Dynamics of Bose-Einstein condensates

Benjamin Schlein, LMU München and University of Cambridge

Focus Week on Nonequilibrium Systems , ESI Vienna

June 5th, 2008

Joint work with L. Erdős and H.-T. Yau

1. Bose-Einstein condensation

N-boson systems: described by a wave function

$$\psi_N \in L^2(\mathbb{R}^{3N}), \quad \text{symmetric w.r.t. permutations.}$$

$$|\psi_N(x_1,\dots,x_N)|^2 = \text{probability density} \qquad \Rightarrow \qquad \|\psi_N\| = 1$$

Dynamics governed by Schrödinger equation

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t} \quad \Rightarrow \quad \psi_{N,t} = e^{-iH_N t}\psi_N \quad \Rightarrow \quad \|\psi_{N,t}\| = 1 \quad \forall t$$

H_N is the Hamiltonian of the system:

$$H_N = \sum_{j=1}^N \left(-\Delta_{x_j} + V_{\text{ext}}(x_j)\right) + \sum_{i<j}^N V(x_i - x_j) \quad \text{acts on} \quad L^2(\mathbb{R}^{3N}).$$

Trivial example of condensate wave function:

$$\psi_N(\mathbf{x}) = \prod_{j=1}^{N} \varphi(x_j) \qquad \text{for all} \quad \mathbf{x} = (x_1, \ldots, x_N) \in \mathbb{R}^{3N}$$

Condensation for general $\psi_N \in L^2_s(\mathbb{R}^{3N})$:

- Density Matrix: orthogonal projection onto ψ_N,

$$\gamma_N = |\psi_N\rangle\langle\psi_N| \quad \Rightarrow \gamma_N(\mathbf{x};\mathbf{x}') = \psi_N(\mathbf{x})\,\overline{\psi}_N(\mathbf{x}').$$

- Marginal Densities: Define the k-particle marginal density

$$\gamma_N^{(k)}(\mathbf{x}_k;\mathbf{x}_k') = \int \mathrm{d}\mathbf{x}_{N-k}\,\gamma_N(\mathbf{x}_k,\mathbf{x}_{N-k};\mathbf{x}_k',\mathbf{x}_{N-k}) \qquad \mathrm{Tr}\,\gamma_N^{(k)} = 1$$

$$\mathbf{x}_k = (x_1,\ldots,x_k), \quad \mathbf{x}_{N-k} = (x_{k+1},\ldots,x_N)$$

In other words: $\quad \gamma_N^{(k)} = \mathrm{Tr}_{k+1,k+2,\ldots N}\ |\psi_N\rangle\langle\psi_N|$

- One particle density:

$$\gamma^{(1)} = \sum_j \lambda_j |\phi_j\rangle\langle\phi_j| \quad \phi_j \in L^2(\mathbb{R}^3), \quad 0 < \lambda_j \leq 1, \quad \sum_j \lambda_j = 1$$

$\lambda_j =$ Probability that a randomly chosen particle is in the one-particle state ϕ_j.

- Definition: a family $\{\psi_N\}_{N\in\mathbb{N}}$ exhibits BEC iff

$$\lim \inf_{N\to\infty} \max \text{spec } \gamma_N^{(1)} > 0$$

Interpretation: BEC exists if a macroscopic number of particles is in the same one-particle state.

- Example: if $\psi_N(\mathbf{x}) = \prod_{j=1}^N \varphi(x_j)$, then $\gamma_N^{(1)} = |\varphi\rangle\langle\varphi|$ and thus

$$\max \text{spec}\, \gamma_N^{(1)} = 1 \qquad \text{for all } N \quad \Rightarrow \quad \text{(complete) BEC}$$

Condensation of trapped Bose gases: N bosons in volume of order one, range of interaction N^{-1}.

$$H_N = \sum_{j=1}^{N} \left(-\Delta_j + V_{\text{ext}}(x_j)\right) + \sum_{i<j}^{N} N^2 V(N(x_i - x_j))$$

• Lieb-Seiringer-Yngvason (2000): the ground state energy is given by

$$\lim_{N\to\infty} \frac{E_{\text{GS}}(N)}{N} = \min_{\varphi:\|\varphi\|=1} \mathcal{E}_{\text{GP}}(\varphi)$$

with

$$\mathcal{E}_{\text{GP}}(\varphi) = \int \mathrm{d}x \left(|\nabla\varphi(x)|^2 + V_{\text{ext}}(x)|\varphi(x)|^2 + 4\pi a_0 |\varphi(x)|^4\right)$$

$$a_0 = \text{scattering length of } V(x)$$

• Lieb-Seiringer (2002): complete condensation of ground state

$$\gamma_N^{(1)} \to |\phi_{\text{GP}}\rangle\langle\phi_{\text{GP}}|, \qquad \phi_{\text{GP}} = \text{minimizer of } \mathcal{E}_{\text{GP}}$$

2. Main result

Scattering length of $V \in L^1(\mathbb{R}^3)$: defined by one-body equation

$$\left(-\Delta + \frac{1}{2}V(x)\right) f(x) = 0 \qquad \text{with } f(x) \to 1 \text{ as } |x| \to \infty \,.$$

$$f(x) \simeq 1 - \frac{a_0}{|x|} \qquad \Longleftrightarrow \qquad 8\pi a_0 = \int \mathrm{d}x V(x) f(x)$$

Rescaled potential: we consider the Gross-Pitaevskii scaling

$$V_N(x) = N^2 V(Nx) \quad \Rightarrow \quad V_N \quad \text{has scattering length } a = a_0/N$$

The dynamics: is governed by

$$i\partial_t \psi_{N,t} = H_N \psi_{N,t}$$

with Hamiltonian

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \sum_{j<k}^{N} V_N(x_j - x_k)$$

Theorem [Erdös, S., Yau, 2008]:

Suppose $V \geq 0$, $|V(x)| \leq C(1+x^2)^{-\sigma/2}$, for some $\sigma > 5$.

Assume that ψ_N has finite energy per particle $\langle \psi_N, H_N \psi_N \rangle \leq CN$ and that it exhibits complete BEC

$$\gamma_N^{(1)} \to |\varphi\rangle\langle\varphi| \qquad \text{for some } \varphi \in L^2(\mathbb{R}^3)$$

Denote by $\psi_{N,t} = e^{-iH_N t}\psi_N$ the time evolution of ψ_N. Then, for every $t \in \mathbb{R}$,

$$\gamma_{N,t}^{(1)} \to |\varphi_t\rangle\langle\varphi_t| \qquad \text{as } N \to \infty\,,$$

where φ_t is the solution to the time-dependent Gross-Pitaevskii equation

$$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t$$

with $\varphi_{t=0} = \varphi$.

Remark: we have $\gamma_{N,t}^{(k)} \to |\varphi_t\rangle\langle\varphi_t|^{\otimes k}$ as $N \to \infty$, for any $k \geq 1$.

Application:

$$H_N^{\text{trap}} = \sum_{j=1}^{N} \left(-\Delta_j + V_{\text{ext}}(x_j)\right) + \sum_{i<j}^{N} V_N(x_i - x_j)$$

Lieb-Seiringer $\Rightarrow \quad \gamma_N^{(1)} \to |\phi_{\text{GP}}\rangle\langle\phi_{\text{GP}}| \quad$ as $N \to \infty$

Here ϕ_{GP} is the minimizer of the Gross-Pitaevskii functional

$$\mathcal{E}_{\text{GP}}(\varphi) = \int \mathrm{d}x \, \left(|\nabla\varphi|^2 + V_{\text{ext}}|\varphi|^2 + 4\pi a_0|\varphi|^4\right)$$

Corollary: Suppose the initial wave function ψ_N is the ground state vector of H_N^{trap}. Then, for every $t \in \mathbb{R}$,

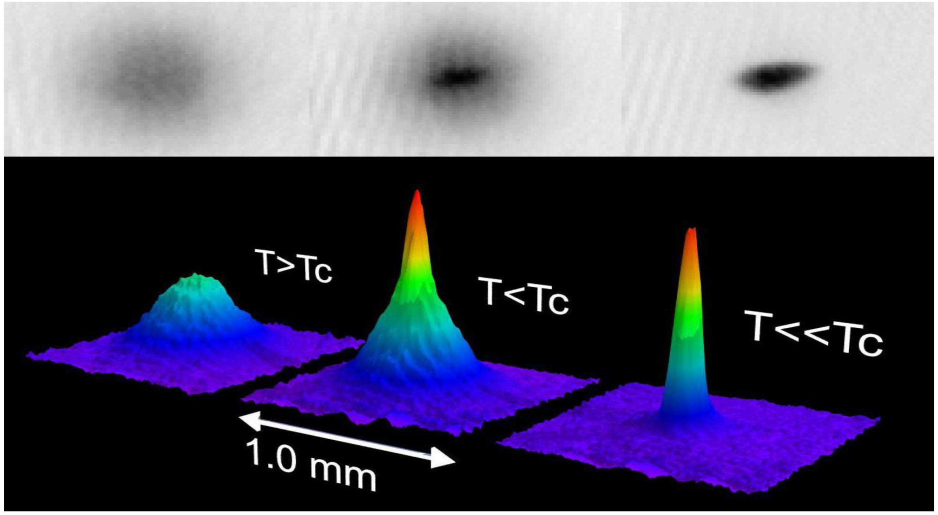
$$\gamma_{N,t}^{(1)} \to |\varphi_t\rangle\langle\varphi_t| \qquad \text{as } N \to \infty$$

where φ_t solves

$$i\partial_t\varphi_t = -\Delta\varphi_t + 8\pi a_0|\varphi_t|^2\varphi_t \qquad \text{with} \quad \varphi_{t=0} = \phi_{\text{GP}}.$$

Experiments on BEC: in 2001, Cornell-Ketterle-Wieman received Nobel prize in physics for experiments which first proved the existence of BEC for trapped Bose gas.

In the experiments gases are trapped in small volumes by strong magnetic fields, and cooled down at very low temperatures. Then one observes the dynamical evolution of the condensate when the trap is removed.



3. General strategy of the proof

Evolution of marginal densities: Recall that the k-particle marginal associated with $\psi_{N,t}$ is:

$$\gamma^{(k)}_{N,t}(\mathbf{x}_k;\mathbf{x}'_k) = \int \mathrm{d}\mathbf{x}_{N-k}\, \gamma_{N,t}(\mathbf{x}_k,\mathbf{x}_{N-k};\mathbf{x}'_k,\mathbf{x}_{N-k})$$
$$= \int \mathrm{d}\mathbf{x}_{N-k}\, \psi_{N,t}(\mathbf{x}_k,\mathbf{x}_{N-k})\overline{\psi}_{N,t}(\mathbf{x}'_k,\mathbf{x}_{N-k})$$

The family $\{\gamma^{(k)}_{N,t}\}_{k=1}^{N}$ satisfies the BBGKY Hierarchy

$$i\partial_t \gamma^{(k)}_{N,t} = \sum_{j=1}^{k}\left[-\Delta_{x_j},\gamma^{(k)}_{N,t}\right] + \sum_{1\le i<j\le k}\left[V_N(x_i-x_j),\gamma^{(k)}_{N,t}\right]$$
$$+ (N-k)\sum_{j=1}^{k}\mathrm{Tr}_{k+1}\left[V_N(x_j-x_{k+1}),\gamma^{(k+1)}_{N,t}\right]$$

where

$$\left(\mathrm{Tr}_{k+1}\, A^{(k+1)}\right)(\mathbf{x}_k;\mathbf{x}'_k) = \int \mathrm{d}x_{k+1}\, A^{(k+1)}(\mathbf{x}_k,x_{k+1};\mathbf{x}'_k,x_{k+1})$$

Three step strategy:

Compactness: the sequence $\{\gamma_{N,t}^{(k)}\}_{k=1}^{N}$ is compact w.r.t. an appropriate weak topology.

Convergence to infinite hierarchy: every limit point $\{\gamma_{\infty,t}^{(k)}\}_{k\geq 1}$ satisfies

$$i\partial_t \gamma_{\infty,t}^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \gamma_{\infty,t}^{(k)}\right] + 8\pi a_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma_{\infty,t}^{(k+1)}\right]$$

Observe that $\gamma_{\infty,t}^{(k)} = |\varphi_t\rangle\langle\varphi_t|^{\otimes k}$ solves infinite hierarchy iff

$$i\partial_t \varphi_t = -\Delta\varphi_t + 8\pi a_0 |\varphi_t|^2 \varphi_t$$

Uniqueness: in certain Sobolev spaces, the infinite hierarchy has a unique solution.

$$\Rightarrow \quad \gamma_{N,t}^{(k)} \to |\varphi_t\rangle\langle\varphi_t|^{\otimes k} \qquad \text{as } N \to \infty$$

This strategy has been used to derive the Hartree equation

$$i\partial_t \varphi_t = -\Delta + (V * |\varphi_t|^2)\varphi_t$$

in mean field systems with Hamiltonian

$$H_N = \sum_{j=1}^{N} -\Delta_{x_j} + \frac{1}{N} \sum_{j<k}^{N} V(x_j - x_k)$$

- Spohn, 1980: bounded potential.

- Erdős - Yau, 2000: Coulomb potential (partial results by Bardos - Golse - Mauser).

- Elgart - S., 2005: relativistic Coulomb potential.

- Adami - Bardos - Golse - Teta, 2006: one-dim δ-function.

- Erdős - S. - Yau, 2006: three-dim δ-function.

4. Proof of the convergence

We start from BBGKY hierarchy

$$i\partial_t \gamma_{N,t}^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \gamma_{N,t}^{(k)}\right] + \sum_{1\leq i<j\leq k} \left[V_N(x_i - x_j), \gamma_{N,t}^{(k)}\right]$$

$$+ (N-k) \sum_{j=1}^{k} \text{Tr}_{k+1} \left[V_N(x_j - x_{k+1}), \gamma_{N,t}^{(k+1)}\right]$$

Assuming $\gamma_{N,t}^{(k)} \to \gamma_{\infty,t}^{(k)}$ as $N \to \infty$, we want to prove that

$$i\partial_t \gamma_{\infty,t}^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \gamma_{\infty,t}^{(k)}\right] + 8\pi a_0 \sum_{j=1}^{k} \text{Tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma_{\infty,t}^{(k+1)}\right]$$

Problem: formally, we have, as $N \to \infty$,

$$(N-k)V_N(x_j - x_{k+1}) \simeq N^3 V(N(x_j - x_{k+1})) \to b_0 \delta(x_j - x_{k+1})$$

$$\text{with} \qquad b_0 = \int \mathrm{d}x\, V(x) \neq 8\pi a_0$$

Solution: correlation structure. For example, for $k=1$,

$$\gamma^{(2)}_{N,t}(x_1,x_2;x_1',x_2') \simeq f_N(x_1-x_2)f_N(x_1'-x_2')\gamma^{(2)}_{\infty,t}(x_1,x_2;x_1',x_2')$$

$$\text{with} \qquad \left(-\Delta + \frac{1}{2}V_N\right) f_N = 0 \quad \Rightarrow \quad f_N(x) = f(Nx)$$

Then

$$\text{Tr}_2\, NV_N(x_1-x_2)\gamma^{(2)}_{N,t} \to 8\pi a_0 \text{Tr}_2\, \delta(x_1-x_2)\gamma^{(2)}_{\infty,t}$$

In fact

$$\begin{aligned}
&\left(\text{Tr}_2\, NV_N(x_1-x_2)\gamma^{(2)}_{N,t} \right)(x_1;x_1') \\
&= \int \mathrm{d}x_2\, N^3 V(N(x_1-x_2))\, \gamma^{(2)}_{N,t}(x_1,x_2;x_1',x_2) \\
&\simeq \int \mathrm{d}x_2\, N^3 V(N(x_1-x_2))\, f(N(x_1-x_2))\, \gamma^{(2)}_{\infty,t}(x_1,x_2;x_1',x_2) \\
&\simeq 8\pi a_0 \int \mathrm{d}x_2\, \delta(x_1-x_2)\, \gamma^{(2)}_{\infty,t}(x_1,x_2;x_1',x_2) \\
&= 8\pi a_0 \left(\text{Tr}_2\, \delta(x_1-x_2)\gamma^{(2)}_{\infty,t} \right)(x_1;x_1')
\end{aligned}$$

First attempt, for small potentials [Erdös, S., Yau, 2006]:

Derive the a-priori bound

$$\int \mathrm{d}\mathbf{x} \left| \nabla_{x_1} \nabla_{x_2} \frac{\psi_{N,t}(\mathbf{x})}{f_N(x_1 - x_2)} \right|^2 \leq C \qquad \text{uniformly in } N \text{ and } t$$

Proof obtained through an energy estimate

$$\langle \psi_N, \; H_N^2 \, \psi_N \rangle \geq \; C N^2 \int \mathrm{d}\mathbf{x} \; \left| \nabla_{x_1} \nabla_{x_2} \frac{\psi_N(\mathbf{x})}{f_N(x_1 - x_2)} \right|^2$$

Note that

$$\left. \begin{array}{ll} \int \mathrm{d}\mathbf{x} \, |\nabla_{x_1} \nabla_{x_2} \psi_{N,t}(\mathbf{x})|^2 & \simeq N \\ \int \mathrm{d}x \, |\nabla^2 f_N(x)| & \simeq N \end{array} \right\} \quad \Rightarrow \text{cancelations are crucial!!}$$

Therefore $\psi_{N,t} = f_N(x_1 - x_2) \phi_{N,t}(\mathbf{x})$ with a $\phi_{N,t}$ regular in x_1, x_2.

Second attempt [Erdös, S., Yau, 2008]: Define the wave operator

$$W = \lim_{t\to\infty} e^{i\mathfrak{h}t}\, e^{i\Delta t} \qquad \text{with } \mathfrak{h} = -\Delta + \frac{1}{2}V(x)\,.$$

W exists and is complete, that is

$$W^{-1} = W^* = \lim_{t\to\infty} e^{-i\Delta t} e^{-i\mathfrak{h}t}$$

It satisfies the intertwining relation

$$W^*\,\mathfrak{h}\,W = -\Delta.$$

For arbitrary $N \in \mathbb{N}$, we also define

$$W_N = \lim_{t\to\infty} e^{i\mathfrak{h}_N t} e^{i\Delta t} \qquad \text{with } \mathfrak{h}_N = -\Delta + \frac{1}{2}V_N(x)\,.$$

$$\Rightarrow \quad W_N^*\,\mathfrak{h}_N\,W_N = -\Delta$$

$$\Rightarrow \quad W_N(x;x') = N^3 W(Nx;Nx') \qquad (\|W_N\|_{L^p\to L^p} = \|W\|_{L^p\to L^p} < \infty)$$

Proposition (a-priori estimate): if $V \geq 0$, $V \in L^1(\mathbb{R}^3) \cap L^2(\mathbb{R}^3)$, we have

$$\int \mathrm{d}\mathbf{x} \left| (\nabla_{x_1} \cdot \nabla_{x_2}) \; W^*_{N,(x_1-x_2)} \; \psi_{N,t}(\mathbf{x}) \right|^2 \leq C, \quad \text{uniformly in } N \text{ and } t$$

Remarks:

• The wave operator acts only in the $x_1 - x_2$ variable;

$$\left(W^*_{N,(x_1-x_2)} \psi \right)(x_1, x_2, \mathbf{x}_{N-2}) = \int \mathrm{d}v \; W^*_N \, (x_1 - x_2; v) \psi \left(\frac{x_1 + x_2}{2} + \frac{v}{2}, \frac{x_1 + x_2}{2} - \frac{v}{2}, \mathbf{x}_{N-2} \right)$$

• $\psi_{N,t}$ is regularized through the wave operator: compared with previous approach, this is not a regularization in configuration space. Effect is similar, since (in a weak sense) $W^*_N f_N = 1$.

• The new estimate only controls the expectation of the dot-product $(\nabla_{x_1} \cdot \nabla_{x_2})$: momenta in orthogonal directions may grow! Fortunately, this weaker estimate is nevertheless enough.

Model computations:

- For $\psi \in L^2(\mathbb{R}^3, \mathrm{d}x)$,

$$\langle \psi, V(x)\psi \rangle = \int \mathrm{d}x\, V(x)\,|\psi(x)|^2 \leq C\|V\|_{L^1}\,\langle \psi, (1-\Delta)^2\psi \rangle$$

- For $\psi \in L^2(\mathbb{R}^3 \times \mathbb{R}^3, \mathrm{d}x_1\mathrm{d}x_2)$,

$$\begin{aligned}\langle \psi, V(x_1-x_2)\psi \rangle &= \int \mathrm{d}x_1\mathrm{d}x_2\, V(x_1-x_2)|\psi(x_1,x_2)|^2 \\ &\leq C\|V\|_{L^1}\,\langle \psi, (1-\Delta_{x_1})(1-\Delta_{x_2})\psi \rangle\end{aligned}$$

(the mixed second derivative is enough).

- It turns out that we also have the bound

$$\langle \psi, V(x_1-x_2)\psi \rangle \leq C\|V\|_{L^1}\,\langle \psi, \Big((\nabla_{x_1}\cdot\nabla_{x_2})^2 - \Delta_{x_1} - \Delta_{x_2} + 1\Big)\,\psi \rangle$$

This is still not enough: need a Poincaré type inequality to control the convergence to a delta-function.

5. Uniqueness of the infinite hierarchy

I) Proof of the a-priori bounds

We can prove uniqueness of the infinite hierarchy in the class of densities $\{\gamma_t^{(k)}\}_{k\geq 1}$ such that

$$\text{Tr}\,(1-\Delta_{x_1})\dots(1-\Delta_{x_k})\gamma_t^{(k)} \leq C^k$$

with a constant C independent of $k \geq 1$ and t.

Need to prove that any limit point $\{\gamma_{\infty,t}^{(k)}\}_{k\geq 1}$ of the marginals $\{\gamma_{N,t}^{(k)}\}_{k=1}^N$ satisfies these a-priori bounds.

Problem: the estimates

$$\text{Tr}\,(1-\Delta_{x_1})\dots(1-\Delta_{x_k})\gamma_{N,t}^{(k)} \leq C^k$$

cannot be true uniformly in N (because of short scale structure).

Choose a length scale ℓ with $N\ell^2 \gg 1$ and $N\ell^3 \ll 1$. For $j = 1, \dots, N$ define

$$\theta_j(\mathbf{x}) \simeq \begin{cases} 1 & \text{if } |x_i - x_j| \gg \ell \quad \forall i \neq j \\ 0 & \text{otherwise} \end{cases}$$

Proposition (higher order energy estimates):

$$\langle \psi_N, (H_N + N)^k \psi_N \rangle \geq C^k N^k \int \mathrm{d}\mathbf{x}\, \theta_1(\mathbf{x}) \dots \theta_{k-1}(\mathbf{x})\, |\nabla_{x_1} \dots \nabla_{x_k} \psi_N(\mathbf{x})|^2$$

$$\Rightarrow \int \mathrm{d}\mathbf{x}\, \theta_1(\mathbf{x}) \dots \theta_{k-1}(\mathbf{x})\, |\nabla_{x_1} \dots \nabla_{x_k} \psi_{N,t}(\mathbf{x})|^2 \leq C^k$$

The cutoff $\theta_j(\mathbf{x})$ is effective only when x_j falls into a volume of order $N\ell^3$ in $\mathbb{R}^3$.

Since $N\ell^3 \to 0$ as $N \to \infty$, the cutoff can be removed in the limit $N \to \infty$, and we obtain the a-priori bounds

$$\text{Tr } (1 - \Delta_{x_1}) \dots (1 - \Delta_{x_k}) \gamma^{(k)}_{\infty,t} \leq C^k.$$

II) Proof of the uniqueness

Theorem: given a family $\{\gamma^{(k)}\}_{k\geq 1}$ with

$$\text{Tr}(1-\Delta_{x_1})\dots(1-\Delta_{x_k})\gamma^{(k)} \leq C^k$$

there exists at most one solution $\{\gamma_t^{(k)}\}_{k\geq 1}$ of

$$i\partial_t \gamma_t^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \gamma_t^{(k)}\right] + 8\pi a_0 \sum_{j=1}^{k} \text{Tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma_t^{(k+1)}\right]$$

such that

$$\text{Tr}(1-\Delta_1)\dots(1-\Delta_k)\gamma_t^{(k)} \leq C^k \quad \text{for all} \quad t \in \mathbb{R}\,.$$

Hierarchy in integral form: rewrite infinite hierarchy

$$i\partial_t \gamma_t^{(k)} = \sum_{j=1}^{k} \left[-\Delta_{x_j}, \gamma_t^{(k)} \right] + 8\pi a_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \left[\delta(x_j - x_{k+1}), \gamma_t^{(k+1)} \right]$$

as

$$\gamma_t^{(k)} = \mathcal{U}^{(k)}(t)\gamma_0^{(k)} + \int_0^t \mathrm{d}s\, \mathcal{U}^{(k)}(t-s) B^{(k)} \gamma_s^{(k+1)}, \qquad k \geq 1$$

with

$$\mathcal{U}^{(k)}(t)\gamma^{(k)} = \exp\left(it \sum_{j=1}^{k} \Delta_{x_j} \right) \gamma^{(k)} \exp\left(-it \sum_{j=1}^{k} \Delta_{x_j} \right)$$

$$B^{(k)}\gamma^{(k+1)} = -i8\pi a_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1} \Big[\delta(x_j - x_{k+1}), \gamma^{(k+1)} \Big]$$

Duhamel series: expand arbitrary solution $\gamma_t^{(k)}$ as

$$\gamma_t^{(k)} = \mathcal{U}^{(k)}(t)\gamma_0^{(k)} + \sum_{m=1}^{n-1} \xi_{m,t}^{(k)} + \eta_{n,t}^{(k)}$$

with

$$\xi_{m,t}^{(k)} = \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{m-1}} \mathrm{d}s_m \, \mathcal{U}^{(k)}(t-s_1)\, B^{(k)} \mathcal{U}^{(k+1)}(s_1-s_2)\, B^{(k+1)} \dots$$
$$\dots \mathcal{U}^{(k+m-1)}(s_{m-1}-s_m) B^{(k+m-1)} \mathcal{U}^{(k+m)}(s_m) \gamma_0^{(k+m)}$$

$$\eta_{n,t}^{(k)} = \int_0^t \mathrm{d}s_1 \dots \int_0^{s_{n-1}} \mathrm{d}s_n \, \mathcal{U}^{(k)}(t-s_1) B^{(k)} \mathcal{U}^{(k+1)}(s_1-s_2) B^{(k+1)} \dots$$
$$\dots \mathcal{U}^{(k+n-1)}(s_{n-1}-s_n) B^{(k+n-1)} \gamma_{s_n}^{(k+n)}$$

with

$$B^{(k)}\gamma^{(k+1)} = -i8\pi a_0 \sum_{j=1}^{k} \mathrm{Tr}_{k+1}\Big[\delta(x_j - x_{k+1}), \gamma^{(k+1)}\Big]$$

For example:

$$\xi_{m,t}^{(k)} = (-8\pi i a_0)^m \sum_{j_1=1}^{k} \sum_{j_2=1}^{k+1} \cdots \sum_{j_m=1}^{k+m-1} \int_0^t \mathrm{d}s_1 \ldots \int_0^{s_{m-1}} \mathrm{d}s_m$$

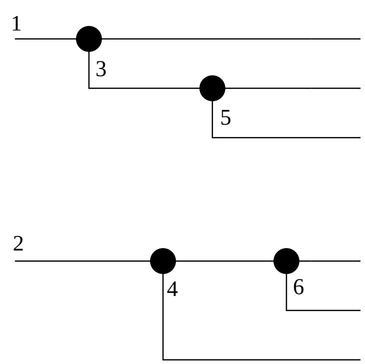
$$\times\, \mathcal{U}^{(k)}(t-s_1)\, \mathrm{Tr}_{k+1}\Big[\delta(x_{j_1}-x_{k+1}),$$

$$\times\, \mathcal{U}^{(k+1)}(s_1-s_2) \mathrm{Tr}_{k+2}\Big[\delta(x_{j_2}-x_{k+2}), \ldots$$

$$\ldots$$

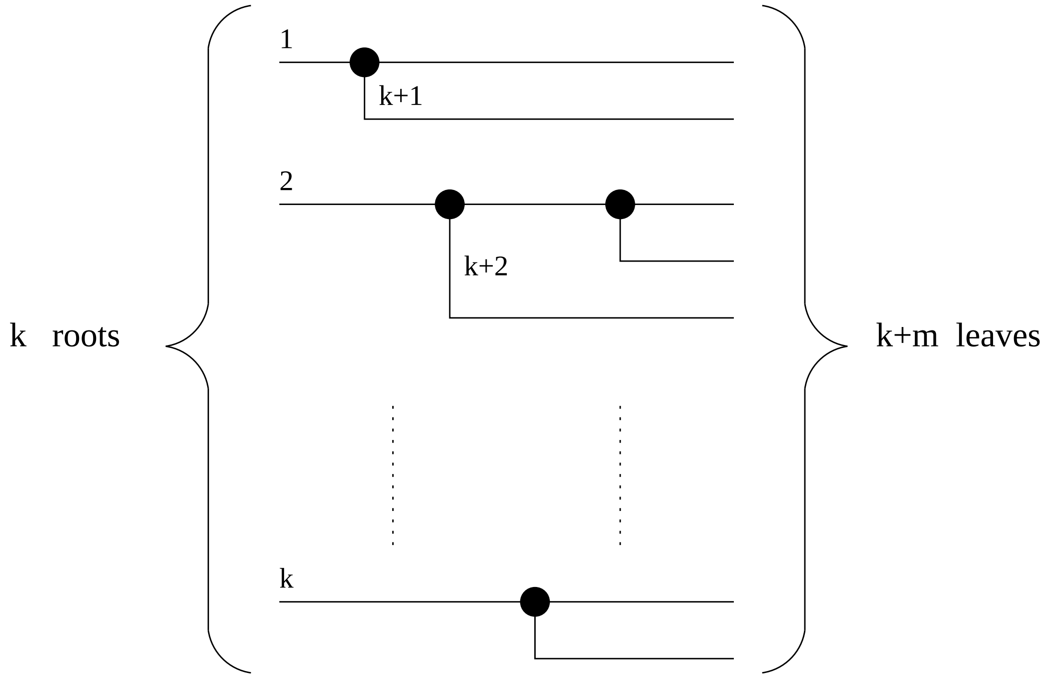
$$\times\, \mathcal{U}^{(k+m-1)}(s_{m-1}-s_m)\, \mathrm{Tr}_{k+m}\Big[\delta(x_{j_m}-x_{k+m}), \mathcal{U}^{(k+m)}(s_m)\gamma_0^{(k+m)}\Big] \ldots \Big]$$

Classical graphs: the graphs should describe the collision history of the different terms. For example, for $k=2$, $m=4$,



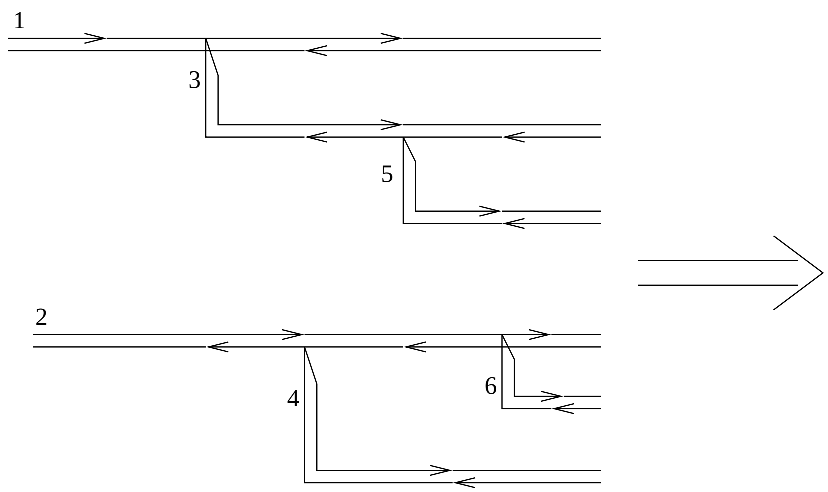
$$\Longrightarrow \quad \delta(x_1-x_3)\ \delta(x_2-x_4)\ \delta(x_3-x_5)\ \delta(x_2-x_6)$$

More generally, contributions to $\xi^{(k)}_{m,t}$ can be represented by ordered forests of k disjoint trees with m vertices



$$\text{Number of ordered graphs } = \frac{(k+m)!}{k!}$$

Doubled graphs: because of the commutators, for every collision we have a binary choice. To represent all contributions we double the classical graphs. For example ($k = 2$, $m = 4$)

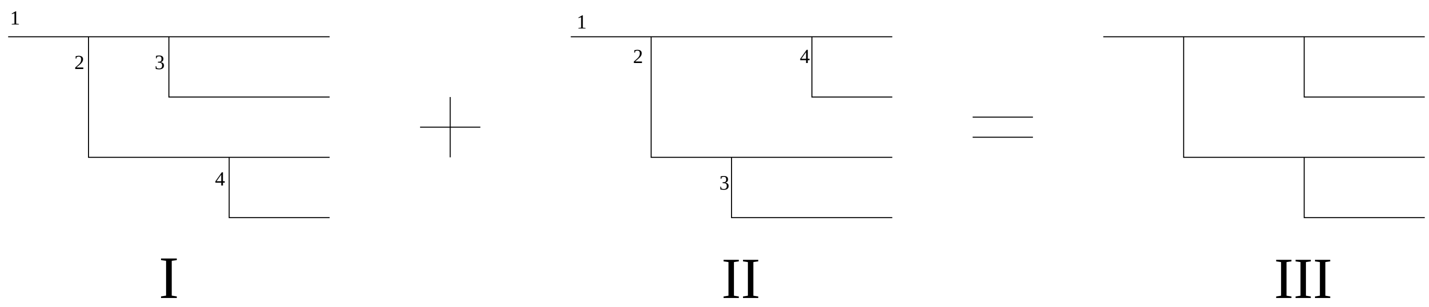


$$\delta(x_1 - x_3)\ \delta(x_2 - x_6)\ \gamma^{(6)}\ \delta(x_3 - x_5)\ \delta(x_2 - x_4)$$

The vertices are still completely ordered, and

$$\text{number of doubled graphs} = 2^m \frac{(k+m)!}{k!}$$

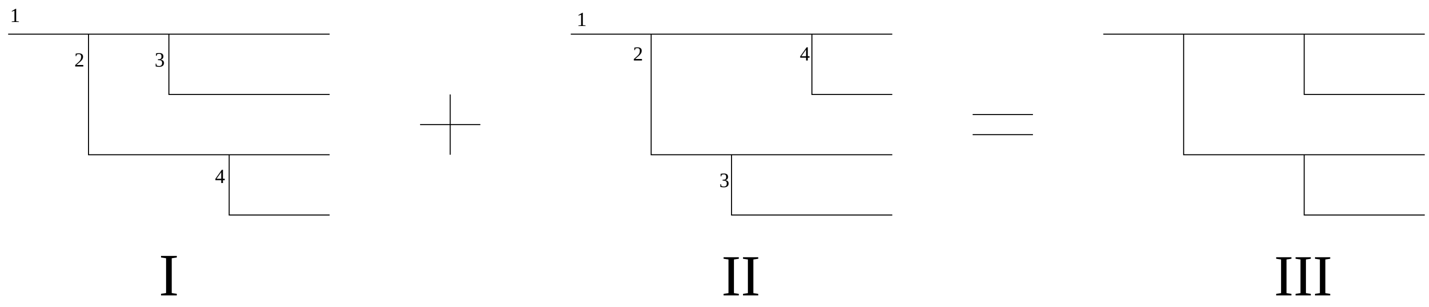
Removing the order: next we combine the contributions of topologically equivalent ordered graphs.



$$\mathrm{I)} = \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \int_0^{s_2} \mathrm{d}s_3\, \mathcal{U}^{(1)}(t-s_1)\mathrm{Tr}_{2,3,4}\, \delta(x_1-x_2)\mathcal{U}^{(2)}(s_1-s_2) \times \delta(x_1-x_3)\mathcal{U}^{(3)}(s_2-s_3)\delta(x_2-x_4)\mathcal{U}^{(4)}(s_3)\gamma_0^{(4)}$$

$$\begin{aligned}\mathrm{II)} &= \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \int_0^{s_2} \mathrm{d}s_3\, \mathcal{U}^{(1)}(t-s_1)\mathrm{Tr}_{2,3,4}\, \delta(x_1-x_2)\mathcal{U}^{(2)}(s_1-s_2) \\ &\quad \times \delta(x_2-x_3)\mathcal{U}^{(3)}(s_2-s_3)\delta(x_1-x_4)\mathcal{U}^{(4)}(s_3)\gamma_0^{(4)} \\ &= \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_3 \int_0^{s_3} \mathrm{d}s_2\, \mathcal{U}^{(1)}(t-s_1)\mathrm{Tr}_{2,3,4}\, \delta(x_1-x_2)\mathcal{U}^{(2)}(s_1-s_2) \\ &\quad \times \delta(x_1-x_3)\mathcal{U}^{(3)}(s_2-s_3)\delta(x_2-x_4)\mathcal{U}^{(4)}(s_3)\gamma_0^{(4)}\end{aligned}$$

Removing the order: next we combine the contributions of topologically equivalent ordered graphs.



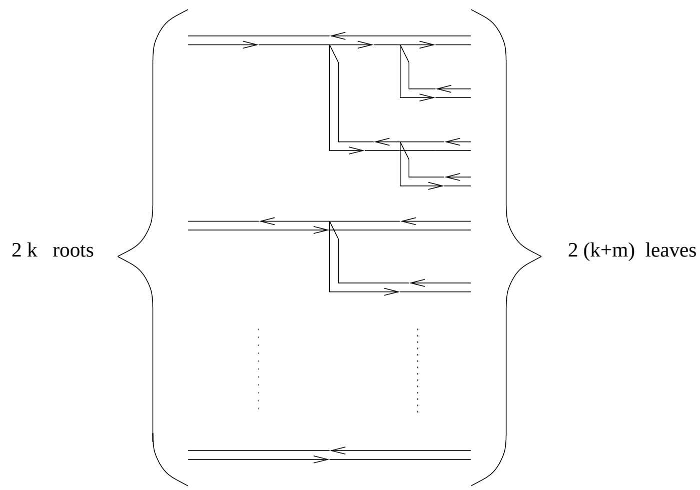
$$\text{III}) := \text{I}) + \text{II})$$

$$= \int_0^t \mathrm{d}s_1 \int_0^{s_1} \mathrm{d}s_2 \int_0^{s_1} \mathrm{d}s_3\, \mathcal{U}^{(1)}(t-s_1)\mathrm{Tr}_{2,3,4}\, \delta(x_1-x_2)\mathcal{U}^{(2)}(s_1-s_2)$$

$$\times\, \delta(x_1-x_3)\mathcal{U}^{(3)}(s_2-s_3)\delta(x_2-x_4)\mathcal{U}^{(4)}(s_3)\gamma_0^{(4)}$$

Feynman graphs: different contributions to $\xi^{(k)}_{m,t}$ will be represented by graphs in

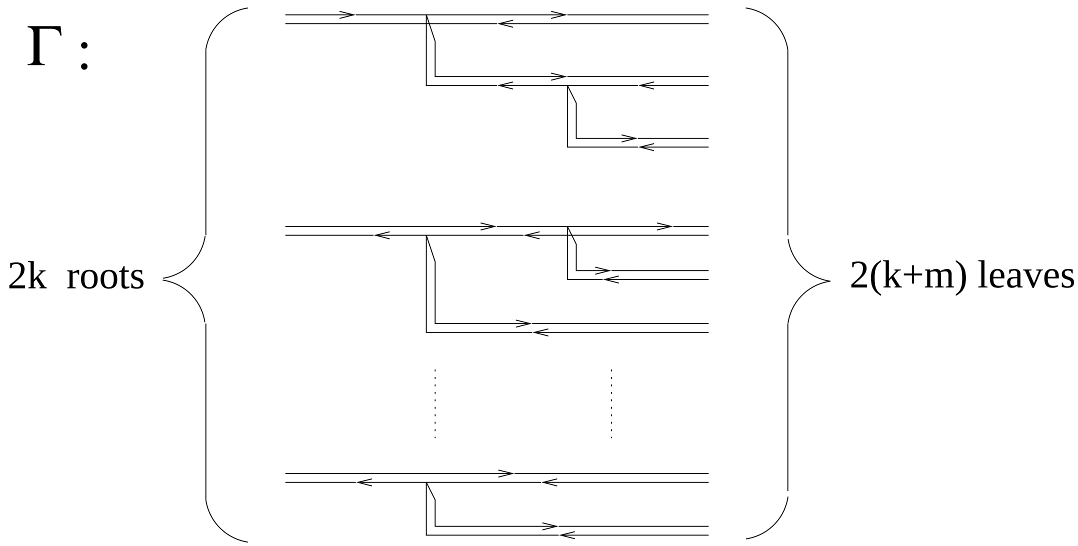
$$\mathcal{F}_{m,k} = \text{set of forests with } 2k \text{ disjoint paired trees with } m \text{ partially ordered vertices}$$



Number of graphs in $\mathcal{F}_{m,k} \leq C^{m+k}$.

Diagrammatic expansion of $\xi^{(k)}_{m,t}$: we expand

$$\text{Tr} J^{(k)} \xi^{(k)}_{m,t} = \sum_{\Gamma \in \mathcal{F}_{m,k}} \text{Tr } J^{(k)} K_{\Gamma,t} \gamma_0^{(k+m)}$$



E (Γ) = set of edges of Γ

V (Γ)= set of vertices of Γ

R (Γ)= set of roots of Γ

L (Γ)= set of leaves of Γ

$$\mathrm{Tr}\, J^{(k)}\, K_{\Gamma,t}\, \gamma_0^{(k+m)} =$$

$$= \int \prod_{e \in E(\Gamma)} \frac{d\alpha_e dp_e}{\alpha_e - p_e^2 + i\tau_e \eta_e} \prod_{v \in V(\Gamma)} \delta\left(\sum_{e \in v} \pm \alpha_e \right) \delta\left(\sum_{e \in v} \pm p_e \right)$$

$$\times J^{(k)}\left(\{(p_e, p_e')\}_{e \in R(\Gamma)} \right) \gamma_0^{(k+m)}\left(\{(p_e, p_e')\}_{e \in L(\Gamma)} \right)$$

$$\times \exp(-it \sum_{e \in R(\Gamma)} \tau_e(\alpha_e + i\tau_e \eta_e)), \qquad \tau_e = \pm 1$$

Control of the integral: use $\langle x\rangle = (1+x^2)^{1/2}$.

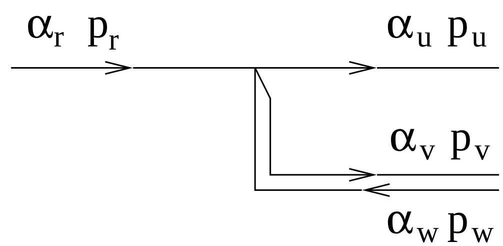
$$\left|\mathrm{Tr}\, J^{(k)}\, K_{\Gamma,t}\,\gamma_0^{(k+m)}\right| \leq\; C^m t^{m/4}$$
$$\times \int \prod_{e\in E(\Gamma)} \frac{\mathrm{d}\alpha_e \mathrm{d}p_e}{\langle \alpha_e - p_e^2\rangle} \prod_{v\in V(\Gamma)} \delta\left(\sum_{e\in v} \pm\alpha_e\right)\delta\left(\sum_{e\in v} \pm p_e\right)$$
$$\times \left|J^{(k)}\left(\{(p_e,p_e')\}_{e\in R(\Gamma)}\right)\right| \left|\gamma_0^{(k+m)}\left(\{(p_e,p_e')\}_{e\in L(\Gamma)}\right)\right|$$

Singularity at $x=0 \Rightarrow$ large momentum problem!!

From a-priori estimates $\Rightarrow$ decay in the momenta of leaves.

Perform integration over all α and p, starting from the leaves and moving towards the roots. At each vertex, we propagate the decay from the son-edges to the father-edge.

Typical example:



Integrate first the α-variables of the son-edges

$$\int \mathrm{d}\alpha_u \mathrm{d}\alpha_v \mathrm{d}\alpha_w \frac{\delta(\alpha_r = \alpha_u + \alpha_v - \alpha_w)}{\langle \alpha_u - p_u^2 \rangle \langle \alpha_v - p_v^2 \rangle \langle \alpha_w - p_w^2 \rangle} \leq \frac{\text{const}}{\langle \alpha_r - p_u^2 - p_v^2 + p_w^2 \rangle^{1-\varepsilon}}$$

Then integrate over the momenta of the son-edges

$$\int \frac{\mathrm{d}p_u \mathrm{d}p_v \mathrm{d}p_w}{|p_u|^{2+\lambda} |p_v|^{2+\lambda} |p_w|^{2+\lambda}} \frac{\delta(p_r = p_u + p_v - p_w)}{\langle \alpha_r - p_u^2 - p_v^2 + p_w^2 \rangle^{1-\varepsilon}} \leq \frac{\text{const}}{|p_r|^{2+\lambda}}$$

After integrating out all vertices

$$\Rightarrow \left| \mathrm{Tr}\, J^{(k)} K_{\Gamma,t} \gamma_0^{(k+m)} \right| \leq C^m t^{m/4} \qquad \forall \Gamma \in \mathcal{F}_{m,k}$$

Convergence of the expansion: Since $|\mathcal{F}_{m,k}| \leq C^m$, we find

$$\left|\text{Tr } J^{(k)} \xi^{(k)}_{m,t}\right| \leq \sum_{\Gamma \in \mathcal{F}_{m,k}} \left|\text{Tr } J^{(k)} K_{\Gamma,t} \gamma_0^{(k+m)}\right| \leq C^m t^{m/4}.$$

Analogously, we prove that $\left|\text{Tr } J^{(k)} \eta^{(k)}_{n,t}\right| \leq C^n t^{n/4}$.

$\Rightarrow$ if $\gamma^{(k)}_{1,t}$, $\gamma^{(k)}_{2,t}$ are two solutions with same initial data

$$\left|\text{Tr } J^{(k)} \left(\gamma^{(k)}_{1,t} - \gamma^{(k)}_{2,t}\right)\right| \leq C^n t^{n/4}$$

Since $n \in \mathbb{N}$ is arbitrary $\Rightarrow$ uniqueness for short time.

A-priori estimates are uniform in time $\Rightarrow$ uniqueness for all times.