

$$a_n(x,a,b)=(1+a\partial)(x-b-na)^n. \tag{0.4}$$

By Taylor's theorem which may be stated as $p(y+x)=e^{x\partial_y}p(y)$ for polynomials $p(y)$ formula (0.3) is equivalent with

$$e^{x\partial_y}p(y)=\sum_k\frac{a_k(x,a,b)}{k!}e^{(b+ka)\partial_y}\left(\partial_y\right)^k p(y) \tag{0.5}$$

for all polynomials $p(y)$. This gives the operator identity

$$e^{x\partial_y}=\sum_k\frac{a_k(x,a,b)}{k!}e^{(b+ka)\partial_y}\left(\partial_y\right)^k \tag{0.6}$$

which by the isomorphism $\partial \leftrightarrow z$ is equivalent with the identity of formal power series

$$e^{xz} = \sum_k \frac{a_k(x,a,b)}{k!} z^k e^{(b+ka)z}. \tag{0.7}$$

The linear operator

$$Q = \partial e^{a\partial}, \tag{0.8}$$

called Abel operator, satisfies

$$Qa_n(x,a,b) = na_{n-1}(x,a,b). \tag{0.9}$$

Since $a_n(0,a) = [n=0]$ we get

$$LQ^k a_n(x,a) = L\partial^k e^{ka\partial} a_n(x,a) = n![k=n], \tag{0.10}$$

where L denotes the linear functional on the polynomials $p(x)$ defined by $Lp = p(0)$ and $[k=n]$ is Knuth's symbol defined by $[k=n]=1$ if $k=n$ is true and $[k=n]=0$ if $k \neq n$.

This implies that the coefficients of the expansion

$$f(z) = \sum_k \frac{c_k}{k!} z^k e^{kaz} \tag{0.11}$$

are given by the Lagrange formula

$$c_n = L\partial^{n-1} e^{-nax} f'(x). \tag{0.12}$$

In the same way we get that the coefficients of the expansion

$$\frac{f(z)}{1+az} = \sum_k \frac{c_k}{k!} z^k e^{kaz} \tag{0.13}$$

are given by the formula of Lagrange-Bürmann

$$c_n = L\partial^n e^{-nax} f(x). \tag{0.14}$$

A special case is

$$\frac{e^{xz}}{1-az} = \sum_{k\geq 0} \frac{(x+ak)^k}{k!} \left(ze^{-az}\right)^k. \tag{0.15}$$

This can also be written in the form

$$\frac{1}{1-az} = \sum_{k\geq 0} \frac{(x+ak)^k}{k!} z^k e^{-(ak+x)z}. \tag{0.16}$$

Expanding $e^{-(ak+x)z}$ into a power series and comparing coefficients of z^n shows that this is equivalent with

$$n!a^n = \sum_{k=0}^{n} (-1)^{n-k} \binom{n}{k} (x+ak)^n. \tag{0.17}$$

Historical notes about some of these formulas can be found in [10].
The first $q-$analogues of some of these results have been given by F. H. Jackson ([5]) in 1910. In [3] I have given another proof depending on Rota's Finite Operator Calculus ([8]). The case $b \neq 0$ has first been considered by J. Hofbauer in [4]. It also appeared in papers by W.P. Johnson ([6]), B. Bhatnagar and St.C. Milne ([2]), C. Krattenthaler and M. Schlosser ([7]) and M. Schlosser ([9]). In the following I want to give a self-contained exposition and some simplifications of these results. I want to thank Michael Schlosser for some useful comments.

1. q-Abel polynomials

Let D or D_x be the $q-$differentiation operator, defined by $Df(x)=\frac{f(x)-f(qx)}{(1-q)x}.$ Instead of $Df(x)$ we also write $f'(x).$

In order to simplify notation we set $(y \dotplus x)^n := \prod_{j=0}^{n-1}(y+q^j x)$ and $(y \dot{-} x)^n := \prod_{j=0}^{n-1}(y-q^j x).$

The other notations from $q-$calculus are the usual ones. The $q-$binomial coefficients are denoted by $\begin{bmatrix} n \\ k \end{bmatrix} = \frac{[n]!}{[k]![n-k]!}$ for $0 \le k \le n.$ Here $[n]=\frac{1-q^n}{1-q}$ and $[n]!=\prod_{j=1}^{n}[j].$

We need the analogues of the exponential series $e(z)=\sum_{k\ge 0}\frac{z^k}{[k]!}$ and $E(z)=\sum_{k\ge 0} q^{\binom{k}{2}}\frac{z^k}{[k]!},$ which are related by $e(z)E(-z)=1$ and the well-known facts that $\sum_{k\ge 0}\frac{(x \dot{-} y)^k}{[k]!}z^k = \frac{e(xz)}{e(yz)}$ and $E(aD)y^n = (y+a)\cdots(y+q^{n-1}a).$

The following $q-$analogue of (0.3) holds:

$$\prod_{j=0}^{n-1}(y+q^j x)=\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} A_k(x,a,b)\prod_{j=0}^{n-k-1}(y+q^j[k]a+q^{k+j}b). \tag{1.1}$$

Here

$$A_n(x,a,b)=(x-b)\prod_{j=1}^{n-1}\left(q^j x - a[n] - q^n b\right) \tag{1.2}$$

is a $q-$analogue of the general Abel polynomial $a_n(x,a,b).$
We will write (1.1) in the form

$$(y \dotplus x)^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} A_k(x,a,b)(y \dotplus ([k]a+q^k b))^{n-k} \tag{1.3}$$

or

$$E(xD_y)y^n = \sum_k \frac{A_k(x,a,b)}{[k]!}E\left(\left([k]a+q^k b\right)D_y\right)D_y^k y^n. \tag{1.4}$$

By the isomorphism $D \leftrightarrow z$ it is also equivalent with the identity of formal power series

$$E(xz) = \sum_k \frac{A_k(x,a,b)}{[k]!} E(([k]a + q^k b)z) z^k . \qquad (1.5)$$

For $b = 0$ this $q-$Abel theorem has been found by F. H. Jackson ([5]). The general case has been considered by J. Hofbauer ([4]), W.P. Johnson ([6]) and is also contained in more general results by B. Bhatnagar and St.C. Milne ([2]) and by C. Krattenthaler and M. Schlosser ([7]) and M. Schlosser ([9]).

In order to prove (1.2) we write (1.3) with unknown coefficients $A_k(x,a,b)$ and try to determine their values.
We consider this formula for $n \to n-1$ and multiply with $c[n]$ for some constant c.
We thus get

$$c[n](y \dotplus x)^{n-1} = c\sum_{k=0}^{n} \begin{bmatrix} n-1 \\ k \end{bmatrix} [n] A_k(x,a,b)(y \dotplus ([k]a + q^k b))^{n-k-1}$$
$$= \sum_{k=0}^{n} \begin{bmatrix} n-1 \\ n-1-k \end{bmatrix} \frac{[n]}{[n-k]} A_k(x,a,b)(y \dotplus ([k]a + q^k b))^{n-k} \frac{c[n-k]}{y + q^{n-k-1}([k]a + q^k b)} .$$

Comparing with (1.3) we see that the first $n-1$ terms coincide if $\dfrac{c[n-k]}{y + q^{n-k-1}([k]a + q^k b)} = 1.$

This gives $y = c[n-k] - q^{n-1-k} a[k] - q^{n-1} b = \dfrac{c(1-q^{n-k}) - q^{n-1-k}(1-q^k)a}{1-q} - q^{n-1} b$.

This is independent on k if $q^{n-k} c + q^{n-1-k} a = 0$, i.e. $c = -\dfrac{a}{q}$.

We then get

$$y = -\frac{a}{q} \frac{(1-q^{n-k}) + q^{n-k}(1-q^k)}{1-q} - q^{n-1} b = -\frac{a}{q}[n] - q^{n-1} b.$$

Therefore

$$A_n(x,a,b) = q^{n-1}(x-b) \prod_{j=0}^{n-2} \left(q^j x - \frac{a}{q}[n] - q^{n-1} b \right) = (x-b) \prod_{j=1}^{n-1} \left(q^j x - a[n] - q^n b \right).$$

For $b = 0$ we get

$$A_n(x,a,0) = x \prod_{j=1}^{n-1} \left(q^j x - [n]a \right) \qquad (1.6)$$

for $n > 0$ and $A_0(x,a,q) = 1$.

This implies Jackson's identity.

By letting $a \to a+(1-q)b$ we get

$$G_n(x,a,b) = A_n(x,a+(1-q)b,b) = (x-b)\prod_{j=1}^{n-1}(q^j x-[n]a-b). \qquad (1.7)$$

This is another form of the general $q-$Abel polynomials. Whereas $A_n(x,a,b)$ and $G_n(x,a,b)$ are equivalent some formulas become simpler by using $G_n(x,a,b)$.

The polynomials $G_n(x,a,b)$ have first been considered by J. Hofbauer ([4]). They are also the special case $h=0$ of the polynomials $a_n(x;b,h,w,q)$, which have been studied by W.P. Johnson in [6].

The corresponding identity is

$$\prod_{j=0}^{n-1}(y+q^j x) = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix} G_k(x,a,b)\prod_{j=0}^{n-k-1}(y+q^j[k]a+q^j b). \qquad (1.8)$$

This is equivalent with identity (8.4) in [7]. There it is stated in the form

$$(c;q)_n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k}{2}} c^k \frac{1-(a+b)}{1-(q^{-k}a+b)}\left(q^{-k}a+b;q\right)_k \left(c(q^k a+b);q\right)_{n-k},$$

where as usual $(x;q)_n = (1-x)(1-qx)\cdots(1-q^{n-1}x)$. To obtain (1.8) make the substitutions $a \to \frac{b(1-q)-a}{(1-q)x}, b \to \frac{a}{(1-q)x}, c \to -\frac{x}{y}$.

The corresponding formula

$$E(xz) = \sum_k \frac{G_k(x,a,b)}{[k]!}E(([k]a+b)z)z^k \qquad (1.9)$$

has been obtained in [4] and is also equivalent with formula [7], (7.4).

2. Abel expansions

If we apply the operator D to (1.9) we get

$$zE(qxz) = DE(xz) = \sum_{n\geq 1}\frac{DG_n(x,a,b)}{[n]!}z^n E((b+[n]a)z)$$

or

$$E(qxz) = \sum_{n\geq 0}\frac{DG_{n+1}(x,a,b)}{[n+1]!}z^n E((b+[n+1]a)z)).$$

On the other hand (1.9) also gives

$$E(qxz)=\sum_{n\geq 0}\frac{G_n(qx,qa,b+a)}{[n]!}z^nE((b+[n+1]a)z)).$$

Comparing these two formulas we get

$$DG_{n+1}(x,a,b)=[n+1]G_n(qx,qa,b+a).$$

By induction this implies

$$D^kG_n(x,a,b)=q^{\binom{k}{2}}\frac{[n]!}{[n-k]!}G_{n-k}(q^kx,q^ka,b+[k]a). \tag{2.1}$$

This means that

$$D^nG_n(x,a,b)=q^{\binom{n}{2}}[n]!$$

and for $k<n$

$$D^kG_n(x,a,b)=q^{\binom{k}{2}}\frac{[n]!}{[n-k]!}\left(q^kx-[k]a-b\right)\prod_{j=k+1}^{n-1}\left(q^jx-[n]a-b\right).$$

An important consequence is

$$G_n^{(k)}(\frac{b+[k]a}{q^k},a,b)=q^{\binom{k}{2}}[k]![k=n]. \tag{2.2}$$

Thus each polynomial $f(x)$ has the following Abel expansion

$$f(x)=\sum_k\frac{f^{(k)}(q^{-k}(b+[k]a))}{[k]!}q^{-\binom{k}{2}}G_k(x,a,b). \tag{2.3}$$

Choosing $f(x)=\prod_{j=0}^{n-1}(y+q^jx)$ we get again (1.8).

By choosing $f(x)=G_n(x,-a,-y-b)$ we get

$$G_n(x,-a,-y-b)=\sum_{k=0}^{n}\begin{bmatrix}n\\k\end{bmatrix}G_k(x,-a,-b)y\prod_{j=1}^{n-k-1}\left(y+(1-q^j)b+([n]-q^j[k])a\right). \tag{2.4}$$

This is the special case $h=0$ of Theorem 4 by W.P. Johnson ([6]).

The expansion (2.3) also holds for formal power series. If we choose $f(x)=E(xz)$ we get (1.9).

From (1.9) we get e.g.

$$z^n=\sum_{k\geq 0}\frac{(-1)^k}{[k]!}\left([n]a+b\right)\left([n+k]a+b\right)^{k-1}z^{n+k}E(([n+k]a+b)z)$$

by applying $q-$differentiation k times and setting $x=0$.

3. A q - analogue of the Abel operator

Let

$$w_n(x,a,b)=\prod_{j=0}^{n-1}\left(q^j x-[n]a-b\right). \qquad (3.1)$$

This can be written in the form

$$w_n(x,a,b)=\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{n-k}{2}}\left([n]a+b\right)^k x^{n-k}=q^{\binom{n}{2}}\sum_{k=0}^{n}(-1)^k\begin{bmatrix}n\\k\end{bmatrix}q^{\binom{k}{2}}\left(\frac{[n]a+b}{q^{n-1}}\right)^k x^{n-k}$$

$$=q^{\binom{n}{2}}E\left(-\frac{[n]a+b}{q^{n-1}}D\right)x^n.$$

In analogy to (0.4) we get

$$G_n(x,a,b)=(1+aD)\prod_{j=0}^{n-1}\left(q^j x-[n]a-b\right)=(1+aD)w_n(x,a,b). \qquad (3.2)$$

Therefore we have

$$G_n(x,a,b)=(1+aD)w_n(x,a,b)=(1+aD)q^{\binom{n}{2}}E\left(-\frac{[n]a+b}{q^{n-1}}D\right)x^n$$

$$=q^{\binom{n}{2}}E\left(-\frac{[n]a+b}{q^{n-1}}D\right)\left(x^n+[n]ax^{n-1}\right).$$

If we write $S_n(x,a)=q^{-\binom{n}{2}}e\left(\frac{[n]a+b}{q^{n-1}}D\right)G_n(x,a,b)=x^n+[n]ax^{n-1}$, then

$DS_n(x,a)=[n]S_{n-1}(x,a).$

Let now Q_n be the operator

$$Q_n=\frac{D}{q^{n-1}}\frac{e\left(\frac{[n]a+b}{q^{n-1}}D\right)}{e\left(\frac{[n-1]a+b}{q^{n-2}}D\right)}. \qquad (3.3)$$

Then we get as a $q-$analogue of (0.9)

$$Q_nG_n(x,a,b)=[n]G_{n-1}(x,a,b). \qquad (3.4)$$

Therefore the operators Q_n can be interpreted as $q-$analogues of the Abel operator Q. Unfortunately they are depending on n.

(3.4) is a consequence of

$$\frac{D}{q^{n-1}}\frac{e\left(\frac{[n]a+b}{q^{n-1}}D\right)}{e\left(\frac{[n-1]a+b}{q^{n-2}}D\right)}G_n(x,a,b)=\frac{1}{q^{n-1}}E\left(-\frac{[n-1]a+b}{q^{n-2}}D\right)q^{\binom{n}{2}}DS_n(x,a,b)$$

$$=q^{\binom{n-1}{2}}E\left(-\frac{[n-1]a+b}{q^{n-2}}D\right)[n]S_{n-1}(x,a,b)=[n]G_{n-1}(x,a,b).$$

The operator Q_n can also be written as

$$Q_n=\sum_{k\geq 0}\frac{\prod_{j=0}^{k-1}\left(([j+1]-q^n[j])a+(1-q^{j+1})b\right)}{[k]!}\left(\frac{D}{q^{n-1}}\right)^{k+1}. \tag{3.5}$$

For

$$[j+1]-q^n[j]=\frac{1-q^{j+1}-q^n(1-q^j)}{1-q}=\frac{(1-q^n)-q^{j+1}(1-q^{n-1})}{1-q}=[n]-q^{j+1}[n-1]$$

and therefore

$$([j+1]-q^n[j])a+(1-q^{j+1})b=([n]-q^{j+1}[n-1])a+(1-q^{j+1})b=([n]a+b]-q^{j+1}\left([n-1]a+b\right).$$

This implies

$$\prod_{j=0}^{k-1}\left(([j+1]-q^n[j])a+(1-q^{j+1})b\right)=\left(([n]a+b)\dot{-}q\left([n-1]a+b\right)\right)^k.$$

4. A q - Lagrange formula

For a formal power series $f(z)$ we want to find the coefficients in the expansion

$$f(z)=\sum_n\frac{c_n}{[n]!}z^nE([n]az). \tag{4.1}$$

Consider first the expansion of $f(z)=e(xz)$. Let

$$e(xz)=\sum_k\frac{B_k(x,a)}{[k]!}z^kE([k]az). \tag{4.2}$$

We know from (1.9) that $E(xz)=\sum_k \frac{A_k(x,a,0)}{[k]!}E([k]az)z^k.$

If we let V be the linear operator defined by $Vq^{\binom{n}{2}}x^n = x^n$, then it is clear that

$B_n(x,a)=VA_n(x,a,0)=V\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}(-1)^k\left([n]a\right)^k q^{\binom{n-k}{2}}x^{n-k} = \sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}(-1)^k\left([n]a\right)^k x^{n-k}.$

Therefore we get

$$B_n(x,a)=x\sum_{j=0}^{n-1}(-1)^j\begin{bmatrix} n-1 \\ j \end{bmatrix}a^j x^{n-1-j}[n]^j = xe(-[n]aD)x^{n-1}. \tag{4.3}$$

Here we have a direct analogue of the formula $a_n(x,a)=xe^{-na\partial}x^{n-1}.$
A possible disadvantage is that there is no simple factorization.
The property we are interested in is

$$LE([k]aD)D^kB_n(x,a)=[n]![k=n], \tag{4.4}$$

which generalizes (0.10).

To prove it observe that $D^ke(xz)=z^ke(xz).$ Therefore

$E([k]aD)D^ke(xz)=\sum_k \frac{E([k]aD)D^kB_k(x,a)}{[k]!}z^kE([k]az)$
and for $x=0$
$E([k]az)z^k=\sum_k \frac{LE([k]aD)D^kB_k(x,a)}{[k]!}z^kE([k]az).$
Comparing coefficients we get (4.4).

This implies the following

$q-$**Lagrange formula** ([3])**:**

The coefficients c_n *in the expansion*

$$f(x)=\sum_n \frac{c_n}{[n]!}x^nE([n]ax) \tag{4.5}$$

are given by

$$c_n = Lf'(D)e(-[n]aD)x^{n-1} = LD^{n-1}e(-[n]ax)f'(x). \tag{4.6}$$

For by (4.4) we have $c_n = Lf(D)B_n(x,a)=Lf(D)xe(-[n]aD)x^{n-1}=LD^{n-1}e(-[n]aD)f'(x).$

The last equation follows from $L(D^k x^n) = [n]![k=n] = L(D^n x^k)$ and the $q-$Pincherle derivative

$$f(D)\underline{x} - \underline{x}f(qD) = f'(D). \tag{4.7}$$

Here $\underline{x}$ denotes the operator multiplication by x.
This well-known fact follows from $\left(D\underline{x} - q\underline{x}D\right)x^n = Dx^{n+1} - qxDx^n = \left([n+1] - q[n]\right)x^n = x^n$, which implies $D\underline{x} - \underline{x}qD = 1$ by induction $D^n\underline{x} - \underline{x}q^n D^n = nD^{n-1}$.

More generally let $B_n(x,a,b) = VA_n(x,a,b)$. *Then the coefficients of the expansion*

$$f(z) = \sum_n \frac{c_n}{[n]!} z^n E\left(\left([n]a + q^n b\right)z\right) \tag{4.8}$$

are given by

$$c_n = Lf(D)B_n(x,a,b) = Lf(D)xe\left(-\left([n]a+q^n b\right)D\right)x^{n-1} - q^{n-1}bLf(D)e\left(-\left(\frac{[n]a+q^n b}{q}\right)D\right)x^{n-1}$$

$$= LD^{n-1}e\left(-\left([n]a+q^n b\right)x\right)f'(x) - q^{n-1}bLD^{n-1}e\left(-\left(\frac{[n]a+q^n b}{q}\right)x\right)f(x).$$

For the proof observe that

$$LE\left(\left(q^k b + [k]a\right)D\right)D^k B_n(x,a,b) = [n]![k=n] \tag{4.9}$$

by the same argument as above and that

$$B_n(x,a,b) = VA_n(x,a,b) = V(x-b)\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}(-1)^k\left([n]a+q^n b\right)^k q^{\binom{n-k}{2}} x^{n-1-k}$$

$$= V\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}(-1)^k\left([n]a+q^n b\right)^k q^{\binom{n-k}{2}} x^{n-k} - bV\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}(-1)^k\left([n]a+q^n b\right)^k q^{\binom{n-1-k}{2}} q^{n-1-k} x^{n-1-k}$$

$$= \sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}(-1)^k\left([n]a+q^n b\right)^k x^{n-k} - q^{n-1}b\sum_{k=0}^{n-1}\begin{bmatrix} n-1 \\ k \end{bmatrix}(-1)^k\left(\frac{[n]a+q^n b}{q}\right)^k x^{n-1-k}$$

$$= xe\left(-\left([n]a+q^n b\right)D\right)x^{n-1} - q^{n-1}be\left(-\left(\frac{[n]a+q^n b}{q}\right)D\right)x^{n-1}.$$

$B_n(x,a,b)$ can also be written in the form

$$B_n(x,a,b) = \sum_{k=0}^{n}(-1)^k x^{n-k}\begin{bmatrix} n \\ k \end{bmatrix}(q^n b + [n]a)^{k-1}(q^{n-k}b + [n-k]a).$$

In order to obtain an analogue of the **Lagrange-Bürmann formula** we note that

$$\left(1+\frac{a}{q}D\right)e(-\frac{q^n b+[n]a}{q}D)x^n = xe\left(-\left([n]a+q^n b\right)D\right)x^{n-1}-q^{n-1}be\left(-\left(\frac{[n]a+q^n b}{q}\right)D\right)x^{n-1} \quad (4.10)$$

$= B_n(x,a,b).$

This follows from the $q-$Pincherle derivative

$$\left(e\left(-\frac{q^n b+[n]a}{q}D\right)\underline{x}-\underline{x}e(-(q^n b+[n]a)D\right)x^{n-1} = -\frac{q^n b+[n]a}{q}e\left(-\frac{q^n b+[n]a}{q}D\right)x^{n-1}.$$

Thus we get a

q - Lagrange - Bürmann type formula:

The coefficients of

$$\frac{f(z)}{1+\frac{az}{q}} = \sum_k \frac{c_k}{[k]!}z^k E\left(\left(q^k b+[k]a\right)z\right) \quad (4.11)$$

are given by

$$c_n = LD^n e(-\frac{q^n b+[n]a}{q}x)f(x). \quad (4.12)$$

This is an immediate consequence of

$$c_n = Lf(D)\frac{1}{1+\frac{aD}{q}}B_n(x,a,b) = Lf(D)e(-\frac{q^n b+[n]a}{q}D)x^n = LD^n e(-\frac{q^n b+[n]a}{q}x)f(x).$$

If we choose $f(z)=E(-yz)$ we get

$$c_n = LD^n e(-\frac{q^n b+[n]a}{q}x)E(-xy) = LD^n\frac{e(-\frac{q^n b+[n]a}{q}x)}{e(xy)} = \left(-\frac{q^n b+[n]a}{q}\dot{-}y\right)^n$$

This is equivalent with

$$\frac{E(xz)}{1-az} = \sum_k \frac{(q^k b+[k]a\dot{+}x)^k}{[k]!}z^k E(-q(q^k b+[k]a)z). \quad (4.13)$$

This $q-$analogue of (0.15) has been found by C. Krattenthaler and M. Schlosser (cf. [7] and [9], (5.4)) in another context.

5. Other methods of proof

We give now another proof of formula (4.13) by using a $q-$analogue of the difference operator:

Let U be the linear operator on the polynomials in q^n defined by

$$Uq^{in} = q^{i(n-1)} \tag{5.1}$$

for all $i \in \mathbb{N}$. As a special case we get $U[n]^m = [n-1]^m$.

Define now

$$\Delta^k = (1-qU)\cdots(1-q^kU). \tag{5.2}$$

Then it is clear that

$$\Delta^k q^{in} = 0 \tag{5.3}$$

for $k \geq i > 0$ and

$$\Delta^k 1 = (1-q)^k [k]!. \tag{5.4}$$

Furthermore

$$\Delta^k \left[n\right]^m = [k]!(1-q)^{k-m} \tag{5.5}$$

for $k \geq m$ and

$$\Delta^k q^{in} \left[n\right]^m = 0 \tag{5.6}$$

for $k \geq m+i$ if $i > 0$.

It suffices to show (5.5), which follows from

$$\Delta^k \left[n\right]^m = \frac{1}{(1-q)^m}\Delta^k (1-q^n)^m = \frac{1}{(1-q)^m}\sum (-1)^j \binom{m}{j} \Delta^k q^{nj} = \frac{\Delta^k 1}{(1-q)^m} = \frac{(1-q)^k[k]!}{(1-q)^m}.$$

This again implies

$$\Delta^n \left(q^{nj}(q^n x + [n]a)^{n-j}\right) = 0 \tag{5.7}$$

if $j > 0$ and

$$\Delta^n \left((q^n x + [n]a)^n\right) = [n]!a^n. \tag{5.8}$$

We conclude that

$$\sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} (-1)^k \left(q^{n-k}b + [n-k]a \dotplus x\right)^{n-k} \left(x \dotplus q(q^{n-k}b + [n-k]a)\right)^k = [n]!a^n. \tag{5.9}$$

For

$$\sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k \left(q^{n-k}b + a\left[n-k\right] \underset{\cdot}{+} x\right)^{n-k} \left(x \underset{\cdot}{+} q(q^{n-k}b + a\left[n-k\right])\right)^k$$

$$= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k+1}{2}} \left(q^{n-k}b + a\left[n-k\right] \underset{\cdot}{+} q^{-k}x\right)^n$$

$$= \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}(-1)^k q^{\binom{k+1}{2}} \sum_{j=0}^{n} q^{-kj} x^j q^{\binom{j}{2}} \left(q^{n-k}b + a\left[n-k\right]\right)^{n-j}$$

$$= \sum_{j=0}^{n} q^{-nj} x^j q^{\binom{j}{2}} \Delta^n \left(q^{nj}\left(q^n b + a\left[n\right]\right)^{n-j}\right) = \Delta^n \left(\left(q^n b + a\left[n\right]\right)^n\right) = [n]!a^n.$$

By comparing coefficients we see that (5.9) is equivalent with

$$\frac{1}{1-az} = \sum_k \frac{(q^k b + \left[k\right]a \underset{\cdot}{+} x)^k}{\left[k\right]!} z^k \sum_j \frac{\left(x \underset{\cdot}{+} q(a[k] + q^k b)\right)^j}{[j]!}(-z)^j$$

$$= \sum_k \frac{(q^k b + \left[k\right]a \underset{\cdot}{+} x)^k}{\left[k\right]!} z^k \frac{e(-xz)}{e(q(q^k b + \left[k\right]a)z)}.$$

This is equivalent with (4.13).

By applying the $q-$differentiation operator k times and then setting $x=0$ we get from (4.13)

$$\frac{z^n}{1-az} = \sum_{k\geq 0}\frac{1}{[k]!}\left([n+k]a + q^{n+k}b\right)^k z^{n+k} E\left(-q\left([n+k]a + q^{n+k}b\right)z\right). \qquad (5.10)$$

By considering the isomorphism $z \to D_y$ and applying it to y^n (4.13) gives

$$(y \underset{\cdot}{+} x)^n = \sum_{k=0}^{n}\begin{bmatrix} n \\ k \end{bmatrix}\left(q^k b + [k]a \underset{\cdot}{+} x\right)^k v(n,k,a,b,y), \qquad (5.11)$$

with $v(n,k,a,b,y) = \left(y \underset{\cdot}{-} q(q^k b + [k]a)\right)^{n-k-1}(y - q^n b - [n]a)$ for $k < n$ and $v(n,n,a,b,y) = 1.$

If we substitute $a \to a + (1-q)b$ in (4.13) we get

$$\frac{E(xz)}{1-\left(a+(1-q)b\right)z} = \sum_k \frac{(b + \left[k\right]a \underset{\cdot}{+} x)^k}{\left[k\right]!} z^k E(-q(b + \left[k\right]a)z). \qquad (5.12)$$

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