Some remarks and conjectures about Hankel determinants of polynomials which are related to Motzkin paths

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Abstract

This note collects some results and conjectures for the generating functions of the Hankel determinants of certain polynomials which are related to Motzkin paths.

1. For non-negative integers n,k let $\mathbf{M}_{n,k}$ be the set of all Motzkin paths from $(0,0)$ to $(n,k),$ i.e. all lattice paths from $(0,0)$ to (n,k) consisting of up-steps $U=(1,1)$, down-steps $D=(1,-1)$ and horizontal steps $H=(1,0),$ which never run below the $x-$axis. For each path P we define the weight $w(P)$ as the product of the weights of its steps, where the horizontal steps H have weight t and the up-steps U and down-steps D have weight 1.

Let $M_{n,k}(t)=\sum\limits_{P\in\mathbf{M}_{n,k}} w(P)$ be the weight of all Motzkin paths from $(0,0)$ to $(n,k).$

These weights satisfy

$$M_{n,k}(t)=M_{n-1,k-1}(t)+tM_{n-1,k}(t)+M_{n-1,k+1}(t) \tag{1.1}$$

with $M_{n,k}(t)=0$ for $k<0$ and $M_{0,k}(t)=[k=0].$

For $k=0$ we get the Motzkin polynomials

$$M_n(t)=M_{n,0}(t)=\sum_{j=0}^{\left\lfloor\frac{n}{2}\right\rfloor}\binom{n}{2j}C_j t^{n-2j}. \tag{1.2}$$

The right-hand side follows because there are $\binom{n}{2j}$ ways to choose $2j$ positions for the up- and down-steps and $C_j=\frac{1}{j+1}\binom{2j}{j}$ ways to construct Dyck paths on them.

For $t=1$ we get the Motzkin numbers $M_n=M_n(1)$, $(M_n)_{n\geq 0}=(1,1,2,4,9,21,51,127,323,\ldots)$, (cf. [9], A001006), for $t=0$ the aerated Catalan numbers $(M_n(0))_{n\geq 0}=(1,0,1,0,2,0,5,0,\cdots)$, (cf. [9] A126120), and for $t=2$ the shifted Catalan numbers $(M_n(2))_{n\geq 0}=(1,2,5,14,\cdots)$, (cf. [9], A000108).

For positive k we get the formula

$$M_{n,k}(t)=\sum_{j=0}^{\left\lfloor\frac{n-k}{2}\right\rfloor} a(k+2j,k)\binom{n}{2j+k}t^{n-2j-k}, \tag{1.3}$$

where $a(n,k)$ denotes the number of non-negative paths with up- and down-steps from $(0,0)$ to (n,k). By the reflection principle we get $a(k+2j,k)=\binom{k+2j}{j}-\binom{k+2j}{j-1}$, i.e. the number of all paths $(0,0)\to(k+2j,k)$ minus the number of those paths which cross the $x-$axis. If we reflect the latter paths on the axis $x=-1$ after the first crossing we get a bijection with all paths $(0,0)\to\left(k+2j,-k-2\right)$, which have $j-1$ up-steps.

The first terms of the matrix $\left(M_{n,k}(t)\right)_{n,k\geq 0}$ are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ t & 1 & 0 & 0 & 0 & 0 \\ 1+t^2 & 2t & 1 & 0 & 0 & 0 \\ t\left(3+t^2\right) & 2+3t^2 & 3t & 1 & 0 & 0 \\ 2+6t^2+t^4 & 4t\left(2+t^2\right) & 3\left(1+2t^2\right) & 4t & 1 & 0 \\ t\left(10+10t^2+t^4\right) & 5\left(1+4t^2+t^4\right) & 5t\left(3+2t^2\right) & 2\left(2+5t^2\right) & 5t & 1 \end{pmatrix}$$

For $t=0,1,2,3$ these triangles occur in OEIS[9] as A053121, A064189, A039598, A091965.

We want to obtain some information about the Hankel determinants

$$d_m^{(k)}(n,t)=\det\left(M_{m+i+j,k}(t)\right)_{i,j=0}^{n-1} \tag{1.4}$$

of the columns of this triangle.

For $k=0$ the determinants $d_m(n,t)=d_m^{(0)}(n,t)$ have been considered in [1], [2] and [4] with different methods. For $k>0$ and small m the determinants $d_m^{(k)}(n,t)$ have been obtained and proved in [2]. We give an overview of these results and state conjectures for the remaining cases. In order to obtain nice results we shall always set $d_m^{(k)}(0,t)=1$.

The approach with orthogonal polynomials shows that the Motzkin polynomials $M_n(t)$ are the moments of the polynomials $p_n(x,t)$ which satisfy

$$p_n(x,t)=(x-t)p_{n-1}(x,t)-p_{n-2}(x,t) \tag{1.5}$$

with initial values $p_{-1}(x,t)=0$ and $p_0(x,t)=1$. Thus

$$p_n(x,t)=F_n(x-t), \tag{1.6}$$

where $F_n(x)$ are the Fibonacci polynomials defined by

$$F_n(x)=xF_{n-1}(x)-F_{n-2}(x) \tag{1.7}$$

with $F_0(x)=1$ and $F_1(x)=x$.

This implies $d_0(n,t)=1$ and

$$d_1(n,t)=(-1)^n p_n(0,t)=(-1)^n F_n(-t)=F_n(t). \tag{1.8}$$

The first terms are $d_1(0,t)=1,\ d_1(1,t)=t,\ d_1(2,t)=t^2-1,\ d_1(3,t)=t^3-2t,$ $d_1(4,t)=t^4-3t^2+1.$

The higher determinants can be computed using Dodgson's condensation.

The first terms of $d_2(n,t)$ are $1,\ t^2+1,\ t^4-t^2+2,\ t^6-3t^4+3t^2+2,\ t^8-5t^6+8t^4-3t^2+3.$

There is no obvious regularity in the coefficients, but there are nice expressions in terms of Fibonacci polynomials.

In [1] it is shown that

$$d_2(n,t)=\sum_{j=0}^{n} F_j(t)^2 \tag{1.9}$$

and in [4] that

$$d_2(n,t)=\det\begin{pmatrix} F_n(t) & F_n'(t) \\ F_{n+1}(t) & F_{n+1}'(t) \end{pmatrix}. \tag{1.10}$$

For higher orders explicit expressions become more complicated, but there are nice generating functions.

Let

$$D_m(x,t)=\sum_{n\geq 0} d_m(n,t)x^n \tag{1.11}$$

denote the generating function of $d_m(n,t).$

For small m it can be proved that

$$D_0(x,t)=\frac{1}{1-x}, \tag{1.12}$$

$$D_1(x,t)=\frac{1}{1-tx+x^2}. \tag{1.13}$$

$$D_2(x,t)=\frac{1+x}{(1-x)^2\left(1-\left(t^2-2\right)x+x^2\right)}, \tag{1.14}$$

$$D_3(x,t)=\frac{\left(1-x^2\right)\left(1+3tx+x^2\right)}{(1-(t^3-3t)x+x^2)\left(1-tx+x^2\right)^3}. \tag{1.15}$$

It turns out that there is a close relation with the Lucas polynomials $L_n(t).$ These are defined by

$$L_n(t) = tL_{n-1}(t) - L_{n-2}(t) \tag{1.16}$$

with initial values $L_0(t) = 2$ and $L_1(t) = t.$

Let

$$\begin{aligned} \alpha(t) &= \frac{t+\sqrt{t^2-4}}{2}, \\ \beta(t) &= \frac{t-\sqrt{t^2-4}}{2} \end{aligned} \tag{1.17}$$

be the roots of $x^2 - tx + 1 = 0.$

Binet's formulas show that

$$\begin{aligned} L_n(t) &= \alpha(t)^n + \beta(t)^n, \\ F_n(t) &= \frac{\alpha(t)^{n+1} - \beta(t)^{n+1}}{\alpha(t) - \beta(t)}. \end{aligned} \tag{1.18}$$

Comparing the denominators $h_m(x,t)$ of $D_m(x,t)$ in the above examples we see that
$h_0(x,t) = 1 - x,$

$$h_1(x,t) = 1 - tx + x^2 = (\alpha(t) - x)(\beta(t) - x) = \left(1 - \frac{x}{\alpha(t)}\right)\left(1 - \frac{x}{\beta(t)}\right) = h_0\left(\frac{x}{\alpha(t)}, t\right) h_0\left(\frac{x}{\beta(t)}, t\right),$$

$$h_2(x,t) = h_1\left(\frac{x}{\alpha(t)}, t\right) h_1\left(\frac{x}{\beta(t)}, t\right) = \big(\alpha(t)\alpha(t) - x\big)\big(\alpha(t)\beta(t) - x\big)\big(\beta(t)\alpha(t) - x\big)\big(\beta(t)\beta(t) - x\big).$$

This leads to the conjecture that $D_m(x,t)$ can be written as a fraction with denominator $h_m(x,t) = \prod_{i_1,\cdots,i_m} \left(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_m} - x\right),$ where the product is taken over all $\left(i_1, i_2, \cdots, i_m\right) \in \{0,1\}^m$ with $\gamma_0 = \gamma_0(t) = \alpha(t), \quad \gamma_1 = \gamma_1(t) = \beta(t).$

I am indebted to Christian Krattenthaler for pointing out that a proof of this conjecture follows from his paper [8], Corollary 9, by setting there $s = t, \quad t = 1$ and $x_i = 0.$

For $2j \neq m$ there are $\binom{m}{j}$ factors $\left(\alpha^{m-j}\beta^j - x\right) = \left(\alpha^{m-2j} - x\right)$ and $\left(\beta^{m-j}\alpha^j - x\right) = \left(\beta^{m-2j} - x\right).$
Therefore $h_m(x,t)$ has $\binom{m}{j}$ factors $\left(\alpha^{m-2j} - x\right)\left(\beta^{m-2j} - x\right) = x^2 - L_{m-2j}(t)x + 1.$

For even $m = 2\ell$ we get $\binom{m}{\ell}$ factors $1 - x.$

Setting

$$A_n(x,t) = x^2 - L_n(t)x + 1 = \left(x - \alpha(t)^n\right)\left(x - \beta(t)^n\right) \text{ for } n > 0,$$
$$A_0(x,t) = 1 - x, \tag{1.19}$$

we get $h_m(x,t) = \prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} A_{m-2j}(x,t)^{\binom{m}{j}}$. This implies

Theorem 1.1

For $m > 0$

$$D_m(x,t) = \frac{R_m(x,t)}{\prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} A_{m-2j}(x,t)^{\binom{m}{j}}} \tag{1.20}$$

with $R_m(x,t) \in \mathbb{Z}[x,t]$.

For $t \in \{0,1,2\}$ some (conjectured) simplifications are possible due to the periodicity of $\left(L_k(t)\right)$.

For example, for $k = 0$ and $t = 1$ we get

$$D_{2m}(x,1) = \frac{r(2m,x)}{\left(1-x^3\right)^{m^2}(1-x)} \tag{1.21}$$

where $r(2m,x)$ is a palindromic polynomial with positive coefficients of degree $\deg r(2m,x) = m\left(3m-2\right)$ and

$$D_{2m+1}(x,1) = \frac{\left(1+(-1)^m x\right)\left(1+x\right)^2 r(2m+1,x,1)}{\left(1+x^3\right)^{m^2+m+1}} \tag{1.22}$$

where $r(2m+1,x)$ is a palindromic polynomial with degree $3m^2 + m + 1$.

For example, $r(1,x) = r(2,x) = 1 + x,\ r(3,x) = \left(1-x\right)\left(1+x\right)^2\left(1+3x+x^2\right),$

$r(4,x) = 1 + 8x + 9x^2 + 14x^3 + 32x^4 + 14x^5 + 9x^6 + 8x^7 + x^8,$
$r(5,x) = \left(1+x\right)^3\left(1 + 18x + 9x^2 - 115x^3 - 203x^4 + 132x^5 + 384x^6 + 132x^7 - 203x^8 - 115x^9 + 9x^{10} + 18x^{11} + x^{12}\right).$

Let us now consider $d_m^{(k)}(n,t)$ and their generating functions

$$D_m^{(k)}(x,t) = \sum_{n \geq 0} d_m^{(k)}(n,t) x^n \tag{1.23}$$

for $k>0$.

In order to stress the analogy with the case $k=0$ we consider generalized Fibonacci polynomials $F_n^{(k)}(x)$ defined by

$$F_n^{(k)}(x)=L_{k+1}(x)F_{n-1}^{(k)}(x)-F_{n-2}^{(k)}(x) \tag{1.24}$$

with initial values $F_{-1}^{(k)}(x)=0$ and $F_0^{(k)}(x)=1$.

For $k=0$ these reduce to $F_n^{(0)}(x)=F_n(x)$.

Since α^{k+1} and β^{k+1} are roots of $\left(x-\alpha^{k+1}\right)\left(x-\beta^{k+1}\right)=x^2-L_{k+1}(t)x+1$ we get as in (1.18)

$$F_n^{(k)}(t)=\frac{\alpha^{(k+1)(n+1)}-\beta^{(k+1)(n+1)}}{\alpha^{k+1}-\beta^{k+1}}. \tag{1.25}$$

By [2], Theorem 1, we know that $d_0^{(k)}((k+1)n,t)=(-1)^{n\binom{k+1}{2}}$ and $d_0^{(k)}(n,t)=0$ else

and by [2], Theorem 2, and (1.25)

$d_1^{(k)}((k+1)n,t)=(-1)^{n\binom{k+1}{2}}F_n^{(k)}(t)$, $d_1^{(k)}((k+1)n+k,t)=(-1)^{\binom{k}{2}}F_n^{(k)}(t)$, and $d_1^{(k)}(n,t)=0$ else.

Therefore we have

$$D_0^{(k)}(x,t)=\sum_{n\geq 0}(-1)^{n\binom{k+1}{2}}x^{(k+1)n}=\frac{1}{1-(-1)^{\binom{k+1}{2}}x^{k+1}} \tag{1.26}$$

and

$$D_1^{(k)}(x,t)=\frac{1+(-1)^{\binom{k}{2}}x^k}{1-(-1)^{\binom{k+1}{2}}L_{k+1}(t)x^{k+1}+x^{2(k+1)}}. \tag{1.27}$$

Let $h_m^{(k)}(x,t)$ be the denominator of $D_m^{(k)}(x,t)$. Then we have $h_0^{(k)}(x,t)=1-(-1)^{\binom{k+1}{2}}x^{k+1}$ and

$h_1^{(k)}(x,t)=1-(-1)^{\binom{k+1}{2}}L_{k+1}(t)x^{k+1}+x^{2(k+1)}=\left(\alpha(t)^{k+1}-(-1)^{\binom{k+1}{2}}x^{k+1}\right)\left(\beta(t)^{k+1}-(-1)^{\binom{k+1}{2}}x^{k+1}\right)$

$=h_0^{(k)}\left(\dfrac{x}{\alpha(t)},t\right)h_0^{(k)}\left(\dfrac{x}{\beta(t)},t\right).$

Further computations suggest that $D_m^{(k)}(x,t)$ can always be written as a fraction with

denominator $h_m^{(k)}(x,t)=\prod_{i_1,\cdots,i_m}\left(\left(\gamma_{i_1}\gamma_{i_2}\cdots\gamma_{i_m}\right)^{k+1}-(-1)^{\binom{k+1}{2}}x^{k+1}\right).$

Setting

$$A_{k,0}(x,t) = 1 + (-1)^{\binom{k-1}{2}} x^{k+1},$$
$$A_{k,n}(x,t) = 1 + (-1)^{\binom{k-1}{2}} L_n(t) x^{k+1} + x^{2(k+1)} \tag{1.28}$$

we get

Conjecture 1.2

$$D_m^{(k)}(x,t) = \frac{R_m^{(k)}(x,t)}{\prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} A_{k,(k+1)(m-2j)}^{\binom{m}{j}}(x,t)} \tag{1.29}$$

with $R_m^{(k)}(x,t) \in \mathbb{Z}[t,x]$.

A closer look at $D_m^{(k)}(x,t)$ suggests that some factors can be cancelled. This leads to

Conjecture 1.3

$$D_m^{(k)}(x,t) = \frac{r_m^{(k)}(x,t)}{\prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} A_{k,(k+1)(m-2j)}^{1+j(m-j)}(x,t)} \tag{1.30}$$

where $r_m^{(k)}(x,t)$ *is a polynomial of degree* $\deg_x r_m^{(k)}(x,t) = \binom{m+1}{3} + k\left(\binom{m}{1} + \binom{m}{2} + \binom{m}{3}\right).$

For $t = 2$ the matrix $\left(M_{n,k}(2)\right)_{n,k \geq 0}$ is the Catalan triangle [9], A039598, whose first terms are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 2 & 1 & 0 & 0 & 0 & 0 & 0 \\ 5 & 4 & 1 & 0 & 0 & 0 & 0 \\ 14 & 14 & 6 & 1 & 0 & 0 & 0 \\ 42 & 48 & 27 & 8 & 1 & 0 & 0 \\ 132 & 165 & 110 & 44 & 10 & 1 & 0 \\ 429 & 572 & 429 & 208 & 65 & 12 & 1 \end{pmatrix}.$$

Here we guess that

$$D_m^{(k)}(x,2) = \frac{A_m^{(k)}(x,2)}{\left(1 - (-1)^{\binom{k+1}{2}} x^{k+1}\right)^{1+\binom{m+1}{2}}} \tag{1.31}$$

where $A_m^{(k)}(x,2)$ is a polynomial of degree $\deg A_m^{(k)}(x,2)=\dfrac{m((k+1)m+k-1)}{2}$ which satisfies

$$x^{\frac{m((k+1)m+k-1)}{2}}A_m^{(k)}\left(\frac{1}{x},2\right)=\varepsilon(k,m)A_m^{(k)}(x,2) \tag{1.32}$$

with $\varepsilon(k,m)=1$ if $k\equiv 0(\operatorname{mod}4)$, $\varepsilon(k,m)=(-1)^m$ if $k\equiv -1(\operatorname{mod}4)$, $\varepsilon(k,m)=(-1)^{\binom{m}{2}}$ if $k\equiv 1(\operatorname{mod}4)$, and $\varepsilon(k,m)=(-1)^{\binom{m+1}{2}}$ if $k\equiv 2(\operatorname{mod}4)$.

For $k=0$ formula (1.31) is a known result. The Hankel determinants of the shifted Catalan numbers satisfy $\det\left(C_{m+i+j}\right)_{i,j=0}^{n-1}=\prod_{1\le i\le j\le m-1}\dfrac{2n+i+j}{i+j}$. I owe to Sam Hopkins [5] the observation that the right-hand side can be interpreted as the number of plane partitions of the form $(m-1,m-2,\cdots,1)$ of non-negative integers $\le n.$ (A simple proof due to Christian Krattenthaler can be found in [3]). Formula (1.31) can be deduced from [10], Theorem 3.15.8.

The polynomials $A_m^{(0)}(x,2)$ are palindromic with positive coefficients of degree $\binom{m}{2}$. For example, $A_0^{(0)}(x,2)=1$, $A_1^{(0)}(x,2)=1$, $A_2^{(0)}(x,2)=1+x$, $A_3^{(0)}(x,2)=1+7x+7x^2+x^3$, $A_4^{(0)}(x,2)=1+31x+187x^2+330x^3+187x^4+31x^5+x^6$.

For $k>0$ we get for example

$$A_1^{(k)}(x,2)=1+(-1)^{\binom{k}{2}}x^k,$$

$$A_2^{(k)}(x,2)=1+(-1)^{\binom{k-1}{2}}x^{k-1}+(-1)^{\binom{k}{2}}(k+1)^2x^k\left(1+(-1)^{\binom{k+1}{2}}x^{k+1}\right)-x^{2k+2}\ \text{ for } k\ge 2, \text{and}$$

$$A_2^{(1)}(x,2)=\left(1-x^2\right)\left(1+4x+x^2\right).$$

Apparently there are also analogs of (1.9) and (1.10):

Conjecture 1.4

$$\begin{aligned}
&(-1)^{\binom{k+1}{2}n}d_2^{(k)}\left((k+1)n,t\right)=F_n^{(k)}(t)^2,\\
&(-1)^{\binom{k+1}{2}n+\binom{k-1}{2}}d_2^{(k)}\left((k+1)n+k-1,t\right)=F_n^{(k)}(t)^2,\\
&(-1)^{\binom{k+1}{2}n+\binom{k}{2}}d_2^{(k)}\left((k+1)n+k,t\right)=L_{k+1}(t)'\sum_{j=0}^{n}F_j^{(k)}(t)^2=\det\begin{pmatrix}F_n^{(k)}(t) & F_n^{(k)}(t)'\\ F_{n+1}^{(k)}(t) & F_{n+1}^{(k)}(t)'\end{pmatrix},\\
&d_2^{(k)}(n,t)=0\ \text{ else.}
\end{aligned} \tag{1.33}$$

2. Now let us consider a slight generalization by changing the weight of the horizontal steps H on height 0 to s instead of t. Let $M_{n,k}(t,s)$ denote these weights of the Motzkin paths.

Here we get $d_0(n,t,s)=1$ and

$$\sum_{n\geq 0} d_1(n,t,s)x^n = \frac{1+(s-t)x}{1-tx+x^2} \tag{2.1}$$

$$\sum_{n\geq 0} d_2(n,t,s)x^n = \frac{1+\left(1+s^2-t^2\right)x+(s-t)^2x^2}{(1-x)^2\left(1+\left(2-t^2\right)x+x^2\right)}. \tag{2.2}$$

Conjecture 2.1

$$\sum_{n\geq 0} d_m(n,t,s)x^n = \frac{R_m(x,t,s)}{\prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} A_{0,m-2j}^{1+j(m-j)}(x,t)} \tag{2.3}$$

where $R_m(x,t,s)$ *is a polynomial in* x,s,t *with integer coefficients with* $\deg_x R_m(x,t,s)=\binom{m+1}{3}+1.$

Conjecture 2.2

For all s *the sequences* $\left(d_m^{(k)}(n,t,0)\right)_{n\geq 0}$ *and* $\left(d_m^{(k)}(n,t+s,s)\right)_{n\geq 0}$ *satisfy the same recurrence of order* 2^{k+m} *for all* $k,m>0$. *In the special case* $m=0$ *we even get*

$$d_0^{(k)}(n,t+s,s)=d_0^{(k)}(n,t,0). \tag{2.4}$$

Let us mention some explicit formulas for some small m and k:

$$\sum_{n\geq 0} d_0^{(1)}(n,t,0)x^n = \frac{1-tx}{1-tx+x^2} \tag{2.5}$$

$$\sum_{n\geq 0} d_0^{(2)}(n,t,0)x^n = \frac{1+x+t^2x^2}{1+x+t^2x^2+x^3+x^4} \tag{2.6}$$

$$\sum_{n\geq 0} d_0^{(3)}(n,t,0)x^n = \frac{1+tx+6\binom{t+1}{3}x^3+\left(t^4+t^2-1\right)x^4+t^3x^5}{1+tx+6\binom{t+1}{3}x^3+\left(t^2-1\right)\left(t^2+2\right)x^4+6\binom{t+1}{3}x^5+tx^7+x^8}. \tag{2.7}$$

The denominator for $k=4$ is

$$1-x+t^2x^3-t^2\left(2t^2-1\right)x^4+\left(t^2-1\right)\left(t^2+3\right)x^5-\left(2t^2-3\right)x^6-t^2\left(t^4-t^2+1\right)x^7+\left(t^2-1\right)t^2\left(t^4+t^2+2\right)x^8$$
$$-t^2\left(t^4-t^2+1\right)x^9-\left(2t^2-3\right)x^{10}+\left(t^2-1\right)\left(t^2+3\right)x^{11}-t^2\left(2t^2-1\right)x^{12}+t^2x^{13}-x^{15}+x^{16}$$

$$\sum_{n\geq 0} d_1^{(1)}(n,t,s)x^n = \frac{1+\left(1+t(s-t)\right)x+(s-t)^2x^2}{1+(s-t)x+\left(L_2(t)+(s-t)^2\right)x^2+(s-t)x^3+x^4} \tag{2.8}$$

$$\sum_{n\geq 0} d_1^{(2)}(n,t,0)x^n$$
$$= \frac{1+tx+t^2\left(t^2-2\right)x^2+t\left(2t^2-3\right)x^3+\left(t^4+t^2-1\right)x^4+t^3x^5}{1+tx+t^2\left(t^2-2\right)x^2+t\left(2t^2-3\right)x^3+t^2\left(2t^2-3\right)x^4++t\left(2t^2-3\right)x^5+t^2\left(t^2-2\right)x^6+tx^7+x^8}.$$
(2.9)

For $(t,s)=(2,1)$ we get another Catalan triangle (cf. [9], A039599), whose first terms are

$$\begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 & 0 & 0 \\ 2 & 3 & 1 & 0 & 0 & 0 & 0 \\ 5 & 9 & 5 & 1 & 0 & 0 & 0 \\ 14 & 28 & 20 & 7 & 1 & 0 & 0 \\ 42 & 90 & 75 & 35 & 9 & 1 & 0 \\ 132 & 297 & 275 & 154 & 54 & 11 & 1 \end{pmatrix}.$$

In this case we get more information about the generating functions:

$$\sum_{n\geq 0} d_m^{(k)}(n,2,1)x^n = \frac{a_m^{(k)}(x)}{\left(1-(-1)^k x^{2k+1}\right)^{\binom{m}{2}+1}}, \tag{2.10}$$

with $\deg a_m^{(k)}(x) = \left(m(m-1)+1\right)k + \binom{m-1}{2}$ and

$$\operatorname{sgn}\left(\frac{a_m^{(k)}\left(\frac{1}{x}\right)}{a_m^{(k)}(x)}\right) = (-1)^{\binom{k}{2}} \text{ if } k \equiv 1,2 \pmod 4 \text{ and } \operatorname{sgn}\left(\frac{a_m^{(k)}\left(\frac{1}{x}\right)}{a_m^{(k)}(x)}\right) = (-1)^{\binom{k+1}{2}} \text{ if } k \equiv 3,0 \pmod 4.$$

3. Addendum

3.1. As another generalization we define for $P \in \mathbf{M}_{n,k}$ a weight $w_{t,s}(P)$ as the product of the weights of its steps, where $w_{t,s}(H)=t$, $w_{t,s}(D)=s$ and $w_{t,s}(U)=1$.

Let $M_{n,k,s}(t) = \sum_{P\in \mathbf{M}_{n,k}} w_{t,s}(P)$ be the weight of all paths from $(0,0)$ to (n,k).

These weights satisfy

$$M_{n,k,s}(t) = M_{n-1,k-1,s}(t) + tM_{n-1,k,s}(t) + sM_{n-1,k+1,s}(t) \tag{3.1}$$

with $M_{n,k,s}(t)=0$ for $k<0$ and $M_{0,k,s}(t)=[k=0]$.

The same argument as above gives

$$M_{n,k,s}(t)=\sum_{j=0}^{\left\lfloor\frac{n-k}{2}\right\rfloor}\binom{n}{2j+k}\binom{k+2j}{j}\frac{k+1}{k+j+1}t^{n-2j-k}s^{j}. \tag{3.2}$$

We want to give a generalization of Conjecture 1.3 to this case. In the same way as above, we see that

$$\det\left(M_{i+j,0,s}(t)\right)_{i,j=0}^{n-1}=s^{\binom{n}{2}}, \tag{3.3}$$

$$\det\left(M_{1+i+j,0,s}(t)\right)_{i,j=0}^{n-1}=s^{\binom{n}{2}}\sqrt{s^n}F_n\left(\frac{t}{\sqrt{s}}\right). \tag{3.4}$$

Therefore, we will consider instead of $\det\left(M_{m+i+j,k,s}(t)\right)_{i,j=0}^{n-1}$ the ratios

$$S_m^{(k)}(n,t,s)=\frac{\det\left(M_{m+i+j,k,s}(t)\right)_{i,j=0}^{n-1}}{s^{\binom{n}{2}}} \tag{3.5}$$

with $S_m^{(k)}(0,t,s)=1$. From (3.4) we get

$$\sum_{n\geq 0}S_1^{(0)}(n,t,s)=\frac{1}{1-tx+sx^2}, \tag{3.6}$$

where $1-tx+sx^2=\left(1-\alpha(t,s)x\right)\left(1-\beta(t,s)x\right)$ with

$$\begin{aligned}\alpha(t,s)&=\frac{t+\sqrt{t^2-4s}}{2},\\ \beta(t,s)&=\frac{t-\sqrt{t^2-4s}}{2}.\end{aligned} \tag{3.7}$$

Let

$$L_n(t,s)=\alpha(t,s)^n+\beta(t,s)^n \tag{3.8}$$

and

$$\begin{aligned}A_{k,n}(x,t,s)&=\left(1-\frac{\alpha(t,s)^n}{(-s)^{\binom{k+1}{2}}}x^{k+1}\right)\left(1-\frac{\beta(t,s)^n}{(-s)^{\binom{k+1}{2}}}x^{k+1}\right)=1-\frac{L_n(t,s)}{(-s)^{\binom{k+1}{2}}}x^{k+1}+\frac{s^n}{s^{k(k+1)}}x^{2(k+1)},\\ A_{k,0}(x,t,s)&=1-\frac{x^{k+1}}{(-s)^{\binom{k+1}{2}}}.\end{aligned} \tag{3.9}$$

Then computer experiments suggest

Conjecture 3.1

$$\sum_{n\geq 0} S_m^{(k)}(n,t,s)x^n = \frac{U_m^{(k)}(x,t,s)}{\prod_{j=0}^{\left\lfloor \frac{m}{2} \right\rfloor} A_{k,(k+1)(m-2j)}^{1+j(m-j)}(s^j x,t,s)} \tag{3.10}$$

where $U_m^{(k)}(x,t,s)$ has degree $\deg_x U_m^{(k)}(x,t,s) = \binom{m+1}{3} + k\left(\binom{m}{1}+\binom{m}{2}+\binom{m}{3}\right).$

3.2. Finally let us consider the set $\mathbf{T}_n$ of ALL lattice paths from $(0,0)$ to $(n,0)$ consisting of up-steps $U=(1,1)$, down-steps $D=(1,-1)$ and horizontal steps $H=(1,0)$ with the weight $w_{t,s}$. Then we get

$$w_{t,s}\left(\mathbf{T}_n\right) = T_n(t,s) = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j}\binom{2j}{j} t^{n-2j}s^j = \left[x^n\right]\left(1+tx+sx^2\right)^n. \tag{3.11}$$

The right-hand identity follows from

$$\left(1+tx+sx^2\right)^n = \left(1+\left(tx\left(1+\frac{sx}{t}\right)\right)\right)^n = \sum_k \binom{n}{k}(tx)^k\left(1+\frac{sx}{t}\right)^k = \sum_{k,j}\binom{n}{k}\binom{k}{j} t^{k-j}s^j x^{k+j}$$

$$= \sum_n x^n \sum_j \frac{n!}{(j)!\,j!(n-2j)!} t^{n-2j}s^j = \sum_n x^n \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j}\binom{2j}{j} t^{n-2j}s^j.$$

The numbers $T_n = \left|\mathbf{T}_n\right|$ of these paths are therefore the central trinomial coefficients

$$T_n = \left[x^n\right]\left(1+x+x^2\right)^n = \sum_{j=0}^{\left\lfloor \frac{n}{2} \right\rfloor} \binom{n}{2j}\binom{2j}{j}. \tag{3.12}$$

Let now $\mathbf{d}_m(n,t,s) = \det\left(T_{m+i+j}(t,s)\right)_{i,j=0}^{n-1}$ with $\mathbf{d}_m(0,t,s)=1$ for $m>0$ and $\mathbf{d}_0(0,t,s)=\frac{1}{2}.$

Then $\mathbf{d}_0(n,t,s) = 2^{n-1}s^{\binom{n}{2}}$ for $n \in \mathbb{N}.$

For

$$\mathbf{D}_m(x,t,s) = \sum_{n\geq 0} \frac{\mathbf{d}_m(n,t,s)}{2\mathbf{d}_0(n,t,s)} x^n \tag{3.13}$$

we get

Conjecture 3.2

$$\mathbf{D}_m(x,t,s)=\frac{\mathbf{U}_m(x,t,s)}{\prod_{j=0}^{\left\lfloor \frac{m}{2}\right\rfloor} A_{0,m-2j}^{1+j(m-j)}(s^j x,t,s)} \tag{3.14}$$

with $\deg_x \mathbf{U}_m(x,t,s)=\binom{m}{3}+\binom{m-1}{2}+m+1=\frac{m^3-m+12}{6}$. for $m>0$.

For $T_n(2,1)=\binom{2n}{n}$ this simplifies to

Conjecture 3.3

$$\mathbf{D}_m(x,2,1)=\frac{\mathbf{V}_m(x,2,1)}{(1-x)^{\binom{m}{2}+1}}, \tag{3.15}$$

where $\mathbf{V}_m(x,2,1)$ is a symmetric polynomial with positive coefficients and
$\deg \mathbf{V}_m(x,2,1)=\binom{m-1}{2}+1$ for $m>0$.

The first numerators are $\mathbf{V}_1(x,2,1)=1$, $\mathbf{V}_2(x,2,1)=1+x$, $\mathbf{V}_3(x,2,1)=1+6x+x^2$, $\mathbf{V}_4(x,2,1)=1+28x+70x^2+28x^3+x^4$.

The computations have been made with Mathematica and the Mathematica packages Guess by Manuel Kauers [6] and RATE by Christian Krattenthaler [7].

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