A class of Rogers-Ramanujan type recursions

Johann Cigler

Abstract

In a previous paper [4] we generalized the Rogers-Ramanujan identities by proving formulas for the Carlitz $q-$Fibonacci polynomials $F_n(t)$ which reduce to the finite version of the Rogers-Ramanujan identities obtained by I. Schur for $t=1$. The $q-$Fibonacci polynomials can be interpreted as the weight of a set of lattice paths in $\mathbb{R}^2$ which are contained in the strip $-2 \le y \le 1$. In this paper we extend these results to lattice paths contained in more general strips. We determine the recursions satisfied by the corresponding polynomials and derive identities of the Rogers-Ramanujan type which are related to some identities by Kirillov [6] and Foda and Quano [5].

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0. Introduction

In a previous paper [4] we proved a finite version of the identity

$$F(t)=\sum_{l\ge 0}\frac{q^{l^2}}{(q)_l}t^l=\sum_{i\in\mathbb{Z}}(-1)^i q^{\frac{i(5i-1)}{2}}\sum_{l\ge|i|}\frac{q^{l^2-i^2}}{(q)_{l-i}(q)_{l+i}}t^l,$$

which for $t=1$ reduces to Schur's polynomial analog [8] of the first Rogers-Ramanujan identity(cf. [1]). In this paper we consider more generally the chain of formal power series

$$a_k(t,q)=\sum_{l\ge 0}d_k(l)t^l=\sum_{i\in\mathbb{Z}}(-1)^i q^{\frac{i((2k+3)i-1)}{2}}\sum_{l\ge|i|}\frac{q^{l^2-i^2}}{(q)_{l-i}(q)_{l+i}}t^l, k\ge 0,$$

which for $t=1$ reduces to $\frac{1}{(q)_\infty}\sum_{i\in\mathbb{Z}}(-1)^i q^{\frac{i((2k+3)i-1)}{2}}$.

We give a polynomial version of $a_k(t,q)$ which satisfies a recursion of order $2k$ and determine this recursion explicitly.
Then we do the same with the chain of formal power series

$$b_k(t,q)=\sum_{l\ge 0}c_k(l)t^l=\sum_{i\in\mathbb{Z}}(-1)^i q^{(k+1)i^2}\sum_{l\ge|i|}\frac{q^{l^2-i^2}}{(q)_{l-i}(q)_{l+i}}t^l, k\ge 1.$$

For $t=q=1$ some of these results have already been proved in [3].
For $t=1$ these chains also occur via Bailey's lemma (cf.[7]).
I want to thank Ole Warnaar for some useful remarks concerning the fermionic form of $a_k(n,t,q)$.

1. The combinatorial background.

We consider lattice paths in $\mathbb{R}^2$ of finite length, which start at the origin $(0,0)$, where at each step only two moves are allowed, a northeast move $(i,j)\to(i+1,j+1)$ and a southeast move $(i,j)\to(i+1,j-1)$. Define a **peak** as a vertex preceded by a northeast step and followed by a southeast step, and a **valley** as a vertex preceded by a southeast step and followed by a northeast step. The **height** of a vertex is its $y-$coordinate. The peaks with height at least 1 and the valleys with height at most -2 are called **extremal points**. Let $D(v)$ be the set of the $x-$coordinates of the extremal points of the path v. Let $d(v)=|D(v)|$ and $\iota(v)=\sum_{i\in D(v)} i$. Then the weight of the path v is defined by $w_t(v)=q^{\iota(v)}t^{d(v)}$. The weight of a set of paths is the sum of the weight of all paths.

Let in the terminology of [4] A_n^0 denote the set of all lattice paths with $k=\left\lfloor\frac{n}{2}\right\rfloor$ northeast steps and $l=\left\lfloor\frac{n+1}{2}\right\rfloor$ southeast steps, which start at $(0,0)$. Let $A_n(r,-s)$ be the set of those paths which are contained in the strip $-s<y<r$. Let further $A_n^m(r,-s)$ (resp. $A_n^m(-s,r)$) be the set of all paths in A_n^0 such that at least m points are outside the strip satisfying the following condition: the height of the first (from left to right) such point is $\geq r$ (resp. $\leq -s$), the height of the second point is $\leq -s$ (resp. $\geq r$), the height of the next point is again $\geq r$ (resp. $\leq -s$), and so on. Thus each path in $A_n^m(r,-s)$ and $A_n^m(-s,r)$ leaves the strip $-s<y<r$ at least m times and oscillates between points above the strip and below the strip. By a simple inclusion-exclusion argument it is clear that

$$w_t(A_n(r,-s))=w_t(A_n^0)+\sum_{m\geq 1}(-1)^m w_t(A_n^m(r,-s))+\sum_{m\geq 1}(-1)^m w_t(A_n^m(-s,r)). \quad (1.1)$$

Here we have $w_t(A_n^0)=\sum_{l\geq 0} q^{l^2}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor\\ l\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1}{2}\right\rfloor\\ l\end{bmatrix}t^l$ by [4], (3.7).

By [4] Lemma 3.1 we have

$$w_t(A_n^{2i+1}(r,-s))=$$

$$=q^{y_{2i+1}}\sum_{l\geq 2i+1} q^{(l+2i+1)(l-2i-1)}\begin{bmatrix}\left\lfloor\frac{n}{2}\right\rfloor-x_{2i+1}+2(2i+1)\\ l+2i+1\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1}{2}\right\rfloor+x_{2i+1}-2(2i+1)\\ l-2i-1\end{bmatrix}t^l$$

where $x_{2i}=(r+s)i, x_{2i+1}=(r+s)i+r, y_i=\dfrac{(r+s)i^2+(r-s)i}{2}$.

Therefore we get

$w_t(A_n^{2i+1}(r,-s)) =$

$$= q^{y_{2i+1}} \sum_{l \geq 2i+1} q^{(l+2i+1)(l-2i-1)} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n-(r+s-4)(2i+1)+s-r}{2} \right\rfloor}{l+2i+1} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n+1+(r+s-4)(2i+1)+r-s}{2} \right\rfloor}{l-2i-1} t^l.$$

In the same way we have

$w_t(A_n^{2i}(r,-s)) =$

$$= q^{y_{2i}} \sum_{l \geq 2i} q^{(l+2i)(l-2i)} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n+1}{2} \right\rfloor - x_{2i} + 2(2i)}{l+2i} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n}{2} \right\rfloor + x_{2i} - 2(2i)}{l-2i} t^l =$$

$$= q^{y_{2i}} \sum_{l \geq 2i} q^{(l+2i)(l-2i)} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n+1-(r+s-4)(2i)}{2} \right\rfloor}{l+2i} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n+(r+s-4)(2i)}{2} \right\rfloor}{l-2i} t^l.$$

Let now $s = r+1$. Then we have in either case

$$w_t(A_n^i(r,-s)) = q^{y_i} \sum_{l \geq i} q^{(l-i)(l+i)} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n-(r+s-4)i+1}{2} \right\rfloor}{l+i} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n+(r+s-4)i}{2} \right\rfloor}{l-i} t^l.$$

Here $y_i = \dfrac{(r+s)i^2 - i}{2}$.

Under the same assumption we get

$w_t(A_n^{2i+1}(-s,r)) =$

$$= q^{\overline{y}_{2i+1}} \sum_{l \geq 2i+1} q^{(l+2i+1)(l-2i-1)} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n+1}{2} \right\rfloor - x_{2i+1} + 2(2i+1)}{l+2i+1} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n}{2} \right\rfloor + x_{2i+1} - 2(2i+1)}{l-2i-1} t^l$$

but now we have $x_{2i+1} = (r+s)i + s$ and $\overline{y}_i = \dfrac{(r+s)i^2 - i}{2}$. Therefore this becomes

$$= q^{\overline{y}_{2i+1}} \sum_{l \geq 2i+1} q^{(l+2i+1)(l-2i-1)} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n-(r+s-4)(2i+1)}{2} \right\rfloor}{l+2i+1} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n+1+(r+s-4)(2i+1)}{2} \right\rfloor}{l-2i-1} t^l.$$

And

$w_t(A_n^{2i}(-s,r)) =$

$$= q^{\overline{y}_{2i}} \sum_{l \geq 2i} q^{(l+2i)(l-2i)} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n}{2} \right\rfloor - x_{2i} + 2(2i)}{l+2i} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n+1}{2} \right\rfloor + x_{2i} - 2(2i)}{l-2i} t^l =$$

$$= q^{\overline{y}_{2i}} \sum_{l \geq 2i} q^{(l+2i)(l-2i)} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n-(r+s-4)(2i)}{2} \right\rfloor}{l+2i} \genfrac{[}{]}{0pt}{}{\left\lfloor \frac{n+1+(r+s-4)(2i)}{2} \right\rfloor}{l-2i} t^l$$

Thus in each case we get

$$w_t(A_n^i(-s,r)=q^{\overline{y}_i}\sum_{l\geq i}q^{(l-i)(l+i)}\begin{bmatrix}\left\lfloor\frac{n-(r+s-4)i}{2}\right\rfloor\\ l+i\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1+(r+s-4)i}{2}\right\rfloor\\ l-i\end{bmatrix}t^l.$$

This is the same as

$$w_t(A_n^i(-s,r)=q^{y_{-i}}\sum_{l\geq|-i|}q^{(l-i)(l+i)}\begin{bmatrix}\left\lfloor\frac{n+(r+s-4)(-i)}{2}\right\rfloor\\ l-(-i)\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1-(r+s-4)(-i)}{2}\right\rfloor\\ l+(-i)\end{bmatrix}t^l.$$

If we set $w_t(A_n^i(r,-s))=f(i,n)$, then we have therefore $w_t(A_n^i(-s,r))=f(-i,n)$.

Let now $A_{n,k}=A_n(-k-2,k+1)$ be the set of all lattice paths in $\mathbb{R}^2$ which start at the origin $(0,0)$, consist of $\left\lfloor\frac{n}{2}\right\rfloor$ northeast steps and $\left\lfloor\frac{n+1}{2}\right\rfloor$ southeast steps, and are contained in the strip $-k-1\leq y\leq k$. By $a_k(n,t,q)=w_t(A_{n,k})$ we denote its weight. Then the above reasoning gives

$$a_k(n,t,q)=\sum_{i\in\mathbb{Z}}(-1)^i q^{\frac{i((2k+3)i-1)}{2}}\sum_{l\geq|i|}q^{(l-i)(l+i)}\begin{bmatrix}\left\lfloor\frac{n+(2k-1)i}{2}\right\rfloor\\ l-i\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n-(2k-1)i+1}{2}\right\rfloor\\ l+i\end{bmatrix}t^l. \quad (1.2)$$

Here $\begin{bmatrix}n\\ k\end{bmatrix}$ denotes the q-binomial coefficient. We always assume that $\begin{bmatrix}n\\ k\end{bmatrix}=0$ if $n<0$.

For $t=1$ by using the q-Vandermonde formula this reduces to

$$a_k(n,1,q)=\sum_{i\in\mathbb{Z}}(-1)^i q^{\frac{i((2k+3)i-1)}{2}}\begin{bmatrix}n\\ \left\lfloor\frac{n+(2k+3)i}{2}\right\rfloor\end{bmatrix}. \quad (1.3)$$

In accordance with the language used by physicists we call (1.2) the „bosonic" representation of the polynomial $a_k(n,t,q)$.
For the form of the corresponding "fermionic" representation I am indebted to Ole Warnaar.

Theorem 1.1 (O. Warnaar)
The "fermionic" representation of $a_k(n,t,q)$ *is given by*

$$a_k(n,t,q)=\sum_{n_1,\cdots,n_k\geq 0}t^{N_1}q^{N_1^2+\cdots+N_k^2}\prod_{j=1}^{k}\begin{bmatrix}n+n_j-2\sum_{i=1}^{j}N_i\\ n_j\end{bmatrix}, \quad (1.4),$$

where $N_j=n_j+n_{j+1}+\cdots+n_k$.

For $t=1$ by the q-Vandermonde formula this theorem simplifies to a polynomial identity of Foda and Quano [5] and Kirillov [6]:

$$\sum_{i\in\mathbb{Z}}(-1)^i q^{\frac{i((2k+3)i-1)}{2}}\begin{bmatrix} n \\ \left\lfloor \frac{n+(2k+3)i}{2}\right\rfloor \end{bmatrix} = \sum_{n_1,\cdots,n_k\geq 0} q^{N_1^2+\cdots+N_k^2}\prod_{j=1}^{k}\begin{bmatrix} n+n_j-2\sum_{i=1}^{j}N_i \\ n_j \end{bmatrix}. \quad (1.5)$$

In order to prove this we give first another lattice path representation of $a_k(n,t,q)$.

Lemma 1.1
*$a_k(n,t,q)$ is the weight of the set of all **non-negative** lattice paths starting at $(0,0)$ and ending in $(n,0)$, where besides a northeast move $(i,j)\to(i+1,j+1)$ and a southeast move $(i,j)\to(i+1,j-1)$ also a **horizontal move** $(i,0)\to(i+1,0)$ is allowed and the maximal height of the peaks is k. Here the **weight** of a vertex is $q^m t$ where m is its $x-$coordinate and the weight of a path is defined as the product of the weight of its peaks.*

It is easy to find a bijection between these two lattice path models. Starting from the first model we map a northeast move from $(i,-1)\to(i+1,0)$ and a southeast move from $(i,0)\to(i+1,-1)$ into a horizontal move $(i,0)\to(i+1,0)$. The nonnegative paths remain the same and the negative paths from $(i,-1)\to(j,-1)$ are reflected on the line $y=-\frac{1}{2}$ into a nonnegative path $(i,0)\to(j,0)$.
This map obviously has a unique inverse.

We now follow the argument used in Lemma 2.1 of Bressoud [2] and apply it to our case: $q^{k^2}=q^{1+3+\cdots+(2k-1)}$ is the weight of the unique path of length $2k$ with k peaks of height 1. The factor $\begin{bmatrix} n-k \\ k \end{bmatrix}$ is the generating function for partitions into at most k parts or, equivalently, into exactly k parts where zeroes are permitted, each of which is $\leq n-2k$ (cf. [1]). If these parts are denoted by $a_1\geq a_2\geq\cdots\geq a_k\geq 0$, we insert a_k horizontal steps in front of the northeast step of the first peak and a_j-a_{j+1} horizontal steps in front of the northeast step of the $(k-j+1)-$th peak.
Let $F(n;n_1,\cdots,n_{k-1})$ be the weight of all paths of length n, where each path has n_j peaks of "relative height" j (for the definition see [2]). If we replace for each peak the set consisting of the northeast path leading to the peak followed by the southeast path which leaves the peak with the set consisting of two northeast paths followed by two southeast paths (called "volcanic uplift" in [2]) then we get the set of all paths of length $n+2N_1$ where each path has $n_{j+1}^*=n_j$ peaks of relative height $j+1$. For the weight $F(n+2N_2^*;n_2^*,\cdots,n_k^*)$, where $N_2^*=n_2^*+\cdots+n_k^*$, we get in this way $F(n+2N_2^*;n_2^*,\cdots,n_k^*)=F(n;n_1,\cdots,n_{k-1})q^{(N_2^*)^2}$. As shown in [2], we can now insert

n_1^* peaks of relative height 1 in all possible ways into such a path such that

$$F(n+2N_2^*+2n_1^*;n_1^*,n_2^*,\cdots,n_k^*)=F(n;n_1,\cdots,n_{k-1})q^{(N_1^*)^2}\begin{bmatrix} n+2N_1^*-n_1^* \\ n_1^* \end{bmatrix}.$$

This implies

$$F(n;n_1,\cdots,n_k)=q^{N_1^2}\begin{bmatrix} n+n_1-2N_1 \\ n_1 \end{bmatrix}F(n-2N_1;n_2,\cdots,n_k). \tag{1.6}$$

Starting with $F(n;k)=q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}$ we get by induction

$$q^{N_1^2+\cdots+N_k^2}\prod_{j=1}^{k}\begin{bmatrix} n+n_j-2\sum_{i=1}^{j}N_i \\ n_j \end{bmatrix}.$$

For $k=0$ we have $a_0(n,t,q)=1$, a well-known result of I. Schur [8].

For $k=1$ we have shown in [4] that $a_1(n,t,q)=F_{n+1}(qt)=\sum_{k<n}q^{k^2}\begin{bmatrix} n-k \\ k \end{bmatrix}t^k$.

For $t=1$ this is Schur's finite version of the Rogers-Ramanujan identities [8].
These polynomials are a q-analog of the Fibonacci polynomials and satisfy the recurrences

$a_1(n,t,q)=a_1(n-1,qt,q)+qta_1(n-2,q^2t,q)$

and

$a_1(n,t,q)=a_1(n-1,t,q)+q^{n-1}ta_1(n-2,t,q)$.

For $k>1$ we show that $a_k(n,t,q)$ satisfies a recurrence of order $2k$ and determine this recurrence explicitly.
Since for $n<2k$ no path in $A_{n,k}$ reaches the boundary we see that

$$a_k(n,t,q)=\sum_{l\geq 0}q^{l^2}\begin{bmatrix} \left\lfloor\frac{n}{2}\right\rfloor \\ l \end{bmatrix}\begin{bmatrix} \left\lfloor\frac{n+1}{2}\right\rfloor \\ l \end{bmatrix}t^l \tag{1.7}$$

for $n<2k$.

Remark 1.1
Theorem 1.1 can slightly be generalized to give an extension of the whole result of Foda and Quano [5] to arbitrary t:
Let $A_{n,k,r}$ be the set of all lattice paths in $\mathbb{R}^2$ with northeast steps and southeast steps which start at the point $(0,k+1-r)$, are contained in the strip $-k-1\leq y\leq k$ and end in either $(n,0)$ or $(n,-1)$. By $a_{k,r}(n,t,q)=w_t(A_{n,k,r})$ we denote its weight.

Then for $1 \le r \le k+1$ we have

$$a_{k,r}(n,t,q) = \sum_{i \in \mathbb{Z}} (-1)^i q^{\frac{i((2k+3)i-(2k+3-2r))}{2}} \sum_{l \ge |i|} q^{(l-i)(l+i)} \begin{bmatrix} \left\lfloor \frac{n+(2k-1)i+r-k-1}{2} \right\rfloor \\ l-i \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n-(2k-1)i+k+2-r}{2} \right\rfloor \\ l+i \end{bmatrix} t^l =$$

$$= \sum_{n_1,\cdots,n_k \ge 0} t^{N_1} q^{N_1^2+\cdots+N_k^2+N_r+\cdots+N_k} \prod_{j=1}^{r-1} \begin{bmatrix} n+n_j-2\sum_{i=1}^{j} N_i \\ n_j \end{bmatrix} \prod_{j=r}^{k} \begin{bmatrix} n+r-j-1+n_j-2\sum_{i=1}^{j} N_i \\ n_j \end{bmatrix}.$$

In order to prove this we shift each such lattice path one unit downward, such that the new path starts in $(0,k-r)$. In order that the new path has the same weight as the old path we have to exchange in the new path the peaks of height 0 and the valleys of height -2. For under this map all extremal points are again mapped onto extremal points, except the peaks of height 1 which are mapped onto peaks of height 0. If we replace the latter peaks with the corresponding valleys of height -2, we get again an extremal point. On the other hand the image of a valley of height -1, which is not an extremal point, is mapped onto a valley of height -2, which is an extremal point. So we replace it with the corresponding peak of height 0, such that the weights are preserved. In this way we obtain the set of all lattice paths starting at $(0,k-r)$ which remain in the strip $-k-2 \le y \le k-1$. We iterate this operation till we come to the set of paths starting at $(0,0)$ which remain in the strip $-2k+r-2 \le y \le r-1$. Then the computation given above leads to the bosonic form of $a_{k,r}(n,t,q)$.

The fermionic form follows again by induction starting with

$a_{1,1}(n,t,q) = \sum_{k=0}^{\left\lfloor \frac{n-1}{2} \right\rfloor} q^{k^2+k} \begin{bmatrix} n-k-1 \\ k \end{bmatrix} t^k$ which is equivalent with the extension of the second Rogers-Ramanujan identity in [4].

2. Some useful polynomials.

Now we define some polynomials which we will need later.
Let

$$r_n(x,t,q) = r_{n-1}(x,t,q) + x(1-q^{n-1}t) r_{n-1}(x,q^2t,q) - x r_{n-2}(x,q^2t,q) \qquad (2.1)$$

with initial values $r_0(x,t,q)=0, r_1(x,t,q)=1.$
It can be shown that

$$r_n(x,t,q) = 1 + x r_{n-1}(x,q^2t,q) - tx \sum_{j=0}^{n-1} q^j r_j(x,q^2t,q),$$

but we shall not need this.
We have $r_n(x,t,q) = \sum_{k=0}^{n-1} x^k \sum_{j=0}^{k} d(n,k,j) t^j$ for some coefficients $d(n,k,j)$.

Comparing coefficients we get the recursion
$d(n,k,j)-d(n-1,k,j)-q^{2j}d(n-1,k-1,j)+q^{n+2j-3}d(n-1,k-1,j-1)+q^{2j}d(n-2,k-1,j)=0.$

Now it is easy to verify that $d(n,k,j)=(-1)^j q^{kj+\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}\begin{bmatrix}n-1-k+j\\j\end{bmatrix}$ satisfies this recursion.

To prove this observe that if we set $d_0(n,k,j)=(-1)^j q^{kj+\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}\begin{bmatrix}n-1-k+j\\j\end{bmatrix}$, then

$$d_0(n,k,j)-d_0(n-1,k,j)=(-1)^j q^{kj+\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}\left(\begin{bmatrix}n-1-k+j\\j\end{bmatrix}-\begin{bmatrix}n-2-k+j\\j\end{bmatrix}\right)=$$

$$=(-1)^j q^{kj+\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}q^{n-1-k}\begin{bmatrix}n-2-k+j\\j-1\end{bmatrix}.$$

Therefore

$$d_0(n-1,k-1,j)-d_0(n-2,k-1,j)=(-1)^j q^{(k-1)j+\binom{j}{2}}\begin{bmatrix}k-1\\j\end{bmatrix}q^{n-1-k}\begin{bmatrix}n-2-k+j\\j-1\end{bmatrix}.$$

This gives
$$d_0(n,k,j)-d_0(n-1,k,j)-q^{2j}(d_0(n-1,k-1,j)-d_0(n-2,k-1,j))=$$

$$=(-1)^j q^{kj+\binom{j}{2}+n-1-k}\begin{bmatrix}k-1\\j-1\end{bmatrix}\begin{bmatrix}n-2-k+j\\j-1\end{bmatrix}.$$

And this is the same as $q^{n+2j-3}d_0(n-1,k-1,j-1)$.
Therefore $d_0(n,k,j)$ satisfies the recurrence. Since for $n=0,1$ we obviously have $d_0(n,k,j)=d(n,k,j)$, the two sequences coincide.

Therefore we get the explicit formula

$$r_n(x,t,q)=\sum_{k=0}^{n-1}r(n,k,t)x^k \tag{2.2}$$

with

$$r(n,k,t)=\sum_{j=0}^{k}(-1)^j t^j q^{kj+\binom{j}{2}}\begin{bmatrix}k\\j\end{bmatrix}\begin{bmatrix}n-1-k+j\\j\end{bmatrix}. \tag{2.3}$$

This can also be written in the form

$$r(n,k,t)=\sum_{j=max(0,2k-n+1)}^{k}(-1)^{k-j}q^{\binom{k-j+1}{2}}\begin{bmatrix}k\\j\end{bmatrix}\begin{bmatrix}n-1-k+j\\k\end{bmatrix}\prod_{l=1}^{j}(1-q^{2k-l}t).$$

The first terms of the sequence $r_n(x,t,q)$ are

$r_0=0,$

$r_1=1,$

$r_2=1+(1-qt)x,$

$r_3=1+(1-q\begin{bmatrix}2\\1\end{bmatrix}t)x+(1-q^2\begin{bmatrix}2\\1\end{bmatrix}t+q^5t^2)x^2=1+(-q+\begin{bmatrix}2\\1\end{bmatrix}(1-qt))x+(1-q^2t)(1-q^3t)x^2,$

$r_4=1+(1-q\begin{bmatrix}3\\1\end{bmatrix}t)x+(1-q^2\begin{bmatrix}2\\1\end{bmatrix}\begin{bmatrix}2\\1\end{bmatrix}t+q^5\begin{bmatrix}3\\2\end{bmatrix}t^2)x^2+(1-q^3\begin{bmatrix}3\\1\end{bmatrix}t+q^7\begin{bmatrix}3\\2\end{bmatrix}t^2-q^{12}t^3)x^3=$

$=1+(-q\begin{bmatrix}2\\1\end{bmatrix}+\begin{bmatrix}3\\1\end{bmatrix}(1-qt))x+(-q\begin{bmatrix}2\\1\end{bmatrix}(1-q^3t)+\begin{bmatrix}3\\2\end{bmatrix}(1-q^2t)(1-q^3t))x^2+$

$+(1-q^3t)(1-q^4t)(1-q^5t)x^3.$

For $q=1$ these polynomials are intimately connected with the Fibonacci polynomials: Let $F_n(x,s)$ be the Fibonacci polynomial, defined by $F_n(x,s)=xF_{n-1}(x,s)+sF_{n-2}(x,s)$ with initial values $F_0(x,s)=0, F_1(x,s)=1$. Then we see from the recursion

$r_n(x,t,1)=(1+(1-t)x)r_{n-1}(x,t,1)-xr_{n-2}(x,t,1)$

that

$r_n(x,t,1)=F_n(1+(1-t)x,-x)$.

Consider now

$$p_n(x,t,q)=r_{n+1}(x,t,q)-xr_n(x,q^2t,q). \tag{2.4}$$

Then we can recover $r_n(x,t,q)$ by

$$r_n(x,t,q)=\sum_{j=0}^{n-1}x^jp(n-1-j,q^{2j}t,x). \tag{2.5}$$

This is easily seen by induction. For this holds for $n=0$ and $n=1$. From $r_{n+1}(x,t,q)=p_n(x,t,q)+xr_n(x,q^2t,q)$ we see that if it holds for n then it also holds for $n+1$.

Now $r_n(x,t,q)=r_{n-1}(x,t,q)+x(1-q^{n-1}t)r_{n-1}(x,q^2t,q)-xr_{n-2}(x,q^2t,q)$ can be reformulated as

$$p_n(x,t,q)-p_{n-1}(x,t,q)=-q^ntxr_n(x,q^2t,q)=-q^ntx\sum_{j=0}^{n-1}x^jp(n-1-j,q^{2j+2}t,x).$$

Therefore we see that $p_k(x,t,q)$ is also characterized by the recursion

$$p_k(x,t,q)=p_{k-1}(x,t,q)-q^kt\sum_{j=1}^{k}x^jp_{k-j}(x,q^{2j}t,q) \tag{2.6}$$

with $p_0(x,t,q)=1$.

From (2.2) we get the explicit formula

$$p_k(x,t,q) = 1 + \sum_{i=1}^{k} x^i \sum_{j=0}^{i} (-1)^j t^j q^{ij+\binom{j}{2}} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \begin{bmatrix} k-i+j \\ j \end{bmatrix}. \qquad (2.7)$$

We also need the polynomials

$$h_k(x,t,q) = p_k(x^2,t,q) - xp_{k-1}(x^2,qt,q), k \geq 1, \qquad (2.8)$$

and $h_0(x,t,q) = 1$.

Thus $h_k(x,t,q)$ can be written in the form

$$h_k(x,t,q) = 1 - x + \sum_{i=1}^{2k} c_k(i,t) x^i, \qquad (2.9)$$

where

$$c_k(2i,t) = \sum_{j=0}^{i} (-1)^j t^j q^{ij+\binom{j}{2}} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \begin{bmatrix} k-i+j \\ j \end{bmatrix} \qquad (2.10)$$

and

$$c_k(2i+1,t) = \sum_{j=0}^{i} (-1)^{j+1} t^j q^{ij+j+\binom{j}{2}} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \begin{bmatrix} k-i+j-1 \\ j \end{bmatrix}. \qquad (2.11)$$

Then these polynomials are uniquely determined by the recursion

$$h_k(x,t,q) = h_{k-1}(x,t,q) - q^k t \sum_{j=1}^{k} x^{2j} h_{k-j}(x, q^{2j}t, q) \qquad (2.12)$$

with initial values $h_0(x,t,q) = 1, h_1(x,t,q) = 1 - x - qtx^2$.

In terms of $r_k(x,t,q)$ for $k \geq 1$ they are given by

$$h_k(x,t,q) = r_{k+1}(x^2,t,q) - xr_k(x^2,qt,q) - x^2 r_k(x^2,q^2t,q) + x^3 r_{k-1}(x^2,q^3t,q). \qquad (2.13)$$

For $q = 1$ we get

$$h_k(x,t,1) = F_{k+1}(x^2(1-t)+1,-x^2) - x(x+1)F_k(x^2(1-t)+1,-x^2) + x^3 F_{k-1}(x^2(1-t)+1,-x^2).$$

For $t = 1$ this reduces to

$$F_{k+2}(1,-x^2) - xF_{k+1}(1,-x^2) = x^{k+1}\left(F_{k+2}\left(\frac{1}{x},-1\right) - F_{k+1}\left(\frac{1}{x},-1\right)\right).$$

Let now

$$j_k(x,t,q) = p_k(x,t,q) - xp_{k-2}(x,q^2t,q) \qquad (2.14)$$

for $k \geq 2$.

This can also be written in the form

$$j_k(x,t,q) = r_{k+1}(x,t,q) - xr_k(x,q^2t,q) - xr_{k-1}(x,q^2t,q) + x^2 r_{k-2}(x,q^4t,q). \qquad (2.15)$$

Therefore for $k \geq 2$ we get

$$j_k(x,t,q)=\sum_{i=0}^{k}s(k,i,t)x^i \quad (2.16)$$

with

$$s(k,i,t)=r(k+1,i,t)-r(k,i-1,q^2t)-r(k-1,i-1,q^2t)+r(k-2,i-2,q^4t). \quad (2.17)$$

If we introduce the polynomials
$l_k(x,t,q)=r_{k+1}(x,t,q)-xr_{k-1}(x,q^2t,q)$,
then we get
$j_k(x,t,q)=l_k(x,t,q)-xl_{k-1}(x,q^2t,q)$.
These polynomials are analogs of the Lucas polynomials: Define now the Lucas polynomials $L_n(x,s)$ by the recurrence $L_n(x,s)=xL_{n-1}(x,s)+sL_{n-2}(x,s)$ with the initial values $L_0(x,s)=2, L_1(x,s)=x$.
For $q=1$ and $k\geq 2$ $j_k(x,t,1)$ reduces to
$j_k(x,t,1)=L_k(x(1-t)+1,-x)-xL_{k-1}(x(1-t)+1,-x)$
For $t=1$ we get $L_k(1,-x)-xL_{k-1}(1,-x)$.

In the same way as above we see that the sequence $j_k(x,t,q)$ satisfies also the recurrences

$$j_k(x,t,q)-j_{k-1}(x,t,q)-\frac{x}{q}(1-q^{k+1}t)j_{k-1}(x,q^2t,q)+\frac{x}{q}j_{k-2}(x,q^2t,q)$$

and

$$j_k(x,t,q)=j_{k-1}(x,t,q)-q^kt\sum_{l=1}^{k-1}x^l j_{k-l}(x,q^{2l}t,q) \quad (2.18)$$

for $k>2$
with the initial conditions

$$j_1(x,t,q)=1+x-qtx, j_2(x,t,q)=1-(1+qt+q^2t)x-q^2t(1-q^3t)x^2. \quad (2.19)$$

3. The main result.

Theorem 3.1
For $k\geq 1$ *the sequence* $(a_k(n,t,q))_{n\geq 0}$ *satisfies the recurrence*

$$h_k(A,t,q)a_k(n,t,q)=0, \quad (3.1)$$

where A *denotes the operator* $A^j w_n(t)=w_{n-j}(q^jt)$.
This means

$$a_k(n,t,q)-a_k(n-1,qt,q)+\sum_{i=1}^{2k}c_k(i,t)a_k(n-i,q^it,q)=0, \quad (3.2)$$

where $c_k(i,t)$ *is defined by* (2.10) *and* (2.11).

It also satisfies a second recursion

$$h_k(E^{-1}, q^n t, \frac{1}{q}) a_k(n,t,q) = 0, \tag{3.3}$$

where $E^{-j} w_n(t) = w_{n-j}(t)$.
This means

$$a_k(n,t,q) - a_k(n-1, qt, q) + \sum_{i=1}^{2k} d_k(i, q^n t) a_k(n-i, t, q) = 0, \tag{3.4}$$

where

$$d_k(2i,t) = \sum_{j=0}^{i} (-1)^j t^j q^{\binom{j}{2} + i - j(i+k)} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \begin{bmatrix} k-i+j \\ j \end{bmatrix} \tag{3.5}$$

and

$$d_k(2i+1,t) = \sum_{j=0}^{i} (-1)^{j+1} t^j q^{\binom{j}{2} + i - j(i+k)} \begin{bmatrix} i-1 \\ j-1 \end{bmatrix} \begin{bmatrix} k-i+j-1 \\ j \end{bmatrix}. \tag{3.6}$$

For $k = 0$ *we have* $a_0(n,t,q) = 1$.

The initial values are $a_k(n,t,q) = \sum_{l \geq 0} q^{l^2} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ l \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor \\ l \end{bmatrix} t^l$ *for* $0 \leq n \leq 2k-1$.

Corollary 3.1
Let

$$a_k(t,q) = \lim_{n \to \infty} a_k(n,t,q) = \sum_{i \in \mathbb{Z}} (-1)^i q^{\frac{i((2k+3)i-1)}{2}} \sum_{l \geq |i|} \frac{q^{(l-i)(l+i)}}{(q)_{l-i}(q)_{l+i}} t^l = \sum_{n_1, \cdots, n_k \geq 0} t^{N_1} \frac{q^{N_1^2 + \cdots + N_k^2}}{(q)_{n_1} \cdots (q)_{n_k}},$$

(3.7)

where we set $(q)_n = (1-q)(1-q^2)\cdots(1-q^n)$.
Then this formal power series satisfies the functional equation

$$a_k(t,q) - a_k(qt,q) + \sum_{i=1}^{2k} c_k(i,t) a_k(q^i t, q) = 0. \tag{3.8}$$

A typical example.

As an example choose $k=2$.
Then

$$a_2(n,t,q)=\sum_{i\in\mathbb{Z}}(-1)^i q^{\frac{7i^2-i}{2}}\sum_{l\geq|i|}q^{l^2-i^2}\begin{bmatrix}\left\lfloor\frac{n+3i}{2}\right\rfloor\\ l-i\end{bmatrix}\begin{bmatrix}\left\lfloor\frac{n+1-3i}{2}\right\rfloor\\ l+i\end{bmatrix}t^l=$$

$$=\sum_{k,l\geq 0}t^{k+l}q^{(k+l)^2+l^2}\begin{bmatrix}n-k-2l\\ k\end{bmatrix}\begin{bmatrix}n-2k-3l\\ l\end{bmatrix}.$$

This sequence begins with

$\{1, 1, 1+qt, 1+qt+q^2t, 1+qt+2q^2t+q^3t+q^4t^2,$
$1+qt+2q^2t+2q^3t+q^4t+q^4t^2+q^5t^2+q^6t^2, 1+qt+2q^2t+$
$2q^3t+2q^4t+q^5t+q^4t^2+2q^5t^2+3q^6t^2+2q^7t^2+q^8t^2+q^9t^3\}$

For $t=1$ this reduces to

$$a_2(n,1,q)=\sum_{i\in\mathbb{Z}}(-1)^i q^{\frac{7i^2-i}{2}}\begin{bmatrix}n\\ \left\lfloor\frac{n+7i}{2}\right\rfloor\end{bmatrix}.$$

The sequence $a_2(n,t,q)$ satisfies the recurrences

$$a_2(n,t,q)-a_2(n-1,qt,q)-q(1+q)ta_2(n-2,q^2t,q)+q^2ta_2(n-3,q^3t,q)-$$
$$-q^2t(1-q^3t)a_2(n-4,q^4t,q)=0$$

and

$$a_2(n,t,q)-a_2(n-1,t,q)-q^{n-2}(1+q)ta_2(n-2,t,q)+q^{n-2}ta_2(n-3,t,q)-$$
$$-q^{n-2}t(1-q^{n-3}t)a_2(n-4,t,q)=0$$

From the second recurrence we get for $t=1$ the recursion

$$a_2(n,1,q)-a_2(n-1,1,q)-q^{n-2}(1+q)a_2(n-2,1,q)+q^{n-2}a_2(n-3,1,q)-$$
$$-q^{n-2}(1-q^{n-3})a_2(n-4,1,q)=0.$$

If we let $n\to\infty$ we get

$$a_2(t,q)=\sum_{i\in\mathbb{Z}}(-1)^i q^{\frac{7i^2-i}{2}}\sum_{l\geq|i|}\frac{q^{l^2-i^2}}{(q)_{l-i}(q)_{l+i}}t^l=\sum_{k,l\geq 0}\frac{q^{(k+l)^2+l^2}}{(q)_k(q)_l}t^{k+l}.$$

For this formal power series we get the functional equation

$$a_2(t,q)-a_2(qt,q)-q(1+q)ta_2(q^2t,q)+q^2ta_2(q^3t,q)-q^2t(1-q^3t)a_2(q^4t,q)=0.$$

If we write

$a_2(t,q)=\sum_{l\geq 0}d(l)q^{l^2}t^l$, then we have

$$d(l)=\sum_{i=-l}^{l}(-1)^i\frac{q^{\frac{5i^2-i}{2}}}{(q)_{l-i}(q)_{l+i}}.$$

Comparing coefficients we see that this sequence satisfies the recursion

$$(1-q^l)d(l)-(1+q+q^{2l-1}-q^l)d(l-1)+qd(l-2)=0.$$

Proof of Theorem 3.1.

Consider lattice paths with $\left\lfloor\frac{n}{2}\right\rfloor$ northeast steps and $\left\lceil\frac{n+1}{2}\right\rceil$ southeast steps. Let $w_n(t)$ be the weight of the set of paths of length n which start at $(0,0)$ or $(0,-1)$. From the properties of the weight function it is clear that these weights are equal. Let $w^+_{n,k}(t)$ be the weight of those paths which begin with at least k northeast steps, $w^-_{n,k}(t)$ be the weight of those paths which begin with at least k southeast steps. Let $w_{n,k}=w^+_{n,k}+w^-_{n,k}$ be the weight of the set of paths which begin with at least k northeast steps or at least k southeast steps.

Consider now the set of lattice paths satisfying $-k-1\le y\le k$. Here we have $w^-_{n,1}(t)=w_{n-1}(qt)$, because of the symmetry of the weight.

Therefore $w_n(t)=w^+_{n,1}(t)+w^-_{n,1}(t)=w^+_{n,1}(t)+w_{n-1}(qt)$,

which gives

$$w^+_{n,1}(t)=w_n(t)-w_{n-1}(qt).$$

We observe that each path which begins with 1^k, i.e. with at least k northeast steps, begins with one of the following steps: $1^{k+1},1^k01,1^k001,\cdots,1^k0^{k-1}1,1^k0^k$.

Here 1 denotes a northeast step and 0 denotes a southeast step.

For $k=1$ this reduces to $11,10$. Therefore we get

$$w^+_{n,1}(t)=w^+_{n,2}(t)+qtw_{n-2}(q^2t)$$ or

$$w^+_{n,2}(t)=w_n(t)-w_{n-1}(qt)-qtw_{n-2}(q^2t).$$

The weight of a path beginning with 1^k0^l1 is the same as q^kt-times the weight of the paths which start at $(0,2l)$ and begin with 1^{k-l+1} for $l<k$, i.e. $q^ktw^+_{n-2l,k-l+1}(q^{2l}t)$.

For $l=k$ we get $q^ktw_{n-2k}(q^{2k}t)$.

Therefore we have

$$w^+_{n,k}(t)=w^+_{n,k+1}(t)+q^kt\sum_{j=0}^{k-2}w^+_{n-2j-2,k-j}(q^{2j+2}t)+q^ktw_{n-2k}(q^{2k}t).$$

This gives

$$w^+_{n,2}(t)=(1-A-qtA)w_n(t)=h_1(A,t,q)w_n(t),$$

$$w^+_{n,3}(t)=w^+_{n,2}(t)-q^2tw^+_{n-2,2}(q^2t)-q^4tw_{n-4}(q^4t)=$$

$$=(h_1(A,t,q)-q^2th_1(A,q^2t,q)A^2-q^4tA^4)w_n(t)=h_2(A,t,q)w_n(t).$$

With induction we see that in general

$w^+_{n,k+1}(t) = h_k(A,t,q)w_n(t)$.

Since for a path with $-k-1 \le y \le k$ we have $w^+_{n,k+1}(t) = 0$ we get the recursion $h_k(A,t,q)a_k(n,t,q) = w^+_{n,k+1}(t) = 0$..

The second recurrence (3.3) can be reduced to the first one by the following observation: Each path ends in $(n,0)$ or $(n,-1)$. From the symmetry of the weight it suffices to consider the first case. Let now (for this proof only) $w^+_{n,k}(t)$ be the weight of those paths which end with at least k southeast steps, $w^-_{n,k}(t)$ be the weight of those paths which end with at least k northeast steps . Then with the same reasoning as above we get

$$w^+_{n,k}(t) = w^+_{n,k+1}(t) + q^{n-k}t\sum_{j=0}^{k-2} w^+_{n-2j-2,k-j}(t) + q^{n-k}tw_{n-2k}(t).$$

This gives the second formula.

4. A related theorem

Let $B_{n,k}$ be the set of all lattice paths in $\mathbb{R}^2$ which start at the origin $(0,0)$, consist of $\left\lfloor \frac{n}{2} \right\rfloor$ northeast steps and $\left\lfloor \frac{n+1}{2} \right\rfloor$ southeast steps, and are contained in the strip $-k \le y \le k$. By $b_k(n,t,q) = w_t(B_{n,k})$ we denote its weight.

Then for $k \ge 1$ it follows in the same way as above that

$$b_k(n,t,q) = \sum_{i\in\mathbb{Z}} (-1)^i q^{i^2(k+1)} \sum_{l\ge|i|} q^{(l-i)(l+i)} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor + i(k-1) \\ l-i \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor - i(k-1)) \\ l+i \end{bmatrix} t^l. \qquad (4.1)$$

For $t = 1$ this reduces to

$$b_k(n,1,q) = \sum_{i\in\mathbb{Z}} (-1)^i q^{i^2(k+1)} \begin{bmatrix} n \\ \left\lfloor \frac{n}{2} \right\rfloor + i(k+1) \end{bmatrix}. \qquad (4.2)$$

Theorem 4.1

For $k \ge 2$ we have the recursion

$$j_k(A^2,t,q)b_k(n,t,q) = 0. \qquad (4.3)$$

For $k = 1$ we get $b_1(n,t,q) = \prod_{0\le i<\left\lfloor \frac{n}{2} \right\rfloor} (1+q^{2i+1}t)$ and therefore the recursion $b_1(n,t,q) - (1+qt)b_1(n-2,q^2t,q) = 0$, which corresponds to the polynomial

$1-(1+qt)x^2$. *For* $n<2k$ *the initial values are given by*

$$b_k(n,t,q)=\sum_{l\geq 0} q^{l^2} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ l \end{bmatrix} \begin{bmatrix} \left\lfloor \frac{n+1}{2} \right\rfloor \\ l \end{bmatrix} t^l .$$

For even numbers we also have a second recurrence: The sequence $b_k(2n,t,q)$ *satisfies*

$j_k(E^{-2},q^{2n}t,\frac{1}{q})b_k(2n,t,q)=0$ *for* $k\geq 2$.

For $k=1$ *we have* $b_1(2n,t,q)-(1+q^{2n-1}t)b_1(2n-2,t,q)=0$.

Proof

In this case we have

$w_{n,1}(t)=w_n(t)$,

$w^+_{n,1}(t)=w^+_{n,2}(t)+qtw_{n-2}(q^2t)$

$w^-_{n,1}(t)=w^-_{n,2}(t)+w_{n-2}(q^2t)$

and therefore we get

$w_{n,2}(t)=w_n(t)-w_{n-2}(q^2t)-qtw_{n-2}(q^2t)$

and

$w^+_{n,2}(t)=w^+_{n,3}(t)+q^2tw^+_{n-2,1}(q^2t)-q^2tq^3tw_{n-4}(q^4t)+q^2tw_{n-4}(q^4t)$

$w^-_{n,2}(t)=w^-_{n,3}(t)+q^2tw^-_{n-2,1}(q^2t)$

This gives

$w_{n,3}(t)=w_n(t)-w_{n-2}(q^2t)-qtw_{n-2}(q^2t)-q^2tw_{n-2}(q^2t)+q^5t^2w_{n-4}(q^4t)-q^2tw_{n-4}(q^4t)$

For $k>3$

$w_{n,k}(t)-w_{n,k+1}(t)-q^kt\sum_{j=0}^{k-3}w_{n-2j-2,k-j}(q^{2j+2}t)$ is the weight of all paths which start with k 1's or k 0's and pass through the points $(2k-1,1)$ or $(2k-1,-1)$.

This can be written in the form

$q^kt(w_{n-2k+2}(q^{2k-2}t)-q^{2k-1}tw_{n-2k}(q^{2k}t))+q^ktw_{n-2k}(q^{2k}t)=$

$=q^ktA^{2k-2}(1-qq^{2k-2}tA^2+A^2)w_n(t)=q^ktA^{2k-2}j_1(A^2,q^{2k-2}t,q)w_n(t).$

From this the theorem follows immediately.

For the second recursion observe that for even numbers n the path ends at $(n,0)$ and therefore the same reasoning as above is possible.

For odd numbers n we could not find a general formula for the minimal recurrences.

We have only computed the first recurrences, which are given by the following formulas:

$b_1(2n+1,t,q)-(1+q^{2n-1}t)b_1(2n-1,t,q)=0,$

$b_2(2n+1,t,q)-(1+q^{2n-1}(1+q)t)b_2(2n-1,t,q)-q^{2n-2}t(1-q^{2n-1}t)b_2(2n-3,t,q)=0,$

$b_3(2n+1,t,q)-(1+q^{2n-3}(1+q^2+q^3)t)b_3(2n-1,t,q)+$

$+q^{2n-7}t(q^4-q^5-q^6+q^{2n+1}(1+q+q^3)t)b_3(2n-3,t,q)-$

$-q^{2n-10}t(q^7-(1+q)q^{2n+3}t+q^{4n}t^2)b_3(2n-5,t,q)=0.$

Let $b_k(t,q)=\lim_{n\to\infty} b_k(n,t,q)=\sum_{i\in\mathbb{Z}}(-1)^i q^{ki^2}\sum_{l\geq|i|}q^{l^2}\frac{t^l}{(q)_{l-i}(q)_{l+i}}=\sum_{l\geq 0}d(l,k)q^{l^2}t^l.$

This gives

$$d(l,k)=\sum_{i=-l}^{l}(-1)^i\frac{q^{ki^2}}{(q)_{l-i}(q)_{l+i}}$$

and

$$\sum_{l\geq 0}d(l,k)=\frac{1}{(q)_\infty}\sum_{i\in\mathbb{Z}}(-1)^i q^{(k+1)i^2}.$$

For $k=1$ we get again well-known results:

It is easy to see that $d(l,1)=\frac{q^{l^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2l})}.$

Therefore

$$\sum_{l\geq 0}\frac{q^{l^2}}{(1-q^2)(1-q^4)\cdots(1-q^{2l})}=(1+q)(1+q^3)(1+q^5)\cdots=\frac{1}{(q)_\infty}\sum_{i\in\mathbb{Z}}(-1)^i q^{2i^2}.$$

In this case we have the recursion

$(1-q^{2l})d(l,1)-q^{2l-1}d(l-1,1)=0.$

From the recurrence $b_1(n,t,q)-(1+qt)b_1(n-2,q^2t,q)=0$ we get

$$b_1(2n,t,q)=(1+qt)(1+q^3t)\cdots(1+q^{2n-1}t)=\sum\begin{bmatrix}n\\k\end{bmatrix}_{q^2}q^{k^2}t^k.$$

Therefore we get

$$\sum_{|i|\leq k}(-1)^i q^{i^2}\begin{bmatrix}n\\k+i\end{bmatrix}\begin{bmatrix}n\\k-i\end{bmatrix}=\begin{bmatrix}n\\k\end{bmatrix}_{q^2}.$$

For $n\to\infty$ this becomes

$$\sum_{|i|\leq k}(-1)^i\frac{q^{i^2}}{(q)_{k+i}(q)_{k-i}}=\frac{1}{(1-q^2)(1-q^4)(1-q^6)\cdots(1-q^{2k})}.$$

For $k\to\infty$ this gives

$$\frac{1}{(q)_\infty}\sum_{i\in\mathbb{Z}}(-1)^i q^{i^2}=\frac{1}{(1+q)(1+q^2)(1+q^3)\cdots}.$$

We conclude this paper by determining the fermionic form of $b_k(n,t,q)$.
Here we can apply the same argument as above. We have only to account for the fact that for the relative height k there do not appear all possibilities but only those induced by the weight $F(n;k) = \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ k \end{bmatrix}_{q^2} q^{k^2}$. This follows from

$b_1(n,t,q) = \sum_{k=0}^{n} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor \\ k \end{bmatrix}_{q^2} q^{k^2} t^k$. Therefore from (1.6) we get by induction

$$F(n;n_1,\cdots,n_k) = q^{N_1^2+\cdots+N_k^2} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor - \sum_{i=1}^{k-1} N_i \\ n_k \end{bmatrix}_{q^2} \prod_{j=1}^{k-1} \begin{bmatrix} n + n_j - 2\sum_{i=1}^{j} N_i \\ n_j \end{bmatrix}.$$

This leads to

Theorem 4.2
The fermionic representation of $b_k(n,t,q)$ is given by

$$b_k(n,t,q) = \sum_{n_1,\cdots,n_k \ge 0} t^{N_1} q^{N_1^2+\cdots+N_k^2} \begin{bmatrix} \left\lfloor \frac{n}{2} \right\rfloor - \sum_{i=1}^{k-1} N_i \\ n_k \end{bmatrix}_{q^2} \prod_{j=1}^{k-1} \begin{bmatrix} n + n_j - 2\sum_{i=1}^{j} N_i \\ n_j \end{bmatrix}. \qquad (4.4)$$

Remark 4.2

Theorem 4.2 can also be generalized to give an extension of theorem 1.2 in Foda and Quano [5] to arbitrary t:
Let $B_{n,k,r}$ be the set of all lattice paths in $\mathbb{R}^2$ with northeast steps and southeast steps which start at the point $(0,k+1-r)$, are contained in the strip $-k \le y \le k$ and end in either $(n,0)$ or $(n,-1)$. By $b_{k,r}(n,t,q) = w_t(B_{n,k,r})$ we denote its weight.
Then for $1 \le r \le k+1$ we have

$$b_{k,r}(n,t,q) = \sum_{n_1,\cdots,n_k \ge 0} t^{N_1} q^{N_1^2+\cdots+N_k^2+N_r+\cdots+N_k} \begin{bmatrix} \left\lfloor \frac{n-k-1+r}{2} \right\rfloor - \sum_{i=1}^{k-1} N_i \\ n_k \end{bmatrix}_{q^2} \prod_{j=1}^{k-1} \begin{bmatrix} n + n_j - 2\sum_{i=1}^{j} N_i - max(0,j+1-r) \\ n_j \end{bmatrix}.$$

(4.5)

For this holds for $k=1$ since $b_{1,2}(n,t,q)=\sum_{k=0}^{\left\lfloor \frac{n}{2}\right\rfloor}\begin{bmatrix}\left\lfloor \frac{n}{2}\right\rfloor-1-1+2\\ k\end{bmatrix}_{q^2} q^{k^2}t^k$ and

$$b_{1,1}(n,t,q)=b_{1,2}(n-1,qt,q)=\sum_{k=0}^{\left\lfloor \frac{n-1}{2}\right\rfloor} q^{k^2+k}\begin{bmatrix}\left\lfloor \frac{n-1-1+1}{2}\right\rfloor\\ k\end{bmatrix}_{q^2} t^k .$$

If we know (4.5) already for k, then for $r>1$ it follows immediately from (1.6) . For $r=1$ we obviously have $b_{k,1}(n,t,q)=b_{k,2}(n-1,qt,q)$ from which the corresponding formula follows.

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Prof. Dr. Johann Cigler
Fakultät für Mathematik
Universität Wien
A-1090 Wien
Nordbergstraße 15
Johann.Cigler@univie.ac.at