Persistence of randomness in the macroscopic limit.
II Interface fluctuations.

Errico Presutti

ESI . June 2–6, 2008 Vienna

Mean field explains how phases separate in spinodal decomposition but *no spatial patterns*, (intrinsic to mean field models)

Relax mean field to observe patterns: introduce Kac potentials

Λ a torus in $\mathbb{Z}^d$, $\gamma > 0$:

$$H_{\gamma,\Lambda}(\sigma_\Lambda) = -\frac{1}{2}\sum_{x \neq y} J_\gamma(x,y)\sigma_\Lambda(x)\sigma_\Lambda(y)$$

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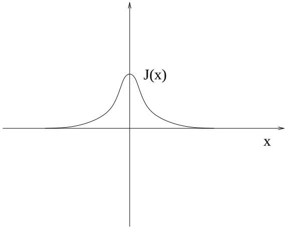

$$J(r) \geq 0. \ J(r) = 0 \text{ for } |r| \geq 1. \ \int J(r)\,dr = 1$$

$\gamma =$ scaling parameter. It controls:

range of interaction γ^{-1}

intensity of pair interaction $= J_\gamma(x, y) \approx \gamma^d$

choice such that the interaction of a spin with all the others is: $\approx \gamma^d$ (interaction with a given spin) times γ^{-d} (number of spins in ball of radius interaction range $= \gamma^{-1}$) ≈ 1.

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If $\gamma^{-1} \approx L$, $L =$ side of Λ, essentially mean field.

If $\gamma > 0$ small and fixed, finite range statistical mechanics model.

If $\gamma \to 0$, $L \to \infty$, $\gamma L \to \infty$, i.e. $\gamma^{-1} \ll L$, "mesoscopic regime".

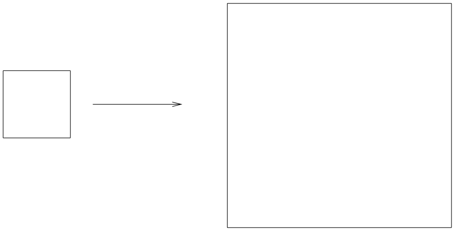

Figure: Mesoscopic region on the left and its blowup by γ^{-1}

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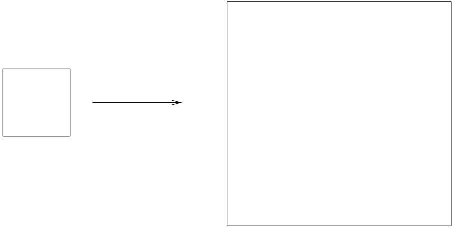

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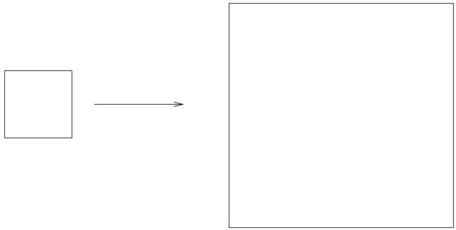

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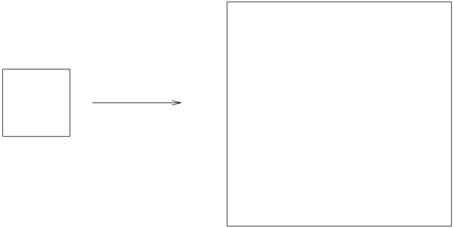

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Glauber dynamics. Mesoscopic limit.

Empirical magnetization:

$$u^R(r,t) = \frac{1}{|B_R(r)|} \sum_{x \in B_R(r) \cap \mathbb{Z}^d} \sigma(x,t)$$

$B_R(r) = \{r' \in \mathbb{R}^D : |r - r'| \le R\}$.

Let $\lim_{R\to\infty} \lim_{\gamma\to 0} P_\gamma\Big(⨍_{B_N(0)} |u^R(\gamma^{-1} r, 0) - m(r,0)| > \zeta \Big) = 0$, for all $\zeta > 0$ and all $N > 0$.

$$⨍_\Lambda f(r)\,dr = \frac{1}{|\Lambda|}\int_\Lambda f(r)\,dr$$

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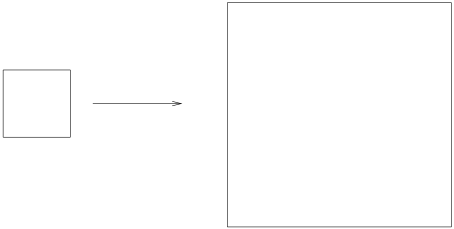
Spinodal decomposition: $\beta > 1$, initial state Bernoulli measure with average 0: spins independent, $P(\sigma(x) = 1) = \frac{1}{2}$.

The mesoscopic picture corresponds to a blow-up by γ^{-1}.

Relevant space scale for spinodal decompositions is



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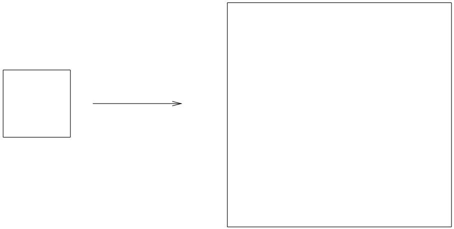

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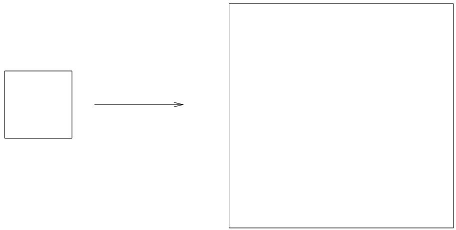

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Mean field times: $t \le \tau_c \log \gamma^{-1}$, $\tau_c = \frac{d}{2\alpha}$, $\alpha = \beta - 1 > 0$

$$\lim_{\gamma \to 0} P_\gamma \Big(⨍_{B_N(0)} |u^{\gamma^{-b}}(\ell_\gamma r, t) - m(r,t)| > \zeta \Big) = 0$$

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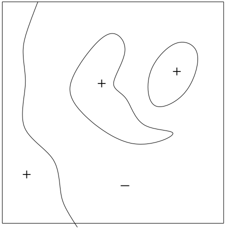
Exists $t^* > \tau_c \log \gamma^{-1}$, $\lim_{\gamma \to 0} \frac{t^* - \tau_c \log \gamma^{-1}}{\log \gamma^{-1}} = 0$ so that:

For any test function ϕ and any N,

$$\lim_{\gamma \to 0} P_\gamma \Big(\int_{B_N(0)} u^{\gamma^{-b}}(\ell_\gamma r, t^*)\phi(r) \Big) = E\Big(\int_{B_N(0)} X(r)\phi(r) \Big)$$

where $X(r) = m_\beta \mathrm{sign} \xi(r)$ and $\{\xi(r), r \in \mathbb{R}^d\}$ is a Gaussian process with mean zero and variance $C(r, r') = e^{-\alpha(r-r')^2/2}$.



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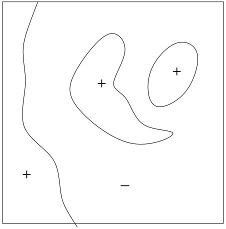

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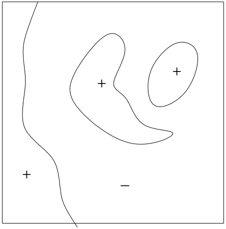

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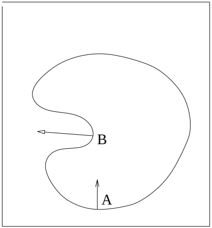
Deterministic interface dynamics.

$$\frac{dm(r,t)}{dt} = -m(r,t) + \tanh\{\beta J * m(r,t)\}, \quad m(\varepsilon r, 0) = u(r,0)$$

$u(r,0) = m_\beta$ outside Ω and $u(r,0) = -m_\beta$ in Ω.

Theorem. $u_\varepsilon(r,t) := m(\varepsilon^{-1} r, \varepsilon^{-2} t) \to u(r,t)$ which moves by mean curvature:



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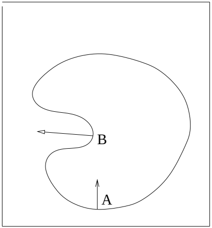

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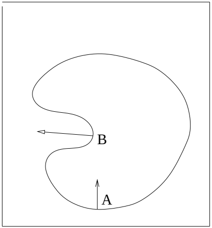

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Stochastic $d = 1$ Allen-Cahn evolution

The local version of the mesoscopic evolution is the Allen-Cahn equation,
to include fluctuations add white noise forcing:

$$\frac{dm(x,t)}{dt} = \frac{d^2}{dx^2} m(x,t) - V'(m(x,t)) + \sqrt{\varepsilon}\dot{w}$$

$V(m) = \dfrac{m^4}{4} - \dfrac{m^2}{2}$, $\dot{w}$ white noise in space and time.

J.B. Walsh: An introduction to stochastic partial differential equation. *Lecture Notes in Mathematics. Springer.* **1180**, 265–437 (1984).

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Interpret m as position, x as time, then it becomes Newton equation for a particle in $d = 1$

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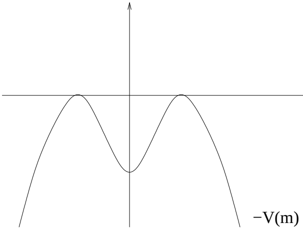

Orbits with energy < 0 are periodic and bounded.
Orbit with energy 0 is bounded and monotone.
Orbit with energy > 0 are unbounded.

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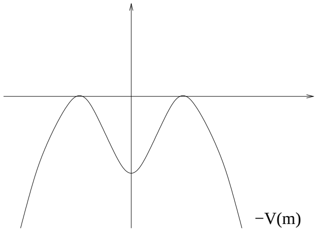

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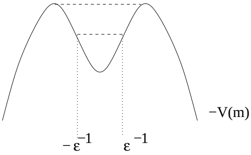

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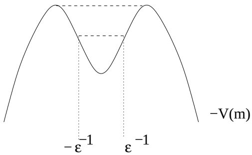
Energy 0 orbit solves: $\dfrac{d^2}{dx^2}\bar{m}(x) = V'(m(x)), \quad \lim_{x\to\pm\infty} \bar{m}(x) = \pm 1$

Orbit in Fig. with negative energy solves:

$$\frac{d^2}{dx^2} m(x) = V'(m(x)), \quad \frac{d}{dx} m(\pm\varepsilon^{-1}) = 0$$



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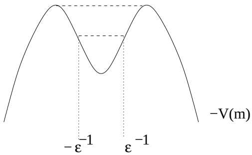

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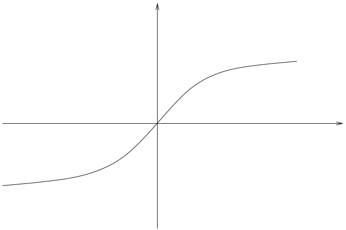

Figure: The instanton $\bar{m}(x)$.

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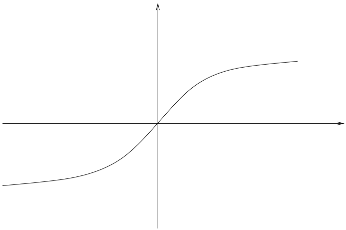

Figure: The instanton $\bar{m}(x)$.

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while $\bar{m}$ is not stable, the manifold $\mathcal{M} = \{\bar{m}_\xi, \xi \in \mathbb{R}\}$ is stable.

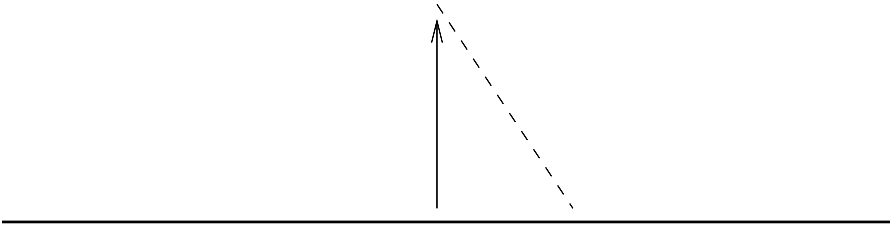

Figure: Thick line is instanton manifold $\mathcal{M}$. Vertical arrow indicates an initial perturbation of an instanton and dashed line its relaxation toward $\mathcal{M}$, in general not to initial instanton.

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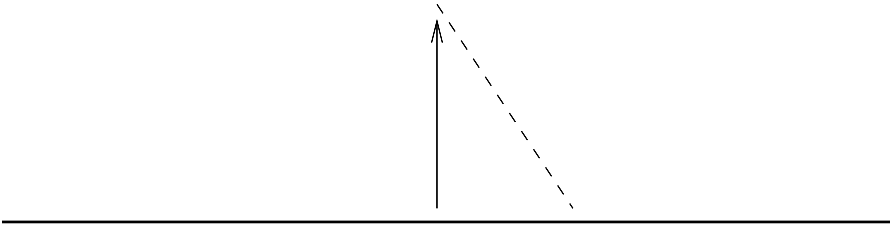

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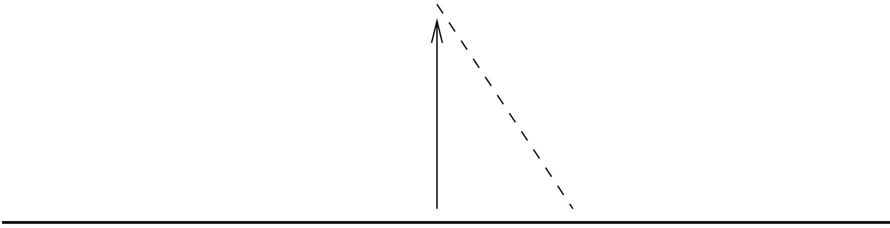

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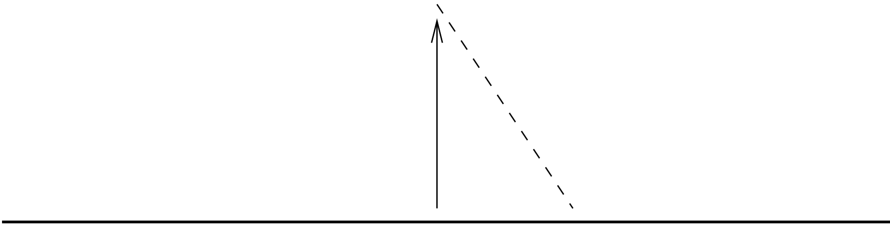

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Ingredients of proof.
Linearized Allen-Cahn evolution around $\bar{m}$:

$$\frac{\partial u}{\partial t} = Au = \frac{\partial^2 u}{\partial x^2} - V''(\bar{m})u$$

A is self-adjoint in $L^2(\mathbb{R}, dx)$.
It has an eigenvalue 0 with eigenvector $\tilde{m}' = \frac{\sqrt{3}}{2}\bar{m}'$.

Proof: Differentiate $\dfrac{d^2\bar{m}}{dx^2} - V'(\bar{m}) = 0$.

Spectrum lies in $\mathbb{R}_-$.

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Theorem. A has a spectral gap namely
0 is a simple eigenvalue and the remaining part of the spectrum lies in $\{\lambda \in \mathbb{R} : \lambda \leq -a,\ a > 0\}$.

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Results (heuristic).

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Analysis in bounded domain: $[-\varepsilon^{-1}, \varepsilon^{-1}]$ with Neumann boundary conditions. $H_t^{(\varepsilon)}(x,y)$ Green function of heat equation.

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