

A special class of Hankel determinants

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Abstract

In this expository paper we compute Hankel determinants of some sequences whose generating functions are given by C-fractions and derive orthogonality properties for associated polynomials.

1. Introduction

Let $(b_k)_{k \geq -1}$ be a non-decreasing sequence of integers such that $b_{k+2} - b_k \geq 1$ with initial values $b_{-1} = -1$ and $b_0 = 0$ and define the formal power series $f(x)$ by the C-fraction

$$f(x) = \sum_{n \geq 0} f_n x^n = \frac{1}{1 - \frac{a_0 x^{b_1 - b_0}}{1 - \frac{a_1 x^{b_2 - b_1}}{1 - \frac{a_2 x^{b_3 - b_2}}{1 - \frac{a_3 x^{b_4 - b_3}}{1 - \ddots}}}}}. \quad (1.1)$$

The Hankel determinants of the coefficients f_n will be denoted by $d(n) = \det(f_{i+j})_{i,j=0}^n$.

Then all non-vanishing Hankel determinants are given by

$$d(b_k) = (-1)^{\binom{b_1 - b_0}{2} + \binom{b_2 - b_1}{2} + \dots + \binom{b_k - b_{k-1}}{2}} a_0^{b_k - b_0} a_1^{b_k - b_1} a_2^{b_k - b_2} a_3^{b_k - b_3} \dots a_{k-1}^{b_k - b_{k-1}}. \quad (1.2)$$

For $b_k = k$ or $b_k = \left\lfloor \frac{k}{2} \right\rfloor$, i.e. $b_k - b_{k-2} = 2$ or $b_k - b_{k-2} = 1$ these are old results which are intimately connected with orthogonal polynomials. For the more general case Paul Barry [2] conjectured that $d(b_k) = \pm 1$ if all $a_n = \pm 1$. This conjecture led me to investigate this problem. Afterwards I learned that an equivalent result had already earlier been found by Victor I. Buslaev [4] with another proof.

If $b_{k+1} - b_k > 1$ for some k the role of orthogonal polynomials is played by polynomials $r_k(x)$ which satisfy the three-term recurrence $r_k(x) = x^{b_{k-1} - b_{k-2}} r_{k-1}(x) - a_{k-2} r_{k-2}(x)$ with initial values $r_0(x) = 1$ and $r_1(x) = x$. They satisfy $\Lambda(r_k(x)x^n) = 0$ for $n < b_k$ and $\Lambda(r_k(x)x^{b_k}) = a_0 \dots a_{k-1}$, if Λ denotes the linear functional on the vector space of polynomials defined by $\Lambda(x^n) = f_n$.

We show that the polynomials $r_k(x)$ are related to the polynomials $p_n(x) = \det(f_{i+j} x - f_{i+j+1})_{i,j=0}^{n-1}$ and study some typical examples.

2. Some preliminary results about continued fractions and Hankel determinants

Let us begin with some well-known facts about continued fractions.

If we write

$$\frac{1}{c_0 + \frac{a_0}{c_1 + \frac{a_1}{c_2 + \frac{\ddots}{c_{n-1}}}}} = \frac{1}{c_0 + c_1 + \frac{a_0 \dots a_{n-2}}{c_{n-1}}} = \frac{A_n}{B_n} \quad (2.1)$$

then $A_0 = 0, A_1 = 1$ and $B_0 = 1, B_1 = c_0$
and

$$\begin{aligned} A_n &= c_{n-1}A_{n-1} + a_{n-2}A_{n-2}, \\ B_n &= c_{n-1}B_{n-1} + a_{n-2}B_{n-2}. \end{aligned} \quad (2.2)$$

Moreover we get by induction

$$A_{k+1}B_k - B_{k+1}A_k = (-1)^k a_0 a_1 \dots a_{k-1}. \quad (2.3)$$

Let us assume that $b_{-1} = -1, b_0 = 0$, and that b_n is not decreasing with $b_{n+2} - b_n \geq 1$. If $b_n = b_{n+1} = b$ for some n we say that b occurs with multiplicity 2. Otherwise it has multiplicity 1. Since $b_{n+2} - b_n \geq 1$ there are no triplets $b_n = b_{n+1} = b_{n+2}$.

If each non-negative integer occurs in the sequence (b_k) the above results are intimately connected with orthogonal polynomials. This fact has been generalized by E. Frank [7] to more general cases.

The above choice of (b_k) leads to the following facts:

Let

$$f_n(x) = \frac{1}{1 - \frac{a_0 x^{b_1 - b_{-1}}}{1 - \frac{a_1 x^{b_2 - b_0}}{1 - \frac{\ddots}{1 - a_{n-2} x^{b_{n-1} - b_{n-3}}}}} = \frac{1}{1 - \frac{a_0 x^{b_1 - b_{-1}}}{1 - \frac{a_1 x^{b_2 - b_0}}{1 - \dots \frac{a_{n-3} x^{b_{n-2} - b_{n-4}}}{1 - \frac{a_{n-2} x^{b_{n-1} - b_{n-3}}}{1}}} = \frac{A_n(x)}{B_n(x)} \quad (2.4)$$

with $f_0(x) = 0$ and $f_1(x) = 1$.

Here (2.2) reduces to

$$A_n(x) = A_{n-1}(x) - a_{n-2} x^{b_{n-1} - b_{n-3}} A_{n-2}(x) \quad (2.5)$$

and

$$B_n(x) = B_{n-1}(x) - a_{n-2} x^{b_{n-1} - b_{n-3}} B_{n-2}(x) \quad (2.6)$$

with initial values $A_0(x) = 0, A_1(x) = 1$ and $B_0(x) = 1, B_1(x) = 1$.

As a generalization of orthogonal polynomials we define polynomials $r_k(x)$ by

$$r_k(x) = x^{b_{k-1}-b_{k-2}}r_{k-1}(x) - a_{k-2}r_{k-2}(x) \quad (2.7)$$

with initial values $r_0(x) = 1$ and $r_1(x) = x$.

It is easily verified that $\deg r_k(x) = b_{k-1} + 1$ for $k \geq 0$.

Proposition 2.1

The polynomials $r_k(x)$ and $B_k(x)$ are related by

$$r_k(x) = x^{b_{k-1}+1}B_k\left(\frac{1}{x}\right). \quad (2.8)$$

Proof

Let $U_k(x) = x^{b_{k-1}+1}B_k\left(\frac{1}{x}\right)$. By (2.6) we get

$$\begin{aligned} U_k(x) &= x^{b_{k-1}+1}B_k\left(\frac{1}{x}\right) = x^{b_{k-1}+1}B_{k-1}\left(\frac{1}{x}\right) - a_{k-2}x^{b_{k-1}+1}x^{-b_{k-1}+b_{k-3}}B_{k-2}\left(\frac{1}{x}\right) \\ &= x^{b_{k-1}-b_{k-2}}U_{k-1}(x) - a_{k-2}U_{k-2}(x) \end{aligned}$$

with initial values $U_0(x) = 1$ and $U_1(x) = x$.

Therefore $U_k(x) = r_k(x)$.

It is clear that $\deg A_n(x) \leq \deg B_n(x)$. By induction we get

$$\deg B_n(x) \leq b_{n-1} + 1.$$

More precisely we get $\deg B_{2n} = 1 + b_{2n-1}$. For

$$\deg(B_{2n}(x)) = \max(\deg B_{2n-1}(x), \deg x^{b_{2n-1}-b_{2n-3}}B_{2n-2}(x)) = 1 + b_{2n-1}.$$

If for some m $b_{2m-1} = b_{2m}$ then $\deg B_{2m+1}(x) = 1 + b_{2m-1} = 1 + b_{2m}$ and for all $n \geq m$ also $\deg B_{2n+1}(x) = 1 + b_{2n}$. For $n < m$ we have $\deg B_{2n+1}(x) = b_{2n}$. In this case $r_{2n+1}(0) = 0$.

This follows from

$$B_{2n+1}(x) = B_{2n}(x) - a_{2n-1}x^{b_{2n}-b_{2n-2}}B_{2n-1}(x)$$

by induction.

Proposition 2.2

Let $f(x) = \sum_{n \geq 0} f_n x^n$ satisfy (3.1) and let

$$B_k(x)f(x) - A_k(x) = \sum_k f_n^{(k)} x^n. \quad (2.9)$$

Then $f_n^{(k)} = 0$ for $n \leq b_{k-1} + b_k$ and $f_{b_{k-1}+b_k+1}^{(k)} = a_0 a_1 \cdots a_{k-1}$.

Proof

Observe that by (2.3)

$$A_{k+1}(x)B_k(x) - B_{k+1}(x)A_k(x) = a_0 a_1 \cdots a_{k-1} x^{1+b_{k-1}+b_k}. \quad (2.10)$$

Therefore we get the series expansion

$$\frac{A_{k+1}(x)}{B_{k+1}(x)} - \frac{A_k(x)}{B_k(x)} = \frac{a_0 a_1 \cdots a_{k-1} x^{1+b_{k-1}+b_k}}{B_{k+1}(x)B_k(x)} = a_0 a_1 \cdots a_{k-1} x^{1+b_{k-1}+b_k} + \cdots$$

since $B_k(0) = 1$ for all k .

Since $1 + b_{k-1} + b_k > 1 + b_{k-2} + b_{k-1}$ the series expansion of $f(x)$ coincides with the expansion of

$$\frac{A_k(x)}{B_k(x)}$$

for the first $b_{k-1} + b_k$ terms.

Therefore the coefficients of x^n in $f(x) - \frac{A_k(x)}{B_k(x)}$ vanish for

$$n < b_1 - b_{-1} + b_2 - b_0 + \cdots + b_k - b_{k-2} = b_{k-1} + b_k + 1.$$

Since $\deg A_k(x) \leq b_{k-1}$ we see that the coefficients of $B_k(x)f(x)$ vanish for $b_{k-1} < n \leq b_{k-1} + b_k$.

If we set $B_k(x) = \sum_j v_{k,j} x^j$. Then we get

$$v_{k,0} f_n + v_{k,1} f_{n-1} + \cdots = 0 \text{ for } b_{k-1} < n \leq b_{k-1} + b_k$$

and

$$v_{k,0} f_{b_{k-1}+b_k+1} + v_{k,1} f_{b_{k-1}+b_k} + \cdots = a_0 a_1 \cdots a_{k-1}.$$

Another equivalent formulation gives

Proposition 2.3

Let Λ denote the linear functional on the polynomials defined by

$$\Lambda(x^n) = f_n. \quad (2.11)$$

Then

for $n < b_k$

$$\Lambda(r_k(x)x^n) = \Lambda\left(x^{b_{k-1}+1} B_k\left(\frac{1}{x}\right) x^n\right) = \Lambda\left(\sum_j v_{k,j} x^{b_{k-1}-j+n+1}\right) = \sum_j v_{k,j} f_{b_{k-1}-j+n+1} = 0 \quad (2.12)$$

and

$$\Lambda(r_k(x)x^{b_k}) = \Lambda\left(\sum_j v_{k,j} x^{b_{k-1}-j+b_k+1}\right) = \sum_j v_{k,j} f_{b_{k-1}-j+b_k+1} = a_0 \cdots a_{k-1}. \quad (2.13)$$

Remark

The identities (2.12) and (2.13) generalize the concept of orthogonality and (2.7) is an analogue of the three-term recurrence of orthogonal polynomials.

For some polynomials $r_k(x)$ a simple formula can be found. To this end we define the polynomials

$$p_n(x) = \det \begin{pmatrix} f_0 & f_1 & \cdots & f_{n-1} & 1 \\ f_1 & f_2 & \cdots & f_n & x \\ f_2 & f_3 & \cdots & f_{n+1} & x^2 \\ \vdots & & & & \vdots \\ f_n & f_{n+1} & \cdots & f_{2n-1} & x^n \end{pmatrix} \quad (2.14)$$

for $n \geq 1$ and let $p_0(x) = 1$.

Note that by elementary row operations $p_n(x)$ can also be expressed as

$$p_n(x) = \det (f_{i+j}x - f_{i+j+1})_{i,j=0}^{n-1}. \quad (2.15)$$

We will need some generalizations of

Proposition 2.4 (G.E.Andrews and J. Wimp [1])

Let

$$s(x) = \sum_{n \geq 0} s_n x^n \quad (2.16)$$

with $s_0 = 1$ and

$$t(x) = \frac{1}{s(x)} = \sum_{n \geq 0} t_n x^n. \quad (2.17)$$

Then for $n \geq 1$

$$\det (s_{i+j})_{i,j=0}^n = (-1)^n \det (t_{i+j+2})_{i,j=0}^{n-1}. \quad (2.18)$$

In view of our applications we note the following special case:

Corollary 2.1

$$f(x) = \sum_{n \geq 0} f_n x^n = \frac{1}{1 - ax^2 \sum_{j \geq 0} c_j x^j} \quad (2.19)$$

implies

$$\det (f_{i+j})_{i,j=0}^n = a^n \det (c_{i+j})_{i,j=0}^{n-1}. \quad (2.20)$$

Example 2.1

Let

$$f(x, a_0, a_1, a_2, \dots) = \sum_{n \geq 0} f_n x^n = \frac{1}{1 - \frac{a_0 x^2}{1 - \frac{a_1 x^2}{1 - \frac{a_2 x^2}{1 - \frac{a_3 x^2}{\ddots}}}}}. \quad (2.21)$$

Then

$$d(n) = a_0^n a_1^{n-1} a_2^{n-2} \dots a_{n-1}. \quad (2.22)$$

This well-known result follows from $f(x, a_0, a_1, a_2, \dots) = \frac{1}{1 - a_0 x^2 f(x, a_1, a_2, \dots)}$.

Example 2.1.1

If $a_n = 1$ for all n then $f(x)$ satisfies $1 - f(x) + x^2 f(x)^2 = 0$ which gives

$$f(x) = \frac{1 - \sqrt{1 - 4x^2}}{2x^2} = \sum_{n \geq 0} C_n x^{2n} \quad (2.23)$$

with the Catalan numbers

$$f_{2n} = C_n = \frac{1}{n+1} \binom{2n}{n}. \quad (2.24)$$

Proposition 2.5

Let $s(x) = \sum_{n \geq 0} s_n x^n$ with $s_0 = 1$ and $t(x) = \frac{1}{s(x)} = \sum_{n \geq 0} t_n x^n$.

Define $s_n = 0$ and $\det(t_{i+j+k})_{i,j=0}^n = 1$ for $n < 0$. Let $m \geq 0$.

Then

$\det(s_{i+j-m})_{i,j=0}^n = 0$ for $n < m$ and

$$\det(s_{i+j-m})_{i,j=0}^{n+m} = (-1)^{n+\binom{m+1}{2}} \det(t_{i+j+m+2})_{i,j=0}^{n-1}. \quad (2.25)$$

The proof is almost the same as in [1]. I shall illustrate the method (called O-reduction in [1]) for

$m = 2$ and $n = 3$. Observe that $t_0 = 1$ and $\sum_{j=0}^n s_{n-j} t_j = [n = 0]$.

$$\det(s_{i+j-2})_{i,j=0}^5 = \det \begin{pmatrix} 0 & 0 & s_0 & s_1 & s_2 & s_3 \\ 0 & s_0 & s_1 & s_2 & s_3 & s_4 \\ s_0 & s_1 & s_2 & s_3 & s_4 & s_5 \\ s_1 & s_2 & s_3 & s_4 & s_5 & s_6 \\ s_2 & s_3 & s_4 & s_5 & s_6 & s_7 \\ s_3 & s_4 & s_5 & s_6 & s_7 & s_8 \end{pmatrix}$$

Then we get with elementary column operations

$$\det(s_{i+j-2})_{i,j=0}^5 = \det \begin{pmatrix} 0 & 0 & s_0 & t_0s_1 + t_1s_0 & t_0s_2 + t_1s_1 + t_2s_0 & t_0s_3 + t_1s_2 + t_2s_1 + t_3s_0 \\ 0 & s_0 & s_1 & t_0s_2 + t_1s_1 & t_0s_3 + t_1s_2 + t_2s_1 & t_0s_4 + t_1s_3 + t_2s_2 + t_3s_1 \\ s_0 & s_1 & s_2 & t_0s_3 + t_1s_2 & t_0s_4 + t_1s_3 + t_2s_2 & t_0s_5 + t_1s_4 + t_2s_3 + t_3s_2 \\ s_1 & s_2 & s_3 & t_0s_4 + t_1s_3 & t_0s_5 + t_1s_4 + t_2s_3 & t_0s_6 + t_1s_5 + t_2s_4 + t_3s_3 \\ s_2 & s_3 & s_4 & t_0s_5 + t_1s_4 & t_0s_6 + t_1s_5 + t_2s_4 & t_0s_7 + t_1s_6 + t_2s_5 + t_3s_4 \\ s_3 & s_4 & s_5 & t_0s_6 + t_1s_5 & t_0s_7 + t_1s_6 + t_2s_5 & t_0s_8 + t_1s_7 + t_2s_6 + t_3s_5 \end{pmatrix}$$

$$= \det \begin{pmatrix} 0 & 0 & s_0 & 0 & 0 & 0 \\ 0 & s_0 & s_1 & -t_2s_0 & -t_3s_0 & -t_4s_0 \\ s_0 & s_1 & s_2 & -t_2s_1 - t_3s_0 & -t_3s_1 - t_4s_0 & -t_4s_1 - t_5s_0 \\ s_1 & s_2 & s_3 & -t_2s_2 - t_3s_1 - t_4s_0 & -t_3s_2 - t_4s_1 - t_5s_0 & -t_4s_2 - t_5s_1 - t_6s_0 \\ s_2 & s_3 & s_4 & -t_2s_3 - t_3s_2 - t_4s_1 - t_5s_0 & -t_3s_3 - t_4s_2 - t_5s_1 - t_6s_0 & -t_4s_3 - t_5s_2 - t_6s_1 - t_7s_0 \\ s_3 & s_4 & s_5 & -t_2s_4 - t_3s_3 - t_4s_2 - t_5s_1 - t_6s_0 & -t_3s_4 - t_4s_3 - t_5s_2 - t_6s_1 - t_7s_0 & -t_4s_4 - t_5s_3 - t_6s_2 - t_7s_1 - t_8s_0 \end{pmatrix}$$

Finally with elementary row operations we get

$$= \det \begin{pmatrix} 0 & 0 & s_0 & 0 & 0 & 0 \\ 0 & s_0 & s_1 & -t_2s_0 & -t_3s_0 & -t_4s_0 \\ s_0 & 0 & s_2 - \frac{s_1^2}{s_0} & -t_3s_0 & -t_4s_0 & -t_5s_0 \\ s_1 & 0 & s_3 - \frac{s_2s_1}{s_0} & -t_3s_1 - t_4s_0 & -t_4s_1 - t_5s_0 & -t_5s_1 - t_6s_0 \\ s_2 & 0 & s_4 - \frac{s_3s_1}{s_0} & -t_3s_2 - t_4s_1 - t_5s_0 & -t_4s_2 - t_5s_1 - t_6s_0 & -t_5s_2 - t_6s_1 - t_7s_0 \\ s_3 & 0 & s_5 - \frac{s_4s_1}{s_0} & -t_3s_3 - t_4s_2 - t_5s_1 - t_6s_0 & -t_4s_3 - t_5s_2 - t_6s_1 - t_7s_0 & -t_5s_3 - t_6s_2 - t_7s_1 - t_8s_0 \end{pmatrix}$$

By iterating this procedure we get finally

$$= \det \begin{pmatrix} 0 & 0 & s_0 & 0 & 0 & 0 \\ 0 & s_0 & s_1 & -t_2 s_0 & -t_3 s_0 & -t_4 s_0 \\ s_0 & 0 & * & -t_3 s_0 & -t_4 s_0 & -t_5 s_0 \\ 0 & 0 & * & -t_4 s_0 & -t_5 s_0 & -t_6 s_0 \\ 0 & 0 & * & -t_5 s_0 & -t_6 s_0 & -t_7 s_0 \\ 0 & 0 & * & -t_6 s_0 & -t_7 s_0 & -t_8 s_0 \end{pmatrix} = (-1)^{\binom{3}{2}+3} \det \begin{pmatrix} t_4 & t_5 & t_6 \\ t_5 & t_6 & t_7 \\ t_6 & t_7 & t_8 \end{pmatrix} = (-1)^{\binom{3}{2}+3} \det \left(t_{i+j+2+2} \right)_{i,j=0}^{3-1}.$$

An analogous result also holds for $m = -1$.

Proposition 2.6

Let $s(x) = \sum_{n \geq 0} s_n x^n$ with $s_0 = 1$ and $t(x) = \frac{1}{s(x)} = \sum_{n \geq 0} t_n x^n$.

Then

$$\det \left(s_{i+j+1} \right)_{i,j=0}^n = (-1)^{n+1} \det \left(t_{i+j+1} \right)_{i,j=0}^n. \quad (2.26)$$

Proof

I will illustrate the proof for $n = 2$.

$$\begin{aligned} \det \begin{pmatrix} s_1 & s_2 & s_3 \\ s_2 & s_3 & s_4 \\ s_3 & s_4 & s_5 \end{pmatrix} &= \det \begin{pmatrix} s_1 & s_2 & -s_2 t_1 - s_1 t_2 - s_0 t_3 \\ s_2 & s_3 & -s_3 t_1 - s_2 t_2 - s_1 t_3 - s_0 t_4 \\ s_3 & s_4 & -s_4 t_1 - s_3 t_2 - s_2 t_3 - s_1 t_4 - s_0 t_5 \end{pmatrix} \\ &= \det \begin{pmatrix} s_1 & s_2 & -s_0 t_3 \\ s_2 & s_3 & -s_1 t_3 - s_0 t_4 \\ s_3 & s_4 & -s_2 t_3 - s_1 t_4 - s_0 t_5 \end{pmatrix} = \det \begin{pmatrix} s_1 & -s_1 t_1 - s_0 t_2 & -s_0 t_3 \\ s_2 & -s_2 t_1 - s_1 t_2 - s_0 t_3 & -s_1 t_3 - s_0 t_4 \\ s_3 & -s_3 t_1 - s_2 t_2 - s_1 t_3 - s_0 t_4 & -s_2 t_3 - s_1 t_4 - s_0 t_5 \end{pmatrix} \\ &= \det \begin{pmatrix} s_1 & -s_0 t_2 & -s_0 t_3 \\ s_2 & -s_1 t_2 - s_0 t_3 & -s_1 t_3 - s_0 t_4 \\ s_3 & -s_2 t_2 - s_1 t_3 - s_0 t_4 & -s_2 t_3 - s_1 t_4 - s_0 t_5 \end{pmatrix} = \det \begin{pmatrix} -s_0 t_1 & -s_0 t_2 & -s_0 t_3 \\ -s_1 t_1 - s_0 t_2 & -s_1 t_2 - s_0 t_3 & -s_1 t_3 - s_0 t_4 \\ -s_2 t_1 - s_1 t_2 - s_0 t_3 & -s_2 t_2 - s_1 t_3 - s_0 t_4 & -s_2 t_3 - s_1 t_4 - s_0 t_5 \end{pmatrix}. \end{aligned}$$

The last determinant obviously equals

$$\det \begin{pmatrix} -s_0 t_1 & -s_0 t_2 & -s_0 t_3 \\ -s_0 t_2 & -s_0 t_3 & -s_0 t_4 \\ -s_0 t_3 & -s_0 t_4 & -s_0 t_5 \end{pmatrix} = -\det \begin{pmatrix} t_1 & t_2 & t_3 \\ t_2 & t_3 & t_4 \\ t_3 & t_4 & t_5 \end{pmatrix}.$$

Corollary 2.2

Let

$$f(x) = \frac{1}{1 - axc(x)}. \quad (2.27)$$

Then

$$\det(f_{i+j})_{i,j=0}^n = a^n \det(c_{i+j+1})_{i,j=0}^{n-1} \quad (2.28)$$

and

$$\det(f_{i+j+1})_{i,j=0}^n = a^{n+1} \det(c_{i+j})_{i,j=0}^n. \quad (2.29)$$

Example 2.2

$$f(x, a_0, a_1, a_2, \dots) = \frac{1}{1 - \frac{a_0 x}{1 - \frac{a_1 x}{1 - \frac{a_2 x}{1 - \dots}}}} \quad (2.30)$$

implies

$$\det(f_{i+j}(a_0, a_1, a_2, \dots))_{i,j=0}^n = (a_0 a_1)^n (a_2 a_3)^{n-1} (a_4 a_5)^{n-2} \dots (a_{2n-2} a_{2n-1}). \quad (2.31)$$

Proof

Here we have

$$f(x, a_0, a_1, a_2, \dots) = \frac{1}{1 - a_0 x f(x, a_1, a_2, \dots)}$$

and

$$f(x, a_1, a_2, \dots) = \frac{1}{1 - a_1 x f(x, a_2, a_3, \dots)}.$$

Therefore

$$\det(f_{i+j}(a_0, a_1, a_2, \dots))_{i,j=0}^n = a_0^n \det(f_{i+j+1}(a_1, a_2, \dots))_{i,j=0}^{n-1}$$

and

$$\det(f_{i+j+1}(a_1, a_2, \dots))_{i,j=0}^{n-1} = a_1^n \det(f_{i+j}(a_2, a_3, \dots))_{i,j=0}^{n-1}.$$

This gives

$$\det(f_{i+j}(a_0, a_1, a_2, \dots))_{i,j=0}^n = (a_0 a_1)^n \det(f_{i+j}(a_2, a_3, a_4, \dots))_{i,j=0}^{n-1}.$$

By induction we get (2.31).

If all $a_n = 1$ then $f_n = C_n$.

These Propositions lead us to

Lemma 2.1

Let $m \geq -1$, $p \geq 1$ and let $f(x) = \sum_{n \geq 0} f_n x^n$ and $g(x) = \sum_{n \geq 0} g_n x^n$ be formal power series with $g_0 = 1$ and define $f_n = 0$ for $n < 0$.

For

$$f(x) = \frac{1}{1 - ax^p g(x)} \quad (2.32)$$

the Hankel determinants satisfy

$$\det(f_{i+j-m})_{i,j=0}^n = 0 \quad \text{for } n < m \quad (2.33)$$

$$\det(f_{i+j-m})_{i,j=0}^m = (-1)^{\binom{m+1}{2}} \quad (2.34)$$

and

$$\det(f_{i+j-m})_{i,j=0}^{n+m} = (-1)^{\binom{m+1}{2}} a^n \det(g_{i+j+m-p+2})_{i,j=0}^{n-1} \quad (2.35)$$

for $n > 0$ if we define $g_n = 0$ for $-p < n < 0$.

3. The main results

Theorem 3.1 (V. I. Buslaev [4])

Let $(b_n)_{n \geq -1}$ be a non-decreasing sequence of integers such that $b_{n+2} - b_n \geq 1$ with initial values $b_{-1} = -1$ and $b_0 = 0$ and let

$$f(x) = \sum_{n \geq 0} f_n x^n = \frac{1}{1 - \frac{a_0 x^{b_1 - b_{-1}}}{1 - \frac{a_1 x^{b_2 - b_0}}{1 - \frac{a_2 x^{b_3 - b_1}}{1 - \frac{a_3 x^{b_4 - b_2}}{1 - \ddots}}}}}. \quad (3.1)$$

Then the Hankel determinants

$$d(n) = \det(f_{i+j})_{i,j=0}^n \quad (3.2)$$

satisfy

$$d(b_k) = (-1)^{\binom{b_1 - b_0}{2} + \binom{b_2 - b_1}{2} + \dots + \binom{b_k - b_{k-1}}{2}} a_0^{b_k - b_0} a_1^{b_k - b_1} a_2^{b_k - b_2} a_3^{b_k - b_3} \dots a_{k-1}^{b_k - b_{k-1}} \quad (3.3)$$

and vanish for all other values of $n > 0$.

Proof

Let

$$f^{(k)}(x) = \frac{1}{1 - a_k x^{b_{k+1} - b_{k-1}} f^{(k+1)}(x)}. \quad (3.4)$$

Then $f(x) = f^{(0)}(x)$.

We want to prove that

$$\det \left(f_{i,j=0}^{(0)} \right)^{n+b_k} = (-1)^{\sum_{j=0}^{k-1} \binom{b_{j+1} - b_j}{2}} \prod_{j=0}^k a_j^{n+b_k - b_j} \det \left(f_{i,j=0}^{(k+1)} \right)^{n-1} \quad (3.5)$$

if we set $\det \left(f_{i,j=0}^{(k+2)} \right)^{n-1} = 1$ for $n = 0$.

This is true for $k = 0$. In this case it reduces to

$$\det \left(f_{i,j=0}^{(0)} \right)^n = a_0^n \det \left(f_{i,j=0}^{(1)} \right)^{n-1}.$$

Now suppose that (3.5) is true for k . This means that

$$\det \left(f_{i,j=0}^{(0)} \right)^{n+b_k} = (-1)^{\sum_{j=0}^{k-1} \binom{b_{j+1} - b_j}{2}} \prod_{j=0}^k a_j^{n+b_k - b_j} \det \left(f_{i,j=0}^{(k+1)} \right)^{n-1}.$$

Therefore

$$\det \left(f_{i,j=0}^{(0)} \right)^{n+b_{k+1}} = (-1)^{\sum_{j=0}^{k-1} \binom{b_{j+1} - b_j}{2}} \prod_{j=0}^k a_j^{n+b_{k+1} - b_j} \det \left(f_{i,j=0}^{(k+1)} \right)^{n-1+b_{k+1} - b_k}.$$

By Lemma 2.1 we have

$$\det \left(f_{i,j=0}^{(k+1)} \right)^{n-1+b_{k+1} - b_k} = (-1)^{\binom{b_{k+1} - b_k}{2}} a_{k+1}^n \det \left(f_{i,j=0}^{(k+2)} \right)^{n-1}.$$

Therefore (3.5) is proved.

Theorem 3.1 follows since $\det \left(f_{i,j=0}^{(k+2)} \right)^{n-1} = 0$ for $0 < n < b_{k+2} - b_{k+1} - 1$.

Remark

This theorem has also been proved by V.I. Buslaev [4] with another method.

It would also be interesting to have formulae for the Hankel determinants of

$$f(x) = \sum_{n \geq 0} f_n x^n = \frac{1}{1 - \frac{a_0 x^{m_0}}{1 - \frac{a_1 x^{m_1}}{1 - \frac{a_2 x^{m_2}}{1 - \frac{a_3 x^{m_3}}{\ddots}}}}}$$

for an arbitrary sequence of positive integers m_n instead of $b_n - b_{n-2}$. This has been attempted in [3], where it is claimed that all such Hankel determinants are products of a_n 's. But there are simple counter examples.

Let for example $(m_n) = (1, 2, 1, 1, 1, \dots)$. Then the Hankel determinants are

$$1, 0, -(a_0 a_1)^2, -(a_0 a_1)^3 (a_2 a_3) (a_3 + a_4), -(a_0 a_1)^4 (a_2 a_3)^2 (a_4 a_5) (a_3 a_5 + a_3 a_6 + a_4 a_6), \dots$$

The general terms will be given in (3.19). The corresponding sequence $(b_n)_{n \geq 0} = (0, 0, 2, 1, 3, 2, 4, 3, 5, 4, \dots)$ does not satisfy our assumptions.

If each $a_k = \pm 1$ in (3.1) then each non-zero Hankel determinant is also ± 1 .

Note that $m_n = b_{n+1} - b_{n-1}$ implies $b_{2n} = \sum_{j=0}^{n-1} m_{2n-1-2j}$ and $b_{2n+1} = \sum_{j=0}^n m_{2n-2j} - 1$.

The set $\{b_n\}_{n \geq 0}$ is the set of all $n \in \mathbb{N}$ such that $d(n) \neq 0$. If $b_i = b_{i+1}$ then $d(b_k)$ contains the term $(a_i a_{i+1})^{b_k - b_i}$, i.e. the powers of a_j in $d(n)$ have a similar "pattern" as the sequence $(b_n)_{n \geq 0}$.

Let for example $(b_n)_{n \geq 0} = (0, 0, 1, 2, 5, 6, 6, 8, 9, 10, \dots)$ i.e. $(p_n)_{n \geq 0} = (1, 1, 2, 4, 4, 1, 2, 3, 2, 2, 2, \dots)$.

Then the sequence of Hankel determinants is

$$1, (a_0 a_1), (a_0 a_1)^2 a_2, 0, 0, -(a_0 a_1)^5 a_2^4 a_3^3, -(a_0 a_1)^6 a_2^5 a_3^4 a_4, 0, (a_0 a_1)^8 a_2^7 a_3^6 a_4^3 (a_5 a_6)^2, (a_0 a_1)^9 a_2^8 a_3^7 a_4^4 (a_5 a_6)^3 a_7, \dots$$

Theorem 3.2

Let $(b_k)_{k \geq -1}$ be a non-decreasing sequence of integers such that $b_{k+2} - b_k \geq 1$ with initial values $b_{-1} = -1$ and $b_0 = 0$ and let $(B_n)_{n \geq 0}$ be the elements of the set $\{b_k\}_{k \geq -1}$ in increasing order.

If $B_n = b_{k-1}$ has multiplicity 1 or if $B_n = b_{k-2} = b_{k-1}$ has multiplicity 2 then $P_n(x) := \frac{p_{B_n+1}(x)}{d(B_n)} = r_k(x)$.

For $B_n + 1 < m < B_{n+1} - 1$ we get $p_m(x) = 0$ and for $m = B_{n+1} - 1$ we get

$$p_{B_{n+1}}(x) = \pm (a_0 a_1 \cdots a_{k-1})^{B_{n+1} - B_n - 1} p_{B_n+1}(x).$$

Proof

Consider

$$\Lambda \left(\frac{1}{\det(f_{i+j})_{i,j=0}^{b_{k-1}}} \det \begin{pmatrix} f_0 & f_1 & \cdots & f_{b_{k-1}} & 1 \\ f_1 & f_2 & \cdots & f_{b_{k-1}+1} & x \\ f_2 & f_3 & \cdots & f_{b_{k-1}+2} & x^2 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{b_{k-1}+1} & f_{b_{k-1}+2} & \cdots & f_{2b_{k-1}+1} & x^{b_{k-1}+1} \end{pmatrix} x^\ell \right)$$

$$= \frac{1}{\det(f_{i+j})_{i,j=0}^{b_{k-1}}} \det \begin{pmatrix} f_0 & f_1 & \cdots & f_{b_{k-1}} & f_\ell \\ f_1 & f_2 & \cdots & f_{b_{k-1}+1} & f_{\ell+1} \\ f_2 & f_3 & \cdots & f_{b_{k-1}+2} & f_{\ell+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{b_{k-1}+1} & f_{b_{k-1}+2} & \cdots & f_{2b_{k-1}+1} & f_{\ell+b_{k-1}+1} \end{pmatrix}$$

By (2.12) we have $\sum_j v_{k,j} f_{b_{k-1}-j+\ell+1} = 0$ for $\ell < b_k$. This means that the last column can be reduced to the 0-column by elementary column operations. Therefore the determinant vanishes for these ℓ . For $\ell = b_k$ we get by (2.13)

$\sum_j v_{k,j} f_{b_{k-1}-j+b_k+1} = a_0 \cdots a_{k-1}$ and therefore by elementary column operations

$$\Lambda \left(\frac{p_{B_n+1}(x)}{d(B_n)} x^{b_k} \right) = \frac{1}{\det(f_{i+j})_{i,j=0}^{b_{k-1}}} \det \begin{pmatrix} f_0 & f_1 & \cdots & f_{b_{k-1}} & 0 \\ f_1 & f_2 & \cdots & f_{b_{k-1}+1} & 0 \\ f_2 & f_3 & \cdots & f_{b_{k-1}+2} & 0 \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{b_{k-1}+1} & f_{b_{k-1}+2} & \cdots & f_{2b_{k-1}+1} & a_0 \cdots a_{k-1} \end{pmatrix} = a_0 \cdots a_{k-1}.$$

Therefore the polynomials $q(x) := \frac{p_{B_n+1}(x)}{d(B_n)}$ which has degree $b_{k-1} + 1$ and $r_k(x)$ which is of the same degree satisfy $\Lambda((q(x) - r_k(x))x^n) = 0$ for $0 \leq n \leq b_k$. This implies $q(x) = r_k(x)$ since the

rank of the matrix $\begin{pmatrix} f_0 & f_1 & \cdots & f_{b_{k-1}} & f_{b_k} \\ f_1 & f_2 & \cdots & f_{b_{k-1}+1} & f_{b_k+1} \\ f_2 & f_3 & \cdots & f_{b_{k-1}+2} & f_{b_k+2} \\ \vdots & \vdots & \vdots & \vdots & \vdots \\ f_{b_{k-1}+1} & f_{b_{k-1}+2} & \cdots & f_{2b_{k-1}+1} & f_{b_k+b_{k-1}+1} \end{pmatrix}$

is $b_{k-1} + 1$.

To prove the other assertion observe that (2.12) implies that at least one column in (2.14) can be reduced to the zero column and therefore $p_m(x) = 0$ for $B_n + 1 < m < B_{n+1} - 1$.

For $m = B_{n+1} = b_k$ column reduction reduces column $b_{k-1} + 1$ which is

$$\begin{pmatrix} f_{b_{k-1}+1} \\ f_{b_{k-1}+2} \\ \vdots \\ f_{b_k-1} \\ f_{b_k} \end{pmatrix} \text{ to } \begin{pmatrix} 0 \\ \vdots \\ 0 \\ a_0 \cdots a_{k-1} \end{pmatrix}.$$

Now expand the determinant with respect to this column and iterate $b_k - b_{k-1} - 1$ times.

As an example consider $(b_k)_{k \geq -1} = (-1, 0, 0, 3, 3, 7, 8, 9, \dots)$ and therefore $(B_n + 1)_{n \geq 0} = (0, 1, 4, 8, 9, 10, \dots)$.

For $B_{-1} = b_{-1} = -1$ we set $p_0(x) = r_0(x) = 1$.

For $B_0 = 0 = b_0 = b_1$ we get $\frac{p_1(x)}{d(0)} = r_2(x)$,

for $B_1 = 3 = b_2 = b_3$ we get $\frac{p_4(x)}{d(3)} = r_4(x)$,

for $B_2 = 7 = b_4$ we get $\frac{p_8(x)}{d(7)} = r_5(x)$, etc.

The remaining polynomials $p_n(x)$ are $p_2(x) = 0$,

$p_3(x) = -(a_0 a_1)^2 r_2$, because $p_3(x) = p_{B_2}(x)$ and $B_1 = b_0 = b_1$,

$p_5(x) = p_6(x) = 0$,

$p_7(x) = p_{B_3}(x)$ and since $B_2 = b_2 = b_3$ we have

$p_7(x) = p_{B_3}(x) = (a_0 a_1 a_2 a_3)^3 p_4(x) = (a_0 a_1 a_2 a_3)^3 d(3) r_4(x)$, etc.

Let us show in detail how to derive $p_3(x)$.

$$p_3(x) = \det \begin{pmatrix} 1 & a_0 & a_0^2 & 1 \\ a_0 & a_0^2 & a_0^3 & x \\ a_0^2 & a_0^3 & a_0^4 + a_0 a_1 & x^2 \\ a_0^3 & a_0^4 + a_0 a_1 & a_0^5 + 2a_0^2 a_1 & x^3 \end{pmatrix} = \det \begin{pmatrix} 1 & 0 & 0 & 1 \\ a_0 & 0 & 0 & x \\ a_0^2 & 0 & a_0 a_1 & x^2 \\ a_0^3 & a_0 a_1 & a_0 a_1 & x^3 \end{pmatrix} = -(a_0 a_1)^2 \det \begin{pmatrix} 1 & 1 \\ a_0 & x \end{pmatrix}.$$

First we give two well-known examples from this point of view.

Example 3.1

The special case $b_n = n$ and therefore $m_n = 2$ gives again (2.21) and therefore

$$d(n) = a_0^n a_1^{n-1} a_2^{n-2} \cdots a_{n-1}.$$

The polynomials $r_k(x)$ have degree $\deg r_k = k$ and satisfy $r_k(x) = x r_{k-1}(x) - a_{k-2} r_{k-2}(x)$ with

$$r_{-1}(x) = 0 \text{ and } r_0(x) = 1. \text{ They are orthogonal with respect to } \Lambda \text{ and satisfy } r_k(x) = \frac{P_k(x)}{d(k-1)} = P_k(x).$$

By (2.14) we get $d_1(k) = \det(f_{i+j+1})_{i,j=0}^n = p_{k+1}(0)$ and therefore $\frac{d_1(k)}{d(k)} = r_{k+1}(0)$.

It is clear that $r_{2k+1}(0) = 0$ and therefore $d_1(2n) = 0$.

Furthermore we get $r_{2k}(0) = -a_{2k-2} r_{2k-2}(0) = \cdots = (-1)^{k-1} a_0 a_2 \cdots a_{2(k-1)}$.

This gives for $n > 0$

$$d_1(2n-1) = (-1)^n a_0^{2n} (a_1 a_2)^{2n-2} (a_3 a_4)^{2n-4} \cdots (a_{2n-3} a_{2n-2})^2. \quad (3.6)$$

The most important special case and prototype for many papers on Hankel determinants is

Example 3.1.1

Let $b_n = n$ and all $a_n = 1$. Then by (2.24) $f(x) = \sum_{n \geq 0} C_n x^{2n}$. In this case $d(n) = 1$ for all n . By this property the Catalan numbers are uniquely determined.

If we define the Fibonacci polynomials $Fib_n(x) = \sum_{k=0}^{\lfloor \frac{n-1}{2} \rfloor} \binom{n-1-k}{k} (-1)^k x^{n-1-2k}$ by

$Fib_n(x) = x Fib_{n-1}(x) - Fib_{n-2}(x)$ with initial values $Fib_0(x) = 0$ and $Fib_1(x) = 1$, then

$$r_k(x) = \det(f_{i+j} x - f_{i+j+1})_{i,j=0}^{k-1} = Fib_{k+1}(x) \quad (3.7)$$

if $f_{2n} = C_n$ and $f_{2n+1} = 0$.

In this case (2.12) and (2.13) are equivalent with

$$\Lambda(Fib_{2k}(x) x^{2n}) = \sum_{j=0}^k (-1)^j \binom{2k-j}{j} C_{k+n-j} = 0 \text{ for } n < k \text{ and } = 1 \text{ for } n = k,$$

and to

$$\Lambda(Fib_{2k+1}(x) x^{2n+1}) = \sum_{j=0}^k (-1)^j \binom{2k+1-j}{j} C_{k+n+1-j} = 0 \text{ for } n < k \text{ and } = 1 \text{ for } n = k.$$

Example 3.2

For $(b_n)_{n \geq 0} = (0, 0, 1, 1, 2, 2, 3, 3, \dots)$ we get

$$f(x) = \sum_{n \geq 0} f_n x^n = \frac{1}{1 - \frac{a_0 x}{1 - \frac{a_1 x}{1 - \frac{a_2 x}{1 - \ddots}}}}. \quad (3.8)$$

and therefore as in (2.31)

$$d(n) = (a_0 a_1)^n (a_2 a_3)^{n-1} \cdots (a_{2n-2} a_{2n-1}). \quad (3.9)$$

In this case

$$d_1(n) = a_0^{n+1} (a_1 a_2)^n (a_3 a_4)^{n-1} \cdots (a_{2n-1} a_{2n}). \quad (3.10)$$

This follows immediately from $\det(f_{i+j+1})_{i,j=0}^n = a_0^{n+1} \det(f_{i+j}^{(1)})_{i,j=0}^n$ and (3.9).

Later we shall need $d_2(n) = \det(f_{i+j+2})_{i,j=0}^n$. To compute this determinant we can use the condensation formula (cf. [8], (2.16))

$$d_2(n)d(n) = d_2(n-1)d(n+1) + d_1(n)^2, \quad (3.11)$$

which by (3.9) and (3.10) reduces to

$$\frac{d_2(n)}{d_1(n)} = a_{2n+1} \frac{d_2(n-1)}{d_1(n-1)} + a_0 a_2 \cdots a_{2n} = \sum_{i_1 < i_2 - 1 < i_3 - 2 < \cdots < i_n - n + 1 \leq n} a_{i_1} a_{i_2} \cdots a_{i_n}.$$

Therefore we get

$$d_2(n) = a_0^{n+1} (a_1 a_2)^n (a_3 a_4)^{n-1} \cdots (a_{2n-1} a_{2n}) \sum_{i_1 < i_2 - 1 < i_3 - 2 < \cdots < i_n - n + 1 \leq n} a_{i_1} a_{i_2} \cdots a_{i_n}. \quad (3.12)$$

If (3.8) holds we have $b_{2k} = b_{2k+1} = k$ and $B_n = n - 1$. Thus $B_n = n - 1 = b_{2n-2} = b_{2n-1}$ and

$$r_{2n}(x) = \frac{P_n(x)}{d(n-1)} = P_n(x).$$

In this case there is also a simple recursion for $P_n(x)$.

We have $P_1(x) = x - a_0$ and $P_n(x) = (x - a_{2n-2} - a_{2n-3})P_{n-1}(x) - a_{2n-4}a_{2n-3}P_{n-2}(x)$ for $n \geq 2$.

This follows from

$$\begin{aligned} r_{2n}(x) &= r_{2n-1} - a_{2n-2}r_{2n-2} = xr_{2n-2}(x) - a_{2n-2}r_{2n-2} - a_{2n-3}r_{2n-3} \\ &= xr_{2n-2}(x) - a_{2n-2}r_{2n-2}(x) - a_{2n-3}(r_{2n-2}(x) + a_{2n-4}r_{2n-4}(x)). \end{aligned}$$

Corollary 3.1

A series $f(x) = \sum_{n \geq 0} f_n x^n$ has a representation as a continued fraction of the form (3.8) if and only if $d(n) \neq 0$ and $d_1(n) \neq 0$ for all $n \in \mathbb{N}$.

Proof

If no determinant vanishes then the numbers a_n can be uniquely computed from (3.9) and (3.10).

This is clear from

$$\frac{d_1(n)}{d(n)} = a_0 a_2 \cdots a_{2n} \quad \text{and} \quad \frac{d(n)}{d_1(n-1)} = a_1 a_3 \cdots a_{2n-1}.$$

Example 3.2.1

The simplest example is $f(x) = \sum_{n \geq 0} C_n x^n$ where $d(n) = d_1(n) = 1$.

Since in this case $d_2(n) = n + 2$ by (3.11) we see that $f(x) = \sum_{n \geq 0} C_{n+1} x^n$ has a representation of the

form (3.8) with $a_{2n} = \frac{n+2}{n+1}$ and $a_{2n+1} = \frac{n+1}{n+2}$.

Example 3.2.2

G. Eisenstein has shown (cf. Perron [9], §59, (32)) that $f(x) = \sum_{n \geq 0} q^{\binom{n+1}{2}} x^n$ has an expansion of the form (3.8) with $a_{2n} = q^{2n+1}$ and $a_{2n+1} = (q^{n+1} - 1)q^{n+1}$. (A simple proof can also be found in [6], (3.3) and (3.6)). For $q \rightarrow 1$ we get $a_{2n} \rightarrow 1$ and $a_{2n+1} \rightarrow 0$. So in some sense the continued fractions converge to $\frac{1}{1-x} = \sum_{n \geq 0} x^n$.

The Hankel determinants are $d(n) = q^{\frac{n(n+1)^2}{2}} \prod_{j=1}^n (q^j - 1)^{n+1-j}$.

The polynomials $r_k(x)$ are given by $r_{2k}(x) = \sum_{j=0}^k (-1)^j q^{kj} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j}$ and

$$r_{2k+1}(x) = \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{j(k+1)} x^{k+1-j}, \quad \text{where} \quad \begin{bmatrix} n \\ k \end{bmatrix} = \prod_{j=0}^{k-1} \frac{1-q^{n-j}}{1-q^{k-j}}$$

is a q -binomial coefficient.

These assertions are by (2.12) and (2.13) equivalent with

$$\Lambda(r_{2k}(x)x^m) = \sum_{j=0}^k (-1)^j q^{kj} \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{k-j+m+1}{2}} = 0 \text{ for } m < k \text{ and}$$

$$\Lambda(r_{2k}(x)x^k) = \sum_{j=0}^k (-1)^j q^{kj} \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{2k-j+1}{2}} = q^{k^2 + \binom{k+1}{2}} (q-1)(q^2-1)\cdots(q^k-1)$$

and

$$\Lambda(r_{2k+1}(x)x^m) = \sum_{j=0}^k (-1)^j q^{(k+1)j} \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{k-j+m+2}{2}} = 0 \text{ for } k < m \text{ and}$$

$$\Lambda(r_{2k+1}(x)x^k) = \sum_{j=0}^k (-1)^j q^{(k+1)j} \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{2k-j+2}{2}} = q^{(k+1)^2 + \binom{k+1}{2}} (q-1)(q^2-1)\cdots(q^k-1).$$

It is easy to show these identities directly by using the well-known formula

$$\sum_{j=0}^k \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2}} z^j = (1+z)(1+qz)\cdots(1+q^{k-1}z).$$

We get

$$\sum_{j=0}^k (-1)^j q^{kj} \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{k-j+m+1}{2}} = q^{\binom{k+1}{2} + \binom{m+1}{2} + km} \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2} - jm} = q^{\binom{k+1}{2} + \binom{m+1}{2} + km} (1-q^{-m})(1-q^{1-m})\cdots(1-q^{k-1-m}).$$

For $m < k$ the right-hand side vanishes and for $m = k$ get $q^{\binom{k+1}{2} + k^2} (q-1)\cdots(q^k-1)$

For $r_{2k+1}(x)$ we get

$$\begin{aligned} \sum_{j=0}^k (-1)^j q^{(k+1)j} \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{k-j+m+2}{2}} &= q^{\binom{k+2}{2} + \binom{m+2}{2} + km-1} \sum_{j=0}^k (-1)^j \begin{bmatrix} k \\ j \end{bmatrix} q^{\binom{j}{2} - jm} \\ &= q^{\binom{k+2}{2} + \binom{m+2}{2} + km-1} (1-q^{-m})(1-q^{1-m})\cdots(1-q^{k-1-m}). \end{aligned}$$

For $m = k$ this gives $q^{\binom{k+1}{2} + (k+1)^2} (q-1)\cdots(q^k-1)$.

As a special case we get that

$$\frac{\det \left(q^{\binom{i+j+1}{2}} x - q^{\binom{i+j+2}{2}} \right)_{i,j=0}^{k-1}}{d(k-1)} = r_{2k}(x) = \sum_{j=0}^k (-1)^j q^{kj} \begin{bmatrix} k \\ j \end{bmatrix} x^{k-j}.$$

For $q \rightarrow 1$ these polynomials converge to $r_{2k}(x) = (x-1)^k$ and $r_{2k+1}(x) = x(x-1)^k$, which are no longer in an analogous relation to the finite continued fraction $\frac{1}{1-x}$.

Example 3.2.3

For the sequence of Motzkin numbers $(M_n)_{n \geq 0} = (1, 1, 2, 4, 9, 21, 51, \dots)$ whose generating function

$$f(x) = \sum_{n \geq 0} M_n x^n = \frac{1 - x - \sqrt{1 - 2x - 3x^2}}{2x^2} \text{ satisfies} \quad f(x) = 1 + xf(x) + x^2 f(x)^2 \quad (3.13)$$

the Hankel determinants are $d(n) = 1$ and $(d_1(n))_{n \geq 0} = (1, 0, -1, -1, 0, 1, 1, 0, -1, -1, 0, 1, \dots)$ which is periodic with period 6.

Since some determinants vanish there is no representation of the form (3.8).

We have instead

$$f(x) = \frac{1}{1 - \frac{x}{1 - \frac{x}{1 - x^2 f(x)}}}.$$

For (3.13) implies $\frac{1}{f(x)} - 1 = -x(1 + xf(x))$ and $\frac{1}{1 + xf(x)} = 1 - \frac{x}{1 - x^2 f(x)}$.

Thus in this case $(b_n)_{n \geq -1} = (-1, 0, 0, 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, \dots)$ and $(m_n)_{n \geq 0} = (1, 1, 2, 1, 1, 2, \dots)$.

Example 3.2.4

If we consider the numbers $M_n(u) = \sum_{k=0}^{\lfloor \frac{n}{2} \rfloor} \binom{2k}{k} \frac{1}{k+1} \binom{n}{2k} u^{n-2k}$ whose generating function $f(x, u)$

satisfies $f(x, u) = 1 + uxf(x, u) + x^2 f(x, u)^2$ then the determinants are $d(n) = 1$ and $d_1(n) = \text{Fib}_{n+2}(u)$ (cf. e.g. [5], Theorem A*).

The sequence $(\text{Fib}_n(1)) = (0, 1, 1, 0, -1, -1, \dots)$ is periodic with period 6.

For $u \rightarrow 1$ the numbers $M_n(u)$ converge to the Motzkin numbers M_n .

Therefore for $u \neq 1$ in some neighborhood of 1 all $f(x, u)$ have a representation of the form (3.8).

It is given by $a_{2n} = \frac{\text{Fib}_{n+2}(u)}{\text{Fib}_{n+1}(u)}$ and $a_{2n-1} = \frac{\text{Fib}_n(u)}{\text{Fib}_{n+1}(u)}$. For $u \rightarrow 1$ we have $a_0 \rightarrow 1$, $a_1 \rightarrow 1$,

and $a_2 \rightarrow 0$. This explains why we have for the Motzkin numbers $m_2 > 1$.

Remark

If the sequence $(b_n)_{n \geq 0}$ contains all non-negative integers then $d(n) \neq 0$ for all n . In this case the above results are special cases of the well-known connection with orthogonal polynomials and continued J-fractions (cf. e.g. [8], (2.30) or [5]). Observe also that in this case all $m_n \in \{1, 2\}$ and each sequence of 1's must have even length.

Example 3.2.5

As in the example with the Motzkin numbers let the power sequence be $(m_n)_{n \geq 0} = (1, 1, 2, 1, 1, 2, \dots)$, i.e. $(b_n)_{n \geq -1} = (-1, 0, 0, 1, 2, 2, 3, 4, 4, 5, 6, 6, 7, \dots)$ with $b_{3k} = b_{3k+1} = 2k$ and $b_{3k+2} = 2k + 1$.

Since $B_{2n} = 2n - 1 = b_{3n-1}$ and $B_{2n+1} = 2n = b_{3n} = b_{3n+1}$ we get

$$P_{2n}(x) = \frac{p_{B_{2n+1}}(x)}{d(B_{2n})} = r_{3n}(x)$$

and

$$P_{2n+1}(x) = \frac{p_{B_{2n+1+1}}(x)}{d(B_{2n+1})} = r_{3n+2}(x).$$

The recurrences are

$$P_{2n+1}(x) = r_{3n+1}(x) - a_{3n}r_{3n}(x) = (xr_{3n}(x) - a_{3n-1}r_{3n-1}(x)) - a_{3n}r_{3n}(x) = (x - a_{3n})P_{2n}(x) - a_{3n-1}P_{2n-1}(x)$$

and since $r_{3n-1}(x) = r_{3n-2}(x) - a_{3n-3}r_{3n-3}(x)$

we get

$$\begin{aligned} P_{2n}(x) &= r_{3n}(x) = xr_{3n-1}(x) - a_{3n-2}r_{3n-2}(x) = xr_{3n-1}(x) - a_{3n-2}(r_{3n-1}(x) + a_{3n-3}r_{3n-3}(x)) \\ &= (x - a_{3n-2})P_{2n-1}(x) - a_{3n-3}a_{3n-2}P_{2n-2}(x). \end{aligned}$$

By Theorem 3.1 the Hankel transform is

$$1, (a_0a_1), (a_0a_1)^2 a_2, (a_0a_1)^3 a_2^2 (a_3a_4), (a_0a_1)^4 a_2^3 (a_3a_4)^2 a_5, \dots$$

or

$$d(2n) = d(b_{3n}) = d(b_{3n+1}) = (a_0a_1)^{2n} a_2^{2n-1} (a_3a_4)^{2n-2} \dots a_{3n-1}$$

and

$$d(2n+1) = d(b_{3n+2}) = (a_0a_1)^{2n+1} a_2^{2n} (a_3a_4)^{2n-1} \dots (a_{3n}a_{3n+1}).$$

Since $P_2(x) = (x - a_1)(x - a_0) - a_0a_1 = x(x - a_0 - a_1)$ we see that $P_2(0) = 0$ and therefore $d_1(1) = 0$.

The transformation into a J-fraction gives

$$\frac{1}{1 - \frac{a_0x}{1 - \frac{a_1x}{1 - \frac{a_2x^2}{1 - \frac{a_3x}{1 - \frac{a_4x}{1 - \frac{a_5x^2}{1 - \dots}}}}}}} = \frac{1}{1 - a_0x - \frac{a_0a_1x^2}{1 - a_1x - \frac{a_2x^2}{1 - a_3x - \frac{a_3a_4x^2}{1 - a_4x - \frac{a_5x^2}{1 - a_6x - \dots}}}}}$$

Example 3.3

Let $(b_k)_{k \geq -1} = (-1, 0, 0, 3, 5, 5, 8, 10, 10, 13, \dots)$ with $b_{3k} = b_{3k+1} = 5k$ and $b_{3k+2} = 5k + 3$.

Here we also have $P_{2n}(x) = r_{3n}(x)$ and $P_{2n+1}(x) = r_{3n+2}(x)$.

We get

$$P_{2n+1}(x) = r_{3n+1}(x) - a_{3n}r_{3n}(x) = (x^2r_{3n}(x) - a_{3n-1}r_{3n-1}(x)) - a_{3n}r_{3n}(x) = (x^2 - a_{3n})P_{2n}(x) - a_{3n-1}P_{2n-1}(x)$$

and from $r_{3n-1}(x) = r_{3n-2}(x) - a_{3n-3}r_{3n-3}(x)$

we get

$$\begin{aligned} P_{2n}(x) &= r_{3n}(x) = x^3r_{3n-1}(x) - a_{3n-2}r_{3n-2}(x) = x^3r_{3n-1}(x) - a_{3n-2}(r_{3n-1}(x) + a_{3n-3}r_{3n-3}(x)) \\ &= (x^3 - a_{3n-2})P_{2n-1}(x) - a_{3n-3}a_{3n-2}P_{2n-2}(x). \end{aligned}$$

Example 3.4

Let $m_n = m \geq 1$ for all n . This is equivalent with $b_{2k} = mk$ and $b_{2k-1} = mk - 1$.

The corresponding polynomials $r_k(x) = P_k^{(m)}(x)$ are given by $r_0(x) = 1$, $r_1(x) = x$,

$$r_{2k}(x) = x^{m-1}r_{2k-1}(x) - a_{2k-2}r_{2k-2}(x)$$

and

$$r_{2k+1}(x) = xr_{2k}(x) - a_{2k-1}r_{2k-1}(x).$$

Then $\deg P_{2k}^{(m)}(x) = mk$ and $\deg P_{2k+1}^{(m)}(x) = mk + 1$.

Let

$$f_2(x) = \frac{1}{1 - \frac{a_0x^2}{1 - \frac{a_1x^2}{1 - \frac{a_2x^2}{1 - \ddots}}}}$$

Then we have

$$f_m(x) = \frac{1}{1 - \frac{a_0x^m}{1 - \frac{a_1x^m}{1 - \frac{a_2x^m}{1 - \ddots}}}}$$

Proposition 3.1

Let $P_{2k}^{(2)}(x) = \sum_{j=0}^k u_{k,j}x^{2j}$ and $P_{2k-1}^{(2)}(x) = x \sum_{j=0}^{k-1} v_{k,j}x^{2j}$.

Then

$$P_{2k}^{(m)}(x) = \sum_{j=0}^k u_{k,j}x^{mj} \quad \text{and} \quad P_{2k-1}^{(m)}(x) = x \sum_{j=0}^{k-1} v_{k,j}x^{mj}.$$

Proof

Define linear functionals Λ_m by $\Lambda_m(x^{mm}) = c_n$ and $\Lambda_m(x^{mn+j}) = 0$ for $0 < j < m$.

The polynomials $P_k^{(m)}(x)$ are uniquely determined by

$$\Lambda_m(P_{2k}^{(m)}(x)x^n) = 0 \text{ for } n < mk \text{ and } \Lambda_m(P_{2k}^{(m)}(x)x^{mk}) = a_0 \cdots a_{2k-1}$$

and

$$\Lambda_m(P_{2k-1}^{(m)}(x)x^n) = 0 \text{ for } n < mk - 1 \text{ and } \Lambda_m(P_{2k-1}^{(m)}(x)x^{mk-1}) = a_0 \cdots a_{2k-2}.$$

Now we have

$$\Lambda_m\left(x^{mi} \sum_{j=0}^k u_{k,j} x^{mj}\right) = \Lambda_2\left(x^{2i} \sum_{j=0}^k u_{k,j} x^{2j}\right) = 0 \text{ for } i < k \text{ and } \Lambda_m\left(x^n \sum_{j=0}^k u_{k,j} x^{mj}\right) = 0 \text{ if } n < mk \text{ and}$$

$n \not\equiv 0 \pmod{m}$

and

$$\Lambda_m\left(x^{mk} \sum_{j=0}^k u_{k,j} x^{mj}\right) = \Lambda_2\left(x^{2k} \sum_{j=0}^k u_{k,j} x^{2j}\right) = a_0 \cdots a_{k-1}.$$

Therefore we must have $P_{2k}^{(m)}(x) = \sum_{j=0}^k u_{k,j} x^{mj}$.

The same argument applies in the second case.

For the Hankel determinants we get

$$d(nm) = d(b_{2n}) = (-1)^{\binom{m-1}{2}n} a_0^{nm} a_1^{1+(n-1)m} a_2^{(n-1)m} a_3^{1+(n-2)m} \cdots a_{2n-2}^m a_{2n-1},$$

$$d(nm-1) = d(b_{2n-1}) = (-1)^{\binom{m-1}{2}n} a_0^{nm-1} a_1^{(n-1)m} a_2^{(n-1)m-1} \cdots a_{2n-3}^m a_{2n-2}^{m-1}.$$

If all $a_i = a$ we get $f_m(x) = \sum_n f_n^{(m)} x^n = \sum_n a^n C_n x^{mn}$, where $C_n = \binom{2n}{n} \frac{1}{n+1}$ are the Catalan numbers.

The corresponding Hankel determinants which have been computed in [5] with another method are

$$\begin{aligned} \det\left(f_{i+j}^{(m)}\right)_{i,j=0}^{mn-1} &= (-1)^{\binom{m-1}{2}n} a^{n(mn-1)}, \\ \det\left(f_{i+j}^{(m)}\right)_{i,j=0}^{mn} &= (-1)^{\binom{m-1}{2}n} a^{n(mn+1)}, \end{aligned} \tag{3.14}$$

and

$$\det\left(f_{i+j}^{(m)}\right)_{i,j=0}^n = 0 \text{ for all other } n.$$

Under the same assumptions we have

$$\det\left(f_{i+j+1}^{(m)}\right)_{i,j=0}^{m-1} = (-1)^{\binom{m}{2}} a^{m^2}. \quad (3.15)$$

Since

$$\det\left(f_{i+j+1}^{(m)}\right)_{i,j=0}^{n-1} = a^n \det\left(f_{i+j-m+1}^{(m)}\right)_{i,j=0}^{n-1}$$

we get $\det\left(f_{i+j+1}^{(m)}\right)_{i,j=0}^{n-1} = 0$ for $0 < n < m-1$ and

$$\det\left(f_{i+j+1}^{(m)}\right)_{i,j=0}^{n+m-1} = a^{n+m} \det\left(f_{i+j-m+1}^{(m)}\right)_{i,j=0}^{n+m-1} = (-1)^{\binom{m}{2}} a^{2n+m} \det\left(f_{i+j+1}^{(m)}\right)_{i,j=0}^{n-1}.$$

In general we get with induction

$$\det\left(f_{i+j+1}^{(m)}\right)_{i,j=0}^{n+km-1} = (-1)^{\binom{m}{2}k} a^{2kn+k^2m} \det\left(f_{i+j+1}^{(m)}\right)_{i,j=0}^{n-1}.$$

This implies (3.15).

Example 3.5

If $b_n = m+n$ for $n > 0$ then the power sequence is $m+2, m+2, 2, 2, \dots$ and

$$f(x) = \sum_{n \geq 0} f_n x^n = \frac{1}{1 - \frac{a_0 x^{2+m}}{1 - \frac{a_1 x^{2+m}}{1 - \frac{a_2 x^2}{1 - \frac{a_3 x^2}{1 - \ddots}}}}}$$

Then $d(0) = 1$, $d(n) = 0$ for $0 < n < m+1$ and for $n > 0$

$$d(m+n) = (-1)^{\binom{m+1}{2}} a_0^{n+m} a_1^{n-1} a_2^{n-2} \dots a_{n-1}. \quad (3.16)$$

Example 3.6

Let $(p_n)_{n \geq 0} = (1, 2, 2, 1, 2, 2, 1, 2, 2, \dots)$. This gives

$$(b_n)_{n \geq 0} = (0, 0, 2, 2, 3, 4, 5, 5, 7, 7, 8, 9, 10, 10, 12, 12, 13, 14, \dots) \text{ with } b_{n+6} = b_n + 5.$$

The Hankel transform is

$$1, 0, -(a_0 a_1)^2, -(a_0 a_1)^3 (a_2 a_3), -(a_0 a_1)^4 (a_2 a_3)^2 a_4, -(a_0 a_1)^5 (a_2 a_3)^3 a_4^2 a_5, 0, (a_0 a_1)^7 (a_2 a_3)^5 a_4^3 (a_6 a_7)^2, (a_0 a_1)^8 (a_2 a_3)^6 a_4^5 a_5^4 (a_6 a_7)^3 (a_8 a_9), (a_0 a_1)^9 (a_2 a_3)^7 a_4^6 a_5^5 (a_6 a_7)^4 (a_8 a_9)^2 a_{10}, \dots$$

We conclude this paper with some examples where the assumptions about (b_k) are not satisfied.

Example 3.7

Let

$$f(x) = \frac{1}{1 - \frac{x^2}{1 - \frac{x}{1 - \frac{x^2}{1 - \dots}}}}$$

with power sequence $(2, 1, 2, 1, 2, 1, \dots)$. Then the Hankel transform is

$$(d(n))_{n \geq 0} = (1, 1, 0, -1, -1 - 1, 0, 1, 1, 1, 0, -1, \dots). \quad (3.17)$$

Proof

Here we have

$$f(x) = \frac{1}{1 - x^2 g(x)} \text{ with } g(x) = \frac{1}{1 - x f(x)}.$$

This implies $f(x) - x^2 f(x)g(x) = 1$ and $g(x) - x f(x)g(x) = 1$ and $f(x) - 1 = x(g(x) - 1)$.

Therefore the coefficients satisfy

$$f_{n+2} = g_{n+1}. \quad (3.18)$$

This gives by Lemma 2.1

$$\det(f_{i+j})_{i,j=0}^n = \det(g_{i+j})_{i,j=0}^{n-1} = \det(f_{i+j+1})_{i,j=0}^{n-2} = \det(g_{i+j-1})_{i,j=0}^{n-2} = -\det(f_{i+j+2})_{i,j=0}^{n-4}.$$

Using (3.18) we get

$$\det(f_{i+j})_{i,j=0}^n = -\det(f_{i+j+2})_{i,j=0}^{n-4} = -\det(g_{i+j+1})_{i,j=0}^{n-4} = -\det(f_{i+j})_{i,j=0}^{n-4}.$$

Since the initial values are

$$d(0) = 1, d(1) = 1, d(2) = 0, d(3) = -1 \text{ we get (3.17).}$$

For arbitrary coefficients a_n the first terms of the Hankel transform $(d(n))_{n \geq 0}$ are

$$1, a_0, 0, -a_0^3 a_1^2 a_2^2, -a_0^4 a_1^3 a_2^3 a_4, -a_0^5 a_1^4 a_2^4 a_3^3, -a_0^6 a_1^5 a_2^5 a_3^3 a_5 a_6 (a_4 a_7 - a_5 a_6), \\ a_0^7 a_1^6 a_2^6 a_3^4 a_4^2 a_5^2 (a_4 a_6^2 + a_5 a_6 a_7 a_8 - a_4 a_7 a_8), \dots$$

Note that in this case $(b_n)_{n \geq 0} = (0, 1, 1, 3, 2, 5, 3, 7, 4, \dots)$.

Example 3.8

Finally we compute all Hankel determinants for the power sequence $(p_n) = (1, 2, 1, 1, 1, \dots)$ which we already used for a counter example.

We write

$$f(x) = \frac{1}{1 - a_0 x g(x)}, \quad g(x) = \frac{1}{1 - a_1 x^2 h^{(0)}(x)}, \quad h^{(0)}(x) = \frac{1}{1 - a_2 x h^{(1)}(x)}$$

where $g(x)$ has power sequence $(2, 1, 1, 1, \dots)$ and $h^{(0)}(x)$ and $h^{(1)}(x)$ have power sequence $(1, 1, 1, \dots)$.

Then

$$d(n) = \det \left(f_{i+j} \right)_{i,j=0}^n = a_0^n \det \left(g_{i+j+1} \right)_{i,j=0}^{n-1}$$

Thus $d(0) = 1$ and $d(1) = 0$.

For $n > 1$ we get

$$\det \left(g_{i+j+1} \right)_{i,j=0}^{n-1} = a_1^n \det \left(h_{i+j-1}^{(0)} \right)_{i,j=0}^{n-1}$$

and

$$\det \left(h_{i+j-1}^{(0)} \right)_{i,j=0}^{n-1} = -a_2^{n-2} \det \left(h_{i+j+2}^{(1)} \right)_{i,j=0}^{n-3}.$$

Therefore $d(2) = -(a_0 a_1)^2$

and by (3.12) for $n \geq 0$

$$d(n+3) = -(a_0 a_1)^{n+3} a_2^{n+1} a_3^{n+1} (a_4 a_5)^n (a_6 a_7)^{n-1} \cdots (a_{2n+2} a_{2n+3}) \sum_{i_1 < i_2 - 1 < i_3 - 2 < \cdots < i_n - n + 1 \leq n} a_{i_1+3} a_{i_2+3} \cdots a_{i_n+3}. \quad (3.19)$$

If all $a_n = 1$ then $h^{(0)}(x) = h^{(1)}(x) = \sum_{n \geq 0} C_n x^n$.

Therefore

$$d(n+3) = \det \left(g_{i+j+1} \right)_{i,j=0}^{n+2} = \det \left(h_{i+j-1}^{(0)} \right)_{i,j=0}^{n+2} = -\det \left(h_{i+j+2}^{(1)} \right)_{i,j=0}^n = -(n+2).$$

Thus in this case we get

$$(d(n))_{n \geq 0} = (1, 0, -1, -2, -3, \dots).$$

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