Hankel determinants of q-exponential polynomials

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Abstract
We give simple proofs for the Hankel determinants of $q-$exponential polynomials.

Let $S(n,k)$ be the Stirling numbers of the second kind. Christian Radoux ([6]) has shown that the Hankel determinants of the exponential polynomials $B_n(x)=\sum_{k=0}^{n} S(n,k)x^k$ are given by

$$\det\left(B_{i+j}(x)\right)_{i,j=0}^{n-1} = x^{\binom{n}{2}} \prod_{j=0}^{n-1} j!. \tag{1}$$

In [2] I have proved some $q-$analogues of this result. Then Richard Ehrenborg [4] has given a combinatorial proof of one of these $q-$analogues.
In this paper I want to show that these $q-$analogues in some sense have simpler proofs than the original case.

We use the usual notations: For $n\in\mathbb{N}$ let $[n]=\frac{1-q^n}{1-q}$. The $q-$factorial is the product $[1]\cdot[2]\cdots[n]$ and the $q-$binomial coefficient $\begin{bmatrix} n \\ k \end{bmatrix}$ is defined by $\begin{bmatrix} n \\ k \end{bmatrix}=\frac{[n]!}{[k]![n-k]!}$ for $0\le k\le n$ and $\begin{bmatrix} n \\ k \end{bmatrix}=0$ else.

The $q-$Stirling numbers $S[n,k]$ of the second kind are defined by

$$S[n,k]=S[n-1,k-1]+[k]S[n-1,k] \tag{2}$$

with $S[0,k]=\left[k=0\right]$ and $S[n,0]=[n=0]$.

There are two natural $q-$analogues of the exponential polynomials:

$$\varphi_n(x)=\sum_{k=0}^{n} S[n,k]x^k \tag{3}$$

and

$$\Phi_n(x)=\sum_{k=0}^{n} q^{\binom{k}{2}} S[n,k]x^k . \tag{4}$$

Our aim is a simple proof of the following theorems.

Theorem 1
The Hankel determinants of the $q-$exponential polynomials $\varphi_n(x)$ are given by

$$\det\left(\varphi_{i+j}(x)\right)_{i,j=0}^{n-1}=x^{\binom{n}{2}}q^{\binom{n}{3}}\prod_{j=0}^{n-1}[j]! \tag{5}$$

and

$$\det\left(\varphi_{i+j+1}(x)\right)_{i,j=0}^{n-1}=x^{\binom{n+1}{2}}q^{\binom{n+1}{3}}\prod_{j=0}^{n-1}[j]!. \tag{6}$$

Theorem 2
The Hankel determinants of the $q-$exponential polynomials $\Phi_n(x)$ are

$$\det\left(\Phi_{i+j}(x)\right)_{i,j=0}^{n-1}=q^{2\binom{n}{3}}x^{\binom{n}{2}}\prod_{j=0}^{n-1}\left([j]!\left((1-q)x;q\right)_j\right) \tag{7}$$

and

$$\det\left(\Phi_{i+j+1}(x)\right)_{i,j=0}^{n-1}=q^{2\binom{n+1}{3}}x^{\binom{n+1}{2}}\prod_{j=0}^{n-1}\left([j]!\left((1-q)x;q\right)_j\right). \tag{8}$$

Here $(x;q)_n$ is defined by $(x;q)_n=\prod_{j=0}^{n-1}\left(1-q^jx\right)$.

The key for the simpler proofs is the well-known identity

$$(q-1)^{n-k}S[n,k]=\sum_i(-1)^{n-i}\binom{n}{i}\begin{bmatrix} i\\ k\end{bmatrix}. \tag{9}$$

The equivalence of (2) and (9) may be seen from the following computation:

$$\sum_i (-1)^{n-i}\binom{n}{i}\begin{bmatrix} i \\ k \end{bmatrix} = \sum_i (-1)^{n-i}\left(\binom{n-1}{i-1}+\binom{n-1}{i}\right)\begin{bmatrix} i \\ k \end{bmatrix}$$
$$= \sum_i (-1)^{n-i}\binom{n-1}{i-1}\begin{bmatrix} i \\ k \end{bmatrix} + \sum_i (-1)^{n-i}\binom{n-1}{i}\begin{bmatrix} i \\ k \end{bmatrix}$$
$$= \sum_i (-1)^{n-1-i}\binom{n-1}{i}\left(\begin{bmatrix} i+1 \\ k \end{bmatrix}-\begin{bmatrix} i \\ k \end{bmatrix}\right) = \sum_i (-1)^{n-1-i}\binom{n-1}{i} q^{i+1-k}\begin{bmatrix} i \\ k-1 \end{bmatrix}$$

$$= \sum_i (-1)^{n-1-i}\binom{n-1}{i}\begin{bmatrix} i \\ k-1 \end{bmatrix} + \sum_i (-1)^{n-1-i}\binom{n-1}{i}\begin{bmatrix} i \\ k-1 \end{bmatrix}(q^{i+1-k}-1)$$
$$= \sum_i (-1)^{n-1-i}\binom{n-1}{i}\begin{bmatrix} i \\ k-1 \end{bmatrix} + \sum_i (-1)^{n-1-i}\binom{n-1}{i}\begin{bmatrix} i \\ k \end{bmatrix}(q^{k}-1)$$

We need some other well-known results:

Lemma 1
For given sequences $s(n)$ and $t(n)$ define $a(n,k)$ by

$$\begin{aligned} &a(0,k)=[k=0] \\ &a(n,0)=s(0)a(n-1,0)+t(0)a(n-1,1) \\ &a(n,k)=a(n-1,k-1)+s(k)a(n-1,k)+t(k)a(n-1,k+1). \end{aligned} \tag{10}$$

Then the Hankel determinant $\det\left(a(i+j,0)\right)_{i,j=0}^{n-1}$ *is given by*

$$\det\left(a(i+j,0)\right)_{i,j=0}^{n-1} = \prod_{i=1}^{n-1}\prod_{k=0}^{i-1} t(k). \tag{11}$$

For a proof see e.g. [3].

Remark
In most cases we start with $a(n)=a(n,0)$ and want to find the corresponding $s(n)$ and $t(n)$. It is then convenient to compute the first values of the orthogonal polynomials $p(n,x)$ (cf. [3] (1.10)) and their Favard resolution [3] (1.11) and try to guess $s(n)$ and $t(n)$. Then guess the explicit form of $a(n,k)$. Afterwards it remains to verify (10) in order to obtain a rigorous proof.

Lemma 2
Define the binomial transform of a sequence (a_n) *by* $BIN(a_n)=(b_n)$ *with* $b_n=\sum_{k=0}^{n}\binom{n}{k}a_k$.
Then

$$\det\left(a_{i+j}\right)=\det\left(b_{i+j}\right). \tag{12}$$

A simple proof can be found in [7].

Further observe that

$$\det\left(x^{i+j}a_{i+j}\right)_{i,j=0}^{n-1} = x^{n(n-1)}\det\left(a_{i+j}\right)_{i,j=0}^{n-1}. \tag{13}$$

From (9) we get

$$\varphi_n(x) = \sum_{k=0}^{n} S[n,k]x^k = \sum_k x^k (q-1)^{k-n} \sum_i (-1)^{n-i} \binom{n}{i} \begin{bmatrix} i \\ k \end{bmatrix}$$

$$= \sum_i (-1)^{n-i} \binom{n}{i} (q-1)^{-n} \sum_k (q-1)^k x^k \begin{bmatrix} i \\ k \end{bmatrix}.$$

In terms of the Rogers-Szegö polynomials $r_n(x)$ defined by

$$r_n(x) = \sum_{k=0}^{n} \begin{bmatrix} n \\ k \end{bmatrix} x^k \tag{14}$$

this means

$$\varphi_n(x) = \frac{1}{(1-q)^n} \sum_i \binom{n}{i} (-1)^i r_i\left((q-1)x\right). \tag{15}$$

By (13) and (12) this implies that

$$\det\left(\varphi_{i+j}(x)\right)_{i,j=0}^{n-1} = \frac{1}{(q-1)^{\binom{n}{2}}} \det\left(r_{i+j}(x)\right)_{i,j=0}^{n-1}. \tag{16}$$

Therefore we have only to determine the Hankel determinants of the Rogers-Szegö polynomials. These are also well-known (cf. [5]), but can also be obtained in a trivial way from (11).

The Rogers-Szegö polynomials satisfy the recurrence (cf. e.g. [1])

$$r_n(x) = (x+1)r_{n-1}(x) + (q^{n-1}-1)x r_{n-2}(x). \tag{17}$$

Let now $s(k) = q^k(x+1)$ und $t(k) = q^k x(q^{k+1}-1)$.

Then it is easily verified that the corresponding $a(n,k)$ are given by

$$a(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} r_{n-k}(x). \tag{18}$$

We have only to check that (10) holds:

$$\begin{bmatrix} n \\ k \end{bmatrix} r_{n-k}(x) = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} r_{n-k}(x) + q^k(x+1)\begin{bmatrix} n-1 \\ k \end{bmatrix} r_{n-k-1}(x) + q^k x(q^{k+1}-1)\begin{bmatrix} n-1 \\ k+1 \end{bmatrix} r_{n-k-2}(x)$$

or

$$\begin{bmatrix} n-1 \\ k \end{bmatrix} r_{n-k}(x) = (x+1)\begin{bmatrix} n-1 \\ k \end{bmatrix} r_{n-k-1}(x) + x(q^{n-k-1}-1)\begin{bmatrix} n-1 \\ k \end{bmatrix} r_{n-k-2}(x)$$

which is immediate from (17).

Therefore by (11) we get

Lemma 3

$$\det\left(r_{i+j}(x)\right)_{i,j=0}^{n-1} = x^{\binom{n}{2}} q^{\binom{n}{3}} (q-1)^{\binom{n}{2}} \prod_{j=0}^{n-1} [j]!. \tag{19}$$

This immediately implies (5).

Let now D denote the $q-$differentiation operator, defined by $Df(x) = \frac{f(x) - f(qx)}{(1-q)x}.$

Then

$$\varphi_n(x) = x(1+D)\varphi_{n-1}(x). \tag{20}$$

For

$\varphi_n(x) = \sum_k S[n,k]x^k = \sum_k S[n-1,k-1]x^k + \sum_k S[n-1,k][k]x^k$

$= x\varphi_{n-1}(x) + xD\varphi_{n-1}(x).$

This implies

$$\varphi_n(x) = \left(x(1+D)\right)^n 1. \tag{21}$$

Let now ε be the linear operator defined by $\varepsilon f(x) = f(qx).$ Then

$\varepsilon x D \varepsilon^{-1} x^n = \varepsilon x D \frac{x^n}{q^n} = [n]x^n = xDx^n,$ d.h. $\varepsilon x D \varepsilon^{-1} = xD.$

Therefore we get

$\varphi_{n+1}(x) = \left(x(1+D)\right)^{n+1} 1 = x\left((1+D)x\right)^n (1+D)1 = x\left((1+D)x\right)^n 1.$

Now $Dxx^n = [n+1]x^n = (q[n]+1)x^n = \left(qxD+1\right)x^n$ for all $n.$ Thus $Dx = 1 + qxD.$

This implies

$\varphi_{n+1}(x) = x\left((1+D)x\right)^n 1 = x\left(1 + x + qxD\right)^n 1 = x\varepsilon^{-1}\left(1 + qx(1+D)\right)^n 1$

and therefore the recurrence relation

$$\varphi_{n+1}(x) = x\sum_{k=0}^{n} \binom{n}{k} q^k \varphi_k\left(\frac{x}{q}\right). \tag{22}$$

By (12) this implies (6). Therefore Theorem 1 is proved.

Formula (5) can be slightly generalized in the following form:
Let $c \geq 1$ be an integer and

$$\psi_n(x,c) = \left(x^c + xD\right)^n 1. \tag{23}$$

Then we get

$$\psi_n(x,c) = \sum_{k=0}^{n} S[n,k,q^c]x^{kc}\left([c]\right)^{n-k}, \tag{24}$$

where $S[n,k,q^c] = S[n,k]_{q \to q^c}.$

For

$$\left(x^c + xD\right)\sum_{k=0}^{n-1} S[n-1,k,q^c]x^{kc}[c]^{n-k-1} =$$

$$\sum_k S[n-1,k,q^c]x^{(k+1)c}\left([c]\right)^{n-k-1} + \sum_k S[n-1,k,q^c]x^{kc}\left[kc\right]\left([c]\right)^{n-k-1}$$

$$= \sum_k x^{kc}\left([c]\right)^{n-k}\left(S[n-1,k-1,q^c] + \frac{[kc]}{[c]}S[n-1,k,q^c]\right) = \sum_{k=0}^{n} S[n,k,q^c]x^{kc}\left([c]\right)^{n-k}.$$

From

$$\psi_n(x,c) = \sum_{k=0}^{n} S[n,k,q^c]x^{kc}\left([c]\right)^{n-k} = \sum_k x^{ck}(q-1)^{k-n}\sum_i (-1)^{n-i}\binom{n}{i}\begin{bmatrix} i \\ k \end{bmatrix}_c$$

$$= \sum_i (-1)^{n-i}\binom{n}{i}(q-1)^{-n}\sum_k (q-1)^k x^{ck}\begin{bmatrix} i \\ k \end{bmatrix}_c$$

we get in the same way as above

$$\det\left(\psi_{i+j}(x,c)\right)_{i,j=0}^{n-1} = \frac{1}{(q-1)^{2\binom{n}{2}}}\det\left(r_{i+j}((q-1)x^c)\Big|_{q^c}\right)_{i,j=0}^{n-1} = \frac{\left(q^c-1\right)^{\binom{n}{2}}}{(q-1)^{\binom{n}{2}}}x^{c\binom{n}{2}}q^{c\binom{n}{3}}\prod_{j=0}^{n-1}[j]_c!.$$

This gives

$$\det\left(\psi_{i+j}(x,c)\right)_{i,j=0}^{n-1} = [c]^{\binom{n}{2}}x^{c\binom{n}{2}}q^{c\binom{n}{3}}\prod_{j=0}^{n-1}[j]_c!. \qquad (25)$$

In the same way as above we get

$$\psi_{n+1}(x,c) = x^c\sum_{k=0}^{n}\binom{n}{k}q^{kc}\psi_k\left(\frac{x}{q},c\right)[c]^{n-k}. \qquad (26)$$

This implies

$$\det\left(\psi_{i+j+1}(x,c)\right) = x^{nc}q^{c\binom{n}{2}}\det\left(\psi_{i+j}(x,c)\right). \qquad (27)$$

Proof of Theorem 2

Consider now the $q-$exponential polynomials $\Phi_n(x)$.

From (9) we get

$$\Phi_n(x)=\sum_{k=0}^{n} q^{\binom{k}{2}} S[n,k]x^k = \sum_k q^{\binom{k}{2}} x^k (q-1)^{k-n} \sum_i (-1)^{n-i} \binom{n}{i} \begin{bmatrix} i \\ k \end{bmatrix}$$

$$= \sum_i (-1)^{n-i} \binom{n}{i} (q-1)^{-n} \sum_k q^{\binom{k}{2}} (q-1)^k x^k \begin{bmatrix} i \\ k \end{bmatrix}.$$

Observing that $(x;q)_n = \prod_{j=0}^{n-1}\left(1-q^j x\right) = \sum_{k=0}^{n} (-1)^k \begin{bmatrix} n \\ k \end{bmatrix} q^{\binom{k}{2}} x^k$ we see that

$$\Phi_n(x) = \frac{1}{(1-q)^n} \sum_i \binom{n}{i} (-1)^i \left((1-q)x;q\right)_i . \tag{28}$$

This implies that

$$\det\left(\Phi_{i+j}(x)\right)_{i,j=0}^{n-1} = \frac{1}{(1-q)^{2\binom{n}{2}}} \det\left(\left((1-q)x;q\right)_{i+j}\right)_{i,j=0}^{n-1}. \tag{29}$$

Therefore we have only to determine the Hankel determinants of the polynomials $\left(x;q\right)_n$.
Let $s(n) = q^n + q^{n-1}x\left(1-q^n(1+q)\right)$
and $t(n) = q^{2n}(1-q^{n+1})x(1-q^n x)$.
If we define $a(n,k)$ by (10) then we get

$$a(n,k) = \begin{bmatrix} n \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k}.$$

To prove this assertion we have to verify that

$$\begin{bmatrix} n \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \left(q^{k-1} x;q\right)_{n-k} + s(k) \begin{bmatrix} n-1 \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k-1} + t(k) \begin{bmatrix} n-1 \\ k+1 \end{bmatrix} \left(q^{k+1} x;q\right)_{n-k-2}$$

holds. This is equivalent with

$$\begin{bmatrix} n \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \left(q^{k-1} x;q\right)_{n-k} + \left(q^k + q^{k-1}x - q^{2k-1}x - q^{2k}x\right) \begin{bmatrix} n-1 \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k-1}$$

$$+(q^{2k} - q^{n+k-1})x \begin{bmatrix} n-1 \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k-1}$$

or

$$\begin{bmatrix} n \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k} = \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \left(q^{k-1} x;q\right)_{n-k} + \left(q^k + q^{k-1}x - q^{2k-1}x - q^{n+k-1}x\right) \begin{bmatrix} n-1 \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k-1}$$

$$= \begin{bmatrix} n-1 \\ k-1 \end{bmatrix} \left(q^{k-1} x;q\right)_{n-k} + q^k \begin{bmatrix} n-1 \\ k \end{bmatrix} \left(q^{k-1} x;q\right)_{n-k} + q^{k-1}x(1-q^n) \begin{bmatrix} n-1 \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k-1}$$

$$= \begin{bmatrix} n \\ k \end{bmatrix} \left(q^{k-1} x;q\right)_{n-k} + q^{k-1}x(1-q^{n-k}) \begin{bmatrix} n \\ k \end{bmatrix} \left(q^k x;q\right)_{n-k-1}$$

which is obviously true.

Thus we get

Lemma 4

$$\det\left(\left(x;q\right)_{i+j}\right)_{i,j=0}^{n-1} = q^{2\binom{n}{3}}(1-q)^{\binom{n}{2}} x^{\binom{n}{2}} \prod_{j=0}^{n-1}[j]!\left(x;q\right)_j . \tag{30}$$

and

$$\det\left(\left(x;q\right)_{i+j+1}\right)_{i,j=0}^{n-1} = q^{2\binom{n}{3}+\binom{n}{2}}(1-q)^{\binom{n}{2}} x^{\binom{n}{2}} \prod_{j=0}^{n-1}[j]!\left(x;q\right)_{j+1} . \tag{31}$$

The second equation follows from $\left(x;q\right)_{n+1} = (1-x)\left(qx;q\right)_n$.

(30) implies immediately (7).

Remark

The special case $x = q$ gives the well-known result

$$\det\left([i+j]!\right)_{i,j=0}^{n-1} = q^{\frac{n(n-1)(2n-1)}{6}} \prod_{j=0}^{n-1}[j]!^2 . \tag{32}$$

As another special case we consider the $q-$Stirling numbers $s[n,k]$ of the first kind. They satisfy $s[n+1,k] = s[n,k-1]-[n]s[n,k]$ with initial values $s[0,k]=[k=0]$ and $s[n,0]=[n=0]$.

Then it is easily verified that

$$\sum_{k=0}^{n} s[n,k]x^k = \left\langle x\right\rangle_n := \prod_{j=0}^{n-1}\left(x-[j]\right). \tag{33}$$

From $\left\langle x\right\rangle_n = \dfrac{(1+(q-1)x)^n}{(q-1)^n}\left(\dfrac{1}{1+(q-1)x};q\right)_n$

we get

$$\det\left(\left\langle x\right\rangle_{i+j}\right)_{i,j=0}^{n-1} = (-1)^{\binom{n}{2}} q^{2\binom{n}{3}} \prod_{j=0}^{n-1}[j]!\left\langle x\right\rangle_j . \tag{34}$$

In order to obtain the second Hankel determinant we observe that

$$\Phi_n(x)=\sum_k q^{\binom{k}{2}}S[n,k]x^k=\sum_k q^{\binom{k}{2}}S[n-1,k-1]x^k+\sum_k q^{\binom{k}{2}}S[n-1,k][k]x^k$$
$$=x\Phi_{n-1}(qx)+xD\Phi_{n-1}(x)$$

or

$$\Phi_n(x)=\left(x\varepsilon+xD\right)\Phi_{n-1}(x)=\left(x\varepsilon+xD\right)^n 1. \tag{35}$$

This gives

$$\Phi_{n+1}(x)=\left(x\varepsilon+xD\right)^{n+1}1=x\left((\varepsilon+D)x\right)^n\left(\varepsilon+D\right)1=x\left((\varepsilon+D)x\right)^n 1$$
$$=x\left(\varepsilon x+qxD+1\right)^n 1=x\left(q(x\varepsilon+xD)+1\right)^n 1.$$

Therefore

$$\Phi_{n+1}(x)=x\sum_{k=0}^{n}\binom{n}{k}q^k\Phi_k(x). \tag{36}$$

By (12) this implies (8). Therefore Theorem 2 is proved.

This theorem can be generalized in the same way as above.
Let $c\geq 1$ be an integer and

$$\Psi_n(x,c)=\left(x^c\varepsilon+xD\right)^n 1. \tag{37}$$

Then we get

$$\Psi_n(x,c)=\sum_{k=0}^{n}S[n,k,q^c]x^{kc}q^{c\binom{k}{2}}\left([c]\right)^{n-k}, \tag{38}$$

where $S[n,k,q^c]=S[n,k]_{q\to q^c}$.

From

$$\Psi_n(x,c)=\sum_{k=0}^{n}S[n,k,q^c]x^{kc}q^{c\binom{k}{2}}\left([c]\right)^{n-k}=\sum_k x^{ck}q^{c\binom{k}{2}}(q-1)^{k-n}\sum_i(-1)^{n-i}\binom{n}{i}\begin{bmatrix}i\\k\end{bmatrix}_c$$
$$=\sum_i(-1)^{n-i}\binom{n}{i}(q-1)^{-n}\sum_k(q-1)^k x^{ck}q^{c\binom{k}{2}}\begin{bmatrix}i\\k\end{bmatrix}_c$$

we get in the same way as above

$$\det\left(\Psi_{i+j}(x,c)\right)_{i,j=0}^{n-1}=\frac{1}{(q-1)^{2\binom{n}{2}}}\det\left(\left((1-q)x^c;q^c\right)_{i+j}\right)_{i,j=0}^{n-1}=[c]^{\binom{n}{2}}x^{c\binom{n}{2}}q^{2c\binom{n}{3}}\prod_{j=0}^{n-1}[j]_c!\left((1-q)x^c;q^c\right)_j$$

This gives

$$\det\left(\Psi_{i+j}(x,c)\right)_{i,j=0}^{n-1}=[c]^{\binom{n}{2}}x^{c\binom{n}{2}}q^{2c\binom{n}{3}}\prod_{j=0}^{n-1}[j]_c!\left((1-q)x^c;q^c\right)_j. \tag{39}$$

Furthermore we get

$$\Psi_{n+1}(x,c)=x^c\sum_{k=0}^{n}\binom{n}{k}[c]^{n-k}q^{kc}\Psi_k(x,c). \tag{40}$$

This implies finally

$$\det\left(\Psi_{i+j+1}(x,c)\right)_{i,j=0}^{n-1}=x^{nc}q^{2c\binom{n}{2}}\det\left(\Psi_{i+j}(x,c)\right)_{i,j=0}^{n-1}. \tag{41}$$

Remark

The method used for the Rogers-Szegö polynomials can also be applied to the $q-$analogue of the Hermite polynomials defined by

$$H_n(x)=xH_{n-1}(x)-[n-1]H_{n-2}(x) \tag{42}$$

with initial values
$H_0(x)=1$ and $H_1(x)=x.$

If we choose $s(n)=q^n x$ and $t(n)=-q^n[n+1]$ we get

$$a(n,k)=\begin{bmatrix}n\\k\end{bmatrix}H_{n-k}(x).$$

We have only to check that

$$\begin{bmatrix}n\\k\end{bmatrix}H_{n-k}(x)=\begin{bmatrix}n-1\\k-1\end{bmatrix}H_{n-k}(x)+s(k)\begin{bmatrix}n-1\\k\end{bmatrix}H_{n-k-1}(x)+t(k)\begin{bmatrix}n-1\\k+1\end{bmatrix}H_{n-k-2}(x)$$

or

$$q^k\begin{bmatrix}n-1\\k\end{bmatrix}H_{n-k}(x)=q^kx\begin{bmatrix}n-1\\k\end{bmatrix}H_{n-k-1}(x)-q^k[k+1]\begin{bmatrix}n-1\\k+1\end{bmatrix}H_{n-k-2}(x)$$

$$=q^kx\begin{bmatrix}n-1\\k\end{bmatrix}H_{n-k-1}(x)-q^k[n-k-1]\begin{bmatrix}n-1\\k\end{bmatrix}H_{n-k-2}(x).$$

But this is true because of (42).

Therefore we get

$$\det\left(H_{i+j}(x)\right)_{i,j=0}^{n-1}=(-1)^{\binom{n}{2}}q^{\binom{n}{3}}\prod_{j=0}^{n-1}[j]!. \tag{43}$$

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